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Slow pulsar scintillation and the spectrum of interstellar electron density fluctuations[1]

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Summary. Intensity fluctuations in pulsars are commonly ascribed to scintillations produced by electron density inhomogeneities in the interstellar medium. The power spectrum of the inhomogeneities is believed to be a power law, $q^{-\beta}$, with $\beta=11/3$ (Kolmogorov spectrum). The spectrum of intensity fluctuations produced by a thin phase-changing screen has already been worked out for the case $2<\beta<4$ and shows that two length-scales exist – a fast variation due to diffractive scintillation and a slow variation due to refractive effects. In this paper the theory is worked out for $4<\beta<6$ and asymptotic expressions are developed for the intensity fluctuation spectrum in various regimes. Two major differences emerge. First, the magnitude of the refractive intensity fluctuations is much larger for $\beta>4$ than for $\beta<4$. Secondly, for $\beta>4$, the intensity fluctuations are essentially independent of the strength of the phase fluctuations on the screen and thus independent of the wavelength of observation λ or the distance D of the pulsar. In contrast, for $\beta<4$, there is a dependence on both λ and D. The available observations on slow refractive intensity fluctuations seem to support a spectrum for the interstellar medium with $\beta>4$. Several arguments have been presented in the past, based on the observed wavelength dependence of decorrelation bandwidth, image size, scintillation time-scale, etc. to support the Kolmogorov value $\beta=11/3$. We show that these observations are equally consistent with $\beta>4$. If $\beta>4$, the observations require the power-law spectrum to extend over only a small range of scales $\sim10^{12}$–10^{15} cm compared to $\sim10^{9}$–10^{15} cm required in the case of a Kolmogorov spectrum.

1 Introduction

Ever since their discovery, pulsars have proved to be excellent probes of the interstellar medium. Scheuer (1968) identified the short-term variability of pulsars, on time-scales of ~100 s, with interstellar scintillation and developed a simple theory for a Gaussian distribution of electron density fluctuations with a characteristic length scale. Later work suggested that the spatial power

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spectrum of electron density fluctuations has no intrinsic length scale but is power-law in nature (e.g. Rickett 1977), i.e. power at wavevector $q \propto q^{-\beta}$, with β having a value close to 4. Scintillation theory for a power-law spectrum has been developed extensively (Rumsey 1975; Lee & Jokipii 1975; Gochelashvily & Shishov 1975; Rickett 1977). Much of the analytical work has been based on a thin-screen approximation where it is assumed that the inhomogeneities are restricted to a narrow region halfway between the source and the observer. The scaling of the time-scale of scintillation, the pulse width, decorrelation bandwidth and image angular size with wavelength λ and pulsar distance D depend on the value of β. Armstrong, Cordes & Rickett (1981) collected together a number of observations and argued that β is close to the Kolmogorov spectrum value of 11/3.

Two phenomena which arise from *refractive* effects associated with long spatial wavelength fluctuations in the interstellar electron density have strengthened the case for a power-law power spectrum. The first is the well-known drifting of dynamic pulsar scintillation spectra in time and frequency (Ewing *et al.* 1970). Shishov (1974) and Hewish (1980) argued that this phenomenon is caused by large 'prisms' in the medium which refract or steer the signal from the pulsar, thus dispersing the diffractive scintillation pattern. Secondly, Rickett, Coles & Bourgois (1984) have shown, following Sieber (1982), that the slow variability of pulsar flux on time-scales of weeks to years could also be attributed to large features in the medium, in this case caused by 'lenses' which refractively focus and defocus the radiation. Both of these phenomena require considerable power in long-wavelength fluctuations, and thus suggest a steep power-law spectrum.

Rickett *et al.* (1984) have shown that the *time-scale* of slow refractive variability, t_{ref}, observed in several pulsars is consistent with the scaling law $t_{\text{ref}} \propto \lambda^{11/5} D^{8/5}$ predicted by the Kolmogorov spectrum, but the data are unfortunately too meagre to strongly constrain β other than to indicate that $\beta \sim 4$. Blandford & Narayan (1985) showed on the basis of an approximate theory that the *amplitude* of slow variability of a typical pulsar should be $\sim 0.2 \lambda^{-17/30} D^{-11/30}$ for a Kolmogorov spectrum, where λ is in metres and D is in kpc. Instead, the observations seem to indicate that all pulsars have well-developed long-term intensity fluctuations with amplitudes that are relatively insensitive to λ and D. This could imply that β is greater than 4 as speculated by Blandford & Narayan and as we show in greater detail in this paper. In an earlier independent study Roberts & Ables (1982) had also concluded from the occurrence of periodicities in drifting scintillation patterns that β may be >4. In contrast, Smith & Wright (1985) find the *magnitude* of the scintillation drift slopes to be consistent with the Kolmogorov spectrum. Thus, it is probably fair to say that the value of β is not well determined and it is therefore important to develop models with $\beta > 4$ as well as $\beta < 4$ in order to interpret the observations better.

The spectrum of intensity fluctuations in the presence of strong scintillation has been worked out by Gochelashvily & Shishov (1975), within the thin-screen approximation, for $\beta < 4$, and the results have been invoked by Rickett *et al.* (1984) in their discussion of refractive interstellar scintillation. The case $\beta > 4$ seems to have been largely neglected in the astronomical literature (however, see Salpeter 1969 and the discussion of 'type B spectra' by Lovelace 1970), although it is recently gaining attention in other fields (e.g., Jakeman 1982; Rino & Owen 1984). We consider this regime in Section 2 of this paper (along with a brief review of the results for $\beta < 4$ for completeness) and show that there are distinct qualitative differences between the two regimes. In particular, the magnitude of the refractive intensity fluctuations is large and independent of λ and D when $\beta > 4$. We then use these results in Section 3 to identify the scaling laws with λ and D of various pulsar observables in the two regimes of β. We argue that much of the evidence usually quoted in favour of $\beta = 11/3$ is equally consistent with $\beta \sim 4.3$. In fact, some of the observations discussed above on refractive intensity fluctuations and periodicities in scintillation patterns *favour* a value $\beta > 4$. We thus conclude that there is a good case in favour of an interstellar density spectrum that is steeper than Kolmorgorov. Some implications are discussed in Section 4.

2 Theory

Section 2.1 reviews a method for the calculation of the spatial correlation function of received intensities from a plane wave passing through a phase-changing turbulent medium in the thin-screen approximation. In Section 2.2 we review for completeness well-known results (Gochelashvily & Shishov 1975; Rumsey 1975; Marians 1975; Prokhorov *et al.* 1975; Tatarski & Zavorotnyi 1980) for the case $\beta<4$, but we do not rederive them. Readers familiar with this material may wish to go at once to Section 2.3, where we present new results for $\beta>4$. Readers who wish to avoid the mathematical details may read Section 2.4 which discusses the physics underlying the intensity-fluctuation spectra of Figs 1 and 2.

2.1 BASIC ASSUMPTIONS AND REVIEW OF FUNDAMENTAL FORMULAE

In order to simplify the calculations, we adopt the standard thin-screen approximation: we replace the electron density fluctuations in the interstellar medium by an equivalent thin layer that changes the phase of radiation crossing it. The *observer's plane* is defined to be parallel to the thin screen at a distance z_{obs} from it. In fact, the large variation in scattering power between different lines-of-sight through the interstellar medium suggests that the density fluctuations are rather localized (Cordes, Weisberg & Boriakoff 1985) – perhaps in old supernova shock fronts, for example – and so the physical situation may be better approximated by a thin screen than by a homogeneous medium.

The change in phase, θ, varies both with position in the screen and with wavelength; θ depends on time as well, but only slowly compared to the rate ($\sim 100\,\mathrm{km\,s^{-1}}$) at which the line between the Earth and the pulsar sweeps across the interstellar medium, so we treat the phase fluctuations as though they were static. Because we will not be concerned with the interference between different frequencies, we suppress in this section the dependence of θ upon wavelength and write $\theta=\theta(\mathbf{s})$, where $\mathbf{s}$ is a vector in the screen.

As a further simplification, we replace the spherical waves emanating from the pulsar by plane waves incident on the screen; for small-angle scattering, all intensities are preserved if the actual distance between the observer and the screen, z_{obs}, is replaced with an effective distance z_{eff}, and transverse displacements on the observer plane, r_{obs}, are replaced by r_{eff} as follows

$$\frac{1}{z_{\mathrm{eff}}}\equiv\frac{1}{z_{\mathrm{obs}}}+\frac{1}{z_{\mathrm{pulsar}}}, \qquad r_{\mathrm{eff}}=\left(\frac{z_{\mathrm{pulsar}}}{z_{\mathrm{pulsar}}+z_{\mathrm{obs}}}\right)r_{\mathrm{obs}}, \tag{2.1.1}$$

where z_{pulsar} is the distance between the pulsar and the screen. Henceforth, when we write z, we shall mean z_{eff}.

We make the (also standard) assumption that the phase fluctuations are a spatially stationary Gaussian process with a power-law power spectrum of the form $Q_0q^{-\beta}$, whence their statistics are completely determined by their two-point correlation function

$$\begin{aligned}C(\mathbf{s})&\equiv\langle\theta(\mathbf{t})\theta(\mathbf{t}+\mathbf{s})\rangle\\ &=\frac{1}{(2\pi)^2}\int d^2\mathbf{q}\exp(i\mathbf{q}\cdot\mathbf{s})\,Q_0q^{-\beta}\\ &=\frac{1}{2\pi}\int_0^\infty dq\,J_0(qs)\,Q_0q^{1-\beta},\end{aligned} \tag{2.1.2}$$

where Q_0 is a constant. The power-law index, β, of two-dimensional phase fluctuations on the phase screen is the same as the corresponding index of actual electron density fluctuations in three dimensions.

With these assumptions, the normalized two-point correlation function of intensities on the observer's plane, $\langle I(\mathbf{r}_1)I(\mathbf{r}_2)\rangle/\langle I\rangle^2 \equiv W(\mathbf{r}_1-\mathbf{r}_2)$, depends only on $|\mathbf{r}_1-\mathbf{r}_2|$; in the context of scalar diffraction theory, it can be expressed in terms of integrals over the phase screen (e.g. Rumsey 1975 and references therein):

$$W(\mathbf{r})=\left(\frac{k}{2\pi z}\right)^2\int d^2\mathbf{v}\exp(ik\mathbf{r}\cdot\mathbf{v}/z)\int d^2\mathbf{u}\exp(ik\mathbf{u}\cdot\mathbf{v}/z)$$
$$\times\exp[2C(\mathbf{u})+2C(\mathbf{v})-C(\mathbf{u}+\mathbf{v})-C(\mathbf{u}-\mathbf{v})-2C(0)]. \tag{2.1.3}$$

In this formula, **u** and **v** are vectors in the phase screen, while **r** lies in the observer's plane. Because the Earth–pulsar line moves with respect to the interstellar medium, the effective position of the observer in the observer's plane changes; hence $W(\mathbf{r})$ is to be interpreted as a correlation in time over an interval $\Delta t=r/v_{\parallel}$, where $v_{\parallel}$ is the component of velocity parallel to the plane.

The integrals in equation (2.1.2) diverge if $\beta\geq 2$, which includes all of the cases of interest here. Fortunately, in the general formula (2.1.3) for intensity fluctuations, the correlation function appears in the particular combination

$$2C(\mathbf{u})+2C(\mathbf{v})-C(\mathbf{u}+\mathbf{v})-C(\mathbf{u}-\mathbf{v})-2C(0),$$

which is unchanged by the replacement

$$C(\mathbf{s})\rightarrow C(\mathbf{s})+A+B\,s^2$$

for arbitrary constants A and B. In particular, $C(\mathbf{s})$ may be replaced by

$$C(\mathbf{s})-C(0)=\frac{1}{2\pi}\int_0^{\infty}dq[J_0(qs)-1]Q_0q^{1-\beta}$$
$$=-\tfrac{1}{2}\langle[\theta(\mathbf{r}+\mathbf{s})-\theta(\mathbf{r})]^2\rangle, \tag{2.1.4}$$

which is proportional to the structure function of phase fluctuations, or by

$$C(\mathbf{s})-C(0)-\frac{s^2}{2}\frac{d^2}{ds^2}C(0)=\frac{1}{2\pi}\int_0^{\infty}dq[J_0(qs)-1+\tfrac{1}{4}(qs)^2]Q_0q^{1-\beta}$$
$$=\tfrac{1}{2}\langle[\mathbf{s}\cdot\nabla\theta(\mathbf{r})]^2-[\theta(\mathbf{r}+\mathbf{s})-\theta(\mathbf{r})]^2\rangle, \tag{2.1.5}$$

which clearly measures the typical curvature of the phase fluctuations. The combination (2.1.4) converges for $2<\beta<4$, whereas (2.1.5) converges for $4<\beta<6$, and both can be written as

$$C_{\text{eff}}(\mathbf{s})=\frac{2^{2-\beta}}{\pi}\frac{\Gamma[(6-\beta)/2]}{(\beta-4)(\beta-2)\Gamma(\beta/2)}Q_0s^{\beta-2}. \tag{2.1.6}$$

For $\beta>2$, the power spectrum in equation (2.1.2) is not physically realizable without a lower cut-off in $q_{\min}$ – i.e. at some large spatial scale – but the preceding argument shows that the intensity fluctuations on the observer's plane are independent of $q_{\min}$ provided that $q_{\min}$ is sufficiently small and $\beta<6$. (More precisely, one requires that $q_{\min}\ll\eta_{\text{ref}}/\sqrt{\lambda z}$, where η_{ref} is defined below.) In particular, although the phase gradient is formally infinite without a cut-off if $4<\beta<6$, the phase curvature is finite, and it is the curvature which dominates the large-scale intensity fluctuations by focusing and defocusing the radiation. This point was also recognized by Rumsey (1975), Marians (1975), and others; but to the best of our knowledge, we are the first to give explicit expressions for $W(\mathbf{r})$ – or rather for its Fourier transform – in the case that $4<\beta<6$ (*cf.* Section 2.3 below). Analogous results for a one-dimensional screen have been obtained, however, by Jakeman (1982) and by Rino & Owen (1984).

Equation (2.1.3) already has the form of a Fourier transform with respect to $\mathbf{v}$. Thus the normalized power spectrum of intensity fluctuations on the observer screen.

$$\tilde{W}(\mathbf{q}) \equiv \int d^2\mathbf{r} \exp[-i\mathbf{q}\cdot\mathbf{r}] W(\mathbf{r}), \tag{2.1.7}$$

produced by a power-law power spectrum (2.1.2) of phase fluctuations on the scattering screen is given by (Rumsey 1975; Marians 1975)

$$l_F^{-2}\tilde{W}(\boldsymbol{\eta}/l_F) = \int d^2\xi \exp[i\boldsymbol{\eta}\cdot\xi]\exp\{A[2\xi^{\beta-2}+2\eta^{\beta-2}-|\boldsymbol{\eta}+\xi|^{\beta-2}-/\boldsymbol{\eta}-\xi|^{\beta-2}]\}, \tag{2.1.8}$$

where $l_F \equiv \sqrt{z/k} = \sqrt{\lambda z/2\pi}$ and we employ the dimensionless variables $\boldsymbol{\eta} \equiv \mathbf{v}/l_F$, $\xi \equiv \mathbf{u}/l_F$, and

$$A \equiv \frac{2^{2-\beta}}{\pi} \frac{\Gamma[(6-\beta)/2]}{(\beta-4)(\beta-2)\Gamma(\beta/2)} Q_0 l_F^{\beta-2}. \tag{2.1.9}$$

Except for a factor depending on β, the quantity A is essentially the mean square interstellar phase fluctuation at the Fresnel length l_F. A is positive for $\beta>4$ and negative for $\beta<4$. Our parameters Q_0 and A are related to T and U of Rumsey (1975) and Marians (1975) as follows

$$T = Q_0/2\pi \tag{2.1.10}$$

$$U = -2^{\beta-1}\Gamma(\beta/2)\cos[\pi(\beta-2)/4]A. \tag{2.1.11}$$

The argument of the second exponential in equation (2.1.8) is finite and non-positive for all finite ξ and $\boldsymbol{\eta}$ if $0<Q_0<\infty$ and $2<\beta<6$. At $\beta=4$, A is singular and changes sign, but the argument itself – which for brevity we shall refer to below as 'the correlation term' – reduces to

$$\frac{Q_0(l_F)^2}{8\pi}[2\xi^2\ln|\xi|+2\eta^2\ln|\eta|-|\boldsymbol{\eta}+\xi|^2\ln|\boldsymbol{\eta}+\xi|-|\boldsymbol{\eta}-\xi|^2\ln|\boldsymbol{\eta}-\xi|]. \tag{2.1.12}$$

2.2 Review of results for $\beta<4$

The integral (2.1.8) cannot be evaluated exactly in closed form for general β. In the physically interesting case that $|A|\gg 1$, however, one can describe the entire power spectrum $\tilde{W}(\mathbf{q})$ by a series of complementary approximations. The details of these derivations are given by Gochelashvily & Shishov (1975) for the case $\beta<4$; here we cite only their results:

$$l_F^{-2}\tilde{W}(\eta/l_F) \approx$$

$$2^{\beta-2}\pi(\beta-2)(4-\beta)\frac{\Gamma(\beta/2)}{\Gamma[(6-\beta)/2]}|A|\eta^{4-\beta}\exp[-2|A|\eta^{\beta-2}] \qquad \text{if } \eta \ll \eta^*_{\text{ref}};$$

$$\frac{2\pi}{\beta-2}\Gamma[1/(\beta-2)](2|A|)^{-2/(\beta-2)} \qquad \text{if } \eta^*_{\text{ref}} \ll \eta \ll \eta_{\text{diff}};$$

$$2^{\beta}\pi(\beta-2)\frac{\Gamma(\beta/2)}{\Gamma[(4-\beta)/2]}|A|\eta^{-\beta} \qquad \text{if } \eta \gg \eta_{\text{diff}}. \tag{2.2.1}$$

The three scales in the power spectrum (2.2.1) are

$$\eta_{\text{ref}} \equiv [2|A|]^{-1/(\beta-2)},$$

$$\eta^*_{\text{ref}} \equiv \eta_{\text{ref}}\left\{\ln\left[\frac{(\beta-2)^2(4-\beta)\Gamma(\beta/2)|A|^2}{2^{\beta-2}\Gamma[1/(\beta-2)]\Gamma[(6-\beta)/2]}\right]\right\}^{1/(\beta-2)},$$

$$\eta_{\text{diff}} \equiv (2|A|)^{1/(\beta-2)}. \tag{2.2.2}$$

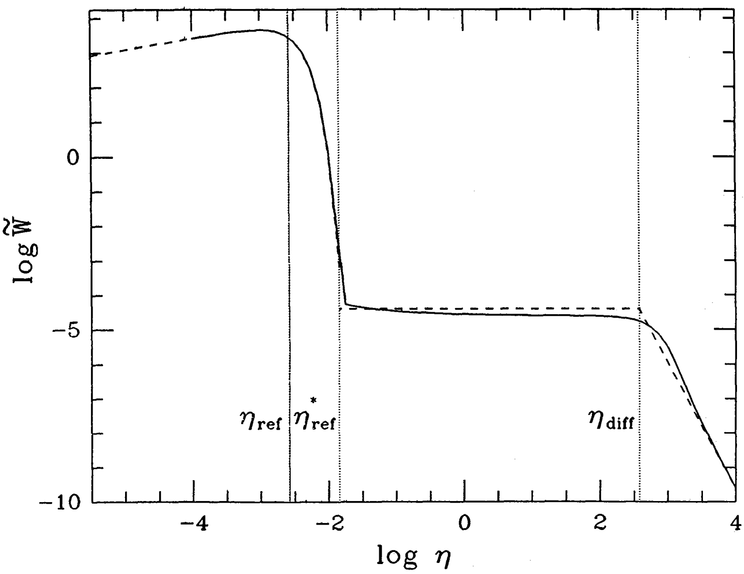

Figure 1. The power spectrum of intensity fluctuations on the observer's plane, $\tilde{W}$, plotted in units such that $\langle I\rangle=2\pi/\lambda z=1$, for $\beta=11/3$, $A=-10^4$. Solid line: numerical evaluation of (2.1.8). Dashed lines: asymptotic approximations (2.2.1). The vertical dotted lines show the scales defined in (2.2.2).

(We have distinguished $\eta_{\rm ref}$ from $\eta^*_{\rm ref}$, even though the two differ by only a logarithmic factor, because $\tilde{W}$ is a power law for $\eta<\eta_{\rm ref}$.) One should add the term $(2\pi)^2\delta(\mathbf{q})$ to these approximations for $\tilde{W}(\mathbf{q})$ because $W(\mathbf{r})\to 1$ as $r\to\infty$.

The general form of the spectrum is exemplified by Fig. 1. The solid line represents a direct numerical evaluation of (2.1.8) for the parameters $\beta=11/3$, $A=-10^4$. The approximations in (2.2.1) are the dashed lines.

We are interested in this paper in the rms intensity fluctuations on the refractive and diffractive scales. Let us define $\Delta I^2_{\rm ref}$ to be the total power in $\tilde{W}(\mathbf{q})$ in the range $0<|\mathbf{q}|<q^*_{\rm ref}\equiv\eta^*_{\rm ref}/l_{\rm F}$ (excluding the δ-function at $\mathbf{q}=0$ due to the non-zero mean value of I); this may be loosely interpreted as the mean-square fluctuation $[\langle I^2\rangle-\langle I\rangle^2]/\langle I\rangle^2$ on the refractive scale $r_{\rm ref}=2\pi/q_{\rm ref}$. By integrating the approximate formula in equation (2.2.1), we find that

$$\Delta I^2_{\rm ref}\equiv\frac{1}{(2\pi)^2}\int_{q<q^*_{\rm ref}} d^2\mathbf{q}\tilde{W}(\mathbf{q})$$

$$=\frac{2^{-\beta(4-\beta)/(\beta-2)}(4-\beta)\Gamma(\beta/2)\Gamma(6-\beta)/(\beta-2)]}{\Gamma[(6-\beta)/2]}|A|^{-2(4-\beta)/(\beta-2]}. \quad (2.2.3)$$

Similarly, $\Delta I^2_{\rm diff}$ is defined to be the total power in the range $q^*_{\rm ref}<|\mathbf{q}|<\infty$. Now, it follows from equation (2.1.8) that

$$\Delta I^2_{\rm ref}+\Delta I^2_{\rm diff}+1=\langle I^2(0)\rangle/\langle I(0)\rangle^2$$

$$=\frac{1}{(2\pi)^2}\iint d^2\boldsymbol{\eta}d^2\xi\exp[i\boldsymbol{\eta}\cdot\xi]\exp\{A[2\xi^{\beta-2}+2\eta^{\beta-2}-|\boldsymbol{\eta}+\xi|d^{\beta-2}-|\boldsymbol{\eta}-\xi|^{\beta-2}]\}. \quad (2.2.4)$$

For very large $|A|$, the integrand is very small unless either $\eta\ll\xi$ or $\xi\ll\eta$; because of the symmetry in $\boldsymbol{\eta}$ and ξ, it is sufficient to calculate the contribution of just one of these regimes and multiply by two. Thus, taking the case $\eta\gg\xi$, we may expand the correlation term as $\sim -2|A|\xi^{\beta-2}\{1+{\rm O}[(\xi/\eta)^{4-\beta}]\}$. Since the leading term is independent of η, the integral over $\boldsymbol{\eta}$

yields approximately $(2\pi)^2\delta^{(2)}(\xi)\exp(-2|A|\xi^{\beta-2})$; the ξ integral is then trivial, and we see that $\Delta I^2_{\rm ref}+\Delta I^2_{\rm diff}+1\to 2$ as $|A|\to\infty$. In view of (2.2.3), this shows that

$$\Delta I^2_{\rm diff}\to 1 \text{ as } |A|\to\infty, \tag{2.2.5}$$

a result that has been derived by Berry (1979).

2.3 RESULTS FOR $4<\beta<6$

In this section we derive analytic approximations to the integral (2.1.8) for the power spectrum of intensity fluctuations on the assumption that $4<\beta<6$ and that $Q_0(l_{\rm F})^{2-\beta}\gg 1$. The main results are given in equations (2.3.9)–(2.3.10).

It is clear that when η is sufficiently small, the main contribution to the integral (2.1.8) comes from $\xi\gg\eta$; we therefore expand the correlation term,

$$A[2\xi^{\beta-2}+2\eta^{\beta-2}-|\boldsymbol{\eta}+\boldsymbol{\xi}|^{\beta-2}-|\boldsymbol{\eta}-\boldsymbol{\xi}|^{\beta-2}]\approx-(\beta-2)A\eta^2\xi^{\beta-4}[1+(\beta-4)\cos^2\psi]+O(\eta^{\beta-2}), \tag{2.3.1}$$

where ψ is the angle between $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$. The absolute magnitude of the correlation term is evidently small compared to unity when $\xi\ll[(\beta-2)A\eta^2]^{-1/(\beta-4)}$. If in addition $[(\beta-2)A\eta^2]^{-1/(\beta-4)}\gg 1/\eta$, i.e. if $\eta\ll[(\beta-2)A]^{-1/(6-\beta)}$, the integral (2.1.8) will be cut off by the oscillation of the complex exponential long before (2.3.1) becomes large (and negative), so we expand the real exponential to first order in its argument. After discarding the zeroth-order term, which contributes only to the delta function in $\tilde{W}(\mathbf{q})$, we are left with

$$-(\beta-2)A\eta^2\int_0^\infty \xi d\xi\int_0^{2\pi} d\psi\exp[i\xi\eta\cos\psi]\xi^{\beta-4}[1+(\beta-4)\cos^2\psi], \tag{2.3.2}$$

which diverges at its upper limit in ξ. The original integral was perfectly convergent, however, and we are justified in evaluating (2.3.2) by the methods of generalized functions (*cf.* Lighthill 1958). Consider the formula (Abramowitz & Stegun 1970)

$$\int_0^\infty \xi d\xi\int_0^{2\pi} d\psi\xi^{\mu-1}\exp[i\xi\eta\cos\psi]=2\pi\eta^{-\mu-1}\int_0^\infty dt\, t^\mu J_0(t)$$
$$=2^{\mu+1}\pi\frac{\Gamma[(\mu+1)/2]}{\Gamma[(1-\mu)/2]}\eta^{-(\mu+1)}. \tag{2.3.3}$$

The intermediate term in (2.3.3) exists as an improper Riemann integral if $-1<\mu<+1/2$, but the final term is well-defined for all μ (even complex values) except odd integers, i.e. except at the poles of the gamma functions; one can use this term to define the integral on the left for general μ as a generalized function. Then, replacing μ with $\mu-2$ and differentiating twice with respect to η, one finds that

$$\int_0^\infty \xi d\xi\int_0^{2\pi} d\psi\xi^{\mu-1}\cos^2\psi\exp[i\xi\eta\cos\psi]=-2^{\mu-1}\pi\frac{\Gamma[(\mu-1)/2]}{\Gamma[(3-\mu)/2]}\mu(\mu-1)\eta^{-(\mu+1)}. \tag{2.3.4}$$

Using (2.3.3) and (2.3.4) with $\mu=\beta-3$, we obtain the first approximation in (2.3.9).

We now consider the case that $\eta\gg[(\beta-2)A]^{-1/(6-\beta)}$, yet η is still small enough so that the integral (2.1.8) is dominated by $\xi\gg\eta$. In this regime the integral is cut off at large ξ by the exponential of the correlation term before the complex exponential begins to oscillate, and we therefore replace the latter by unity. The expansion (2.3.1) is still serviceable, so (2.1.8) reduces approximately to

$$\int_0^\infty \xi d\xi\int_0^{2\pi} d\psi\exp\left[-\tfrac{1}{2}(\beta-2)(\beta-4)A\eta^2\xi^{\beta-4}\left(\cos 2\psi+\frac{\beta-2}{\beta-4}\right)\right]. \tag{2.3.5}$$

The dependence upon η and A can be scaled out of (2.3.5) by the substitution

$t \equiv \tfrac{1}{2}(\beta-2)(\beta-4)A\eta^2\xi^{\beta-4}$.

The resulting expression is given in the second part of equation (2.3.9).

We turn finally to the case $\eta \gg [(\beta-4)A]^{-1/(\beta-2)}$, which means that the correlation term is $\ll -1$ at $\xi=\eta$. In this range the largest contribution to the integral (2.1.8) comes from the interval $0<\xi\ll\eta$. We therefore expand the correlation term in increasing powers of ξ:

$$A[2\xi^{\beta-2}+2\eta^{\beta-2}-|\boldsymbol{\eta}+\boldsymbol{\xi}|^{\beta-2}-|\boldsymbol{\eta}-\boldsymbol{\xi}|^{\beta-2}] \approx -(\beta-2)A\eta^{\beta-4}[\xi^2+(\beta-4)\xi^2\cos^2\psi]+2A\xi^{\beta-2}. \quad (2.3.6)$$

For the moment, we ignore the final term in this expansion, since it is smaller than the rest by the factor $(\xi/\eta)^{\beta-4} \ll 1$; we shall return to this term below. Choosing cartesian coordinates (ξ_1, ξ_2) for $\boldsymbol{\xi}$ such that the first axis is parallel to $\boldsymbol{\eta}$, we find as an approximation to (2.1.8) the following product of Gaussian integrals:

$$\int_{-\infty}^{\infty} d\xi_1 \exp[i\eta\xi_1-(\beta-2)(\beta-3)A\eta^{\beta-4}\xi_1^2]\int_{-\infty}^{\infty} d\xi_2 \exp[-(\beta-2)A\eta^{\beta-4}\xi_2^2]. \quad (2.3.7)$$

The result of evaluating (2.3.7), which is given in the third part of (2.3.9), is exponentially cut off for η greater than the value $\eta_{\rm diff}$ defined in (2.3.10). Therefore, we must ask whether the final term in (2.3.6), which was neglected above, makes a more slowly decreasing contribution to $\tilde{W}$ when $\eta>\eta_{\rm diff}$. In fact it does, because $\xi^{\beta-2}$ has a cusp at the origin: for very large η, the integral (2.1.8) is dominated by the region around this cusp and is well approximated by the generalized Fourier transform

$$\int d^2\xi\, 2A\xi^{\beta-2}\exp(i\boldsymbol{\xi}\cdot\boldsymbol{\eta}) = \frac{2^{\beta-1}\pi(\beta-2)(\beta-4)\Gamma(\beta/2)}{\Gamma[(6-\beta)/2]}A\eta^{-\beta}, \quad (2.3.8)$$

in which we have again made use of the formula (2.3.3).

We summarize here our results for the spatial power spectrum $\tilde{W}(\mathbf{q})$ of intensity fluctuations in the observer's plane in the case that $4<\beta<6$:

$$l_{\rm F}^{-2}\tilde{W}(\boldsymbol{\eta}/l_{\rm F}) \approx$$

$$\begin{aligned}
&2^{\beta-2}\pi\frac{\Gamma(\beta/2)}{\Gamma[(6-\beta)/2]}(\beta-2)(\beta-4)A\eta^{4-\beta} && \text{if } \eta\ll\eta_{\rm ref};\\
&K(\beta)[(\beta-4)A]^{-2/(\beta-4)}\eta^{-4/(\beta-4)} && \text{if } \eta_{\rm ref}\ll\eta\ll\eta_{\rm int};\\
&\frac{\pi}{(\beta-2)\sqrt{\beta-3}}\frac{\eta^{4-\beta}}{A}\exp\left[-\frac{\eta^{6-\beta}}{4(\beta-2)(\beta-3)A}\right] && \text{if } \eta^*_{\rm diff}>\eta\gg\eta_{\rm int};\\
&\frac{2^{\beta-1}\pi(\beta-2)(\beta-4)\Gamma(\beta/2)}{\Gamma[(6-\beta)/2]}A\eta^{-\beta} && \text{if } \eta\gg\eta^*_{\rm diff},
\end{aligned} \quad (2.3.9)$$

in which the scales are

$$\begin{aligned}
&\eta_{\rm ref} \equiv [(\beta-2)A]^{-1/(6-\beta)},\\
&\eta_{\rm int} \equiv [(\beta-4)A]^{-1/(\beta-2)},\\
&\eta_{\rm diff} \equiv [4(\beta-2)(\beta-3)A]^{1/(6-\beta)},\\
&\eta^*_{\rm diff} \equiv \eta_{\rm diff}\left\{\ln\left[A^{2(\beta-4)/(6-\beta)}\left(\frac{\eta^*_{\rm diff}}{\eta_{\rm diff}}\right)^4\frac{\Gamma[(6-\beta)/2][4(\beta-2)(\beta-3)]^{4/(6-\beta)}}{2^{\beta-1}(\beta-2)^2(\beta-4)\sqrt{\beta-3}\,\Gamma(\beta/2)}\right]\right\}^{1/(6-\beta)}
\end{aligned} \quad (2.3.10)$$

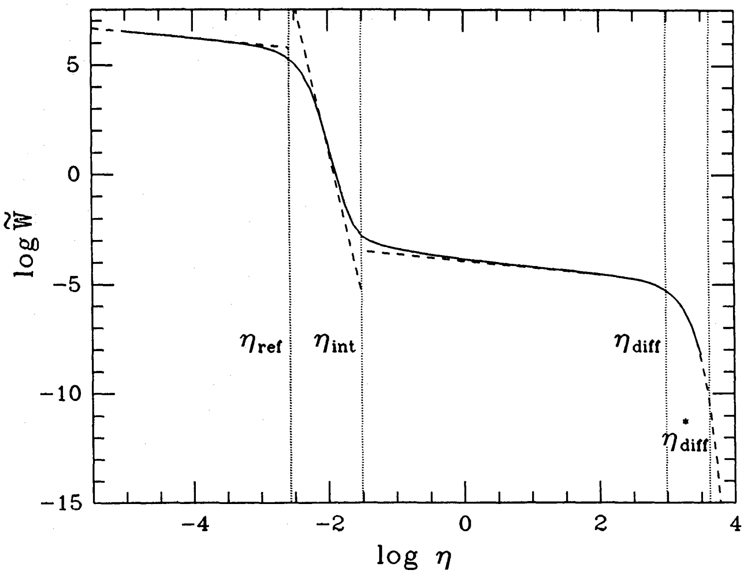

Figure 2. Same as Fig. 1, but for $\beta=4.3$, $A=+10^4$. Asymptotic approximations (dashed lines) and scales (vertical lines) are from (2.3.9) and (2.3.10).

and

$$K(\beta)\equiv \frac{2\pi}{\beta-4}[(\beta-2)/2]^{-2/(\beta-4)}\int_0^\infty dt\, t^{(6-\beta)/(\beta-4)}\exp\left[-\frac{\beta-2}{\beta-4}t\right]I_0(t).$$

(Here I_0 is a modified Bessel function, not an intensity.)

Fig. 2 shows the power spectrum of intensity fluctuations for $\beta=4.3$ and $A=10^4$. As discussed in Section 2.2, the total mean square intensity fluctuation (diffractive as well as refractive) can be written in the form (2.2.4). Once again it is seen that the integral is dominated by two regimes, $|\boldsymbol{\eta}|\ll|\boldsymbol{\xi}|$ and $|\boldsymbol{\eta}|\gg|\boldsymbol{\xi}|$, which clearly correspond to refractive and diffractive fluctuations respectively. By the symmetry between $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ in (2.2.4) these two contributions are equal and it is therefore sufficient to obtain an expression for one. We choose to use the third regime in (2.3.9), corresponding to $\eta_{\rm int}<\eta<\eta^*_{\rm diff}$, to obtain

$$\Delta I^2_{\rm ref}=\Delta I^2_{\rm diff}-1=\frac{1}{4\pi^2}\int_0^\infty 2\pi\eta\, d\eta\frac{\pi\eta^{4-\beta}}{(\beta-2)\sqrt{\beta-3}A}\exp\left[-\frac{\eta^{6-\beta}}{4(\beta-2)(\beta-3)A}\right]$$

$$=\frac{2\sqrt{\beta-3}}{(6-\beta)}-1. \tag{2.3.11}$$

The error from neglecting the power-law regime for $\eta>\eta^*_{\rm diff}$ is negligible for large A. Equation (2.3.11) agrees with a similar result obtained by Jakeman & Jefferson (1984).

2.4 DISCUSSION OF SPECTRA

Figs 1 and 2 reveal that, although the *density/phase* fluctuation spectrum is a scale-free power law of the form $q^{-\beta}$ (equation 2.1.2), the *intensity* fluctuation spectrum $\tilde{W}(\eta)$ has a very rich structure with several regimes and characteristic scales. We discuss here the underlying physics of the various regimes.

We begin by establishing certain basic properties of the wavefront at the screen. The mean square phase fluctuation produced by a given logarithmic interval of q can be obtained by setting $s=0$ in (2.1.2) and goes as $\sim Q_0 q^{2-\beta} d(\ln q)$. This decreases with increasing q for all $\beta>2$ and we can identify $q_{\max} \sim Q_0^{1/(\beta-2)}$ as the wavevector beyond which the phase fluctuations have an rms amplitude $\lesssim 1$. Waves with $q \gg q_{\max}$ are unimportant for our discussion which is mainly concerned with *strong* scattering. Next, the scattering angle at any point on the screen is proportional to the local transverse phase gradient and hence the mean-square scattering angle per logarithmic interval of q goes as $\sim Q_0 q^{4-\beta} d(\ln q)$. We notice that, for $\beta<4$, the dominant scattering comes from large $q \sim q_{\max}$, i.e. small scales, while for $\beta>4$, the dominant scales are large ($\sim q_{\mathrm{ref}} = \eta_{\mathrm{ref}}/l_{\mathrm{F}}$, as we show below, where l_{F} is the Fresnel scale $(\lambda z/2\pi)^{1/2} = (z/k)^{1/2}$). This distinction is at the heart of the differences between the results for $\beta<4$ (Fig. 1) and $\beta>4$ (Fig. 2). Finally, the degree of focusing produced by the screen is proportional to the curvature of the phase front. The mean square curvature per logarithmic interval goes as $\sim Q_0 q^{6-\beta} d(\ln q)$. This decreases at longer scales for all $\beta<6$.

The dimensionless scale η_{ref} in Figs 1 and 2, or equivalently the spatial wavevector $q_{\mathrm{ref}} = \eta_{\mathrm{ref}}/l_{\mathrm{F}}$, corresponds to the size of the image as projected on the scattering screen. For $\beta<4$, the scattering increases at smaller scales and so q_{ref} is given by the scattering angle at $q_{\max}$, i.e. $q_{\mathrm{ref}} l_{\mathrm{F}} \sim A^{-1/(\beta-2)}$ (where the dimensionless quantity $A \propto Q_0 l_{\mathrm{F}}^{\beta-2}$ is defined in 2.1.9). On the other hand, for $\beta>4$, the scattering increases at larger scales and hence the size of the image is given by that scale on the screen which can just focus radiation at the observer, i.e. $q_{\mathrm{ref}} l_{\mathrm{F}} \sim A^{-1/(6-\beta)}$. This scale is called q_{ref} because intensity fluctuations with $q \lesssim q_{\mathrm{ref}}$ are caused by the refractive focusing and defocusing of radiation by large inhomogeneities on the screen.

The intensity fluctuation spectrum $\tilde{W}(\eta)$ goes as $A\eta^{4-\beta}$ for $\eta<\eta_{\mathrm{ref}}$ and hence the mean square intensity fluctuation in a logarithmic interval of η goes as $\sim A\eta^{6-\beta} d(\ln \eta)$. This is understood as follows. Waves on the screen with $\eta<\eta_{\mathrm{ref}}$ have wavelengths larger than the size of the image and hence their primary effect on the intensity is through their phase curvature which causes focusing/defocusing at the observer screen. The intensity fluctuations occur on the same scale η as the waves producing them. The fluctuating component of the intensity squared can be easily shown to be proportional to the mean square phase curvature and should therefore go as $A\eta^{6-\beta}$. Note that the rms *magnitude* of the normalized intensity fluctuations at η_{ref} is ~ 1 for $\beta>4$ and $\sim A^{-(4-\beta)/(\beta-2)}$ for $\beta<4$, as can be also seen from equations (2.2.3) and (2.3.12). We discuss this further in Section 3.

Normal diffractive (~ 100 s) scintillation occurs on the diffractive scale $q_{\mathrm{diff}} = \eta_{\mathrm{diff}}/l_{\mathrm{F}}$ and is caused by interference among rays arriving at the observer plane from different parts of the pulsar image. A simple argument shows that η_{diff} is inversely proportional to the angular size of the image, i.e. $\eta_{\mathrm{diff}} \sim 1/\eta_{\mathrm{ref}}$. Therefore, $q_{\mathrm{diff}} l_{\mathrm{F}} \sim A^{1/(\beta-2)}$ for $\beta<4$ and $\sim A^{1/(6-\beta)}$ for $\beta>4$. The spectrum of diffractive scintillation extends over a range of $\eta<\eta_{\mathrm{diff}}$. Intensity fluctuations with a given η correspond to the interference of rays from the image separated by distances $\sim 1/\eta$. Thus, the intensity spectrum is intimately related to the autocorrelation function $V(\alpha)$ of the image, where α is the angular separation of two rays. We defer a detailed discussion of the properties of $V(\alpha)$ to a later communication but quote some relevant results here. For $\beta<4$ it turns out that the image is a smooth disc so that $V(\alpha)$ is flat out to the angular size of the image, beyond which it cuts off (as a power law, see below). Hence $\tilde{W}(\eta)$ is also independent of η. However, for $\beta>4$, $V(\alpha)$ has a power-law dependence $\alpha^{4-\beta}$ corresponding to a patchy image with a hierarchical clustering, i.e. *fractal geometry*. Consequently, the diffractive scintillation spectrum goes as $\tilde{W}(\eta) \sim A\eta^{4-\beta}$, $\eta<\eta_{\mathrm{diff}}$.

For $\eta \gg \eta_{\mathrm{diff}}$, the intensity spectrum falls off steeply as $\eta^{-\beta}$ in both regimes of β. This is due to weak scattering by very short scales ($q \gg q_{\max}$) which cause the image to have a power-law tail (Lovelace 1970). Correspondingly, $V(\alpha)$ goes as $\sim \alpha^{-\beta}$ for $\alpha \gg$ image diameter and the interference of rays with this angular separation produces the $\eta^{-\beta}$ power-law tail in the intensity spectrum.

We finally come to the power law $\eta^2\tilde{W}(\eta)\sim A\eta^{-2(6-\beta)/(\beta-4)}$ between $\eta_{\rm ref}$ and $\eta_{\rm int}$. This is a regime that is peculiar to $\beta>4$ and does not exist when $\beta<4$. We offer the following explanation based on a refractive interpretation. The scattering angle from any scale $\eta<\eta_{\rm ref}$ on the screen is $\propto\eta^{(4-\beta)/2}$ and therefore gives an intensity distribution on the observer screen that is correlated over a length $l_{\rm obs}\sim l_{\rm F}\eta^{(4-\beta)/2}$. Equivalently, intensity variations on a scale $l_{\rm obs}$ at the observer correspond to scatterers of size $1/\eta\sim(l_{\rm obs}/l_{\rm F})^{2/(\beta-4)}$. The number of independent blobs contributing to the intensity at any point is clearly $N\sim(l_{\rm obs}\eta/l_{\rm F})^2$. Therefore, invoking Poisson statistics, the mean square fluctuation in the intensity on a scale $l_{\rm obs}$ goes as $\sim1/N\propto l_{\rm obs}^{2(6-\beta)/(\beta-4)}$, thus explaining the form of the spectrum. We note that the intensity variations on the scale $\eta_{\rm int}$ itself are produced by scatterers of size $\sim q_{\rm max}^{-1}$. Therefore, if there is a high q cut-off in the phase fluctuation spectrum at some $q_{\rm upper}<q_{\rm max}$, it is the region of the intensity spectrum around $\eta_{\rm int}$ that will be most affected. In fact, the cut-off *increases* the power here. The reason for this is that when the small-scale fluctuations are absent the intensity statistics are dominated by caustics produced by the large scales, leading to a logarithmic divergence in the second moment (Salpeter 1967; Berry 1977 and references therein). The short scales help to break up the caustics and thus control the divergence when there is no cut-off. Another important point to note is that, for the case when $\beta>4$, neither the scale $\eta_{\rm ref}$ nor $\eta_{\rm diff}$ is strongly affected by a high-q cut-off. This is in contrast to $\beta<4$, where the values of both $\eta_{\rm ref}$ and $\eta_{\rm diff}$ are determined entirely by the high-q end of the spectrum. We elaborate on this in Section 4.

The scales $\eta^*_{\rm ref}$ and $\eta^*_{\rm diff}$ contain no new physics and differ only logarithmically from the more fundamental scales $\eta_{\rm ref}$ and $\eta_{\rm diff}$. These are the scales at which a large term with an exponential fall-off in the intensity spectrum equals an underlying weaker power-law term.

3 Scaling laws

3.1 $\beta<4$

For a pulsar at distance D whose signal reaches the observer through a homogeneous medium, the equivalent thin screen should be placed at $D/2$. Since the discussion of Section 2 considers plane waves incident on the screen from a source at infinity, the corresponding value of z is $D/4$ (*cf.* equation 2.1.1). The Fresnel scale is $l_{\rm F}=(z/k)^{1/2}\propto(\lambda D)^{1/2}$. Phase fluctuations in a plasma are $\propto(\lambda^2\times$density fluctuations), and hence for a homogeneous medium the power in phase fluctuations $Q_0\propto(\lambda^2\times$power in density fluctuations$)\propto(\lambda^2 D)$. In what follows we concentrate only on the scaling of various observables with λ and D, neglecting the coefficients except when we discuss the magnitude of refractive intensity fluctuations. The quantity A defined in (2.1.9) varies as

$$|A|\propto\lambda^{(\beta+2)/2}D^{\beta/2}. \tag{3.1.1}$$

The length scale over which diffractive scintillation occurs is given by $r_{\rm diff}\sim l_{\rm F}/\eta_{\rm diff}$ where $\eta_{\rm diff}$ is given by (2.2.2). For an observer moving with a constant velocity $v_\parallel$ through the scintillation pattern, this gives a scintillation time-scale

$$t_{\rm diff}\sim r_{\rm diff}/v_\parallel\propto\lambda^{-2/(\beta-2)}D^{-1/(\beta-2)}. \tag{3.1.2}$$

The rms magnitude of diffractive intensity fluctuations, $\Delta I_{\rm diff}$ has been estimated in equation (2.2.5),

$$\Delta I_{\rm diff}=1 \tag{3.1.3}$$

which is a well-known result for strong scintillation ($|A|\gg1$).

The refractive time-scale is obtained from $\eta_{\rm ref}$ in (2.2.2) to be

$$t_{\rm ref} \sim l_{\rm F}/\eta_{\rm ref} v_{\parallel} \propto \lambda^{\beta/(\beta-2)} D^{(\beta-1)/(\beta-2)}. \tag{3.1.4}$$

The magnitude of refractive intensity fluctuations has been calculated in equation (2.2.3) to be

$$\Delta I_{\rm ref} = \left[\frac{2^{-\beta(4-\beta)/(\beta-2)}(4-\beta)\Gamma(\beta/2)\Gamma[(6-\beta)/(\beta-2)]}{\Gamma[(6-\beta)/2]} \right]^{1/2} |A|^{-(4-\beta)/(\beta-2)}$$
$$\propto \lambda^{-(\beta+2)(4-\beta)/2(\beta-2)} D^{-\beta(4-\beta)/2(\beta-2)}. \tag{3.1.5}$$

The result depends on $|A|$ as well as on β (Fig. 3).

The refractive fluctuations are produced by the focusing and defocusing of the radiation from the source by long-wavelength inhomogeneities on the screen. As discussed in Section 2.4, the dominant effect comes from waves comparable in size to the projected image on the screen. Hence the angular size of the image is

$$\theta \sim r_{\rm ref}/D \propto \lambda^{\beta/(\beta-2)} D^{1/(\beta-2)}. \tag{3.1.6}$$

The finite angular size of the image leads to a broadening τ of the pulsed signal from the pulsar as well as to a decorrelation bandwidth $\Delta\nu$ for the scintillation pattern (e.g. Manchester & Taylor 1977)

$$\Delta\nu \propto \tau^{-1} \propto 1/D\theta^2 \propto \lambda^{-2\beta/(\beta-2)} D^{-\beta/(\beta-2)}. \tag{3.1.7}$$

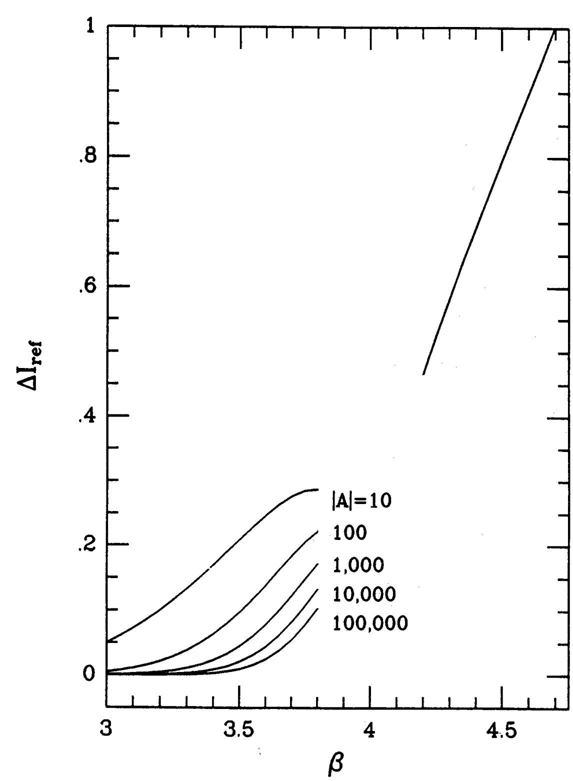

Figure 3. The total strength of refractive intensity fluctuations, $\Delta I_{\rm ref}$, as a function of β and A. The curve for $\beta > 4$ is independent of A, as discussed in the text. The region around $\beta = 4$ has been omitted because our asymptotic expressions are not reliable in this range.

3.2 $\beta>4$

A similar analysis on the formulae of Section 2.3 gives the following scaling laws for this regime

$$t_{\mathrm{diff}} \propto \lambda^{-(\beta-2)/(6-\beta)} D^{-(\beta-3)/(6-\beta)} \tag{3.2.1}$$

$$\Delta I_{\mathrm{diff}} = [2\sqrt{\beta-3}/(6-\beta)]^{1/2} \tag{3.2.2}$$

$$t_{\mathrm{ref}} \propto \lambda^{4/(6-\beta)} D^{3/(6-\beta)} \tag{3.2.3}$$

$$\Delta I_{\mathrm{ref}} = [2\sqrt{\beta-3}/(6-\beta)-1]^{1/2} \tag{3.2.4}$$

$$\theta \propto \lambda^{4/(6-\beta)} D^{(\beta-3)/(6-\beta)} \tag{3.2.5}$$

$$\Delta\nu \propto \tau^{-1} \propto \lambda^{-8/(6-\beta)} D^{-\beta/(6-\beta)}. \tag{3.2.6}$$

Some of these scalings were earlier worked out by Lovelace (1970). Note that ΔI_{ref} is large (~ 1) and independent of A. Its variation with β is shown in Fig. 3.

3.3 COMPARISON OF SCALINGS WITH OBSERVATIONS

Figs 4 and 5 show the variation with β of the exponents of λ and D in the scaling laws (Sections 3.1 and 3.2) of t_{diff}, t_{ref}, ΔI_{ref}, θ and $\Delta\nu$. The most striking feature is that nearly all the exponents have an extremum at $\beta=4$. Thus any measurement of an exponent has two solutions: one for $\beta<4$ and the other for $\beta>4$. This has not been adequately emphasized in earlier discussions, which invariably interpreted observations in terms of the $\beta<4$ solution.

We now attempt to estimate β by comparing observations with the predicted scaling laws. The scaling with D could in principle be studied by observing several pulsars at different distances, but is unfortunately inconclusive since the electron density fluctuations in the Galaxy are quite inhomogeneous (e.g. Manchester & Taylor 1977; Cordes *et al.* 1985). Observations on pulsars with a wide range of dispersion measure *DM* (Sutton 1971; Armstrong & Rickett 1981; Cordes *et*

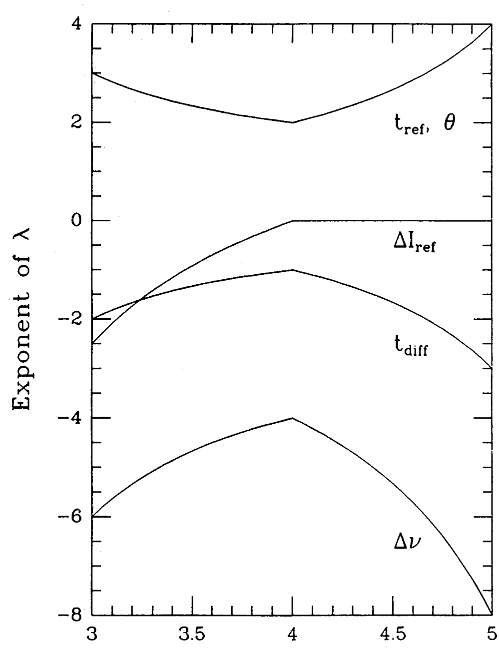

Figure 4. The exponents of wavelength, λ, in the scaling laws (3.1.2)–(3.2.6) for various physical observables, plotted against β.

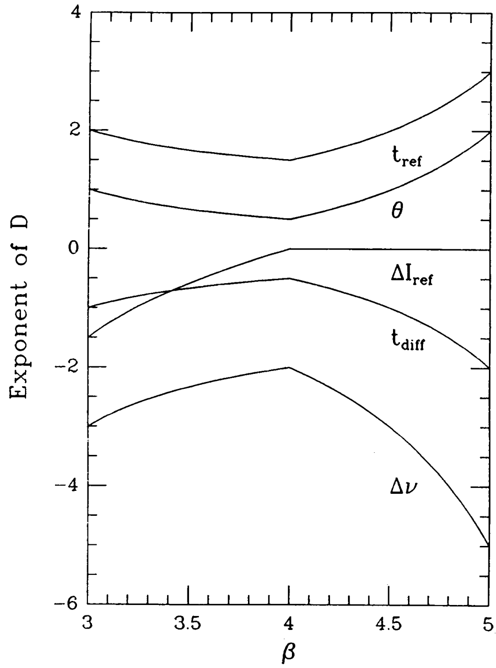

Figure 5. The exponents of distance to the pulsar, D, in the scaling laws (3.1.2)–(3.2.6), plotted against β.

al. 1985) show that $\Delta\nu \propto DM^{-3}$. This requires $\beta \sim 3$ or ~ 4.5. The solution $\beta \sim 3$ is considered too low since all the other evidence suggests $\beta \gtrsim 3.5$ and the usual explanation is to say that distant high-DM pulsars are found in the direction of the inner Galaxy where the electron density fluctuations could be stronger (e.g. Cordes *et al.* 1985). On the other hand, $\beta \sim 4.5$ is not inconsistent with the value ~ 4.3 suggested by the other observations discussed below.

Observations of the same pulsar at several different wavelengths are a cleaner probe of the electron density spectrum since the same line-of-sight through the medium is involved. Backer (1974) measured the variation of t_{diff} with λ in PSR 0833−45 and obtained an exponent 1.0 ± 0.2, which is consistent with values of $\beta \sim 4$. Mutel *et al.* (1974) measured the angular size of the Crab pulsar using VLBI at metre wavelengths. The observations showed that $\theta \propto \lambda^{2.05 \pm 0.25}$ which again suggests $\beta \sim 4$. Slee, Dulk & Otrupcek (1980) found $\tau \sim \lambda^{\sim 3.7}$ in PSR 1749−28. Several groups have verified the variation $\Delta\nu \propto \lambda^{-x}$. Backer (1974) obtained $x = 4.0 \pm 0.2$ in PSR 0833−45 between 837 and 8085 MHz. Armstrong & Rickett (1981) obtained $x = 4.9 \pm 0.8$ in PSR 0329+54 and 5.6(+1.8, −1.3) in PSR 1642−03. Cordes *et al.* (1985) made careful observations on five pulsars to obtain $x = 4.83 \pm 0.63$ in PSR 0329+54, 4.10 ± 0.07 in PSR 0833−45, 4.45 ± 0.37 in PSR 1642−03, 4.67 ± 0.29 in PSR 1749−28 and 4.22 ± 0.16 in PSR 1933±16, averaging to $x = 4.45 \pm 0.3$. Rickett (1970) and Roberts & Ables (1982) have also made similar observations. If we take the mean value $x = 4.5$ measured by Cordes *et al.* (1985), we obtain $\beta = 3.7 \pm 0.2$ or 4.2 ± 0.1.

Rickett *et al.* (1984) showed that the somewhat meagre observations currently available are consistent with t_{ref} increasing with DM and λ as required by theory. However, the data are insufficient to constrain β tightly.

Among all the scaling laws, only ΔI_{ref} discriminates strongly between the two regimes of β. When $\beta < 4$, ΔI_{ref} decreases with increasing λ and D while, when $\beta > 4$, it is independent of both λ and D. Observations appear to favour $\beta > 4$ (Blandford & Narayan 1985). Cole, Hesse & Page (1970) observed the long-term intensity variations in PSR 0950+08 and PSR 1919+21 at 81.5 and 408 MHz and found essentially equally strong fluctuations at the two frequencies. The Crab pulsar PSR 0531+21 is known to have correlated strong intensity variations at 400 MHz (Rankin, Payne & Campbell 1974) in all three of its pulse components. Rickett & Seiradakis (1982) report strong variations at 74 MHz as well suggesting that there is no reduction with increasing λ. Helfand, Fowler & Kuhlman (1977) have presented detailed observations over a four-year period on several pulsars at 156 MHz and over a shorter period at 390 MHz. They found the variations to increase at the higher frequency which could, however, be due to a poorer signal-to-noise ratio.

Quite apart from questions of scaling, the observed *amplitude* of $\Delta I_{\rm ref}$ suggests that $\beta>4$. When $\beta<4$, $\Delta I_{\rm ref}$ is predicted to be quite small and moreover decreases rather rapidly with increasing $|A|$. We can estimate the value of $|A|$ as follows. From equation (2.2.2) we obtain

$$t_{\rm ref}/t_{\rm diff} \sim \eta_{\rm diff}/\eta_{\rm ref} = [2|A|]^{2/(\beta-2)}. \tag{3.3.1}$$

Now, $t_{\rm diff}$ is typically $\sim$100 s while $t_{\rm ref}$ is $\gtrsim$10 days and may be as high as 5 yr as in the case of the 74-MHz observations of the Crab pulsar (Rickett *et al.* 1984). This gives, for $\beta=11/3$, values of $|A|\sim 10^3$–10^5. Equation (3.1.5) and Fig. 3 then show that $\Delta I_{\rm ref}$ should only be of order a few per cent (see also Figs 6 and 7). On the other hand all long-term observations of pulsars find large intensity fluctuations, $\Delta I_{\rm ref}$ of order unity (e.g. Huguenin, Taylor & Helfand 1973; Cole *et al.* 1970; Helfand *et al.* 1977; Rankin *et al.* 1974; Rickett & Seiradakis 1982). If these intensity variations are caused by interstellar propagation effects, as seems likely in the light of the arguments presented by Sieber (1982) and Rickett *et al.* (1984), then it appears that a value of $\beta<4$ is ruled out. On the other hand, a value of $\beta>4$ can naturally explain strong refractive intensity fluctuations.

4 Discussion

Combining all the observations discussed above, we would suggest $\beta\sim 4.3$ as an average value for the spectral exponent in the interstellar medium; there could of course be deviations from this

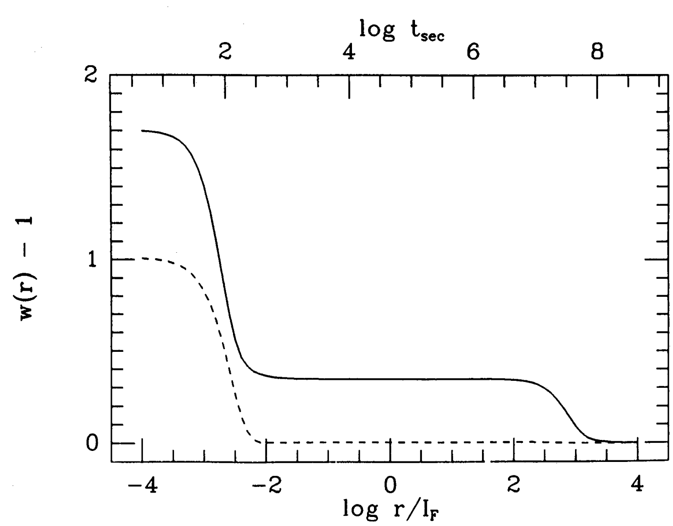

Figure 6. Spatial autocorrelation of the normalized intensity fluctuations as a function of distance lag r for $|A|=10^4$. The dashed line corresponds to a Kolmogorov spectrum, $\beta=11/3$, and the full line to $\beta=4.3$. The particular value of $|A|$ chosen corresponds to a pulsar distance of 2 kpc, observing wavelength of 2 m, and an average line-of-sight through the interstellar medium (i.e. $C_N^2\sim 10^{-4}$, see Rickett 1977 for a definition of C_N^2). The time-scale plotted at the top of the figure is $t=2r_{\rm eff}/v_{\parallel}$, with $l_F=\sqrt{(D/4)\lambda/2\pi}=2.2\times 10^{11}$ cm and $v_{\parallel}=100\,{\rm km\,s^{-1}}$. [The factors $2r_{\rm eff}$ and $D/4$ are obtained from equation (2.1.1) for a screen assumed to be mid-way between the observer and the pulsar.] Note the presence of the two characteristic time-scales, $t_{\rm diff}\sim 10^2$ s and $t_{\rm ref}\sim 10^7$ s, corresponding to diffractive and refractive scintillation respectively. The refractive fluctuations for $\beta=11/3$ are seen to be significantly less than those for $\beta=4.3$. However, it must be pointed out that the difference is exaggerated since $w(r)-1$ is the *square* of the intensity fluctuation.

value in any particular line-of-sight. The observable that is most sensitive to the value of β seems to be the refractive intensity fluctuation ΔI_{ref} which has strong qualitative differences between the two regimes of β. Figs 6 and 7 illustrate these differences in terms of observable quantities. Let us define the one-dimensional intensity autocorrelation function

$$w(v_{\|}t)=\langle I(0)\,I(t)\rangle/\langle I\rangle^2\equiv W(|\mathbf{v}_{\|}t|), \tag{4.1}$$

where t is the time difference between two observations and $\mathbf{v}_{\|}$ is the transverse velocity of the observer through the scintillation pattern. The two-dimensional correlation function $W(\mathbf{r})$ is defined in the discussion preceding equation (2.1.3) and is obtained by inverting equation (2.1.7)

$$W(\mathbf{r})=\frac{1}{(2\pi)^2}\int d^2\mathbf{q}\exp[i\mathbf{q}\cdot\mathbf{r}]\,\tilde{W}(\mathbf{q}). \tag{4.2}$$

It is convenient also to consider the one-dimensional power spectrum of the observed intensity fluctuations

$$\tilde{w}(q)=\int dr\,\exp[-iqr]\;w(r)=\frac{1}{2\pi}\int dq'\,\tilde{W}[(q^2+q'^2)^{1/2}]. \tag{4.3}$$

A numerical evaluation of $w(r)-1$ is shown in Fig. 6 for $|A|=10^4$ and two values of β, namely 11/3 and 4.3. The peak at small values of r (or short times) is due to diffractive scintillation, while the long tail is due to refractive effects. Note the difference in the magnitude of the latter between the two values of β. Fig. 7 shows $dP/d\ln q\equiv q\tilde{w}(q)$, which is the fluctuating power per logarithmic interval of q, for the same two cases. The existence of the two characteristic length scales/time-scales (t_{diff} and t_{ref}) is clearer in this representation, as is the point that the refractive fluctuations are much more pronounced for $\beta>4$ compared to the conventional Kolmogorov spectrum. Long-term ($>t_{\text{ref}}$) observations on the intensity fluctuations of pulsars will clearly be valuable to determine the value of β in the interstellar medium. Blandford & Narayan (1984, 1985) showed that, along with intensity fluctuations, slow variations are also expected in several other observables such as pulse width, decorrelation bandwidth, pulse arrival time, image size, image position, drift slope, etc. many with time-scales $\sim t_{\text{ref}}$. Observations on these will also be useful.

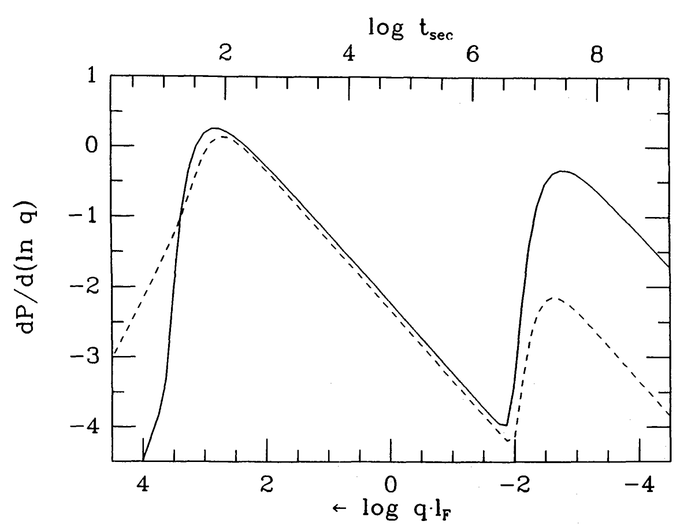

Figure 7. Intensity fluctuation power $dP/d\ln q\equiv q\tilde{w}(q)$ per logarithmic interval of wavevector q corresponding to the case shown in Fig. 6. The dashed line is for $\beta=11/3$ and the full line for $\beta=4.3$. The time-scale at the top is $t=2/qv_{\|}$ with l_F and $v_{\|}$ given the same values as before. Once again note the two time-scales, $t_{\text{diff}}\sim10^2$ s and $t_{\text{ref}}\sim10^7$ s, and the marked reduction in refractive power for $\beta=11/3$.

Ewing *et al.* (1970) and Roberts & Ables (1982) interpreted the occurrence of periodicities in dynamic scintillation spectra as due to the interference of a small number of beams. The implication is that the pulsar image is not the smooth (nearly Gaussian) disc that one normally imagines but that it is 'patchy'. Roberts & Ables suggested on this basis that β may be greater than 4. The argument goes as follows. We have seen in Section 2.4 that, when $\beta>4$, the scattering is maximum at the largest scales, which means that the appearance of the image is dominated by the power in wavelengths of order the size of the image. The image is thus broken up into a small number of patches, with each patch being further fragmented by phase fluctuations on the next smaller scale, etc. (a *fractal geometry*). The contrast in the patches probably increases with increasing β, but no quantitative results are available. We intend to pursue this question in a later paper.

If $\beta>4$, then the random deflection of the position of the source, as well as the slope of drifting scintillation patterns, formally diverge because of the phase gradient from the longest ($\geqslant$image size) waves on the screen. (These waves do not affect the *size* of the image since they coherently move the whole image.) This means that the rms magnitudes of these quantities depend on the low q cut-off in the spectrum. On the other hand, if $\beta<4$, there is no need for a cut-off since the contribution to the phase gradient from the long waves is finite and convergent. Smith & Wright (1985) claim that the observed drift slopes in a number of pulsars are consistent with a Kolmogorov spectrum ($\beta=11/3$). If β is really >4 as we have argued in this paper, then these observations require the low-q cut-off in the spectrum ('outer scale') to occur at a scale comparable to the image size, i.e. $\sim10^{15}$ cm. This is not unreasonable since there are several processes in the interstellar medium which can introduce power at these length scales – for instance, stellar winds, the ionization fronts of H II regions, cosmic ray precursors in supernova fronts, etc. (Blandford, private communication).

There is a crucial difference between the two regimes of β which may have important consequences for our understanding of the interstellar medium. We showed in Section 2.4 that, when $\beta<4$, the scattering angle is largest at small scales. Hence the angular size of the image θ as well as the pulse broadening τ, the decorrelation bandwidth $\Delta\nu$ and the diffraction time-scale t_{diff} are all determined by the electron density fluctuation spectrum at $q\sim q_{max}$. Consequently, measurements of the scaling of these quantities with λ and D directly probe the exponent β of the spectrum near this range of q. On the other hand, we argued in the Introduction that the occurrence of refractive effects means that there is considerable power at $q\sim q_{ref}$. Since q_{max} and q_{ref} typically differ by a factor $\sim10^4$–10^6, the standard description of the interstellar medium in terms of a Kolmogorov power law requires the spectrum to be valid over a very wide range of scales. This requires extremely efficient cascading of the turbulent energy and it is not clear whether this is possible. On the other hand, if $\beta<4$, θ is essentially determined by the fluctuations with $q\sim q_{ref}$. Thus, all the phenomena, both diffractive and refractive, are dominated by a single scale q_{ref} in the spectrum, and so the range of scales needed is much smaller. If there were physical reasons to suspect that the interstellar medium cannot sustain a power-law density fluctuation spectrum at such small scales as $(q_{max})^{-1}\sim10^9$ cm, then this in itself would be a strong incentive to propose that β is greater than 4. (We thank Peter Goldreich for provoking the investigations reported in this paper by asking the crucial question: what firm evidence is there that the ISM really has density fluctuations at $\sim10^9$ cm?).

A direct method of measuring electron density variations is by monitoring variations of the dispersion measure DM with time in a given pulsar. It can be shown (e.g. Lovelace 1970) that $\Delta DM\propto t_{obs}^{(\beta-2)/2}$ where ΔDM is the fluctuation in DM during an observation period t_{obs}. Unfortunately, it has not been possible to measure this yet, although there is good hope that it can be done with a millisec pulsar like PSR 1937+214 which has an extremely sharp pulse profile.

Electron density fluctuations can contribute to pulsar timing residuals at the microsecond level due to group velocity variations and image wander (Lovelace 1970; Armstrong 1984; Blandford, Narayan & Romani 1984). The group velocity effect can in principle be eliminated by accurately monitoring the *DM* fluctuations of the pulsar, whereas the fluctuating geometrical time delay due to image wander cannot be corrected for (unless one does frequent VLBI on the pulsar against a number of reference background sources) and is an unavoidable source of timing noise. When $\beta<4$, the effect is only a small fraction of the pulse broadening τ, but when $\beta>4$, the timing noise has a magnitude $>\tau$ and grows as $t_{\rm obs}^{\beta-4}$ where $t_{\rm obs}$ is the duration of the observations. The noise saturates when the outer scale of the spectrum is reached, which corresponds to ~ 3 yr for 10^{15} cm at $100\,{\rm km\,s^{-1}}$.

The $\beta>4$ regime that we have discussed in detail has recently received some attention from scattering theorists in other fields (Jakeman 1982, 1983; Jakeman & Jefferson 1984; Rino & Owen 1984). When $\beta>4$, the wavefront produced by the thin screen is sometimes described as being *sub-fractal* or having a *fractal slope*, as against the $\beta<4$ case which is described as having a *fractal wavefront*. Unfortunately, much of the previous work on $\beta>4$ scattering has concentrated on a one-dimensional screen (where the analogous condition is $\beta>3$) and therefore a detailed comparison with our results is not possible. Jakeman & Jefferson (1984) do consider a two-dimensional screen in their so-called 'isotropic case', with their ν to be identified with our $\beta-2$. They obtain the total mean square fluctuation in the intensity, in the limit of large A, to be $\langle I^2\rangle/\langle I\rangle^2 \equiv \Delta I_{\rm ref}^2+\Delta I_{\rm diff}^2+1=4\sqrt{\nu-1}/(4-\nu)$, which agrees with our equation (2.3.11).

Jakeman & Jefferson (1984) use their theory to analyse the scintillation pattern in the Fresnel region behind a thermal plume illuminated with laser light. They obtain an excellent fit for $\nu\sim 2.3$, i.e. $\beta\sim 4.3$. In another study, Walker & Jakeman (1984) have investigated the optical scintillation produced by laser light scattered by a turbulently mixing layer of brine and pure water, and again find the spectral index to be $\beta\sim 4.3$. These laboratory systems show that spectra that are steeper than Kolmogorov are not as rare or unusual as previously thought.

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