CSE417: Introduction to Machine Learning
Problem Set 1

Problems

1. LFD Problem 1.3 2. Out of textbook 3. LFD Problem 1.7 4. LFD Problem 1.8

Q1: LFD Problem 1.3

(a)

Since w^* is the optimal set of weights, for all $1 \leq n \leq N$, ${w^*}^T x_n$ must have same sign as y_n since a linear separation is achieved.
Therefore, for all $1 \leq n \leq N$, $y_n({w^*}^T x_n) > 0$, which suggests $min_{1 \leq n \leq N} y_n({w^*}^T x_n) > 0$, or $\rho > 0$

(b)

Given the update rule, $w(t+1) = w(t) + y(t)x(t)$, transpose both sides and multiply by w^*:

$$
\begin{aligned}
w(t+1)^T &= w(t)^T + (y(t)x(t))^T \\
w(t+1)^T w^* &= w(t)^T w^* + (y(t)x(t))^T w^* \\
w(t+1)^T w^* &= w(t)^T w^* + y(t)(x(t)^T w^*) \rightarrow \text{ given y(t) is 1 * 1} \\
w(t+1)^T w^* &= w(t)^T w^* + y(t)({w^*}^T x(t)) \rightarrow \text{ given } {w^*}^T x(t) \text{ is 1*1 so its symmetric}
\end{aligned}
$$

And by the definition of $\rho = min_{1 \leq n \leq N} y_n({w^*}^T x_n)$, for all $1 \leq t \leq N$, we have $\rho \leq y(t)({w^*}^T x(t))$.
Then we could conclude that:

$$w(t+1)^T w^* \geq w(t)^T w^* + \rho$$

Base Case:
t = 0, since we assumed w(0) = 0, we have:

$$w^T(0)w^* = 0 \geq 0\rho = 0$$

Induction:
for t = n, given that the inequality is valid:

$$w^T(n)w^* \geq n\rho$$

and incorporate the inequality we deduced above:

$$
\begin{aligned}
w^T(n+1)w^* &\geq w(n)^T w^* + \rho \\
w^T(n+1)w^* &\geq n\rho + \rho \\
w^T(n+1)w^* &\geq (n+1)\rho
\end{aligned}
$$

so the inequality must also be valid for t = n + 1.
Thus proved.

(c)

Given the update rule, $w(t+1) = w(t) + y(t)x(t)$, transpose both sides and multiply by the original equation:

$$w(t+1) * w(t+1)^T = (w(t) + (y(t)x(t)))(w(t)^T + (y(t)x(t))^T)$$
$$||w(t+1)||^2 = ||w(t)||^2 + 2y(t)x(t)w(t)^T + y(t)^2||x(t)||^2$$

Since for any y(t), we have $y(t)^2 = 1$, and $2y(t)x(t)w(t)^T < 0$ given y(t) represents a misclassified point, we have:

$$||w(t+1)||^2 \leq ||w(t)||^2 + y(t)^2||x(t)||^2$$
$$||w(t+1)||^2 \leq ||w(t)||^2 + ||x(t)||^2$$

(d)

Proof:
Base case:
for t = 0, $||w(t)||^2 = 0 \leq 0 * R^2 = 0$

Induction:
Given that $||w(t)||^2 \leq tR^2$, we have:

$$||w(t+1)||^2 \leq ||w(t)||^2 + ||x(t-1)||^2$$
$$||w(t+1)||^2 \leq ||w(t)||^2 + R^2$$
$$||w(t+1)||^2 \leq ||w(t)||^2 + R^2$$
$$||w(t+1)||^2 \leq ||w(t)||^2 + R^2$$
$$||w(t+1)||^2 \leq tR^2 + R^2$$
$$||w(t+1)||^2 \leq (t+1)R^2$$

□

(e)

Proof: Using the conclusion in part (b):

$$
\begin{aligned}
w^T(t)w^* &\geq (t)\rho \\
\frac{w^T(t)w^*}{||w(t)||} &\geq \frac{t\rho}{||w(t)||} \\
\frac{w^T(t)w^*}{||w(t)||} &\geq \frac{t\rho}{||w(t)||} \geq \frac{t\rho}{\sqrt{tR^2}} \rightarrow \text{given the conclusion in (d)} \\
\frac{w^T(t)w^*}{||w(t)||} &\geq \frac{t\rho}{\sqrt{tR^2}} \\
\frac{w^T(t)w^*}{||w(t)||} &\geq \frac{\sqrt{t}\rho}{R}
\end{aligned}
$$

Since $\frac{w^T(t)w^*}{||w(t)||||w^*||} = \cos\theta$ where θ represents the angle between $w^T(t)$ and w^*, we must have $\frac{w^T(t)w^*}{||w(t)||||w^*||} \leq 1$, then we could rewrite the inequality to be:

$$
\begin{aligned}
\sqrt{t} &\leq \frac{Rw^T(t)w^*}{||w(t)||\rho} \\
\sqrt{t}\frac{w^T(t)w^*}{||w(t)||||w^*||} &\leq \frac{Rw^T(t)w^*}{||w(t)||\rho} \rightarrow \text{Given That} \sqrt{t} \geq 0 \\
\frac{\sqrt{t}}{||w^*||} &\leq \frac{R}{\rho} \\
\sqrt{t} &\leq \frac{R||w^*||}{\rho} \\
t &\leq \frac{R^2||w^*||^2}{\rho^2}
\end{aligned}
$$

□

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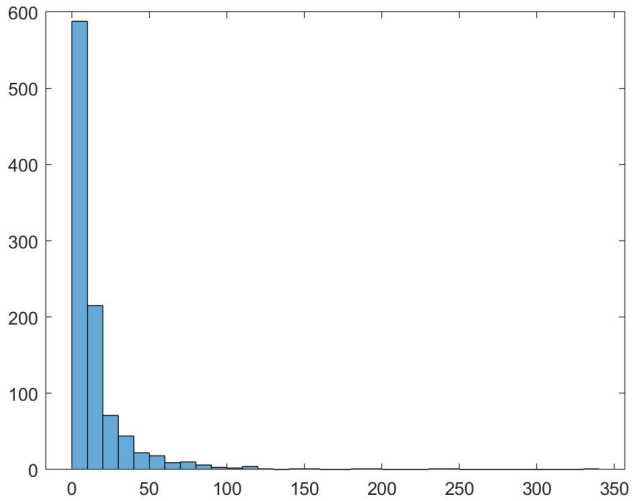
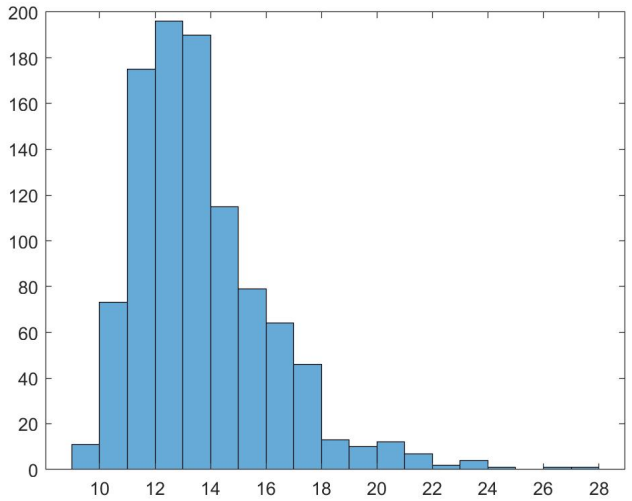
Q2

The first plot is the histogram of number of operation for each iteration and the second plot is a histogram of log difference. Plot attached below:





It looks like the log difference can be approximate by a normal distribution which suggests it is completely random with respect to the algorithm. Also, the bound is loose for the PLA algorithm as it converges fast for such small linearly separable data we generated.

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Q3: LFD Problem 1.7

(a)

For $\mu = 0.05$

$\mathrm{P}[0|10, 0.05] = 0.95^{10} = 0.5987$
1 Coin: P $= 0.5987$
10 Coins: P $= 1 - (1 - 0.95^{10})^{10} = 0.9998$
1000 Coins: P $= 1 - (1 - 0.95^{10})^{1000} = 1$
1000000 Coins: P $= 1 - (1 - 0.95^{10})^{1000000} = 1$

For $\mu = 0.8$

$\mathrm{P}[0, |10, 0.8] = 0.2^{10} = 1.024e - 7$
1 Coin: P = 1.024e-7
10 Coins: P $= 1 - (1 - 0.2^{10})^{10} = 0.000001024$
1000 Coins: P $= 1 - (1 - 0.2^{10})^{1000} = .0001024$
1000000 Coins: P $= 1 - (1 - 0.2^{10})^{1000000} = 0.09733$

(b)

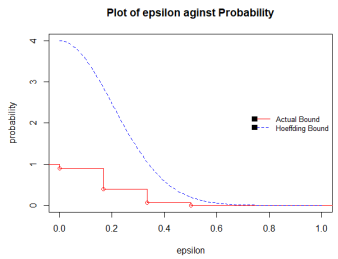
The result should be equivalent to P $= 1 - P(|v_1 - u_1| \leq \epsilon) * P(|v_2 - u_2| \leq \epsilon)$ for each ϵ, which produces a step function satisfy the following condition:

$$\begin{cases} P = 1, & \epsilon = 0 \\ P = 1 - P[3|6, 0.5]^2, & 0 \leq \epsilon < \frac{1}{6} \\ P = 1 - P[2, 3, 4|6, 0.5]^2, & \frac{1}{6} \leq \epsilon < \frac{1}{3} \\ P = 1 - P[1, 2, 3, 4, 5|6, 0.5]^2, & \frac{1}{3} \leq \epsilon < \frac{1}{2} \\ P = 0 & \epsilon > 0.5 \end{cases}$$

Using Hoeffding bound for 2 coins,

$$P[max|v_i - u_i| > \epsilon] = 2 * 2e^{-12x^2} = 4e^{-12x^2}$$

Plot attached below:



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Q4: LFD Problem 1.8

(a)

By definition, we have that $E(t) = \int_0^\infty \frac{tP(t)}{\alpha} dt$
Then for each $\alpha > 0$, we have:

$$\begin{aligned}
\frac{E(t)}{\alpha} &= \int_0^\infty \frac{tP(t)}{\alpha} dt \\
&= \int_\alpha^\infty \frac{tP(t)}{\alpha} + \int_0^\alpha \frac{tP(t)}{\alpha} dt \\
&\geq \int_\alpha^\infty \frac{tP(t)}{\alpha} dt \\
&\geq \int_\alpha^\infty \frac{\alpha P(t)}{\alpha} dt \\
&= \int_\alpha^\infty P(t) dt \\
&= P(t \geq \alpha)
\end{aligned}$$

Proved.

(b)

By definition, $E[(u-\mu)^2] = \sigma^2$, then using the conclusion in part (a), we have:

$$P((u-\mu)^2 \geq \alpha) \leq \frac{E[(u-\mu)^2]}{\alpha}$$

$$P((u-\mu)^2 \geq \alpha) \leq \frac{\sigma^2}{\alpha}$$

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(c)

By definition, for N iid random variables $(u_1, u_2, ...u_n)$ each with $E(u_i) = \mu$ and variance $Var(u_i) = \sigma^2$ for $0 < i \leq n$
Let $u = \frac{u_1+u_2+...+u_n}{n}$ we have that:

$$\begin{aligned} E[u] &= E[\frac{u_1 + u_2 + ... + u_n}{n}] \\ &= \frac{1}{N}E[(u_1 + u_2 + ... + u_n)] \\ &= \frac{1}{N}(E[u_1] + E[u_2] + ...E[u_n]) \\ &= \frac{1}{N}(N\mu) \\ &= \mu \end{aligned}$$

And that:

$$\begin{aligned} Var[u] &= Var[\frac{u_1 + u_2 + ... + u_n}{N}] \\ &= \frac{1}{N^2}Var[(u_1 + u_2 + ... + u_n)] \\ &= \frac{1}{N^2}(Var[u_1] + Var[u_2] + ...Var[u_n]) \\ &= \frac{1}{N^2}(n\sigma^2) \\ &= \frac{\sigma^2}{N} \end{aligned}$$

Then similar to that in part (b):

$$\begin{aligned} P((u-\mu)^2 \geq \alpha) &\leq \frac{E[(u-\mu)^2]}{\alpha} \\ P((u-\mu)^2 \geq \alpha) &\leq \frac{Var[u]}{\alpha} \\ P((u-\mu)^2 \geq \alpha) &\leq \frac{\sigma^2}{N\alpha} \end{aligned}$$

proved.