AIP | Physics of Fluids

Hydromagnetic Surface Waves in a Conducting Liquid Surrounded by a Compressible Gas

Richard Gerwin

Citation: Phys. Fluids **10**, 2164 (1967); doi: 10.1063/1.1762013
View online: http://dx.doi.org/10.1063/1.1762013
View Table of Contents: http://pof.aip.org/resource/1/PFLDAS/v10/i10
Published by the American Institute of Physics.

Additional information on Phys. Fluids

Journal Homepage: http://pof.aip.org/
Journal Information: http://pof.aip.org/about/about_the_journal
Top downloads: http://pof.aip.org/features/most_downloaded
Information for Authors: http://pof.aip.org/authors

ADVERTISEMENT

Downloaded 23 Mar 2012 to 142.150.190.39. Redistribution subject to AIP license or copyright; see http://pof.aip.org/about/rights_and_permissions

Hydromagnetic Surface Waves in a Conducting Liquid Surrounded by a Compressible Gas

RICHARD GERWIN
Boeing Scientific Research Laboratories, Seattle, Washington
(Recived 6 March 1967)

Those surface waves are studied that can propagate along a plane interface separating a conducting (incompressible) liquid from a nonconducting, compressible gas, when there is a static magnetic field parallel to the interface. For any given direction of propagation along the surface (except exactly perpendicular to the magnetic lines), and with the gas mass density small compared to that of the liquid, we find an approximate critical magnetic field strength above which the usual surface Alfvén waves cannot exist. Instead, at these strong magnetic fields, there are surface sound waves in the gas. The critical magnetic field at which the (actually smooth) transition from Alfvén-type to sound-type surface waves takes place is practically independent of the gas density, depending essentially only on its temperature.

I. INTRODUCTION

THE subject of hydromagnetic waves in fluids of infinite extent has been well explored. Less well-known is the propagation of hydromagnetic waves in fluids having boundaries. In addition to bulk waves, such fluids can also support surface waves. Notable contributions to this topic have been made by Melcher,[1] Chandrasekhar,[2] and Kruskal and Schwarzschild.[3]

In this paper, we study those surface waves that can propagate along a plane interface separating an incompressible conducting fluid (that is, a conducting liquid) from a nonconducting compressible gas. The lines of force of the steady magnetic field are assumed to be parallel to the interface.

For sufficiently small gas mass density, and with the magnetic field strengths below a certain critical approximate value that depends on the direction of wave propagation along the surface, we find the usual surface Alfvén waves. For field strengths exceeding this approximate critical value, we find that the surface Alfvén waves no longer exist, but that surface sound waves do. The latter cannot exist in the absence of a magnetic field, in the physical situation in question.

The critical field at which the (actually smooth) transition from Alfvén waves to sound waves takes place increases for propagation directions that approach perpendicularity to the field lines. Furthermore, this critical field depends essentially only on the gas temperature, and is practically independent of its density.

[1] J. R. Melcher, Phys. Fluids **4**, 1348 (1961).
[2] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford, University Press, London, 1961), Chap. 10.
[3] M. Kruskal and M. Schwarzschild, Proc. Roy. Soc. (London) **A223**, 348 (1954).

There also exist Alfvén waves in the bulk of the liquid, and if the magnetic field strength exceeds another critical approximate value, then for certain unique directions of propagation, surface sound waves accompany the Alfvén bulk waves.

II. EQUATIONS FOR THE LIQUID

With the equilibrium quantities denoted by a subscript 0 and small perturbation quantities by small letters, the linearized equations to be used in the liquid are

$$\nabla\cdot\mathbf{v} = 0, \tag{1}$$

$$\rho_0\,\partial_t\mathbf{v} + \nabla p = (1/c)\mathbf{j}\times\mathbf{B}_0, \tag{2}$$

$$\boldsymbol{\varepsilon} + (1/c)\mathbf{v}\times\mathbf{B}_0 = 0, \tag{3}$$

$$(4\pi/c)\mathbf{j} = \nabla\times\mathbf{b}, \tag{4}$$

$$\nabla\times\boldsymbol{\varepsilon} = -(1/c)\partial_t\mathbf{b}. \tag{5}$$

Equation (1) states that the fluid is incompressible, and we refer to this fluid as a liquid; $\mathbf{v}$ is the velocity of the liquid.

Equation (2) states that the fluid acceleration is caused by the pressure gradient ∇p and the magnetic body force $(1/c)\mathbf{j}\times\mathbf{B}_0$, $\mathbf{j}$ representing the current density and $\mathbf{B}_0$ the steady magnetic field. In addition, ρ_0 is the liquid mass density and c is the speed of light.

Equation (3) states that the liquid is a perfect conductor. A simple order-of-magnitude argument indicates that this approximation is justified for wavelengths that are far greater than the skin depth $(c^2/2\pi\sigma\omega)^{\frac{1}{2}}$, σ being the conductivity of the liquid, and ω a characteristic wave frequency.

Equation (4) is Ampere's law. This is a magnetostatic approximation because it neglects the dis-

Downloaded 23 Mar 2012 to 142.150.190.39. Redistribution subject to AIP license or copyright; see http://pof.aip.org/about/rights_and_permissions

placement current $(1/c)\partial_t \boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}$ being the electric field. This approximation is generally valid when the characteristic velocities of the problem are much less than c, and when the energy storage is in magnetic fields rather than electric fields.

Equation (5) is Faraday's law of induction.

As is now apparent, we use unrationalized Gaussian units.

A. Reduction of the Equations

Taking the curl of Eq. (3), noting Eqs. (1) and (5), and using the uniformity of $\mathbf{B}_0$ which points along the z axis, we find

$$\partial_t \mathbf{b} = B_0\, \partial_z \mathbf{v}. \tag{6}$$

Using Eq. (4) to write Eq. (2) in the form

$$\rho_0\, \partial_t \mathbf{v} + \nabla(p + B_0 b_z/4\pi) = B_0\, \partial_z \mathbf{b}/4\pi,$$

we operate on the latter with ∂_t, and then use Eq. (6) to obtain

$$(\partial_t^2 - \alpha^2\, \partial_z^2)\mathbf{v} + \nabla(\rho_0^{-1}\, \partial_t p + \alpha^2\, \partial_z v_z) = 0, \tag{7}$$

where the Alfvén speed α is defined by

$$\alpha^2 \equiv B_0^2/4\pi\rho_0. \tag{8}$$

The z component of Eq. (7) gives

$$\partial_t(\partial_t v_z + \partial_z \rho_0^{-1} p) = 0. \tag{9}$$

Operating on Eq. (7) with ∂_z and then using Eq. (9), we find

$$(\partial_t^2 - \alpha^2\, \partial_z^2)(\partial_z \mathbf{v} - \nabla v_z) = 0. \tag{10}$$

The divergence of Eq. (10), noting Eq. (1), gives

$$(\partial_t^2 - \alpha^2\, \partial_z^2)\nabla^2 v_z = 0. \tag{11}$$

For waves of the form $\exp[i(k_z z - \omega t)]$, with the phase velocity along the z axis taken as

$$a \equiv \omega/k_z, \tag{12}$$

we can rewrite Eqs. (6), (9)–(11) as follows:

$$\mathbf{b} = -B_0\mathbf{v}/a, \tag{13}$$

$$p = \rho_0 a v_z, \tag{14}$$

$$(a^2 - \alpha^2)(ik_z\mathbf{v} - \nabla v_z) = 0, \tag{15}$$

$$(a^2 - \alpha^2)\nabla^2 v_z = 0. \tag{16}$$

III. EQUATIONS FOR THE GAS

Denoting quantities in the nonconducting, compressible gas by a subscript g, the linearized equations are the continuity equation,

$$\partial_t \rho_g + \rho_{g0}\nabla\cdot\mathbf{v}_g = 0; \tag{17}$$

the force equation,

$$\rho_{g0}\, \partial_t \mathbf{v}_g + \nabla p_g = 0; \tag{18}$$

and the adiabatic equation,

$$\partial_t p_g = s^2\, \partial_t \rho_g. \tag{19}$$

Equation (19) is the linearized version of the equation

$$(\partial_t + \mathbf{V}_g\cdot\nabla)(P_g \mathrm{P}_g^{-\gamma}) = 0,$$

where γ is the ratio of specific heats. Also, s is the speed of sound in the gas, and is given by

$$s^2 \equiv \gamma P_{g0}/\rho_{g0}.$$

A. Reduction of the Equations

Operating on Eq. (17) with ∂_t, using Eq. (18) to eliminate the velocity and Eq. (19) to eliminate the density, we find

$$s^{-2}\, \partial_t^2 p_g - \nabla^2 p_g = 0. \tag{20}$$

For waves of the form $\exp[i(k_z z - \omega t)]$, Eqs. (18) and (21) take the form [using Eq. (12)],

$$\mathbf{v}_g = \nabla p_g/(iak_z\rho_{g0}), \tag{21}$$

$$\nabla^2 p_g + \kappa^2 p_g = 0, \tag{22}$$

where

$$\kappa^2 \equiv \omega^2/s^2 \equiv a^2 k_z^2/s^2. \tag{23}$$

B. The Magnetic Field In the Gas

As there are no currents in the gas and the displacement current is neglected, $\nabla\times\mathbf{b}_g = 0$. This means $\mathbf{b}_g$ can be derived from a magnetic potential ϕ, that is,

$$\mathbf{b}_g = \nabla\phi. \tag{24}$$

Since $\nabla\cdot\mathbf{b}_g = 0$, ϕ obeys the equation

$$\nabla^2\phi = 0. \tag{25}$$

IV. THE BOUNDARY CONDITIONS AT THE INTERFACE

We have already taken the steady magnetic field to point along the z axis. We now take the interface between liquid and gas to be the y-z plane. The surface normal then points along the x axis. We take the liquid to fill the space for which $x < 0$, and the gas then fills the space $x > 0$. A derivative with respect to the x coordinate is denoted by a prime, thus, $\partial_x f \equiv f'$.

A. Pressure Balance

The sum of the mechanical pressure P and the magnetic pressure $B^2/8\pi$, must be the same on each

Downloaded 23 Mar 2012 to 142.150.190.39. Redistribution subject to AIP license or copyright; see http://pof.aip.org/about/rights_and_permissions

side of the interface. The linearized form of this pressure balance equation is

$$p + B_0 b_z/4\pi = p_g + B_{g0} b_{gz}/4\pi, \quad \text{at} \quad x = 0. \tag{26}$$

In writing this equation, we have allowed for the possibility that the steady magnetic field in the gas, B_{g0}, differs in strength (but not direction) from the steady magnetic field in the liquid, B_0. This difference, of course, must be supported by a sheet current in the y direction on the interface.

Using Eqs. (13), (14), and Eq. (24) in Eq. (26), the latter becomes

$$(a^2 - \alpha^2)v_z = ap_g/\rho_0 + ik_z a\alpha_g^2\phi/B_{g0}, \quad \text{at} \quad x = 0, \tag{27}$$

where

$$\alpha_g^2 \equiv B_{g0}^2/4\pi\rho_0. \tag{28}$$

B. The Boundary Condition at a Perfect Conductor

The tangential component of the electric field seen by the liquid just at the surface, namely $[\boldsymbol{\varepsilon}_g + (1/c)\mathbf{v}\times\mathbf{B}_{g0}]_{\tan}$, must vanish at the surface of the liquid. Letting $\hat{x}$ be a unit vector in the positive x direction, we note that the expression

$$\hat{x}\cdot\nabla\times[\boldsymbol{\varepsilon}_g + (1/c)\mathbf{v}\times\mathbf{B}_{g0}], \quad \text{at} \quad x = 0$$

involves only components of the bracketed expression lying in the surface, and the derivatives of these components are also taken along the surface. (This statement is true to lowest order in small perturbation amplitudes.) This expression must therefore vanish.

Remembering Eq. (1), and Eq. (5) which is also valid at the surface, and noting also that $\mathbf{B}_{g0}$ does not vary along the surface, we find that the vanishing of the above expression leads to

$$-\partial_t b_{gx} + B_{g0}\,\partial_z v_x = 0, \quad \text{at} \quad x = 0.$$

For perturbation quantities having the factor $\exp[i(k_z z - \omega t)]$, and using Eq. (24), the above equation becomes

$$v_x = -a\phi'/B_{g0}, \quad \text{at} \quad x = 0. \tag{29}$$

Furthermore, if $a^2 \neq \alpha^2$, we can obtain v_x in terms of v_z from (15), so that (29) becomes

$$v_z' = -ik_z a\phi'/B_{g0}, \quad \text{at} \quad x = 0. \tag{30}$$

C. Velocity Matching at the Boundary

It is obvious that the motion of the interface in the x direction must involve the same velocity there, for both the liquid and the gas. Hence, noting Eq. (21),

$$v_x = v_{gx} = p_g'/(iak_z\rho_{g0}), \quad \text{at} \quad x = 0. \tag{31}$$

If $a^2 \neq \alpha^2$, we can again apply Eq. (15), so that Eq. (31) becomes

$$a\rho_{g0}v_z' = p_g', \quad \text{at} \quad x = 0. \tag{32}$$

V. BULK WAVES

By bulk waves, we mean disturbances that do not necessarily die down away from the interface. At the same time, we do not accept solutions that blow up at infinity.

So far, we have only specified a space dependence of the perturbation of the form $\exp(ik_z z)$. Here, and henceforth, we also stipulate that the y dependence takes the form $\exp(ik_y y)$. It is then convenient to define the "total wavenumber" k by the positive square root,

$$k = (k_y^2 + k_z^2)^{\frac{1}{2}}. \tag{33}$$

Another reciprocal length q that appears later, is here defined by

$$q = (k^2 - \kappa^2)^{\frac{1}{2}}, \tag{34}$$

where κ^2 has been defined in Eq. (23).

A. Bulk Alfvén Waves

Suppose

$$a^2 = \alpha^2. \tag{35}$$

Then Eqs. (15) and (16) are satisfied for arbitrary velocities of the liquid $\mathbf{v}$ provided $\nabla\cdot\mathbf{v} = 0$. For example, we can have $\mathbf{v} = [0, v_y(x), 0]$, with $v_y(x)$ being an arbitrary function, and $k_y = 0$. (Here, and always, we interpret the perturbation quantities as including the factor $\exp[i(k_y y + k_z z - \omega t)]$).

We can satisfy the equations in the gas, Eq. (22) and (25), by the simple choices $p_g \equiv 0$ and $\phi \equiv 0$. Then boundary condition (27) is automatically satisfied, and boundary conditions (29) and (31) are satisfied provided the normal velocity of the liquid vanishes at the interface, $v_x(0) = 0$. For example, we can choose $v_x(x)$ to be any function of x ($x \leq 0$) that vanishes at the interface, $x = 0$. Then $v_y(x)$ and $v_z(x)$ are (not uniquely) determined by the condition $\nabla\cdot\mathbf{v} = 0$. Such a disturbance propagates with the phase velocity α along the field lines in the liquid, without coupling to the external gas or the external magnetic field. This disturbance can, therefore, be considered as a bulk Alfvén wave in the liquid. (The actual phase velocity is, of course, ω/k, in the direction $\mathbf{k} = (0, k_y, k_z)$. Nevertheless it is true that surfaces of constant phase, as seen by a field line, propagate with the velocity $\omega/k_z = \alpha$.)

Downloaded 23 Mar 2012 to 142.150.190.39. Redistribution subject to AIP license or copyright; see http://pof.aip.org/about/rights_and_permissions

There also exist bulk Alfvén waves in the liquid whose influence *does* extend beyond the interface, but there are then only certain allowed directions for the propagation vector $\mathbf{k} = (0, k_y, k_z)$. To see this, note that the nonzero solutions to Eqs. (22) and (25) are of the form

$$p_g \propto e^{-qx} \quad \text{and} \quad \phi \propto e^{-kx}, \qquad x > 0. \tag{36}$$

(Here, and henceforth, we only allow solutions such that q has a non-negative real part.)

Using the results (36) in the boundary conditions (27), (29), (31), and recalling Eq. (35), we find

$$q/k = (\rho_{g0}/\rho_0)(\alpha^2/\alpha_g^2). \tag{37}$$

We see that q has a positive real part, as required. Noting Eqs. (23) and (34), we can rewrite Eq. (37) as

$$k^2[1 - (\rho_{g0}/\rho_0)^2(\alpha^2/\alpha_g^2)^2] = \omega^2/s^2, \tag{38}$$

or, alternatively, remembering Eq. (35), as

$$[1 - (\rho_{g0}/\rho_0)^2(\alpha^2/\alpha_g^2)^2](s^2/\alpha^2) = k_z^2/k^2. \tag{39}$$

Now, in some situations, the gas mass density can be very small compared to the liquid density. Thus, ignoring the term involving $(\rho_{g0}/\rho_0)^2$ in Eqs. (38) and (39), we see that Alfvén bulk waves in the liquid ($a^2 = \alpha^2 = \omega^2/k_z^2$) can be associated with surface sound waves in the gas at just those directions for $\mathbf{k}$ such that the phase velocity, ω/k, is practically equal to the sound speed s. These four directions are given approximately by $\cos\theta \approx \pm s/\alpha$.

Clearly, this can occur only if the sound speed is less than the Alfvén speed, $s < \alpha$ (again assuming $\rho_{g0} \ll \rho_0$).

The disturbance in the gas is a surface wave because the gas pressure is proportional to $\exp(-qx)$, and q is here positive. However, this *decaying* exponential extends many wavelengths into the gas, since the inequality $\rho_{g0} \ll \rho_0$ implies $q \ll k$ from (37).

B. Bulk Sound Waves

There appear to be no bulk waves in the gas unless there is a source of sound at $x = \infty$. In this case, there is no dispersion relation except that given by definition (34), but for a given ω and $\mathbf{k}$, one can calculate reflection coefficients. The results are not interesting, so we leave this matter now.

VI. SURFACE WAVES

In Sec. V we discussed those waves whose surfaces of constant phase along the magnetic lines have the velocity α. That is, we assumed $a^2 = \alpha^2$ and found bulk Alfvén waves in the liquid. In Sec. VI we seek solutions for which $a^2 \neq \alpha^2$. In this case, Eq. (16) requires

$$\nabla^2 v_z = 0 \quad \text{or} \quad v_z'' - k^2 v_z = 0, \qquad x < 0. \tag{40}$$

Furthermore, we see from Eqs. (15) and (13) that $\nabla \times \mathbf{b} = 0$, so that there are no volume conduction currents in the liquid, in this case.

A. Derivation of the Dispersion Relation

We could try to satisfy Eq. (40) by having $v_z \equiv 0$, but then it is easily shown by means of the boundary conditions (27), (30), (32), and the solutions (36), that all perturbation quantities must vanish identically. Therefore, we henceforth assume $v_z \not\equiv 0$, in which case, the solution to Eq. (40) has the form

$$v_z \propto e^{kx}, \qquad x \leq 0. \tag{41}$$

Using the solutions (41) and (36) in the boundary conditions (27), (30), (32), we easily obtain the following dispersion relation

$$a^2[1 + (\rho_{g0}/\rho_0)(k/q)] = \alpha^2 + \alpha_g^2. \tag{42}$$

If we set $(\rho_{g0}/\rho_0) = 0$, we obtain just that dispersion relation that would be found in the absence of the gas, namely

$$a^2 \equiv \omega^2/k_z^2 = \alpha^2 + \alpha_g^2,$$

with α^2 given by (8) and α_g^2 by (28). This result is well-known, and can be found as the appropriate limiting cases of the results of Melcher[1] (his problem 2b) and Chandrasekhar.[2] It represents a surface Alfvén wave in an incompressible, inviscid, highly conducting fluid, in the absence of complications such as surface tension and gravity.

B. Method of Solution of the Dispersion Relation

In view of Eqs. (23) and (34), the dispersion relation (42) can be considered as an equation for the unknown, $a^2 \equiv \omega^2/k_z^2$, provided that the propagation vector $\mathbf{k}$ is given. The technique we use is to solve first for q, and then for a^2.

From Eqs. (23) and (34), we have

$$a^2 = (k^2 - q^2)s^2/k_z^2. \tag{43}$$

Defining

$$\epsilon \equiv \rho_{g0}/\rho_0, \qquad S^2 \equiv s^2k^2/k_z^2 \equiv s^2/\cos^2\theta,$$
$$A^2 \equiv \alpha^2 + \alpha_g^2, \quad \text{and} \quad \tau \equiv q/k, \tag{44}$$

Eqs. (42) and (43) become, respectively,

$$a^2 = (1 - \tau^2)S^2 \tag{45}$$

and

$$\tau[\tau^2 - (1 - A^2/S^2)] = \epsilon(1 - \tau^2). \tag{46}$$

Downloaded 23 Mar 2012 to 142.150.190.39. Redistribution subject to AIP license or copyright; see http://pof.aip.org/about/rights_and_permissions

We henceforth refer to A as the "surface Alfvén speed" and to S as the "modified sound speed."

Equation (46) is a cubic equation for τ with real coefficients. There must be either three real roots, or one real and two complex-conjugates. Of these solutions, we only accept those for which τ has a non-negative real part, because q must have a non-negative real part [see Eqs. (36) and (44)].

C. General Properties of the Solutions of the Dispersion Relation (46)

It is immediately evident from relation (46) that τ cannot be purely imaginary; for, if τ were purely imaginary, then the left side of (46) would be also, but the right side would be real. Thus, q, from (44), cannot be purely imaginary. Since the gas pressure is proportional to exp $(-qx)$, we conclude that there can be no solutions involving sound waves radiating out to or coming in from $x = \infty$.

By plotting the functions of τ, namely $(\tau^2 - [1 - A^2/S^2])$, and $\epsilon(\tau^{-1} - \tau)$, on the same graph, we reach the following further conclusions about the solutions of (46):

(i) In both cases $A^2 > S^2$ and $A^2 < S^2$, there is a real, positive root, τ_1, with $\tau_1 < 1$.

(ii) In the case $A^2 < S^2$, there are also two real negative roots if ϵ is sufficiently small. These negative roots are of no physical interest, as stated before.

By writing (46) in the form

$$(\tau - \tau_1)(\tau - \tau_2)(\tau - \tau_3) = 0,$$

and carrying out the multiplication, we further find, by comparing the τ^2 term with that of (46), that

$$-\epsilon = \tau_1 + \tau_2 + \tau_3.$$

Since τ_1 is positive by conclusion (i), this equation leads us to:

(iii) Whether or not τ_2 and τ_3 are real, they always must have negative real parts, and hence are of no physical interest.

Thus, the only root of physical interest in any case is τ_1, with $0 < \tau_1 < 1$. Equation (45) now tells us that a^2, or ω^2, is always positive, so that the solution always represents a purely oscillatory motion with respect to time.

Finally, we discuss the case of no magnetic field. According to Eqs. (8), (28), and (44), this means that $A = 0$. Then, Eq. (46) has the following three roots: $\tau_1 = 1$, $\tau_2 = -1$, and $\tau_3 = -\epsilon$. Of these, only the first is valid, and corresponds to a perturbation of zero frequency [see (45)]. Boundary condition (32) then implies that $p_g' = 0$ at $x = 0$. But since $p_g =$ const exp $(-qx)$, the constant must be zero, so $p_g \equiv 0$. Thus, this type of perturbation involves only a static deformation of the boundary of the liquid, with no oscillations and no disturbance in the neutral gas. These remarks also apply when $A \neq 0$ to the case of exactly perpendicular propagation ($\theta = \pi/2$), since then $(A^2/S^2) \equiv (A^2 \cos^2\theta)/s^2 = 0$.

D. The Case of Relatively Low Gas Density

Here we make the assumption,

$$\rho_{g0} \ll \rho_0, \quad \text{or} \quad \epsilon \ll 1,$$

that is, the gas density is much less than the liquid density. In this case, the dispersion relation predicts the existence of two kinds of surface waves, an Alfvén-type wave for $A^2 < S^2$, and a sound-type wave for $A^2 > S^2$.

To see this, we consider Eq. (46) in the limit $\epsilon \rightarrow 0$, for fixed (A^2/S^2). To lowest order in ϵ, it is easily seen, for example by direct substitution, that the three roots of Eq. (46) are given by

(a) $\tau = -\epsilon(1 - A^2/S^2)^{-1}$;

(b) $\tau = (1 - A^2/S^2)^{\frac{1}{2}} + \frac{1}{2}\epsilon[(S^2/A^2) - 1]^{-1}$;

(c) $\tau = -(1 - A^2/S^2)^{\frac{1}{2}} + \frac{1}{2}\epsilon[(S^2/A^2) - 1]^{-1}$.

For $A^2 < S^2$, only (b) is valid, since the only valid root must be positive (see Sec. VIC). Substituting into Eq. (45), we then find, to lowest order in ϵ,

$$a^2 = A^2[1 - \epsilon(1 - A^2/S^2)^{-\frac{1}{2}}], \quad A^2 < S^2. \tag{47a}$$

Recalling that $A^2 \equiv \alpha^2 + \alpha_g^2$, we see that the wave velocity along the lines of force is practically the surface Alfvén speed, A.

For $A^2 > S^2$, (a) is now the only positive (and hence valid) root. Substituting into (45), we then find, to lowest order in ϵ,

$$a^2 = [1 - \epsilon^2(1 - A^2/S^2)^{-2}]S^2, \quad A^2 > S^2. \tag{47b}$$

We now see that the wave velocity along the lines of force is essentially the modified sound speed, $S \equiv s/\cos\theta$, where θ is the angle of the propagation vector with the magnetic lines. Recalling that $a^2 \equiv \omega^2/k_z^2$ [Eq. (12)], and $S^2 \equiv s^2k^2/k_z^2$ [Eq. (44)], we see that (47b) can be written (since here $\epsilon \ll 1$),

$$\omega^2 \approx s^2k^2,$$

that is, the wave speed along the propagation vector $\mathbf{k}$ is just the speed of sound in the gas.

For the Alfvén surface waves ($A^2 < S^2$), the disturbance is localized within a few wavelengths of the

Downloaded 23 Mar 2012 to 142.150.190.39. Redistribution subject to AIP license or copyright; see http://pof.aip.org/about/rights_and_permissions

interface, being proportional to $\exp(kx)$ in the liquid $(x < 0)$, and approximately to $\exp[-kx(1-A^2/S^2)^{\frac{1}{2}}]$ in the gas $(x > 0)$.

For the surface sound waves $(A^2 > S^2)$, the disturbance extends many wavelengths into the gas, being approximately proportional to

$$\exp[-kx\epsilon(A^2/S^2 - 1)^{-1}],$$

where we recall that $\epsilon \equiv \rho_{g0}/\rho_0 \ll 1$.

We note that for small gas density, the critical magnetic fields at which there is a transition from a surface Alfvén wave to a surface sound wave are approximately determined by

$$A^2 \approx S^2 \quad \text{or} \quad \alpha^2 + \alpha_g^2 \approx s^2/\cos^2\theta.$$

If the internal and external magnetic fields are equal, the transition occurs at a field determined by

$$\alpha^2 \approx s^2/2\cos^2\theta.$$

Thus the critical field strength is required to be larger for propagation angles θ that approach $\frac{1}{2}\pi$.

Finally, we must emphasize that these results are only valid when A^2 is not too close to S^2. The transition from surface Alfvén waves to surface sound waves does not occur suddenly as the magnetic field strength is increased, but gradually, within a small transition region around $A^2 \approx S^2$. We now briefly examine this transition region.

E. The Case $A^2 \approx S^2$: The Surface Alfvén Speed is Close to the Modified Sound Speed

Let us write Eq. (46) as

$$\tau^3 = (1 - A^2/S^2)\tau + \epsilon(1 - \tau^2)$$

and regard $(1 - A^2/S^2)$ as being of the order ϵ or smaller. Then, neglecting the τ and τ^2 terms, which is consistent with the final result, we find the following approximate solutions in the limit $\epsilon \to 0$, namely

$$\tau = \epsilon^{\frac{1}{3}}, \quad \tau = \epsilon^{\frac{1}{3}}\exp(2\pi i/3), \quad \tau = \epsilon^{\frac{1}{3}}\exp(-2\pi i/3).$$

Of these, only the first root is valid, the rest having negative real parts. Substituting the first root into Eq. (45), we have

$$a^2 = (1 - \epsilon^{\frac{2}{3}})S^2, \quad \text{or} \quad \omega^2 = (1 - \epsilon^{\frac{2}{3}})s^2k^2$$

to lowest order in ϵ.

VII. DISCUSSION

It is remarkable that the interface cannot support surface Alfvén waves if the magnetic field is too strong, no matter how small the gas density is. (This statement must, of course, be qualified by the remark that if the gas density is too low, the model equations break down. The fluid-type equations for the gas are only valid when the mean free path of the gas molecules is much smaller than the characteristic macroscopic lengths involved, such as a wavelength. For wavelengths on the order of 1 cm, the limiting gas pressure is about 1 mm Hg at room temperature.) It is also remarkable that the critical magnetic field is practically independent of the gas density at low densities, depending only on the gas temperature. If there were no gas at all, the dispersion relation would be simply $a^2 = A^2$, so that surface Alfvén waves would then exist without an upper limit on the magnetic field strength.

For mercury liquid surrounded by air at room temperature, the critical magnetic field strength is on the order of 400 000 G. This high field can be cut by factors of 2 or 3 if gases such as xenon, radon, or uranium-hexafluoride are used instead of air. (It can be cut again by a factor 3 if liquid sodium is used instead of mercury.)

The present work is being extended to the case when the gas is flowing past the liquid, and the results will be reported shortly. In that case it is found possible to excite surface sound waves at much smaller magnetic fields than those quoted above.

Finally, we observe that, having solved the dispersion relation, one can go back and find all of the perturbation quantities, provided that the amplitude of one of them is given. We recall that the perturbation quantities in the liquid are proportional to $\exp[i(k_yy + k_zz - \omega t) + kx]$, and the perturbation quantities in the gas are proportional to $\exp[i(k_yy + k_zz - \omega t) - qx]$ (except, of course, for the external magnetic field perturbation, which is proportional to $\exp[i(k_yy + k_zz - \omega t) - kx]$), where q is defined by Eq. (34), and k is defined by Eq. (33). It is the amplitudes of these exponentials that we now wish to compare, so that when we now write a perturbation quantity, such as v_z, we shall really mean the amplitude of this quantity that multiplies one of the above exponentials.

Let us take the gas velocity component v_{gz} as having a given amplitude. Then, without going into details, we easily find [basically using the pressure balance boundary condition (27)] the following relations (between amplitudes) to lowest order in $\epsilon \equiv \rho_{g0}/\rho_0$

$$v_z = (1 - A^2/S^2)^{\frac{1}{2}}v_{gz}, \quad A^2 < S^2 \quad \text{(surface Alfvén waves);} \tag{48}$$

$$v_z = \epsilon v_{gz}(1 - A^2/S^2)^{-1}, \quad A^2 > S^2 \quad \text{(surface sound waves).} \tag{49}$$

Downloaded 23 Mar 2012 to 142.150.190.39. Redistribution subject to AIP license or copyright; see http://pof.aip.org/about/rights_and_permissions

The other perturbation amplitudes can now be obtained by means of Eqs. (13), (14), (15) in the liquid, Eq. (21) for the gas, and boundary condition (30).

From Eq. (48), we see that perturbation motions in the liquid due to the Alfvén surface wave produce perturbation motions in the gas of the same order of magnitude. On the other hand, from Eq. (49), we see that perturbation motions in the gas due to the surface sound wave produce perturbation motions in the liquid which are much smaller than those in the gas, smaller by the factor ϵ. This is as we expect: a sound wave in the light gas should have practically no effect on the heavy liquid. Yet the small effect on the conducting liquid must be crucial for the existence of the surface sound wave, as we have seen in Sec. VIC that no such wave is possible without a magnetic field, in the physical model in question.

VIII. THE CASE OF RELATIVELY HIGH GAS DENSITY

There has recently been some discussion in the literature as to the possibility of confining a hot plasma by a cold gas.[4] If the characteristic wave speed a is much less than the speed of sound in the plasma, then the latter should move approximately as an incompressible fluid[5] with respect to the hydromagnetic surface waves of interest here. (The speed of sound in the plasma, $\sim(T_e/m_i)^{\frac{1}{2}}$, would presumably be large because of the high electron temperature.)

Moreover, one might expect the cold gas to be much denser than the hot plasma under some circumstances. In the notation of Eq. (44), this means $\epsilon \gg 1$.[6]

We are thus led to solve the surface wave dispersion relation, Eqs. (45) and (46), in the limit $\epsilon \gg 1$. It is then easily seen that the only valid solution for the wave speed a is

$$a^2 = A^2/\epsilon,$$

regardless of the speed of sound in the gas.

We conclude that, in the case of a hot plasma surrounded by a cold, relatively dense gas, the hydromagnetic surface wave velocity along the magnetic lines a is smaller than the usual surface Alfvén speed A by the factor $(\epsilon)^{\frac{1}{2}} \equiv (\rho_{g0}/\rho_0)^{\frac{1}{2}}$. This is not a surprising result. In the case of increasing gas density, the stored magnetic energy must push around an increasing mass of material, so the oscillation frequency naturally decreases. This is analogous to the case of a mass M bouncing on the end of an ideal spring of force constant K. The characteristic frequency, $(K/M)^{\frac{1}{2}}$, decreases as $M^{-\frac{1}{2}}$, just as in the hydromagnetic problem.

ACKNOWLEDGMENT

I am very grateful to Professor R. Gajewski (of the Institute for Nuclear Research in Warsaw) for many helpful discussions about the formulation of the problem.

[4] C. M. Braams, Phys. Rev. Letters 17, 470 (1966) and references cited therein.

[5] The use of a single-fluid model for the hot plasma is questionable. A Vlasov-equation-description for both electrons and ions would be more accurate. However, if the cyclotron radii and Debye lengths are less than the characteristic wavelengths involved, and if the ion thermal speed is less than the wave velocity along the magnetic lines, one can expect qualitatively meaningful results from the simple fluid model.

[6] If the magnetic field is the same in both fluids, then pressure balance in equilibrium requires $\rho_0 s_0^2 = \rho_{g0} s^2$, where s_0 is the sound speed in the plasma. With $\rho_{g0} \gg \rho_0$, this condition is consistent with regard to the plasma as an incompressible fluid.

Downloaded 23 Mar 2012 to 142.150.190.39. Redistribution subject to AIP license or copyright; see http://pof.aip.org/about/rights_and_permissions