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As there is an internal angle along C_1 for $Z = \pm\frac{1}{2}h$, where infinite stress may be expected, a 'thick-plate' solution is insufficient. The next approximation seems to need some information on the stress distribution in the Z direction from a three-dimensional solution to a problem involving a change of thickness.

In conclusion, I wish to thank Professor L. M. Milne-Thomson for his interest in and criticism of the work.

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Some instabilities of a completely ionized plasma

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Two cases of equilibrium for a highly conducting plasma are investigated for their stability. In the first case, a plasma is supported against gravity by the pressure of a horizontal magnetic field. This equilibrium is found unstable, in close correspondence to the classical case of a heavy fluid supported by a light one. The second case refers to the so-called pinch effect. Here a plasma is kept within a cylinder by the pressure of a toroidal magnetic field which in turn is caused by an electric current within the plasma. This equilibrium is found unstable against lateral distortions.

1. Introduction

In classical hydrodynamics the problem of stability of fluid motions has been solved explicitly for a number of basic cases. Recently, Chandrasekhar (1952, 1953) has investigated and solved several of these basic problems in their hydromagnetic formulations in which electromagnetic fields are introduced and in which the fluid in question is considered electrically highly conductive. In the present paper two more cases of hydromagnetic instability are investigated.

The first case (§ 3) is that of an infinitely conducting plasma at uniform temperature lying above a horizontal plane in a uniform gravitational field directed vertically downwards. There is a horizontal magnetic field uniform in each half-volume with a jump in field strength produced by a uniform horizontal sheet current in the boundary plane. The gravitational force is balanced by a pressure gradient in the plasma and by the jump in magnetic pressure at the plane. This case is somewhat analogous to the familiar unstable equilibrium of a dense fluid supported against gravity by a lighter one (see, for instance, § 231 of Lamb 1932).

The second case (§ 4) is that of an infinitely conducting uniform plasma lying within an infinitely long circular cylinder. There is a uniform sheet current on the cylinder parallel to the axis, which produces a toroidal magnetic field outside the cylinder. There is no gravitational field. The plasma pressure is balanced by the magnetic pressure. This case is an idealization of the well-known pinch effect.

Finally, in § 5, it is investigated how far the approximation of infinite conductivity with the simultaneous introduction of sheet currents and sheet charges on the surface of the plasma is an appropriate representation of the actual cases with large but finite conductivity. This question is studied in a sample case similar to that described in § 4 but of a simpler geometry.

The results are summarized in § 6.

2. Basic equations and approximations

Let p, ρ and $\mathbf{v}$ be the pressure, density and velocity of the plasma respectively. Let $\mathbf{E}$, $\mathbf{B}$, $\mathbf{j}$ and ϵ be the electric field, the magnetic field, the current density, and the electric charge density respectively. Let $\mathbf{g}$ be the acceleration due to gravity, σ the conductivity of the plasma, μ_0 the permeability of free space, κ_0 the permittivity of free space, and γ the ratio of specific heats of the plasma. We shall use the following equations for the interior of the plasma:

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{j} \times \mathbf{B} + \epsilon \mathbf{E} - \nabla p + \rho \mathbf{g}, \tag{1}$$

$$\nabla \cdot (\rho \mathbf{v}) = -\frac{\partial \rho}{\partial t}, \tag{2}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{\sigma}(\mathbf{j} - \mathbf{v}\epsilon), \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \kappa_0 \frac{\partial \mathbf{E}}{\partial t} \right), \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{6}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\kappa_0} \epsilon, \tag{7}$$

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\gamma}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}. \tag{8}$$

These equations should apply to a plasma under the following conditions. First, encounters between particles are sufficiently frequent to permit the representation of the stress tensor by a scalar pressure. Secondly, the perturbations are sufficiently slow and the electron density sufficiently high so that the extra terms which strictly should be added to equation (3) are negligible. Thirdly, heat losses by conduction and gains by Joule heating are negligible so that the simple adiabatic equation (8) holds.

In the vacuum we have $p = 0$ and $\rho = 0$, and equations (4), (5), (6) and (7) hold with $\mathbf{j} = 0$ and $\epsilon = 0$.

We next derive the equations that hold at a surface separating the plasma from the vacuum. At such a surface we allow a sheet current $\mathbf{j}^*$ and a sheet charge ϵ^*, as well as jump discontinuities in the other physical quantities. Let the unit normal to the surface be $\mathbf{n}$ (directed into the plasma), and let the surface have a normal velocity $u\mathbf{n}$. Since the surface moves with the plasma we must have

$$u = \mathbf{n} \cdot \mathbf{v}, \quad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t} = \mathbf{n} \times (\mathbf{n} \times \nabla u). \tag{9}$$

The separating surface is only an approximate representation of a thin layer of plasma of width δ in which $\mathbf{j}$ and ϵ are finite but large of order δ^{-1} and across which the other physical quantities vary continuously. The integrals of $\mathbf{j}$ and ϵ across this layer are given by $\mathbf{j}^*$ and ϵ^* respectively. The integral of any one of the other quantities across the layer is of order δ and therefore vanishes with δ. If q or $\mathbf{q}$ represents any of these other quantities, then the integral of ∇q, $\nabla \cdot \mathbf{q}$, or $\nabla \times \mathbf{q}$ is given, except for terms which vanish with δ, by $\mathbf{n}[q]$, $\mathbf{n} \cdot [\mathbf{q}]$, or $\mathbf{n} \times [\mathbf{q}]$ respectively, where the brackets (in this section only) denote the difference of the quantity within them from one side of the layer to the other, or in other words, the jump of the quantity across the separating surface. Finally, we must observe that the main contribution to $\partial q/\partial t$ or $\partial \mathbf{q}/\partial t$ within the layer comes from the motion of the layer combined with the gradient of q or $\mathbf{q}$ across the layer, and is given by $-u\mathbf{n} \cdot \nabla q$ or $-u(\mathbf{n} \cdot \nabla)\,\mathbf{q}$ respectively.

Our procedure is now to integrate each of the plasma equations (1) to (8) across the layer and then to go to the limit as $\delta \to 0$. Equation (2) yields $\mathbf{n} \cdot [\rho \mathbf{v}] = u[\rho]$, which follows anyway from equation (9), since $\rho = 0$ in the vacuum. Equation (8) yields $0 = 0$. Equations (4), (5), (6) and (7) yield respectively

$$\mathbf{n} \times [\mathbf{B}] = \mu_0(\mathbf{j}^* - \kappa_0 u[\mathbf{E}]), \tag{10}$$

$$\mathbf{n} \cdot [\mathbf{B}] = 0, \tag{11}$$

$$\mathbf{n} \times [\mathbf{E}] = u[\mathbf{B}], \tag{12}$$

$$\mathbf{n} \cdot [\mathbf{E}] = \frac{1}{\kappa_0}\epsilon^*. \tag{13}$$

Equation (3) yields

$$\frac{1}{\sigma}(\mathbf{j}^* - \mathbf{v}\epsilon^*) = 0, \tag{14}$$

unless $\sigma = \infty$, in which case it yields $0 = 0$. Finally, equation (1) yields

$$\mathbf{j}^* \times \overline{\mathbf{B}} + \epsilon^* \overline{\mathbf{E}} - \mathbf{n}[p] = 0, \tag{15}$$

where $\overline{\mathbf{B}}$ and $\overline{\mathbf{E}}$ denote appropriate averages of the values of $\mathbf{B}$ and $\mathbf{E}$ respectively in the layer; the correct treatment of equation (1), in which $\mathbf{j}$ and ϵ are eliminated by equations (4) and (7) before the integration across the layer, shows that the proper average to take is in each case the ordinary arithmetic mean of the values on the two sides of the separating surface.

3. Instability of a plasma supported against gravity by a magnetic field

In this case we take $\sigma = \infty$, $g_x = g_z = 0$, $g_y = -g_0$. In the equilibrium let the plasma lie above the plane $y = 0$, and in it let $p = p_0 \exp(-hy)$, $\rho = \rho_0 \exp(-hy)$, $\mathbf{v} = 0$, $B_x = B_y = 0$, $B_z = B_0^P$, $\mathbf{E} = 0$, $\mathbf{j} = 0$, $\epsilon = 0$. In the vacuum let $B_x = B_y = 0$, $B_z = B_0^V$, $\mathbf{E} = 0$. On the surface $y = 0$ let $j_x^* = j_0^*$, $j_y^* = j_z^* = 0$, $\epsilon^* = 0$. Then the plasma equations (1) to (8), the vacuum equations, and the surface equations (9) to (15) are fulfilled if the constants g_0, p_0, ρ_0, h, B_0^P, B_0^V and j_0^* satisfy

$$h = g_0\rho_0/p_0, \quad B_0^P - B_0^V = \mu_0 j_0^*, \quad j_0^*(B_0^P + B_0^V) + 2p_0 = 0. \tag{16}$$

We now write a tilde over the symbol for a physical quantity to indicate its perturbation, i.e. the difference between that quantity and its value in the equilibrium solution just given; thus $\tilde{p} = p - p_0 \exp(-hy)$, $\tilde{\mathbf{v}} = \mathbf{v}$, etc. We then seek a solution near the equilibrium solution by linearizing all our equations in terms of the perturbations. The resulting homogeneous linear equations have constant coefficients except for plasma equations (1), (2) and (8), which have coefficients proportional to $\exp(-hy)$. If we restrict ourselves to investigating phenomena which in the plasma differ from the equilibrium solution appreciably only near the surface, i.e. for which the perturbations approach zero much faster than $\exp(-hy)$ as y gets large, then we may to a good approximation replace the factor $\exp(-hy)$ in the coefficients by unity.

Every solution of the resulting system of homogeneous linear differential and algebraic equations may be obtained by superposition of elementary solutions, an elementary solution being one in which each perturbation in the plasma is proportional to $\exp(\mathrm{i}k_x x - k_y^P y + \mathrm{i}k_z z + \omega t)$, each perturbation in the vacuum to $\exp(\mathrm{i}k_x x + k_y^V y + \mathrm{i}k_z z + \omega t)$, and at the surface to $\exp(\mathrm{i}k_x x + \mathrm{i}k_z z + \omega t)$. For the solution to make physical sense we require k_x and k_z to be real and k_y^P and k_y^V to have non-negative real parts, and in fact we require $\mathscr{R}[k_y^P] \gg h$ to accord with the restriction made in the previous paragraph.

For each physical quantity we denote the constant amplitude factor by the same symbol except for replacing the tilde by a circumflex. The system of differential equations for the perturbations becomes a purely algebraic system of linear equations for the amplitude factors. Introducing the dimensionless quantities

$$\left.\begin{aligned} &\beta = \mu_0 p_0/B_0^{P2}, \quad G = \beta h/k_x, \quad 1/\Gamma = 1/\gamma + \beta + p_0 k_z^2/\rho_0\omega^2, \\ &P = k_y^P/k_x, \quad V = k_y^V/k_x, \quad Z = (k_z^2 + \mu_0\kappa_0\omega^2)/k_x^2, \\ &T = (\mu_0\rho_0)^{\frac{1}{2}}\,\omega/B_0^P k_x, \end{aligned}\right\} \tag{17}$$

the conditions that the system of homogeneous equations for the amplitude factors have a non-trivial solution are found by successive elimination to be

$$\left.\begin{aligned} &V^2 = 1 + Z, \\ &T^2 + Z = PG - (1+2\beta)(\gamma P - \Gamma G) Z/\gamma V \\ &\qquad + [\gamma - 1 + (\beta + 1/\gamma)\Gamma] G^2/\gamma\beta, \\ &(T^2 + Z)\{P - 1/P - [1 - (\beta + 1/\gamma)\Gamma] G + (1+2\beta)(1-\beta\Gamma) Z/V\} \\ &\qquad = [\gamma - 1 + (\beta + 1/\gamma)\Gamma] G^2/\gamma\beta P - (1+2\beta)\Gamma G Z/\gamma P V. \end{aligned}\right\} \tag{18}$$

We thus have three conditions on the five characteristic parameters k_x, k_y^P, k_y^V, k_z, ω, so we may regard k_x and k_z as given (i.e. the wave-lengths of the perturbation in the two horizontal directions as given) and the others (i.e. the extent of the perturbation upwards into the plasma and downwards into the vacuum and the characteristic time of the perturbation) to be determined. Since $\mu_0\kappa_0$ is the reciprocal of the square of the velocity of light, we have, very nearly, $Z = k_z^2/k_x^2$, so that in equations (18) we may consider G and Z as known and P, V and T to be determined; in terms of these variables

$$1/\Gamma = 1/\gamma + \beta - \mu_0\kappa_0 p_0/\rho_0 + \beta Z/T^2.$$

We now assume that $|G| \ll 1$ and $|Z| \ll 1$ and seek the limiting forms of the solutions of equations (18); we obtain a solution which to lowest order is

$$|P| = 1, \quad |V| = 1, \quad T^2 = Gk_x/|k_x| - 2(1+\beta)Z. \tag{19}$$

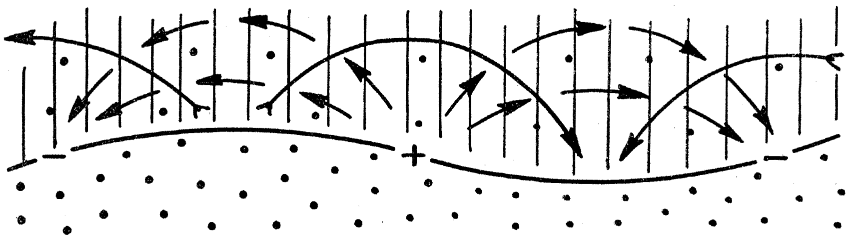

FIGURE 1. Instability of plasma supported against gravity by a magnetic field. ||| plasma, ∴ magnetic field; +, − electric charge; →→ electric field; ⤚→ motion of plasma.

Furthermore, it seems that for any other limiting solution of equations (18) T^2 becomes real and negative as G and Z approach zero, so that solution (19) represents the only possibility for instability. The condition that instability occur in this solution in the approximation of infinite light velocity is $T^2 > 0$, or

$$2(1+1/\beta)\, k_z^2 < h\,|k_x|. \tag{20}$$

Hence instability occurs if the wave-length of the perturbation along the magnetic field (z-direction) is long compared to the geometric mean of the scale height in the plasma and the wave-length across the magnetic field (x-direction).

The amplitude factors are given in table 1 for two limiting cases, one of instability when $Z = 0$, the other of stability when $G = 0$. Each amplitude factor is given to lowest order in G and in Z in the respective cases. However, terms containing the velocity of light in the denominator are omitted, except in $\hat{\mathbf{j}}$ and $\hat{\mathbf{j}}^*$, where they are retained so that the equation $\nabla \cdot \mathbf{j} = -\partial\epsilon/\partial t$ (derivable from equations (4) and (7)) can be seen to be satisfied at the boundary surface.

Finally, we get from equation (19) for the limiting case of $Z = 0$ (no dependence on z)

$$\omega^2 = g_0\,|k_x|, \tag{21}$$

and for the limiting case of $G = 0$ (no gravity)

$$\omega^2 = -k_z^2(B_0^{P^2} + B_0^{V^2})/\mu_0\rho_0, \tag{22}$$

while in both cases

$$k_y^P = k_y^V = |k_x|. \tag{23}$$

The perturbation corresponding to the first limiting case ($Z = 0$) is shown in figure 1. We may describe this perturbation approximately as follows. Electric charges on the boundary of the plasma produce electric fields. These electric fields are perpendicular to the magnetic field and hence cause the plasma to move. These motions are essentially divergence-free so that density and pressure do not vary if one follows an element of matter. The motions carry along the magnetic lines, but neither bend nor compress them, so that the magnetic field, both in the plasma and in the vacuum, stays constant. The accelerations of the plasma and the pressure gradients together produce small electric currents which tend to increase the charges on the plasma boundary, thus making the perturbation unstable. Even though in this perturbation the electromagnetic field produces the plasma motions, the speed of the instability as given by equation (21) is exactly the same as that of the well-known purely hydrodynamic case of a heavy fluid supported against gravity by a lighter one.

TABLE 1

	unstable solution $Z = 0$	stable solution $G = 0$
	plasma	
p	$p_0 h/k_x$	$-\gamma p_0(1+2\beta)\,k_z/(1+\gamma\beta)\,k_x$
ρ	$\rho_0 h/k_x$	$-\rho_0(1+2\beta)\,k_z/(1+\gamma\beta)\,k_x$
v_x	$-\mathrm{i}\omega/k_x$	$-\mathrm{i}\omega/k_z$
v_y	ω/k_x	ω/k_z
v_z	0	0
B_x	0	B_0^P
B_y	0	$\mathrm{i}B_0^P$
B_z	0	$-B_0^P(1+2\beta)\,k_z/(1+\gamma\beta)\,k_x$
E_x	$-B_0^P\,\omega/k_x$	$-B_0^P\,\omega/k_z$
E_y	$-\mathrm{i}B_0^P\,\omega/k_x$	$-\mathrm{i}B_0^P\,\omega/k_z$
E_z	0	0
j_x	$[1+(1+2\beta)/(1+\gamma\beta)]\,\kappa_0 g_0 B_0^P k_x/\lvert k_x\rvert$	$[1+(1+2\beta)/(1+\gamma\beta)]\,B_0^P k_z/\mu_0$
j_y	$\mathrm{i}[1+(1+2\beta)/(1+\gamma\beta)]\,\kappa_0 g_0 B_0^P$	$\mathrm{i}[1+(1+2\beta)/(1+\gamma\beta)]\,B_0^P k_z/\mu_0$
j_z	0	0
ϵ	0	0
	vacuum	
B_x	0	$-B_0^V$
B_y	0	$\mathrm{i}B_0^V$
B_z	0	$-B_0^V k_z/k_x$
E_x	$-B_0^V\,\omega/k_x$	$-B_0^V\,\omega/k_z$
E_y	$\mathrm{i}B_0^V\,\omega/k_x$	$\mathrm{i}B_0^V\,\omega/k_z$
E_z	0	0
	boundary surface	
j_x^*	$\kappa_0 g_0[B_0^V - B_0^P(1+2\beta)/(1+\gamma\beta)]/k_x$	$[B_0^V - B_0^P(1+2\beta)/(1+\gamma\beta)]\,[1-2(1+1/\beta)$ $\times\,\mu_0\kappa_0 p_0/\rho_0]\,k_z/\mu_0 k_x$
j_y^*	$\mathrm{i}(B_0^P - B_0^V)/\mu_0$	$\mathrm{i}(B_0^P - B_0^V)\,k_x/\mu_0 k_z$
j_z^*	0	$-(B_0^P + B_0^V)/\mu_0$
ϵ^*	$-\mathrm{i}\kappa_0(B_0^P + B_0^V)\,\omega/k_x$	$-\mathrm{i}\kappa_0(B_0^P + B_0^V)\,\omega/k_z$

The second limiting case ($G = 0$) represents a hydromagnetic surface wave which travels on the boundary of the plasma along the magnetic lines with the characteristic speed given by equation (22).

The discriminating condition (20) between the stable and the unstable cases may be described as follows. If the wave-length along the magnetic lines is sufficiently short, the restoring force of the magnetic field which resists the bending of the magnetic lines will prevent their sagging. On the other hand, if the wave-length of the perturbation along the magnetic lines is too long, the perturbation will bend the lines only a little and the magnetic restoring force will be too small to counteract gravity, so that the plasma will drop downwards.

4. Lateral instability of plasma cylinder in toroidal magnetic field

In this case we take $\sigma = \infty$, $\mathbf{g} = 0$, and use cylindrical co-ordinates r, θ, z. In the equilibrium let the plasma lie within the cylinder $r = r_0$, and in it let $p = p_0$, $\rho = \rho_0$, and $\mathbf{v}$, $\mathbf{B}$, $\mathbf{E}$, $\mathbf{j}$ and ϵ vanish. In the vacuum let $B_r = B_z = 0$, $B_\theta = B_0 r_0/r$, $\mathbf{E} = 0$. On the surface $r = r_0$ let $j_r^* = j_\theta^* = 0$, $j_z^* = j_0^*$, $\epsilon^* = 0$. Here r_0, p_0, ρ_0, B_0 and j_0^* are constants, and we require

$$B_0 = \mu_0 j_0^*, \quad j_0^* B_0 = 2p_0, \tag{24}$$

so that the equations of § 2 be satisfied.

We now investigate solutions in the neighbourhood of this equilibrium solution as before by indicating the perturbations from the equilibrium by a tilde. In the equations of § 2 we retain only the first powers of these perturbations and obtain a system of homogeneous differential and algebraic equations with coefficients depending only on r. Accordingly, the elementary solutions of this system have the form $\tilde{q} = \hat{q}\exp(\mathrm{i}m\theta + \mathrm{i}kz + \omega t)$, where q is any physical quantity, $\hat{q}$ is a function of r (unless q is a surface quantity), m is an integer, and k is real. We restrict ourselves to the case $m = 1$.

Substitution gives a system of ordinary differential equations for the amplitude factors which reduce, after eliminations, to

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\hat{p}\right) - [(k^2 + \rho_0\omega^2/\gamma p_0)\, r^2 + 1]\,\hat{p} = 0 \tag{25}$$

in the plasma, and to

$$\left.\begin{aligned} r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\hat{B}_z\right) - [(k^2 + \mu_0\kappa_0\omega^2)\, r^2 + 1]\,\hat{B}_z = 0, \\ r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\hat{E}_z\right) - [(k^2 + \mu_0\kappa_0\omega^2)\, r^2 + 1]\,\hat{E}_z = 0 \end{aligned}\right\} \tag{26}$$

in the vacuum. We introduce the functions

$$J(x) = -\mathrm{i}J_1(\mathrm{i}x), \quad H(x) = -H_1^{(1)}(\mathrm{i}x), \tag{27}$$

where J_1 and $H_1^{(1)}$ are the Bessel and Hankel functions of the first kind and of first order. We further introduce the constants

$$\zeta^2 = k^2 + \rho_0\omega^2/\gamma p_0, \quad \eta^2 = k^2 + \mu_0\kappa_0\omega^2. \tag{28}$$

Assuming that ω^2 is real and positive and taking $\zeta \geqslant 0$, $\eta \geqslant 0$, we see from equations (25) and (26) that

$$\hat{p} = p^1 J(\zeta r), \quad \hat{B}_z = B_z^1 H(\eta r), \quad \hat{E}_z = E_z^1 H(\eta r), \tag{29}$$

where p^1, B_z^1, E_z^1 are constants. The other independent solutions of equations (25) and (26) are excluded because they become infinite at $r = 0$ and $r = \infty$ respectively. The surface equations now give three independent linear homogeneous relations among p^1, B_z^1 and E_z^1, and the condition that these have a non-trivial solution is

$$\frac{\rho_0 r_0^2 \eta \omega^2}{2p_0 \zeta} \frac{J(\zeta r_0)}{J'(\zeta r_0)} = \eta r_0 + \frac{k^2}{\eta^2} \frac{H(\eta r_0)}{H'(\eta r_0)} + \mu_0 \kappa_0 r_0^2 \omega^2 \frac{H'(\eta r_0)}{H(\eta r_0)}, \tag{30}$$

TABLE 2

	exact solution in terms of	approximate solution for
	$B_z^1 = \dfrac{p_0 B_0 k \zeta}{\rho_0 r_0 \omega^2 \eta} \dfrac{J'(\zeta r_0)}{H'(\eta r_0)}$	$\mu_0 \kappa_0 \approx 0$ and $\lvert k \rvert r_0 \ll 1$
	$E_z^1 = \dfrac{p_0 B_0 \zeta}{\rho_0 \omega} \dfrac{J'(\zeta r_0)}{H(\eta r_0)}$	
	plasma	
p	$p_0 J(\zeta r)$	$\frac{1}{2} \lvert k \rvert p_0 [1 + 2L/\gamma]^{\frac{1}{2}} r$
ρ	$\rho_0 J(\zeta r)/\gamma$	$\frac{1}{2} \lvert k \rvert \rho_0 [1 + 2L/\gamma]^{\frac{1}{2}} r/\gamma$
v_r	$-p_0 \zeta J'(\zeta r)/\rho_0 \omega$	$-\frac{1}{2}[p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}}$
v_θ	$-\mathrm{i} p_0 J(\zeta r)/\rho_0 \omega r$	$-\frac{1}{2}\mathrm{i}[p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}}$
v_z	$-\mathrm{i} k p_0 J(\zeta r)/\rho_0 \omega$	$-\frac{1}{2}\mathrm{i}k[p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}} r$
	vacuum	
B_r	$-\mathrm{i}\mu_0 \kappa_0 \omega E_z^1 H(\eta r)/\eta^2 r - \mathrm{i}k B_z^1 H'(\eta r)/\eta$	$-\frac{1}{4}\mathrm{i} B_0 r_0 [1 + 2L/\gamma]^{\frac{1}{2}}/\lvert k \rvert L r^2$
B_θ	$k B_z^1 H(\eta r)/\eta^2 r + \mu_0 \kappa_0 \omega E_z^1 H'(\eta r)/\eta$	$-\frac{1}{4} B_0 r_0 [1 + 2L/\gamma]^{\frac{1}{2}}/\lvert k \rvert L r^2$
B_z	$B_z^1 H(\eta r)$	$-\frac{1}{4} k B_0 r_0 [1 + 2L/\gamma]^{\frac{1}{2}}/\lvert k \rvert L r$
E_r	$\mathrm{i}\omega B_z^1 H(\eta r)/\eta^2 r - \mathrm{i}k E_z^1 H'(\eta r)/\eta$	$\frac{1}{2}\mathrm{i}k B_0 r_0 [p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}} \times [L r_0^2/r^2 + L + \ln(r_0/r)]$
E_θ	$k E_z^1 H(\eta r)/\eta^2 r - \omega B_z^1 H'(\eta r)/\eta$	$\frac{1}{2} k B_0 r_0 [p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}} \times [L r_0^2/r^2 - L - \ln(r_0/r)]$
E_z	$E_z^1 H(\eta r)$	$\frac{1}{2} B_0 r_0 [p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}}/r$
	boundary surface	
j_r^*	$-\mathrm{i}k p_0 B_0 \zeta J'(\zeta r_0)/\mu_0 \rho_0 \omega^2$	$-\frac{1}{4}\mathrm{i}k j_0^* [1 + 2L/\gamma]^{\frac{1}{2}}/\lvert k \rvert L$
j_θ^*	$-B_z^1 H(\eta r_0)/\mu_0$	$\frac{1}{4} k j_0^* [1 + 2L/\gamma]^{\frac{1}{2}}/\lvert k \rvert L$
j_z^*	$k B_z^1 H(\eta r_0)/\mu_0 \eta^2 r_0 + \kappa_0 \omega E_z^1 H'(\eta r_0)/\eta + p_0 B_0 \zeta J'(\zeta r_0)/\mu_0 \rho_0 r_0 \omega^2$	$\frac{1}{4} \lvert k \rvert j_0^* r_0 [1 + 2L/\gamma]^{\frac{1}{2}}$
ϵ^*	$\mathrm{i}\kappa_0 \omega B_z^1 H(\eta r_0)/\eta^2 r_0 - \mathrm{i}k \kappa_0 E_z^1 H'(\eta r_0)/\eta$	$\frac{1}{2}\mathrm{i}k \kappa_0 B_0 r_0 [p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}} \times [L r_0^2/r^2 + L + \ln(r_0/r)]$

where J' and H' are the derivatives of J and H. If we normalize by taking $p^1 = p_0$, then B_z^1 and E_z^1 are determined by these relations. Their values are given in table 2 at the head of a column. The remainder of this column contains the exact solution expressed in terms of B_z^1 and E_z^1; $\hat{\mathbf{B}}$, $\hat{\mathbf{E}}$, $\hat{\mathbf{j}}$ and $\hat{\epsilon}$ all vanish in the plasma and are therefore not listed in the table.

We now neglect terms which have the light velocity in the denominator, such as the last term of the second equation (28) and the last term of equation (30). Thus, equations (28) give $\eta = \lvert k \rvert$ and equation (30) gives

$$\frac{\rho_0 r_0^2 \lvert k \rvert \omega^2}{2p_0 \zeta} \frac{J(\zeta r_0)}{J'(\zeta r_0)} = \lvert k \rvert r_0 + \frac{H(\lvert k \rvert r_0)}{H'(\lvert k \rvert r_0)}. \tag{31}$$

It can be proved that, for any fixed value of $k \neq 0$, the left-hand side of this equation varies monotonically from 0 to ∞ as ω^2 goes from 0 to ∞ through real values; since the right-hand side can be shown to be positive, it follows that equation (31) has a unique real positive solution ω^2 for each value of k. Thus the equilibrium is unstable. This instability is in agreement with the earlier results of Lundquist (1951).

In the limiting case in which the wave-length of the perturbation along the cylinder is long compared to the cylinder radius, i.e. in which $|k|r_0 \ll 1$, equation (31) gives for the main term in ω (taken positive)

$$\omega \approx |k|(2Lp_0/\rho_0)^{\frac{1}{2}} \quad \text{with} \quad L = \ln\frac{2}{|k|r_0} - C + 1, \tag{32}$$

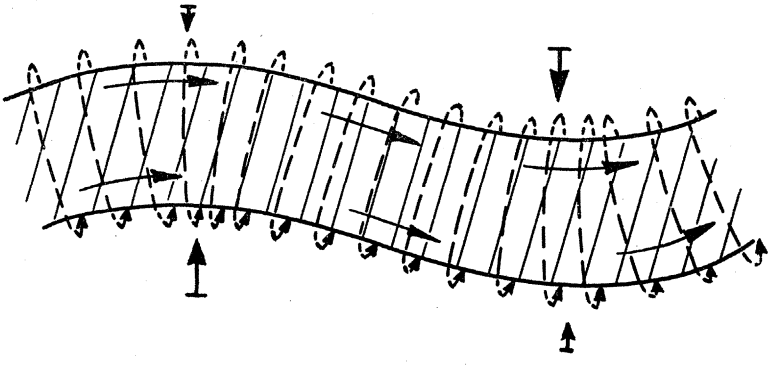

FIGURE 2. Lateral instability of plasma cylinder in toroidal magnetic field. /// plasma; → electric current; magnetic field; magnetic force.

where $C \simeq 0{\cdot}5772$ is Euler's constant. The amplitude factors for this limiting case are given in the last column of table 2. In the opposite limiting case of large $|k|r_0$, equation (31) gives

$$\omega \approx (2|k|p_0/\rho_0 r_0)^{\frac{1}{2}}. \tag{33}$$

Equations (32) and (33) show that these unstable perturbations e-fold in about the time that it takes a sound wave to travel a distance equal to the wave-length of the perturbation or equal to the geometric mean of the wave-length and the cylinder radius, whichever is the larger.

The character of this unstable perturbation of a plasma cylinder is shown in figure 2. (This figure actually shows a lateral sinusoidal perturbation obtained by superposing two oppositely twisted helical perturbations, i.e. two perturbations as given in table 2 with k's of opposite sign.) The cause of the instability can be described as follows. When the plasma cylinder is distorted into the wavy form shown in figure 2, the toroidal magnetic lines are separated from each other at the convex side of the cylinder and pressed together at the concave side. In consequence, the magnetic pressure which holds the cylinder together is put out of balance, being weakened on the convex side and strengthened on the concave side. The net result is that the total magnetic force tends to increase the perturbation and thus leads to instability.

Another problem, similar to the one detailed above, was computed through in which the plasma instead of being confined by a cylindrical surface was confined by two parallel planes. In the equilibrium configuration, the plasma was again kept in its boundary by the pressure of a magnetic field, which was parallel to the planes and larger outside the plasma than inside. In contrast with the foregoing cylindrical case, it was found that this plane case is neutral to perturbations in which the two confining surfaces are distorted into sinusoidal wave-forms. Furthermore, for this simple case, it was easy to compute through the perturbations for two different assumptions regarding the current distribution in the equilibrium configuration. In the first version it was assumed that the equilibrium current was entirely confined to the surface of the plasma—similar to the assumption in the cylindrical case —and in the second version it was assumed that the equilibrium current was evenly distributed throughout the plasma between the two planes. In both versions the same neutrality of equilibrium was found. This latter result may be taken to indicate that the stability question for the type of equilibria here considered is not seriously affected by the form of the assumed current distribution.

5. Effects of finite conductivity

In this case we take σ finite but very large, $\mathbf{g} = 0$. Let the plasma lie between the two planes $x = \pm x_0$, and in it let $p = p_0(1-x^2/x_0^2)$, $\rho = \rho_0(1-x^2/x_0^2)$, $B_y = B_0 x/x_0$, $E_z = E_0$, $j_z = j_0$ and $\mathbf{v}$, ϵ and the other components of $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{j}$ be zero. In the vacuum let $B_x = B_z = 0$, $B_y = B_0$ for $x > x_0$, $B_y = -B_0$ for $x < -x_0$, $\mathbf{E} = 0$. On the surfaces $x = \pm x_0$ let $\mathbf{j}^* = 0$, $\epsilon^* = 0$. We then have an equilibrium system if the constants $x_0, p_0, \rho_0, B_0, E_0, j_0$ and σ satisfy

$$E_0 = j_0/\sigma, \quad B_0 = \mu_0 j_0 x_0, \quad j_0 B_0 x_0 = 2p_0. \tag{34}$$

We now introduce the perturbations as usual and seek solutions of the somewhat restricted form $\tilde{q} = \hat{q}\exp(\mathrm{i}kz + \omega t)$, where the amplitude factor $\hat{q}$ is a function of x and k is real. The resulting system of equations for the amplitude factors splits into two entirely separate systems, one for $\hat{v}_y, \hat{B}_x, \hat{B}_z, \hat{E}_y, \hat{j}_y$ and $\hat{j}_y^*$, the other for the remaining variables. We confine our attention to the latter system.

We introduce the dimensionless constants and functions

$$\left.\begin{aligned} \alpha &= \mu_0\kappa_0 p_0/\rho_0, & w^2 &= \rho_0 x_0^2\omega^2/\gamma p_0, \\ \lambda &= \tfrac{1}{2}(\alpha^{-1}+2)\,\mu_0 x_0^2\sigma\omega, & \nu^2 &= (2\gamma^{-1}+1)\,\mu_0 x_0^2\sigma\omega, \\ \eta^2 &= x_0^2(k^2+\mu_0\kappa_0\omega^2), & R &= 1+2\gamma^{-1}x_0^{-2}x^2, \\ S &= 1-2(1-\gamma^{-1})\,x_0^{-2}x^2, & T &= 1+2\alpha x_0^{-2}x^2, \end{aligned}\right\} \tag{35}$$

choosing $\mathscr{R}[\nu] \geqslant 0$ and $\mathscr{R}[\eta] \geqslant 0$.

Now any solution for the amplitude factors can be written as the sum of an odd and an even solution, where an odd solution is one in which the functions $\hat{v}_x, \hat{B}_y, \hat{E}_x, \hat{j}_x$ are all odd in x, and the functions $\hat{p}, \hat{\rho}, \hat{v}_z, \hat{E}_z, \hat{j}_z, \hat{\epsilon}$ are all even in x, and vice versa for even solutions. For a rough comparison with the cylindrical case, odd solutions correspond to radial perturbations and even solutions to perturbations varying as $\exp(\mathrm{i}\theta)$, which we have discussed in the preceding section. Accordingly, we shall restrict ourselves here to even solutions.

The system of differential (and algebraic) equations for the amplitude factors in the plasma is of fifth order and is singular at $x = 0$. It has three independent even solutions.

We are interested in the asymptotic solutions as $\sigma \to \infty$. We assume that ω has a limit different from zero. If we set $\sigma = \infty$ the system of equations in the plasma reduces to a non-singular system of second order, having one even solution. This solution is given in the first column of table 3, where ϕ is a function of x vanishing at $x = 0$ and satisfying

$$x_0^2 \phi'' + 2x(R^{-1} - 2\alpha T^{-1})\, \phi' = (k^2 x_0^2 + w^2 R^{-1} T)\, \phi, \tag{36}$$

the prime denoting differentiation with respect to x.

TABLE 3

	column 1	column 2 (each entry to be multiplied by $e^{\nu(x-x_0)/x_0}$)	column 3 (each entry to be multiplied by $e^{\lambda(x-x_0)/x_0}$)	column 4 (each entry to be multiplied by $e^{-\eta(x-x_0)/x_0}$)
p	$p_0 R^{-1}(\gamma\phi - 2w^{-2} S T^{-1} x\phi')$	$2p_0$	$\mathrm{i}kx_0 p_0[\gamma(1+2\alpha)\, w^2 + 4(2/\gamma - 1)/(1/\alpha + 2)]/\lambda^2$	
ρ	$\rho_0 R^{-1}(\phi - 2w^{-2} T^{-1} x\phi')$	$2\rho_0/\gamma$	$-2\mathrm{i}kx_0 \rho_0(1 - 1/\gamma)/\lambda$	
v_x	$-\gamma p_0(\rho_0 \omega T)^{-1} \phi'$	$-2x_0 \omega/\gamma\nu$	$-\mathrm{i}kx_0^2 \omega/\lambda$	
v_z	$-\mathrm{i}k\gamma p_0(\rho_0 \omega T)^{-1} \phi$	$-2\mathrm{i}kx_0^2 \omega/\gamma\nu^2$	$x_0 \omega$	
B_y	$B_0(x_0 R)^{-1}(x\phi + x_0^2 w^{-2} S T^{-1} \phi')$	$-B_0$	$-2\mathrm{i}kx_0 B_0(2/\gamma - 1)/(1/\alpha + 2)\, \lambda^2$	$-B_0 \gamma\alpha w^2$
E_x	$-\mathrm{i}k\gamma p_0 B_0(\rho_0 x_0 \omega T)^{-1} x\phi$	$\mathrm{i}kx_0^2 B_0 \omega/\nu^2$	$x_0 B_0 \omega$	$\mathrm{i}kx_0^2 B_0 \omega$
E_z	$\gamma p_0 B_0(\rho_0 x_0 \omega T)^{-1} x\phi'$	$-x_0 B_0 \omega/\nu$	$\mathrm{i}kx_0^2 B_0 \omega/\lambda$	$x_0 B_0 \omega\eta$
j_x	$-\mathrm{i}kj_0[(R^{-1} - \gamma\alpha T^{-1})\, x\phi + x_0^2 w^{-2} R^{-1} S T^{-1} \phi']$	$\mathrm{i}kx_0 j_0$	$-j_0 \gamma\alpha w^2$	
j_z	$j_0[(S+2)\, R^{-1} + 1 + k^2 x_0^2 w^{-2} S T^{-1}]\, \phi$ $-j_0 w^{-2} T^{-1}[2(S+2)\, R^{-1} - (1-\gamma\alpha)\, w^{-2}]\, x\phi'$	$-j_0 \nu$	$-\mathrm{i}kx_0 j_0 \alpha[\gamma w^2 + 2(2/\gamma - 1)/(1+2\alpha)]/\lambda$	
ϵ	$\mathrm{i}k\gamma\alpha j_0(\omega T)^{-1}(1 - 2T^{-1})\, \phi$	$-2\mathrm{i}kj_0(1+\gamma\alpha)\, w^2/(1/\alpha + 2)\, \nu^2$	$(j_0/x_0 \omega)\, \gamma\alpha w^2 \lambda$	

The asymptotic forms of the two remaining even solutions are not obtained by this procedure, since they do not converge uniformly as $\sigma \to \infty$. Instead, they drop off from their values at $x = x_0$ more and more rapidly as σ gets larger and larger. If we rewrite our system of equations in terms of the new independent variable $\xi = (x - x_0)\, \sigma^{\frac{1}{2}}$ and then let $\sigma \to \infty$, the system becomes asymptotically a non-singular system of fourth order. This has one even solution in addition to an asymptotically constant (in ξ) solution corresponding to the solution already found (given in the first column). Written as a function of x, the main term in each amplitude factor of this solution is given in the second column of table 3.

If we write the system of equations in terms of the new independent variable $\zeta = (x - x_0)\, \sigma$ (instead of the variable ξ above) and then let $\sigma \to \infty$, the system becomes asymptotically a non-singular system of fifth order. This has a third even solution in addition to two even asymptotically constant (in ζ) solutions corresponding to the two even solutions already found (given in the first two columns of table 3). Written as a function of x, the main term in each amplitude factor of this solution is given in the third column of table 3.

So much for the equations in the plasma. As for the equations in the vacuum, they are easily solved exactly, the amplitude factors being given in the fourth column of table 3. (The similar mathematical solution with exponent of opposite sign becomes infinite as $x \to \infty$ and is therefore excluded.)

The surface equations give $\hat{u} = \hat{v}_x$, $\hat{n}_z = -\mathrm{i}k\omega^{-1}\hat{v}_x$, $\hat{j}_x^* = \hat{j}_z^* = 0$, $\hat{\epsilon}^* = 0$, and the four joining conditions

$$\hat{p} - \frac{2p_0}{x_0\omega}\hat{v}_x = 0, \quad \hat{E}_x^P = \hat{E}_x^V, \quad \hat{E}_z^P = \hat{E}_z^V, \quad \hat{B}_y^P + \frac{B_0}{x_0\omega}\hat{v}_x = \hat{B}_y^V, \tag{37}$$

where all the amplitude factors are to be evaluated at $x = x_0$, the superscripts P and V distinguishing between plasma and vacuum quantities respectively. Now, the solution in the plasma is a linear combination of the independent solutions given in the first, second and third columns of table 3, say with coefficients a_1, a_2 and a_3 respectively, and the solution in the vacuum is a multiple of the solution given in the fourth column, say by a_4. Equations (37) thus give four linear homogeneous equations for a_1, a_2, a_3 and a_4. Asymptotically as $\sigma \to \infty$ the condition that these have a non-trivial solution becomes

$$\phi + \frac{2\alpha}{1+2\alpha}\frac{x_0}{\eta}\phi' = 0, \tag{38}$$

and the solution itself becomes

$$\left.\begin{aligned} a_2 &= -(2\gamma^{-1}+1)^{-1}(1+2\alpha)^{-1}(2w^{-2}-\gamma\alpha\eta^{-1})\,x_0\phi' a_1,\\ a_3 &= \mathrm{i}kx_0^2(1+2\alpha)^{-2}w^{-2}\eta^{-1}\phi' a_1,\\ a_4 &= (1+2\alpha)^{-1}w^{-2}\eta^{-1}x_0\phi' a_1; \end{aligned}\right\} \tag{39}$$

in equations (38) and (39) ϕ and ϕ' are to be evaluated at $x = x_0$.

The characteristic time constants ω are determined by the eigenvalues w^2 of the differential equation (36) subject to the boundary conditions $\phi = 0$ at $x = 0$ and (38) at $x = x_0$. Any such eigenvalue is real and negative; in fact, it is easily seen that $w^2 < -k^2x_0^2$. Thus ω is purely imaginary and the elementary solutions we have found are oscillations of constant amplitude.

We are interested in these solutions only as examples of perturbations in plasmas with large but finite σ. According to equations (35), ν is a positive real multiple of $(1+\mathrm{i})\sigma^{\frac{1}{2}}$ and λ of $\pm\mathrm{i}\sigma$. Hence, the second particular solution listed in table 3 decreases exponentially from the plasma boundary inwards, and the more rapidly the larger σ is. If our solution were carried through to the next order terms in σ, presumably λ would contain a positive real multiple of $\sigma^{\frac{1}{2}}$, so that the third particular solution listed in table 3 would drop off similarly away from the plasma boundary. Thus, the second and third solutions may be taken as describing skin phenomena. Indeed, in the limit of $\sigma \to \infty$, the only quantities in these two particular solutions which remain of physical interest are j_z of the second solution, which contains a factor ν and hence converges to a finite sheet current equal to $-j_0x_0$. and ϵ of the third solution which contains a factor λ and hence converges to a sheet charge equal to $\gamma\alpha w^2 j_0/\omega$. It may be mentioned that this whole case was also computed through with $\sigma = \infty$ from the start, and the solutions were entirely what would be expected as the limits of the solutions of the general case, including sheet quantities of just the right magnitudes.

This example of a perturbation in a plasma with finite but large conductivity seems therefore to indicate that the essential features of the type of problems here considered are fairly represented if one assumes infinite conductivity but simultaneously allows for electric sheet currents and electric sheet charges at the surface of the plasma.

6. Summary

In this paper, two types of instabilities have been discussed for highly conductive plasmas in magnetic fields. The first type of instability arises in a plasma which is supported against gravity by magnetic pressure. Such a plasma is found to be unstable against perturbations which move the magnetic lines essentially parallel to themselves but do not bend them seriously. This instability is found to be of the same type and of the same speed of development as the well-known instability of a heavy fluid supported against gravity by a lighter one.

The second instability arises in the well-known pinch effect, i.e. for a plasma which is contained within a cylinder by toroidal magnetic fields which in turn are caused by electric currents within the plasma parallel to the cylinder axis. Such a plasma is unstable against perturbations which distort the cylinder into a sinusoidal tube (figure 2). The e-folding time of this perturbation is roughly equal to the time a sound wave travels a distance equal to the wave-length of the perturbation.

These two instabilities are computed through under the approximation of infinite conductivity, with proper allowance for electric currents and charges on the surface of the plasma. The belief that this approximation is in fact adequate has been strengthened by investigating a particularly simple case in which some sample perturbations could be computed through under the assumption of large but finite conductivity.

We are happy to acknowledge our indebtedness to Dr Lyman Spitzer, Jr, for his constant advice and in particular for suggesting the possible existence of the instabilities here investigated.

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