① a)

$K = \mathbb{R}$

$f = x^5 + x^4 - x^3 - 2x - 1$

$g = 3x^4 - 2x^3 + x^2 - 2x - 2$

$h = \text{НОД}(f, g)$

Воспользуемся алгоритмом Евклида:

$$\begin{array}{l|l} x^5 + x^4 - x^3 + 0x^2 - 2x - 1 & 3x^4 - 2x^3 + x^2 - 2x - 2 \\ \hline -\ x^5 - \frac{2}{3}x^4 + \frac{x^3}{3} - \frac{2}{3}x^2 - \frac{2}{3}x + 0 & \frac{1}{3}x + \frac{5}{9} \\ \quad \frac{5}{3}x^4 - \frac{4}{3}x^3 + \frac{2}{3}x^2 - \frac{4}{3}x - 1 & \\ -\ \frac{5}{3}x^4 - \frac{10}{9}x^3 + \frac{5}{9}x^2 - \frac{10}{9}x - \frac{10}{9} & \\ r_1 = -\frac{2}{9}x^3 + \frac{1}{9}x^2 - \frac{2}{9}x + \frac{1}{9} & \end{array}$$

$$f = \left(\frac{1}{3}x + \frac{5}{9}\right) g + r_1 \Rightarrow r_1 = f - \left(\frac{1}{3}x + \frac{5}{9}\right) g$$

$$\begin{array}{l|l} 3x^4 - 2x^3 + x^2 - 2x - 2 & -\frac{2}{9}x^3 + \frac{1}{9}x^2 - \frac{2}{9}x + \frac{1}{9} \\ \hline -\ 3x^4 - \frac{3}{2}x^3 + 3x^2 - \frac{3}{2}x + 0 & -\frac{27}{2}x + \frac{9}{4} \\ \quad -\frac{1}{2}x^3 - 2x^2 - \frac{1}{2}x - 2 & \\ -\ -\frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4} & \\ r_2 = -\frac{9}{4}x^2 + 0x - \frac{9}{4} & \end{array}$$

$$g = \left(-\frac{27}{2}x + \frac{9}{4}\right) r_1 + r_2$$

$$r_2 = g - \left(-\frac{27}{2}x + \frac{9}{4}\right) r_1$$

$$\begin{array}{l|l} -\frac{2}{9}x^3 + \frac{1}{9}x^2 - \frac{2}{9}x + \frac{1}{9} & -\frac{9}{4}x^2 + 0x - \frac{9}{4} \\ \hline -\ -\frac{2}{9}x^3 + 0x^2 - \frac{2}{9}x + 0 & \frac{8}{81}x - \frac{4}{81} \\ \quad \frac{1}{9}x^2 + 0x + \frac{1}{9} & \\ -\ \frac{1}{9}x^2 + 0x + \frac{1}{9} & \\ \quad 0 & \end{array}$$

$$\text{НОД}(f, g) = -\frac{9}{4}x^2 - \frac{9}{4} = r_2$$

$$h = g - \left(-\frac{27}{2}x + \frac{9}{4}\right) r_1 = g + \left(\frac{27}{2}x - \frac{9}{4}\right)\left(f - \left(\frac{1}{3}x + \frac{5}{9}\right) g\right) = \left(\frac{27}{2}x - \frac{9}{4}\right) f +$$

$$+ \left(1 - \left(\frac{27}{2}x - \frac{9}{4}\right)\left(\frac{1}{3}x + \frac{5}{9}\right)\right) g = \left(\frac{27}{2}x - \frac{9}{4}\right) f + \left(-\frac{9x^2}{2} - \frac{27x}{4} + \frac{9}{4}\right) g.$$

$\delta)$ $K = \mathbb{Z}_5$

$f = X^5 + 2X^4 + 4X^3 + 2X^2 + 4$

$g = 3X^3 + 4X^2 + 4X + 1$

$h = \text{НОД}(f, g)$

$$\begin{array}{l|l}
X^5 + 2X^4 + 4X^3 + 2X^2 + 0X + 4 & 3X^3 + 4X^2 + 4X + 1 \\ \cline{2-2}
-\underline{X^5 + 3X^4 + 3X^3 + 2X^2 + 0X + 0} & 2X^2 + 3X + 3 \\
4X^4 + X^3 + 0X^2 + 0X + 4 & \\
-\underline{4X^4 + 2X^3 + 2X^2 + 3X + 0} & \\
4X^3 + 3X^2 + 2X + 4 & \\
-\underline{4X^3 + 2X^2 + 2X + 3} & \\
r_1 = X^2 + 0X + 1 &
\end{array}$$

$f = (2X^2 + 3X + 3)g + r_1 \Rightarrow r_1 = f - (2X^2 + 3X + 3)g$

$$\begin{array}{l|l}
3X^3 + 4X^2 + 4X + 1 & X^2 + 0X + 1 \\ \cline{2-2}
-\underline{3X^3 + 0X + 3X + 0} & 3X + 4 \\
4X^2 + X + 1 & \\
-\underline{4X^2 + 0X + 4} & \\
r_2 X + 2 &
\end{array}$$

$g = (3X + 4)r_1 + r_2 \Rightarrow r_2 = g - (3X + 4)r_1$

$$\begin{array}{l|l}
X^2 + 0X + 1 & X + 2 \\ \cline{2-2}
-\underline{X^2 + 2X + 0} & X + 3 \\
3X + 1 & \\
-\underline{3X + 1} & \\
0 &
\end{array}$$

$\text{НОД}(f, g) = X + 2 = r_2$

$h = g - (3X+4)r_1 = g - (3X+4)(f - (2X^2 + 3X + 3)g) = (-3X-4)f + (\cancel{1 + 6X^3}$

$+ (1 + (3X+4)(2X^2 + 3X + 3))g = (-3X-4)f + (1 + X^3 + 3X^2 + 4X^2 + 2X + 4X + 2)g =$

$= (-3X-4)f + (X^3 + 2X^2 + X + 3)g = (2X + 1)f + (X^3 + 2X^2 + X + 3)g$

$$
\begin{array}{l|l}
x^4 + 6x^3 + x^2 + 3x + 1 & x+2 \\
\underline{-\,x^4 + 2x^3} & \overline{x^3 + 3x^2 + 3} \\
\quad 3x^3 + x^2 + 3x + 1 & \\
\quad \underline{-\,3x^3 + x^2} & \\
\qquad 3x + 1 & \\
\qquad \underline{-\,3x + 1} & \\
\qquad\quad 0 &
\end{array}
$$

$$f = (x^3 + 3x^2 + 3)(x+2)(x+3)$$

x	0	1	2	3	4
f'	3	2	3	4	0

$$f = s(x+2)(x+3)(x+1)$$

$$
\begin{array}{l|l}
x^3 + 3x^2 + 0x + 3 & x+1 \\
\underline{-\,x^3 + x^2} & \overline{x^2 + 2x + 3} \\
\quad 2x^2 + 0x + 3 & \\
\quad \underline{-\,2x^2 + 2x + 6} & \\
\qquad 3x + 3 & \\
\qquad \underline{-\,3x + 3} & \\
\qquad\quad 0 &
\end{array}
$$

$$f = (x^2 + 2x + 3)(x+1)(x+2)(x+3)$$

x	0	1	2	3	4
f'	3	1	4	2	2

$\Rightarrow f$ неприводим

③ $F = \mathbb{Q}[z]/(z^3-z^2-1)$

F поле $\Leftrightarrow$ (z^3-z^2-1) неприводим над $\mathbb{Q}$

$g = z^3-z^2-1$

Докажем, что g неприводим:

Пусть $t = \frac{p}{q}$ - корень $\Rightarrow \begin{cases} -1 \vdots p \\ 1 \vdots q \end{cases} \Rightarrow \left[\begin{array}{l} t=1,\ g(1)=-3 \\ t=-1,\ g(-1)=-1 \end{array}\right. \Rightarrow$

$\Rightarrow g$ неприводим $\Rightarrow F$ поле.

$\alpha = \bar{z} + (z^3-z^2-1)$

$$\frac{3\alpha^2-12\alpha+7}{\alpha^2-3\alpha+1} = A_0 + A_1\alpha + A_2\alpha^2$$

$3\alpha^2-12\alpha+7 = (A_0+A_1\alpha+A_2\alpha^2)(\alpha^2-3\alpha+1) = A_0\alpha^2 - 3A_0\alpha + A_0 + A_1\alpha^3 -$
$-3A_1\alpha^2 + A_1\alpha + A_2\alpha^4 - 3A_2\alpha^3 + A_2\alpha^2$ ⊜

$\alpha^3 - \alpha^2 - 1 = 0$

$\alpha^3 = \alpha^2+1$

$\alpha^4 = \alpha^3+\alpha = \alpha^2+\alpha+1$

⊜ $A_0\alpha^2 - 3A_0\alpha + A_0 + A_1\alpha^2 + A_1 - 3A_1\alpha^2 + A_1\alpha + A_2\alpha^2 + A_2\alpha + A_2 - 3A_2\alpha^2 - 3A_2 +$
$+ A_2\alpha^2 = (A_0 + A_1 - 3A_1 + A_2 - 3A_2 + A_2)\alpha^2 + (-3A_0 + A_1 + A_2)\alpha +$
$+ (A_0 + A_1 + A_2 - 3A_2)$

$\begin{cases} 3 = A_0 + A_1 - 3A_1 + A_2 - 3A_2 + A_2 \\ -12 = (-3A_0 + A_1 + A_2) \\ 7 = A_0 + A_1 - 2A_2 \end{cases} \Rightarrow \begin{cases} 3 = A_0 - 2A_1 - A_2 \\ \cancel{-12 = -3A_0 + A_1 - A_2} \\ 7 = A_0 - A_1 + 2A_2 \end{cases}$ $A_0 = 2A_1 + A_2 + 3$ $\Rightarrow \cancel{2A_1 + A_2} + 3 - A_1 + 2A_2 = 7$

$A_1 + 3A_2 = 4$

$\Rightarrow A_1 = 4 - 3A_2$ $\Rightarrow$ $-12 = -6A_1 - 3A_2 - 9 + 4 - 3A_2 - A_2$

$13A_2 = 7$

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ -3 & 1 & 1 & -12 \\ 1 & 1 & -2 & 7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -5 & -2 & -3 \\ 0 & 0 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 0 & -5 & 0 & -5 \\ 0 & 0 & 1 & -1 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Итоговый вид: $-\alpha^2+\alpha+4$

4) $K[x]/(h)$

$\bar{f} = f+(h)$ $(f \in k[x])$ необратим $\Leftrightarrow$ НОД$(f,h) \neq 1$

$\bar{f} \in K[x]/(h) \Rightarrow f$ также необратим и $f \in (K[x]/(h))$ (по свойству замкнутости фактор кольца)

Домножим f на $\frac{h}{\text{НОД}(f,h)}$,

$\frac{h}{\text{НОД}(f,h)} \neq 0$ ~~(f, н~~

$\frac{fh}{\text{НОД}(f,h)} = \frac{hf}{\text{НОД}(f,h)} = 0 \Rightarrow$ всякий ненулевой необратимый эл-т фактокольца $K[x]/(h)$ является делителем нуля (h эквивалентно 0 в этом фактокольце)