Całkowanie numeryczne metodą Gaussa w przedziale A,B, Całkowanie macierzy H

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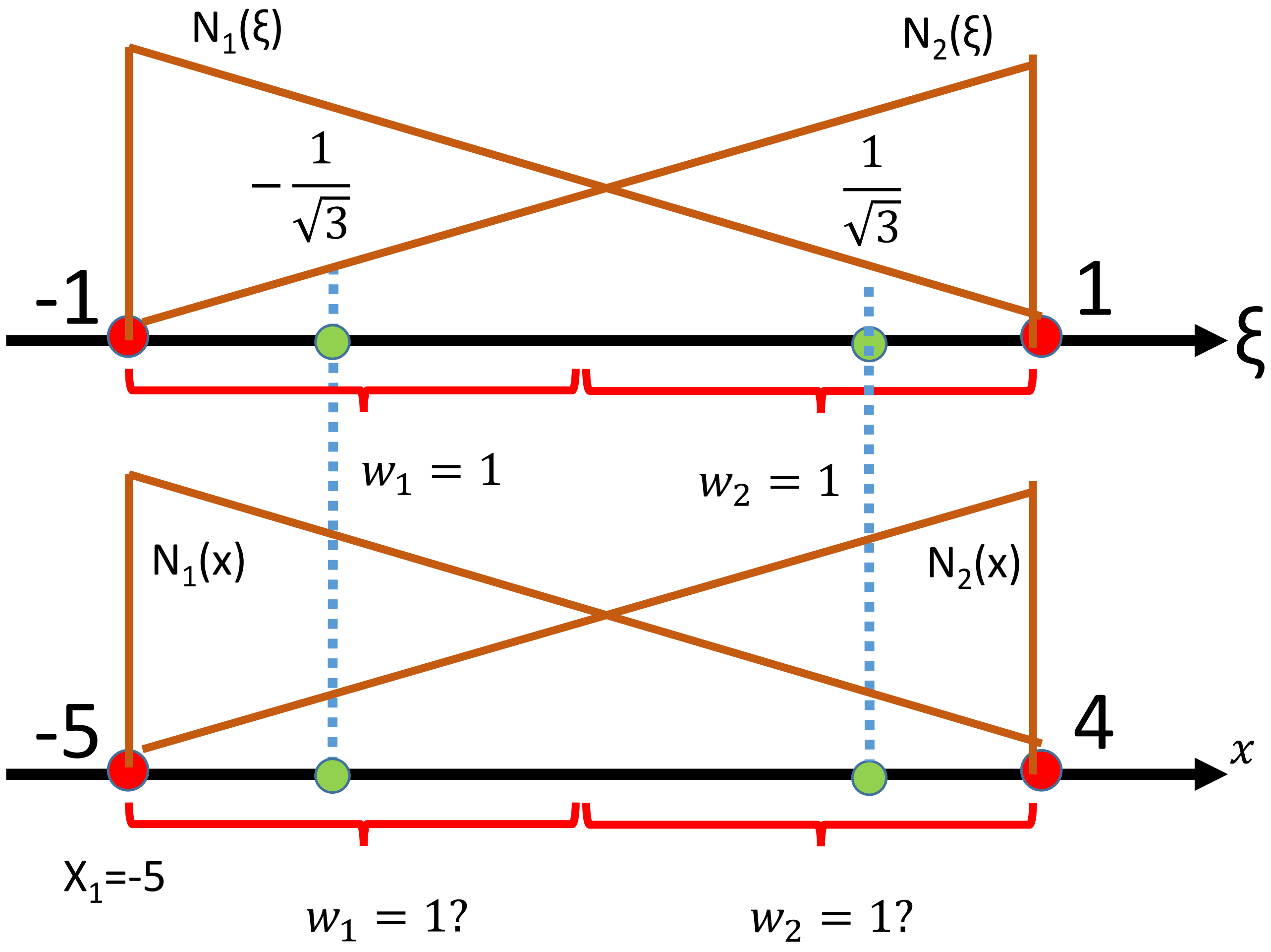
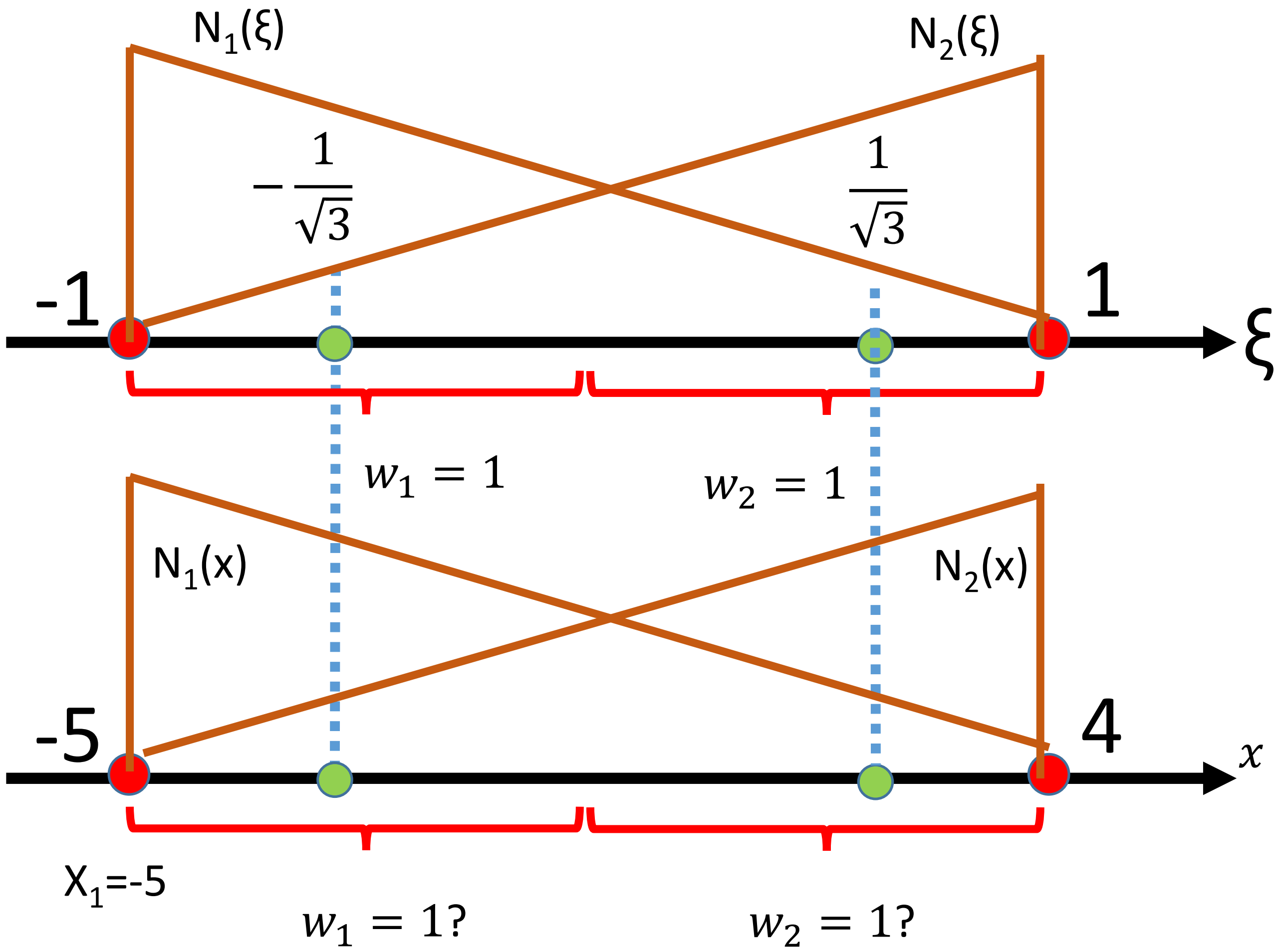
WIMiIP, KISiIM, AGH

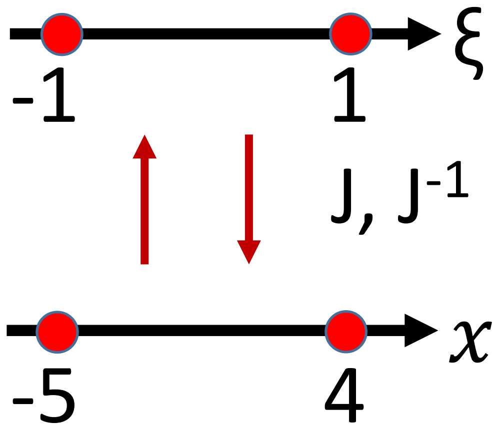
B5, pokój 710

Jakobian przekształcenia 1D

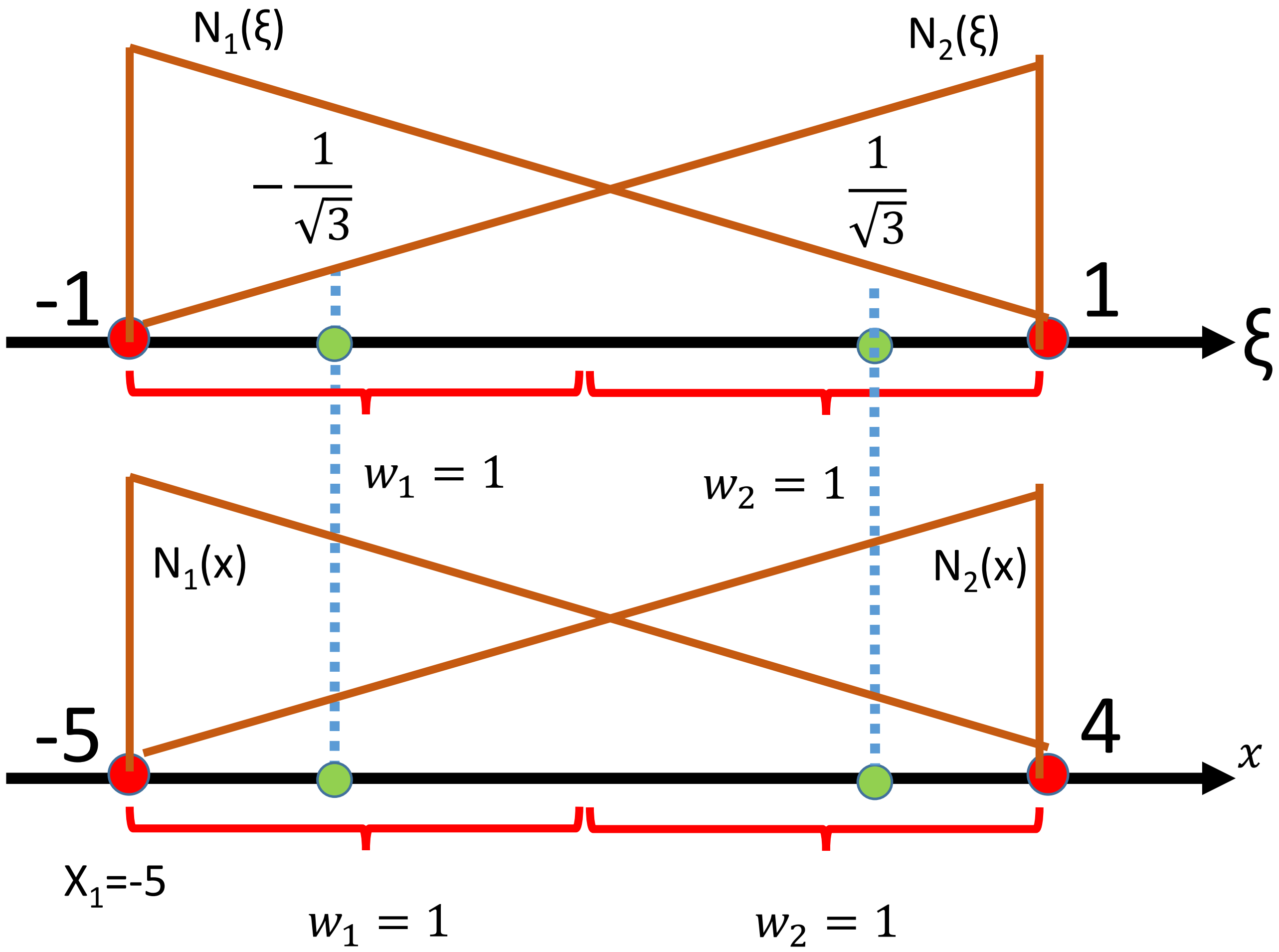


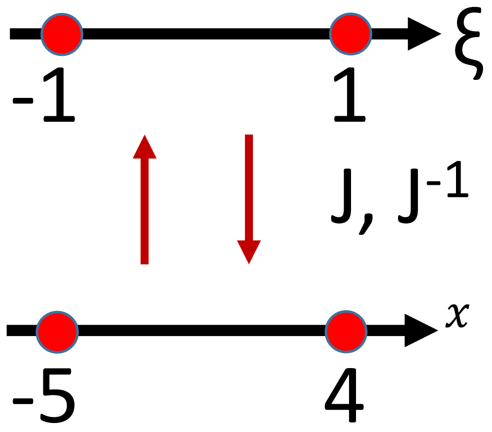
Jakobian przekształcenia 1D





Jakobian przekształcenia 1D

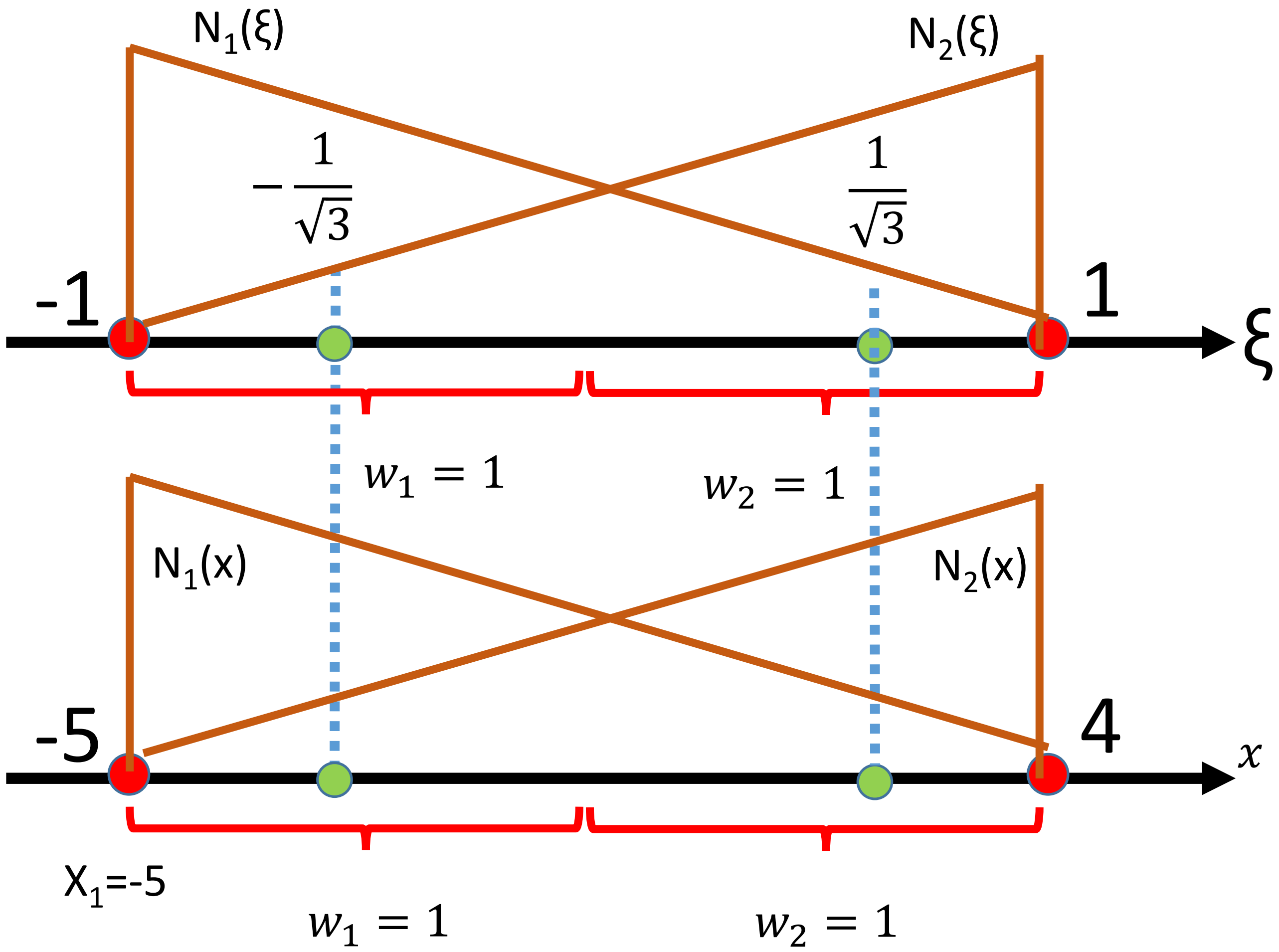




$$N_1(\mathrm{x}) = \frac{x_2 - x}{x_2 - x_1}$$

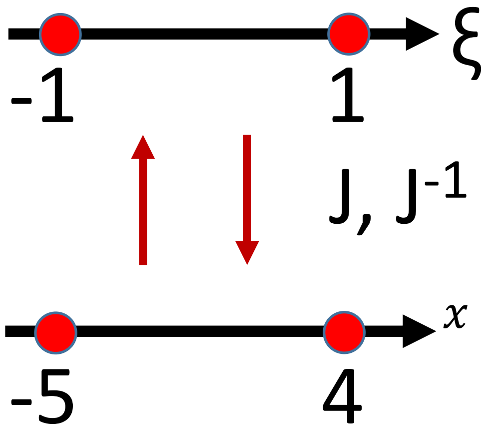
$$N_2(\mathrm{x}) = \frac{x - x_1}{x_2 - x_1}$$

Jakobian przekształcenia 1D



$$N_1(\xi) = \frac{\xi_2 - \xi}{\xi_2 - \xi_1}$$

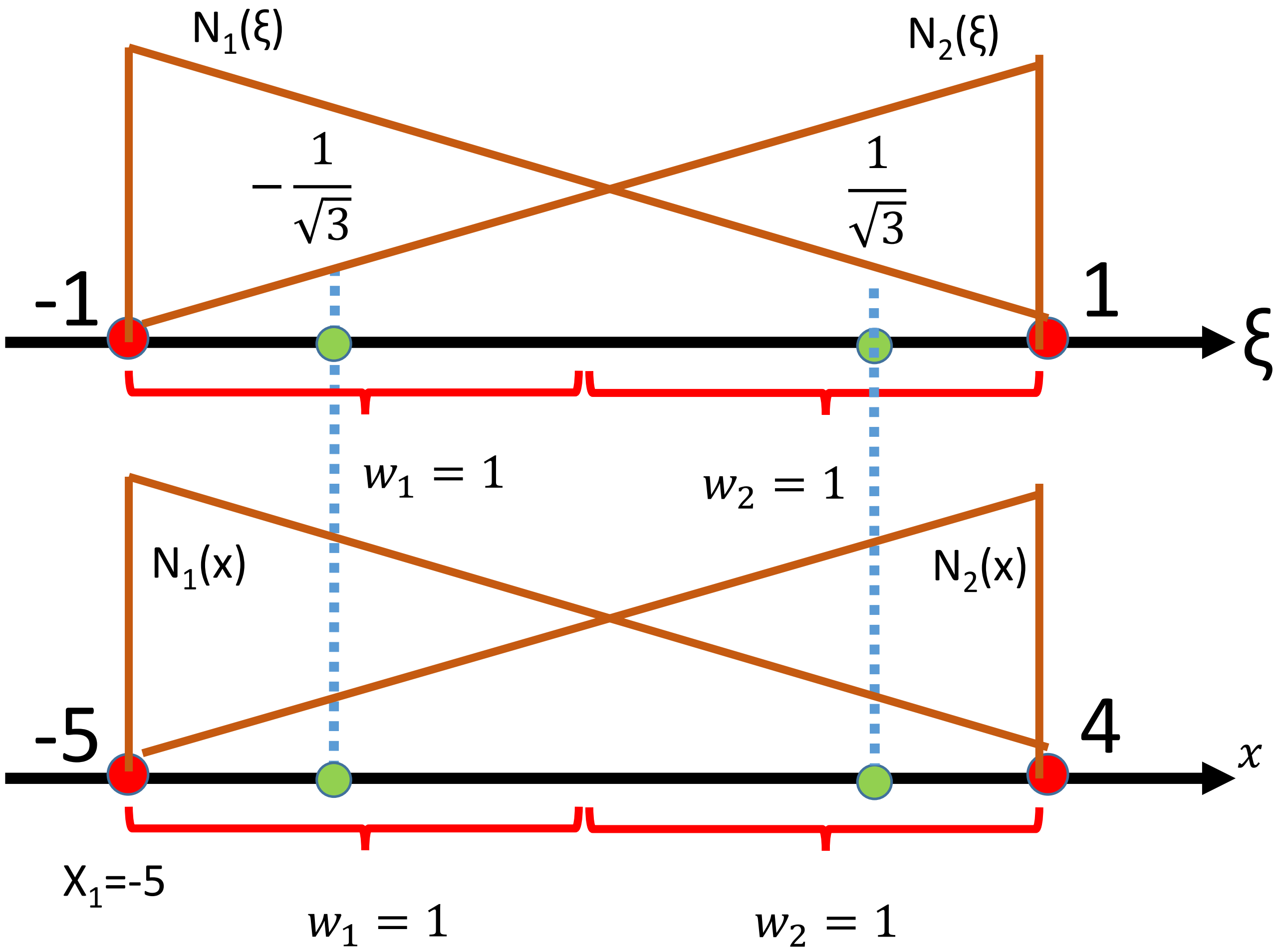
$$N_2(\xi) = \frac{\xi - \xi_1}{\xi_2 - \xi_1}$$



$$N_1(\mathrm{x}) = \frac{x_2 - x}{x_2 - x_1}$$

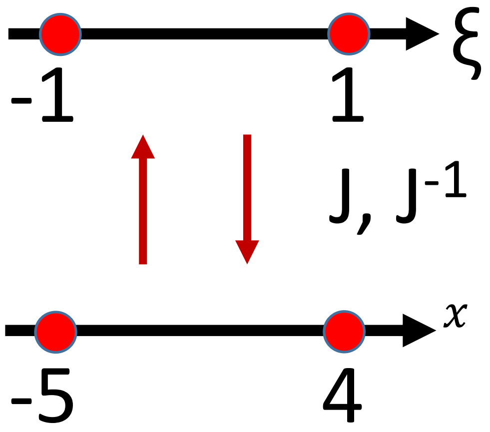
$$N_2(\mathrm{x}) = \frac{x - x_1}{x_2 - x_1}$$

Jakobian przekształcenia 1D



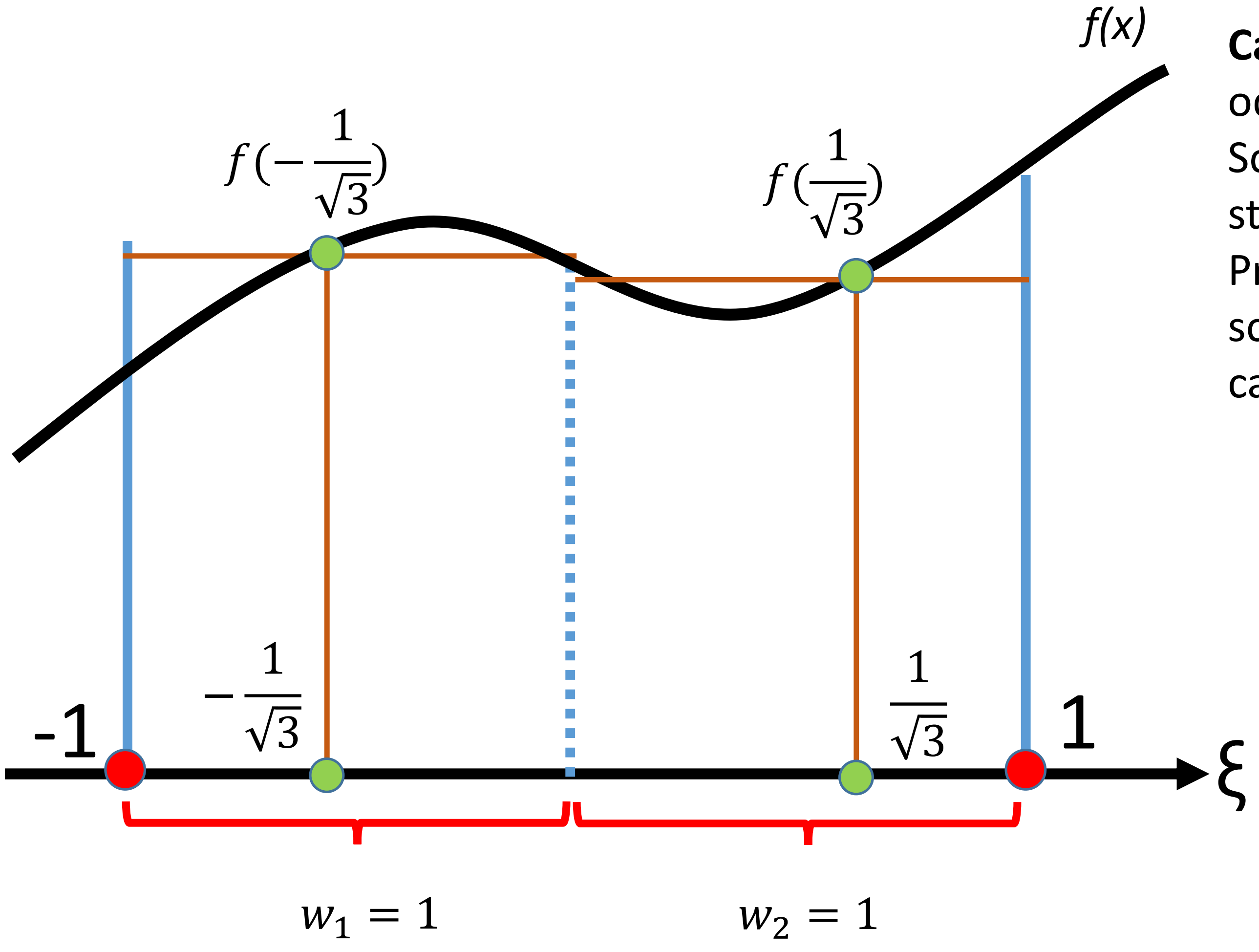
$$N_1(\xi) = \frac{\xi_2 - \xi}{\xi_2 - \xi_1} = \frac{1 - \xi}{1-(-1)} = \frac{1}{2}(1 - \xi)$$

$$N_2(\xi) = \frac{\xi - \xi_1}{\xi_2 - \xi_1} = \frac{1}{2}(1 + \xi)$$



$$N_1(\mathrm{x}) = \frac{x_2 - x}{x_2 - x_1}$$

$$N_2(\mathrm{x}) = \frac{x - x_1}{x_2 - x_1}$$



$$\int_{-1}^{1} f(\xi)d\xi = w_1 * f(\xi_1) + w_2 * f(\xi_2)$$

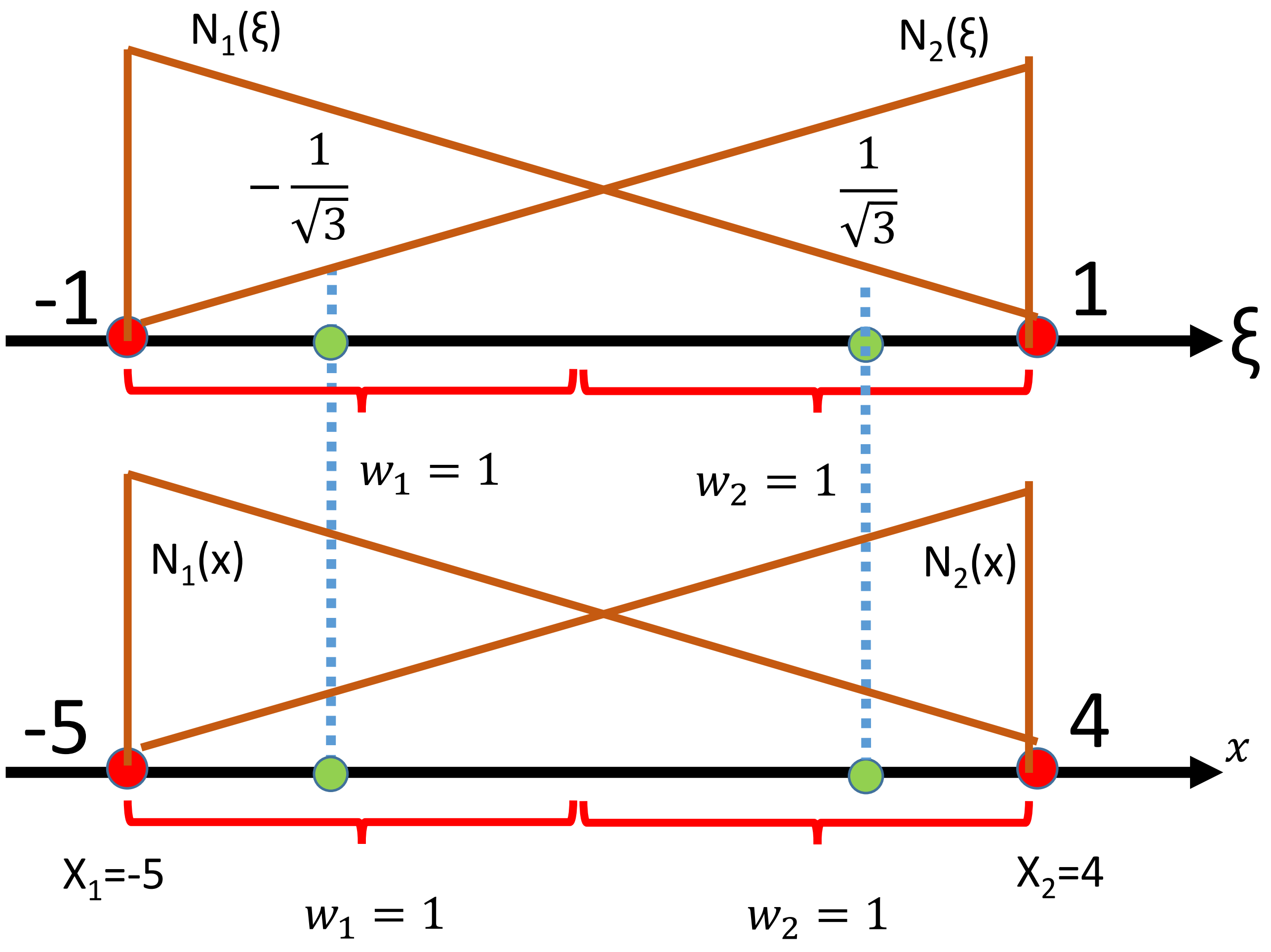
Całkowanie metodą Gaussa

odbywa się w przedziale <-1; 1>

Schematy całkowania zostały opracowane oraz stabelaryzowane. Nazywane są one kwadraturami Gaussa. Przedstawiono przykład dla *n*=1 czyli dwupunktowego schematu całkowania. X_k oznaczono współrzędną punktu całkowania a A_k wagę punktu całkowania.

Węzły i współczynniki kwadratur Gaussa-Legendre'a dla N=1, 2, 3, 4

N	k	Węzły x_k	Współczynniki A_k
1	0; 1	$\mp 1/\sqrt{3}$	1
2	0; 2	$\mp\sqrt{3/5}$	5/9
	1	0	8/9
3	0; 3	∓ 0.861136	0.347855
	1;2	∓ 0.339981	0.652145
4	0;4	∓ 0.906180	0.236927
	1;3	∓ 0.538469	0.478629
	2	0	0.568889



Całkowanie metodą Gaussa

Jeżeli całkowanie realizowane jest w innym przedziale niż <-1;1> punkty całowania znajdują się w innych miejscach. Ich lokalizację należy obliczyć. Funkcje kształtu w układzie lokalnym maja postać:

$N_1(\xi)=0.5*(1-\xi)$

$N_2(\xi)=0.5*(1+\xi)$

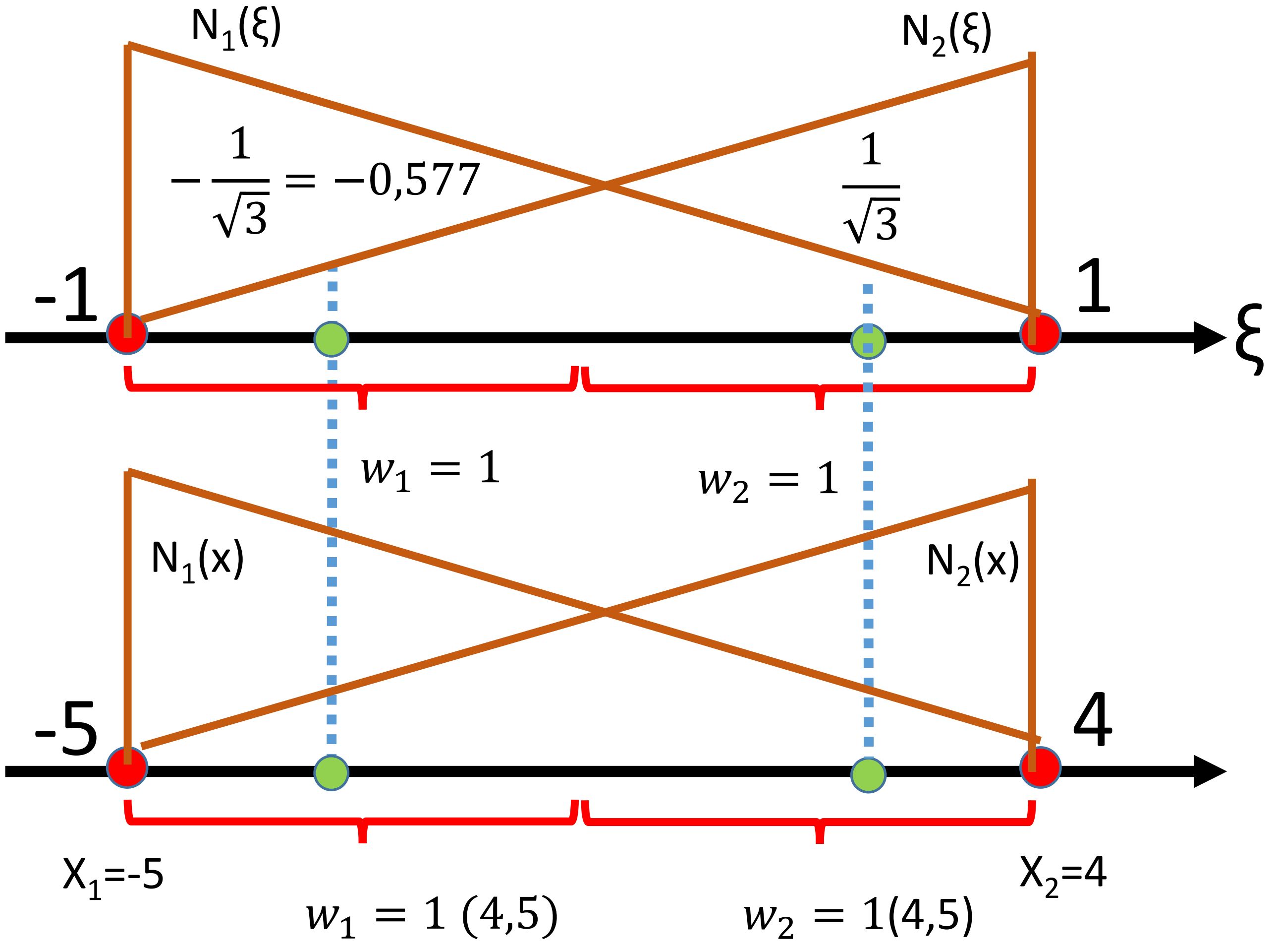
$N_1(x)=(x_2-x)/(x_2-x_1)$

$N_2(x)=(x-x_1)/(x_2-x_1)$

Z przedstawionego schematu wynika, iż bez względu na to ile wynosi x_1 oraz x_2 wartość funkcji kształtu w punktach całkowania w układzie lokalnym oraz globalnym jest taka sama.

$$\int_{-1}^{1} f(\xi)d\xi = w_1 * f(\xi_1) + w_2 * f(\xi_2) = \sum_{i=1}^{n}(f(\xi_i)w_i)$$

Obliczanie położenia punktów całkowania
Interpolacja x



Jak można zauważyć waga w układzie globalnym nie jest równa 1. Wartość ta wynika z geometrii. W układzie lokalnym długość pomiędzy -1 a 1 wynosi 2 dlatego suma wag wynosi 2. W układzie globalnym długość wynosi x_2-x_1 czyli 9. Dlatego waga powinna być równa 4,5.

Ponieważ wartości funkcji N(x) w punktach całowania mają taką samą wartość jak N(ξ) w punktach całkowania interpolację można przeprowadzić w następujący sposób:

$x=N_1(\xi)*x_1+N_2(\xi)*x_2$

Obliczamy wartości funkcji kształtu w pierwszym punkcie całkowania ξ=-0,577:

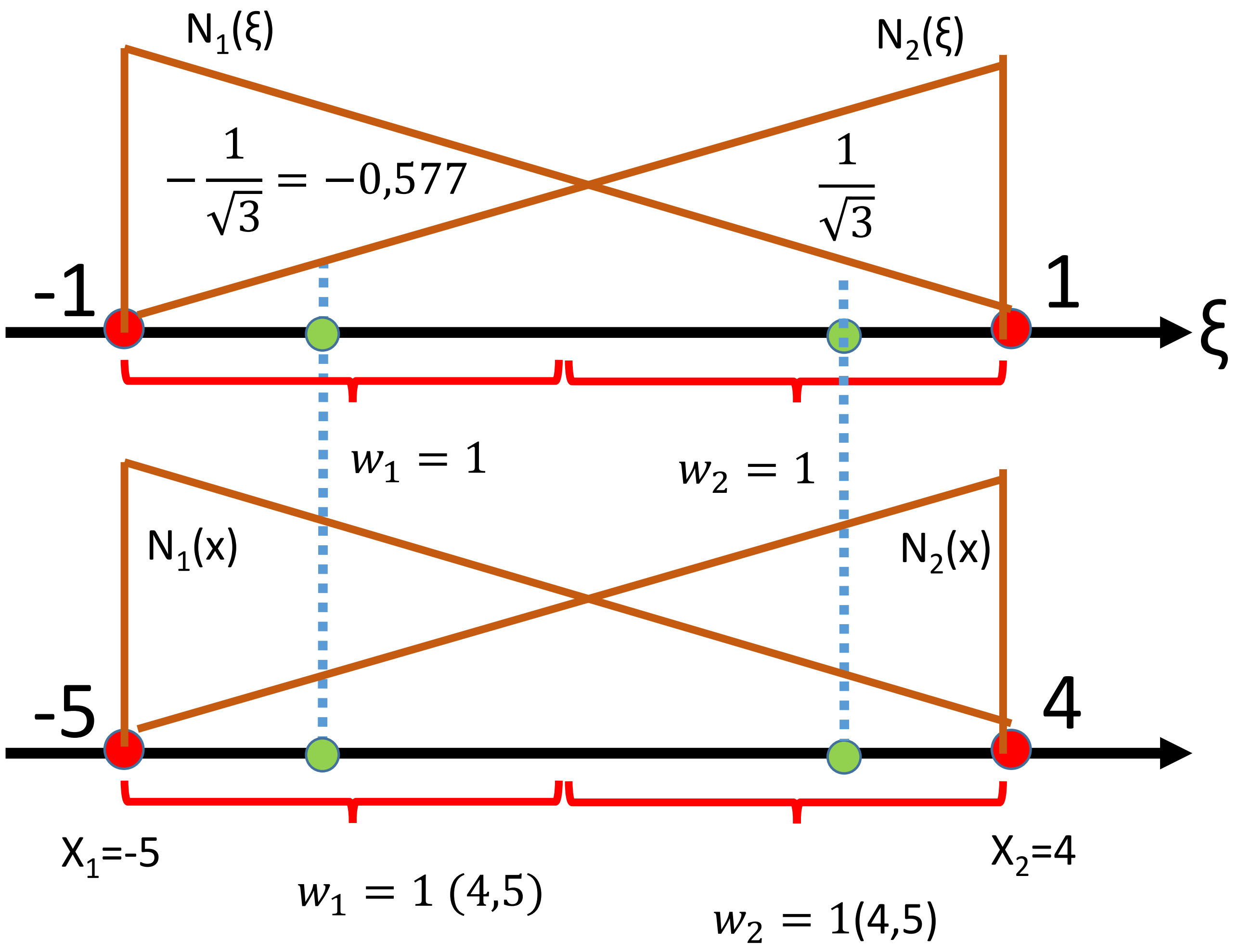
$N_1(\xi)=0.5*(1-\xi)=0.5*(1-(-0{,}577))=0{,}788$

$N_2(\xi)=0.5*(1+\xi)=0.5*(1+(-0{,}577))=0{,}212$

$x_{pc1}=0{,}788*(-5)+0{,}212*4=-3{,}098$

$x_{pc2}=0{,}212*(-5)+0{,}788*4=2{,}098$

$$\int_{-5}^{4} f(x)dx = (w_1 * f(x_{pc1}) + w_2 * f(x_{pc2}))detJ$$



$$\int_{-5}^{4} f(\mathrm{x})d\mathrm{x} = (w_1 * f(\mathrm{x}_{pc1}) + w_2 * f(\mathrm{x}_{pc2})) * \det[J]$$

$$\int_{-5}^{4} f(\mathrm{x})d\mathrm{x} = \sum_{i=1}^{n} (f(\mathrm{x}_{pci}) * w_i) * \det[J]$$

Różnica długości wag związana jest z Jakobianem przekształcenia 1D: http://home.agh.edu.pl/~pkustra/MES/Jakobian.pdf

$x = N_1(\xi) * x_1 + N_2(\xi) * x_2$

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi} * x_1 + \frac{\partial N_2}{\partial \xi} * x_2$$

$$\frac{\partial N_1}{\partial \xi} = -0,5 \qquad \frac{\partial N_1}{\partial \xi} = 0,5$$

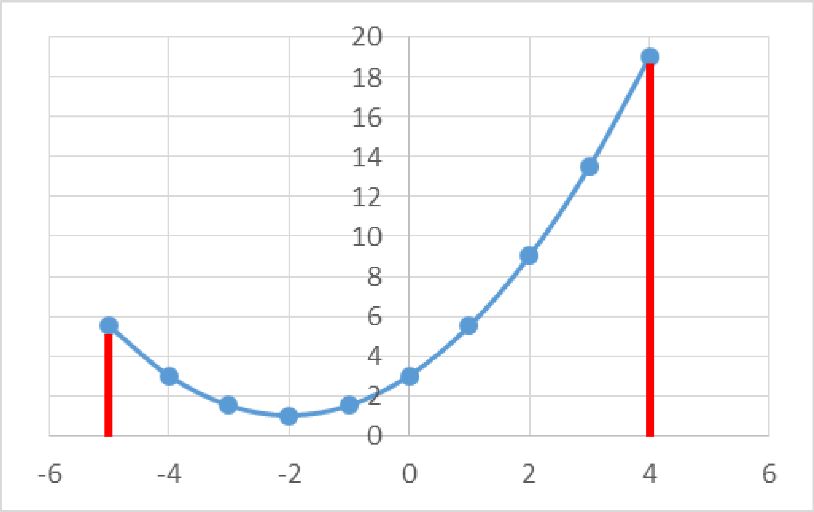
$$\frac{\partial x}{\partial \xi} = -0,5 * x_1 + 0,5 * x_2 = \frac{(x_2 - x_1)}{2} = \frac{\Delta x}{\Delta \xi}$$

$$\frac{\partial x}{\partial \xi} = \frac{(4-(-5))}{2} = \frac{9}{2} = 4,5 = \det[J]$$

Wartość 4,5 mówi o tym jak zmienia się długość układu globalnego względem układu lokalnego. W związku z tym całkowanie realizowane jest w sposób następujący:

Przykład dla dwupunktowego schematu całkowania

$$\int_{-5}^{4} 0.5x^2 + 2x + 3dx$$



$$f(x) = 0.5x^2 + 2x + 3$$

x_{pc1}=0,788*(-5) + 0,212*4=-3,098

x_{pc2}=0,212*(-5) + 0,788*4=2,098

$$f(xpc1) = 0.5(-3{,}098)^2 + 2(-3{,}098) + 3 = 1{,}60288$$

$$f(xpc2) = 0.5(2{,}098\,)^2 + 2(2{,}098\,) + 3 = 9{,}398$$

$$\int_{-5}^{4} f(\mathrm{x})dx = (w_1 * f(\mathrm{x}_{pc1}) + w_2 * f(\mathrm{x}_{pc2})) * \det[J]$$

$$\int_{-5}^{4} f(\mathrm{x})dx = (1 * 1{,}60288 + 1*9{,}938) * 4{,}5 = 49{,}5$$

Węzły i współczynniki kwadratur Gaussa-Legendre'a dla N=1, 2, 3, 4

N	k	Węzły x_k	Współczynniki A_k
1	0; 1	$\mp 1/\sqrt{3}$	1
2	0; 2	$\mp\sqrt{3/5}$	5/9
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4	0;4	∓ 0.906180	0.236927
	1;3	∓ 0.538469	0.478629
	2	0	0.568889

Przykład dla trójpunktowego schematu całkowania

$$\int_{-5}^{4} 0.5x^2 + 2x + 3dx$$

$$f(x) = 0.5x^2 + 2x + 3$$

x_{pc1}=0,88729*(-5) + 0,112702*4=-3,9856

x_{pc2}=-0,5 $\quad$ x_{pc3}=-2,9856

Węzły i współczynniki kwadratur Gaussa-Legendre'a dla N=1, 2, 3, 4

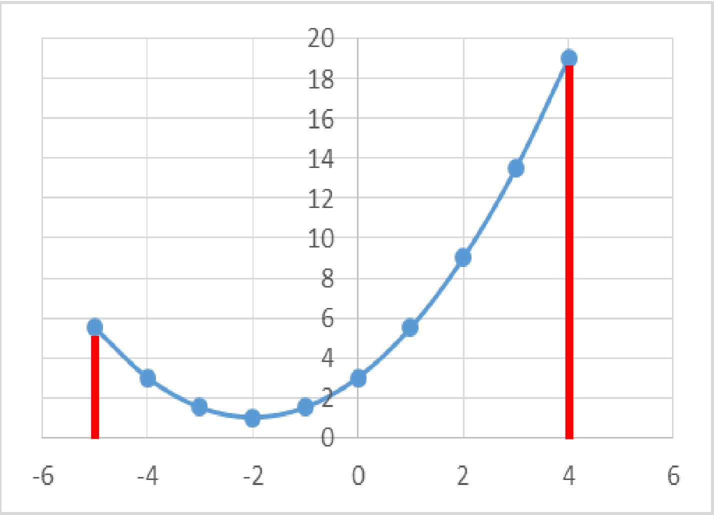
N	k	Węzły x_k	Współczynniki A_k
1	0; 1	$\mp 1/\sqrt{3}$	1
2	0; 2	$\mp\sqrt{3/5}$	5/9
	1	0	8/9
3	0; 3	∓0.861136	0.347855
	1;2	∓0.339981	0.652145
4	0;4	∓0.906180	0.236927
	1;3	∓0.538469	0.478629
	2	0	0.568889

$$f(xpc1) = 0.5(-3{,}9856)^2 + 2(-3{,}9856) + 3 = 2{,}97148$$

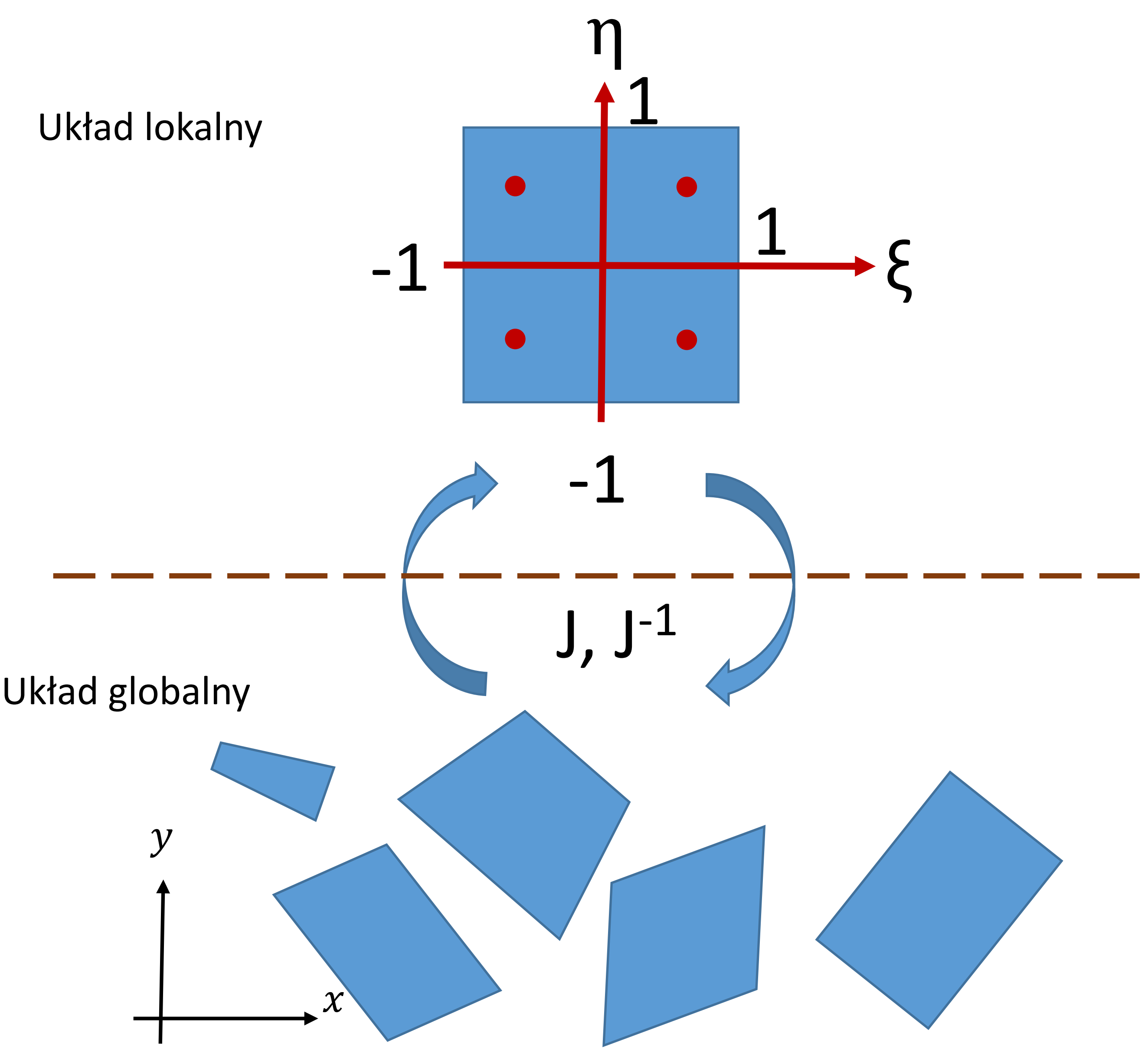
$$f(xpc2) = 2{,}125 \qquad f(xpc3) = 13{,}428$$

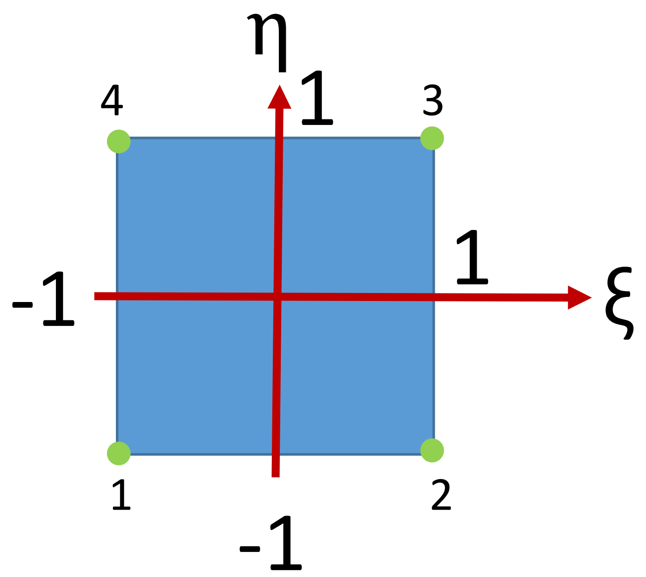
$$\int_{-5}^{4} f(\mathrm{x})dx = (w_1 * f(\mathrm{x}_{pc1}) + w_2 * f\left(\mathrm{x}_{pc2}\right) + w_3 * f(\mathrm{x}_{pc3})) * \det[J]$$

$$\int_{-5}^{4} f(\mathrm{x})dx = (\tfrac{5}{9} * 2{,}697148 + \tfrac{8}{9} * 2{,}125 + \tfrac{5}{9} * 13{,}428) * 4{,}5 = 49{,}5$$



Jakobian przekształcenia 2D





$$N1 = 0.25(1-\xi)(1-\eta)$$

$$N2 = 0.25(1+\xi)(1-\eta)$$

$$N3 = 0.25(1+\xi)(1+\eta)$$

$$N4 = 0.25(1-\xi)(1+\eta)$$

Problem do rozwiązania:

$$[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right)dV$$

http://home.agh.edu.pl/~pkustra/MES/Jakobian.pdf

$$\boxed{[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right) dV}$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{\det[J]} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \boxed{\dfrac{\partial N_i}{\partial \xi}} \\ \boxed{\dfrac{\partial N_i}{\partial \eta}} \end{bmatrix}$$

$$\frac{dN1}{d\xi} = -0.25(1-\eta)$$

$$\frac{dN2}{d\xi} = 0.25(1-\eta)$$

$$\frac{dN3}{d\xi} = 0.25(1+\eta)$$

$$\frac{dN4}{d\xi} = -0.25(1+\eta)$$

$$\frac{dN1}{d\eta} = -0.25(1-\xi)$$

$$\frac{dN2}{d\eta} = -0.25(1+\xi)$$

$$\frac{dN3}{d\eta} = 0.25(1+\xi)$$

$$\frac{dN4}{d\eta} = 0.25(1-\xi)$$

$$N1 = 0.25(1-\xi)(1-\eta)$$

$$N2 = 0.25(1+\xi)(1-\eta)$$

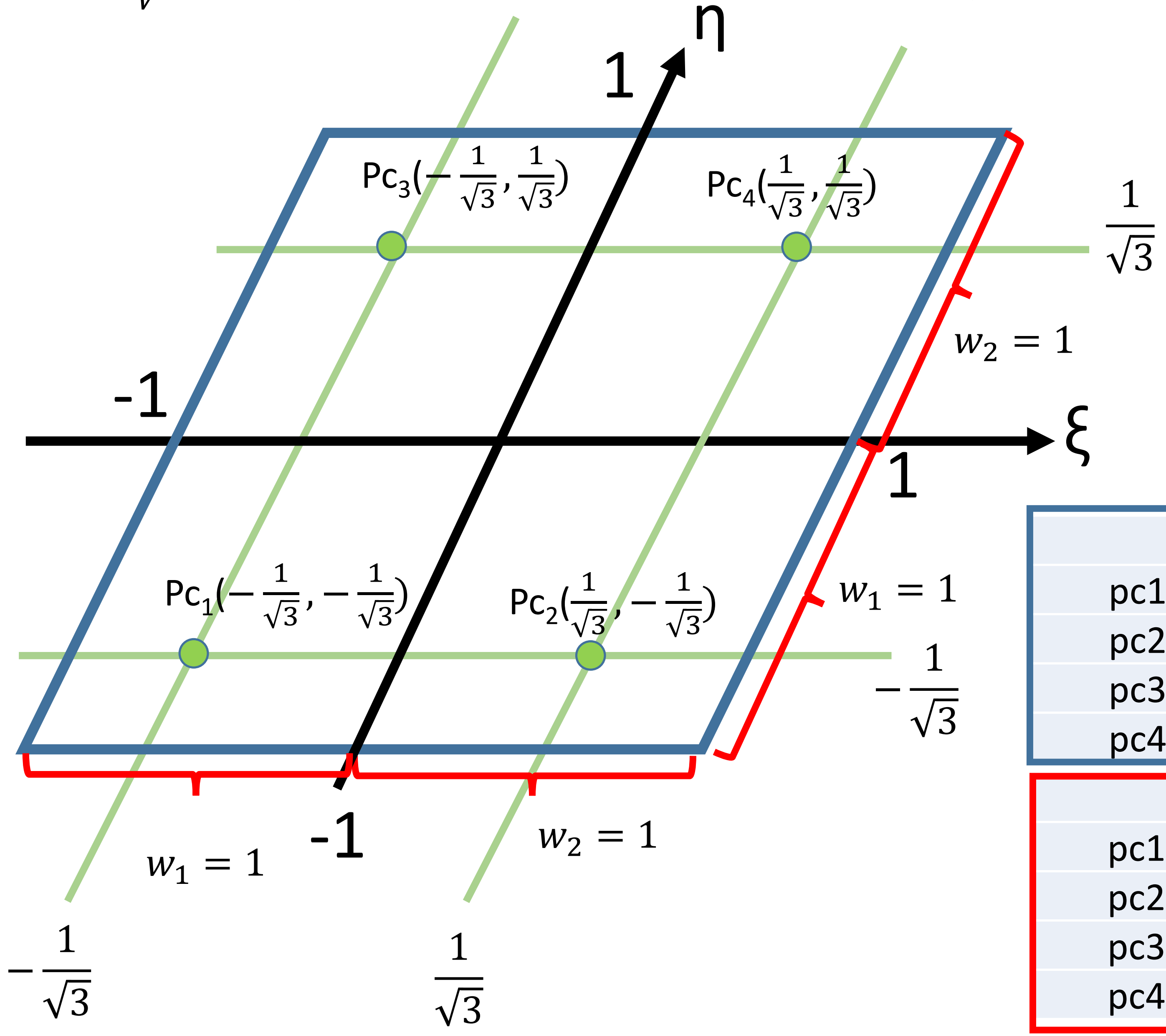
$$N3 = 0.25(1+\xi)(1+\eta)$$

$$N4 = 0.25(1-\xi)(1+\eta)$$

$$x = \sum_{i=1}^{np}(N_i x_i) = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 = \{N\}^T\{x\}$$

$$y = \sum_{i=1}^{np}(N_i y_i) = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 = \{N\}^T\{y\}$$

$$[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right) dV$$



$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{\det[J]} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \boxed{\frac{\partial x}{\partial \xi}} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

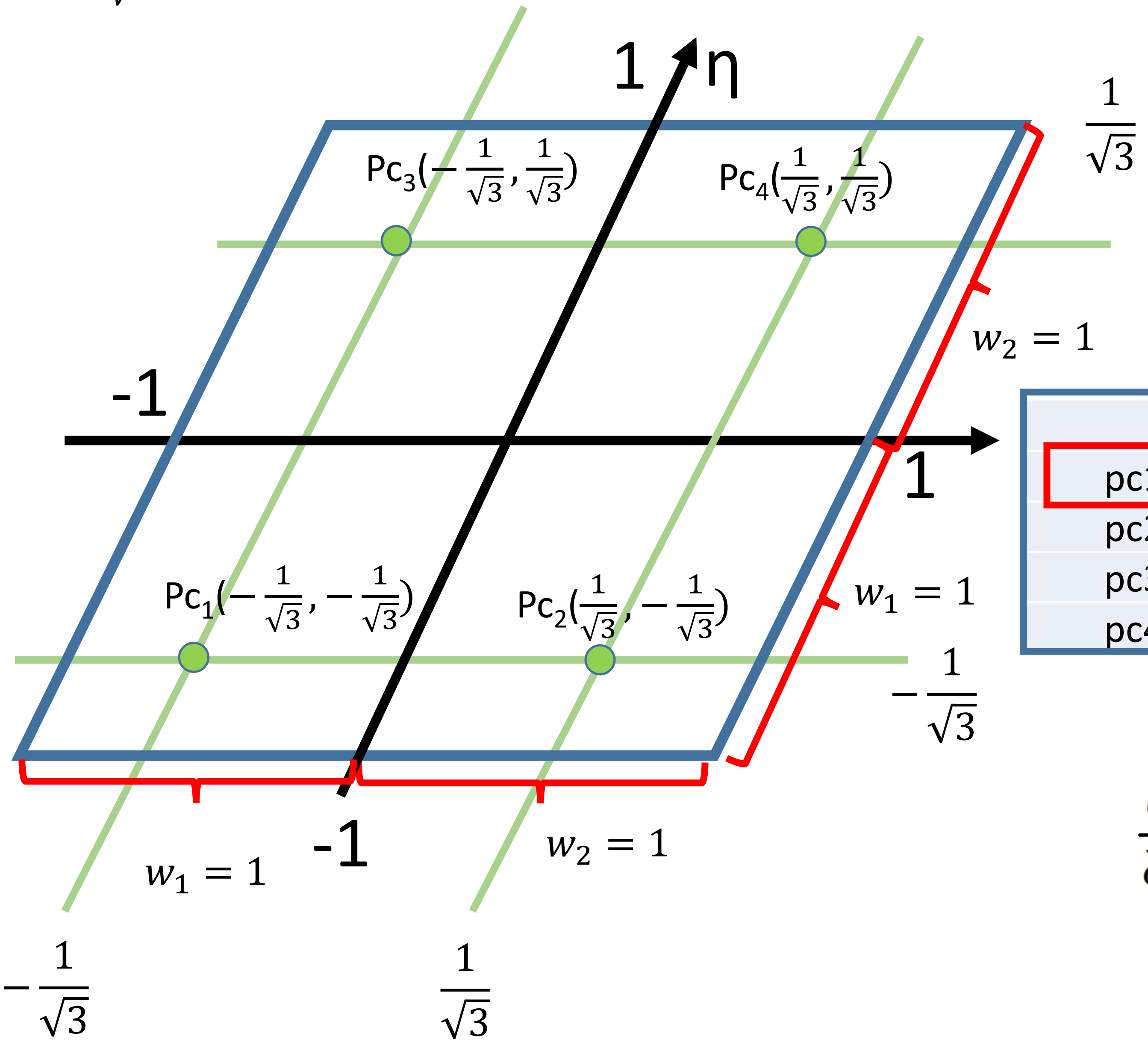
$$x = \sum_{i=1}^{np} (N_i x_i) = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4$$

	dN1/dξ	dN2/dξ	dN3/dξ	dN4/dξ
pc1	-0,39434	0,394338	0,105662	-0,10566
pc2	-0,39434	0,394338	0,105662	-0,10566
pc3	-0,10566	0,105662	0,394338	-0,39434
pc4	-0,10566	0,105662	0,394338	-0,39434

	dN1/dη	dN2/dη	dN3/dη	dN4/dη
pc1	-0,39434	-0,10566	0,105662	0,394338
pc2	-0,10566	-0,39434	0,394338	0,105662
pc3	-0,39434	-0,10566	0,105662	0,394338
pc4	-0,10566	-0,39434	0,394338	0,105662

$$[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right) dV$$



$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{\det[J]} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \boxed{\frac{\partial x}{\partial \xi}} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

ID	1	2	3	4
x	0	0,025	0,025	0
y	0	0	0,025	0,025

	dN1/dξ	dN2/dξ	dN3/dξ	dN4/dξ
pc1	-0,39434	0,394338	0,105662	-0,10566
pc2	-0,39434	0,394338	0,105662	-0,10566
pc3	-0,10566	0,105662	0,394338	-0,39434
pc4	-0,10566	0,105662	0,394338	-0,39434

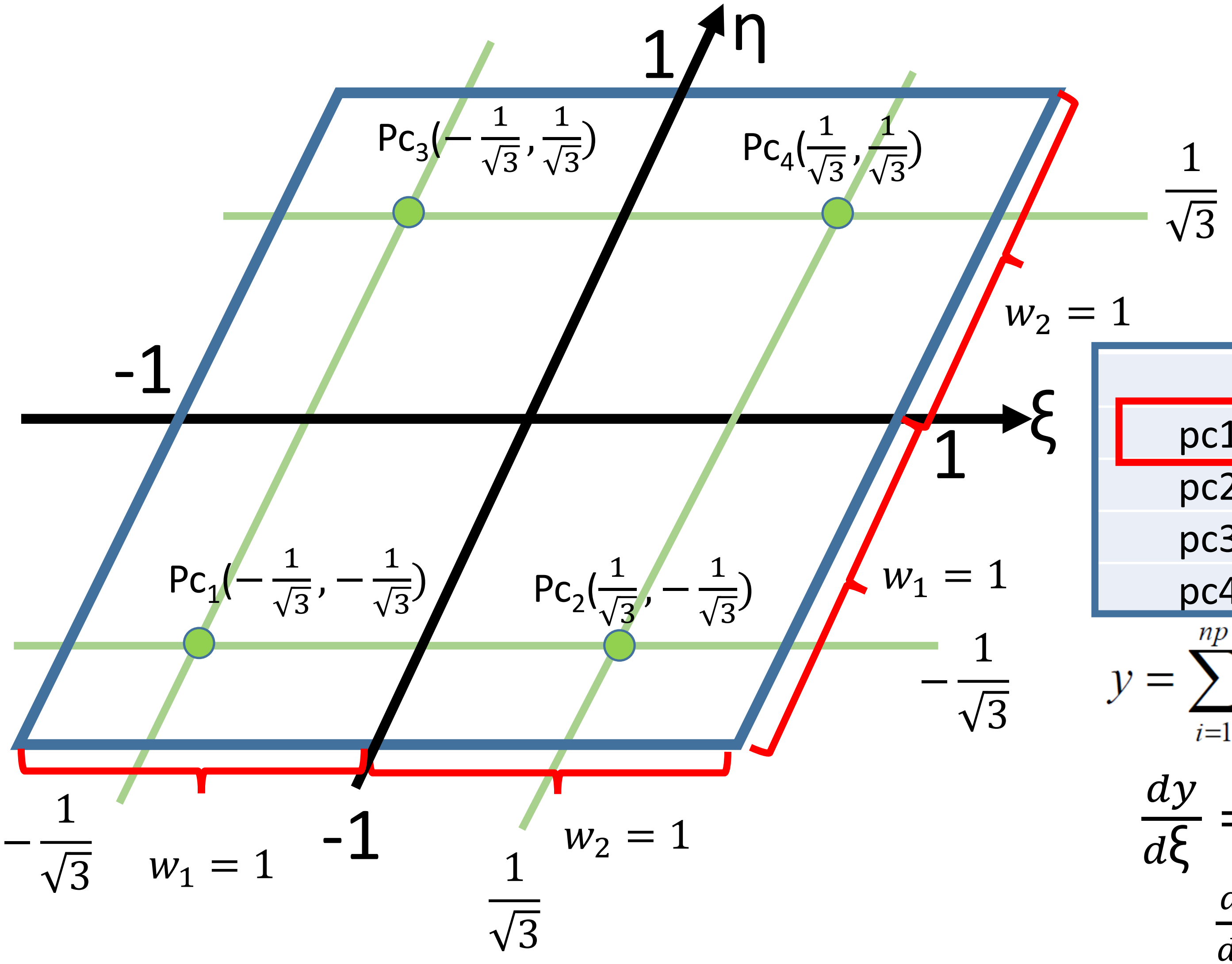
$$x = \sum_{i=1}^{np}(N_i x_i) = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4$$

$\frac{dx}{d\xi} = -0{,}39439 * 0{,}0 + 0{,}39439$*0,025+
0,105662*0,025+(-0,105662)*0,0=0,0125

$$[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right) dV$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{\det[J]} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & \boxed{-\dfrac{\partial y}{\partial \xi}} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}$$



ID	1	2	3	4
x	0	0,025	0,025	0
y	0	0	0,025	0,025

	dN1/dξ	dN2/dξ	dN3/dξ	dN4/dξ
pc1	-0,39434	0,394338	0,105662	-0,10566
pc2	-0,39434	0,394338	0,105662	-0,10566
pc3	-0,10566	0,105662	0,394338	-0,39434
pc4	-0,10566	0,105662	0,394338	-0,39434

$$y = \sum_{i=1}^{np}(N_i y_i) = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 = \{N\}^T\{y\}$$

$$\frac{dy}{d\xi} = \frac{dN_1}{d\xi} y_1 + \frac{dN_2}{d\xi} y_2 + \frac{dN_3}{d\xi} y_3 + \frac{dN_4}{d\xi} y_4$$

$$\frac{dy}{d\xi} = -0{,}39439 * 0{,}0 + 0{,}39439 * 0{,}0 +$$
$0{,}105662 * 0{,}025 + (-0{,}105662) * 0{,}025 = 0{,}0$

$$[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right) dV$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix} = \boxed{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}} \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix}$$

$$\frac{dN1}{d\xi} = -0.25(1-\eta)$$
$$\frac{dN2}{d\xi} = 0.25(1-\eta)$$
$$\frac{dN3}{d\xi} = 0.25(1+\eta)$$
$$\frac{dN4}{d\xi} = -0.25(1+\eta)$$

$$\frac{dN1}{d\eta} = -0.25(1-\xi)$$
$$\frac{dN2}{d\eta} = -0.25(1+\xi)$$
$$\frac{dN3}{d\eta} = 0.25(1+\xi)$$
$$\frac{dN4}{d\eta} = 0.25(1-\xi)$$

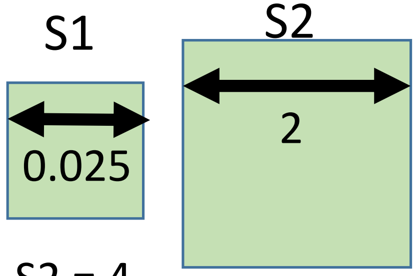
Dla pc1

0,0125	0
0	0,0125

det[j] = 0,00015625

1/det[j] = 6400

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \frac{1}{\det[J]} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \boxed{\dfrac{\partial N_i}{\partial \xi}} \\ \boxed{\dfrac{\partial N_i}{\partial \eta}} \end{bmatrix}$$



$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = 6400 \begin{bmatrix} 0{,}0125 & 0 \\ 0 & 0{,}0125 \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \begin{bmatrix} 80 & 0 \\ 0 & 80 \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{bmatrix}$$

S2 = 4

S1 = 0,000625

S1/S2 = 0,00015625

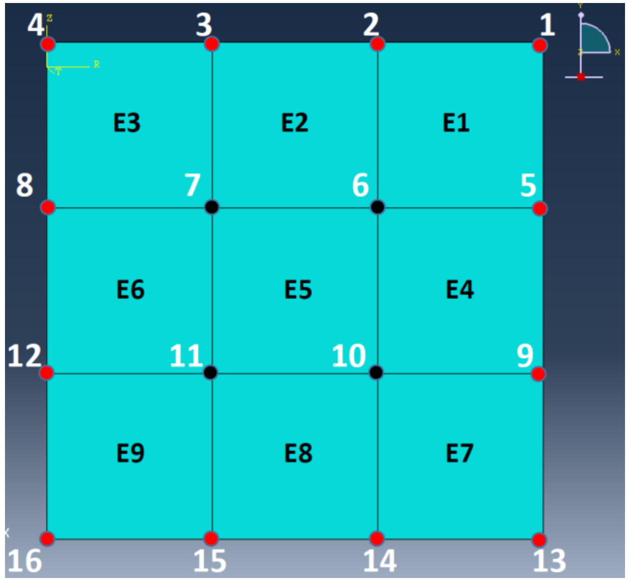
```
struct GlobalData ()
{
}

struct node
{
}

struct element
{
    Jakobian[npc]
}

struct grid
{
}
```

```
struct GlobalData (odczyt z pliku)
{
    npc
}
```

Praca domowa

```
struct ElemUniv
{
   dN_dξ[npc][4]
   dN_dη[npc][4]
}
struct Jakobian
{
     J[2][2]
     J1[2][2] // macierz Jakobiego odwrotna
     detJ
}
```

Siatka elementów skończonych




```
*Node
    1,  0.100000001, 0.00499999989
    2, 0.0666666701, 0.00499999989
   ........................
   14, 0.0666666701, -0.0949999988
   15, 0.0333333351, -0.0949999988
   16,           0., -0.0949999988
*Element, type=DC2D4
 1,  1,  2,  6,  5
 2,  2,  3,  7,  6
 3,  3,  4,  8,  7
```

Całkowanie macierzy H

Dla pc1

$$[H]=\int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T+\left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right)dV$$

	dN1/dξ	dN2/dξ	dN3/dξ	dN4/dξ
pc1	$\boxed{-0{,}39434}$	$\boxed{0{,}394338}$	0,105662	-0,10566
pc2	-0,39434	0,394338	0,105662	-0,10566
pc3	-0,10566	0,105662	0,394338	-0,39434
pc4	-0,10566	0,105662	0,394338	-0,39434

	dN1/dη	dN2/dη	dN3/dη	dN4/dη
pc1	$\boxed{-0{,}39434}$	$\boxed{-0{,}10566}$	0,105662	0,394338
pc2	-0,10566	-0,39434	0,394338	0,105662
pc3	-0,39434	-0,10566	0,105662	0,394338
pc4	-0,10566	-0,39434	0,394338	0,105662

$$\begin{bmatrix}\dfrac{\partial N_i}{\partial x}\\ \dfrac{\partial N_i}{\partial y}\end{bmatrix}=\begin{bmatrix}80 & 0\\ 0 & 80\end{bmatrix}\begin{bmatrix}\dfrac{\partial N_i}{\partial \xi}\\ \dfrac{\partial N_i}{\partial \eta}\end{bmatrix}$$

$$\frac{dN_1}{d\mathrm{x}}=80*\frac{dN_1}{d\xi}+0{,}0*\frac{dN_1}{d\eta}=80*(-0{,}39434)+0{,}0*(-0{,}39434)=-31{,}547$$

$$\frac{dN_2}{d\mathrm{x}}=80*\frac{dN_2}{d\xi}+0{,}0*\frac{dN_2}{d\eta}=80*(0{,}39434)+0{,}0*(-0{,}10566)=31{,}547$$

pc	dN1/dx	dN2/dx	dN3/dx	dN4/dx
1	-31,547	31,54701	8,452995	-8,452995
2	-31,547	31,54701	8,452995	-8,452995
3	-8,453	8,452995	31,54701	-31,54701
4	-8,453	8,452995	31,54701	-31,54701

pc	dN1/dy	dN2/dy	dN3/dy	dN4/dy
1	-31,547	-8,453	8,45299	31,547
2	-8,45299	-31,547	31,547	8,45299
3	-31,547	-8,453	8,45299	31,547
4	-8,45299	-31,547	31,547	8,45299

Obliczanie macierzy H dla pierwszego punktu całkowania

$$[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right) dV$$

pc	dN1/dx	dN2/dx	dN3/dx	dN4/dx
1	-31,547	31,54701	8,452995	-8,452995
2	-31,547	31,54701	8,452995	-8,452995
3	-8,453	8,452995	31,54701	-31,54701
4	-8,453	8,452995	31,54701	-31,54701

pc	dN1/dy	dN2/dy	dN3/dy	dN4/dy
1	-31,547	-8,453	8,45299	31,547
2	-8,45299	-31,547	31,547	8,45299
3	-31,547	-8,453	8,45299	31,547
4	-8,45299	-31,547	31,547	8,45299

Obliczanie macierzy H dla pierwszego punktu całkowania

$$[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right) dV$$

pc	dN1/dx	dN2/dx	dN3/dx	dN4/dx		pc	dN1/dy	dN2/dy	dN3/dy	dN4/dy
1	-31,547	31,54701	8,452995	-8,452995		1	-31,547	-8,453	8,45299	31,547

$$[H] = \int_V 30\left(\begin{Bmatrix} -31{,}54 \\ 31{,}54 \\ 8{,}45 \\ -8{,}45 \end{Bmatrix}\{-31{,}54 \quad 31{,}54 \quad 8{,}45 \quad -8{,}45\} + \begin{Bmatrix} -31{,}54 \\ -8{,}45 \\ 8{,}45 \\ 31{,}54 \end{Bmatrix}\{-31{,}54 \quad -8{,}45 \quad 8{,}45 \quad 31{,}54\}\right) dV$$

$$[H] = \int_V 30(\ \ldots\ +\ \ldots\ )\, dV$$

995,21	-995,21	-266,66	266,66
-995,21	995,21	266,66	-266,66
-266,66	266,66	71,45	-71,45
266,66	-266,66	-71,45	71,45

+

995,21	266,66	-266,66	-995,21
266,66	71,45	-71,45	-266,66
-266,66	-71,45	71,45	266,66
-995,21	-266,667	266,66	995,21

$) \, dV$

Obliczanie macierzy H dla pierwszego punktu całkowania

$$[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right) dV$$

$$[H] = \int_V 30\left(\begin{matrix} 995,21 & -995,21 & -266,66 & 266,66 \\ -995,21 & 995,21 & 266,66 & -266,66 \\ -266,66 & 266,66 & 71,45 & -71,45 \\ 266,66 & -266,66 & -71,45 & 71,45 \end{matrix} + \begin{matrix} 995,21 & 266,66 & -266,66 & -995,21 \\ 266,66 & 71,45 & -71,45 & -266,66 \\ -266,66 & -71,45 & 71,45 & 266,66 \\ -995,21 & -266,667 & 266,66 & 995,21 \end{matrix}\right) dV$$

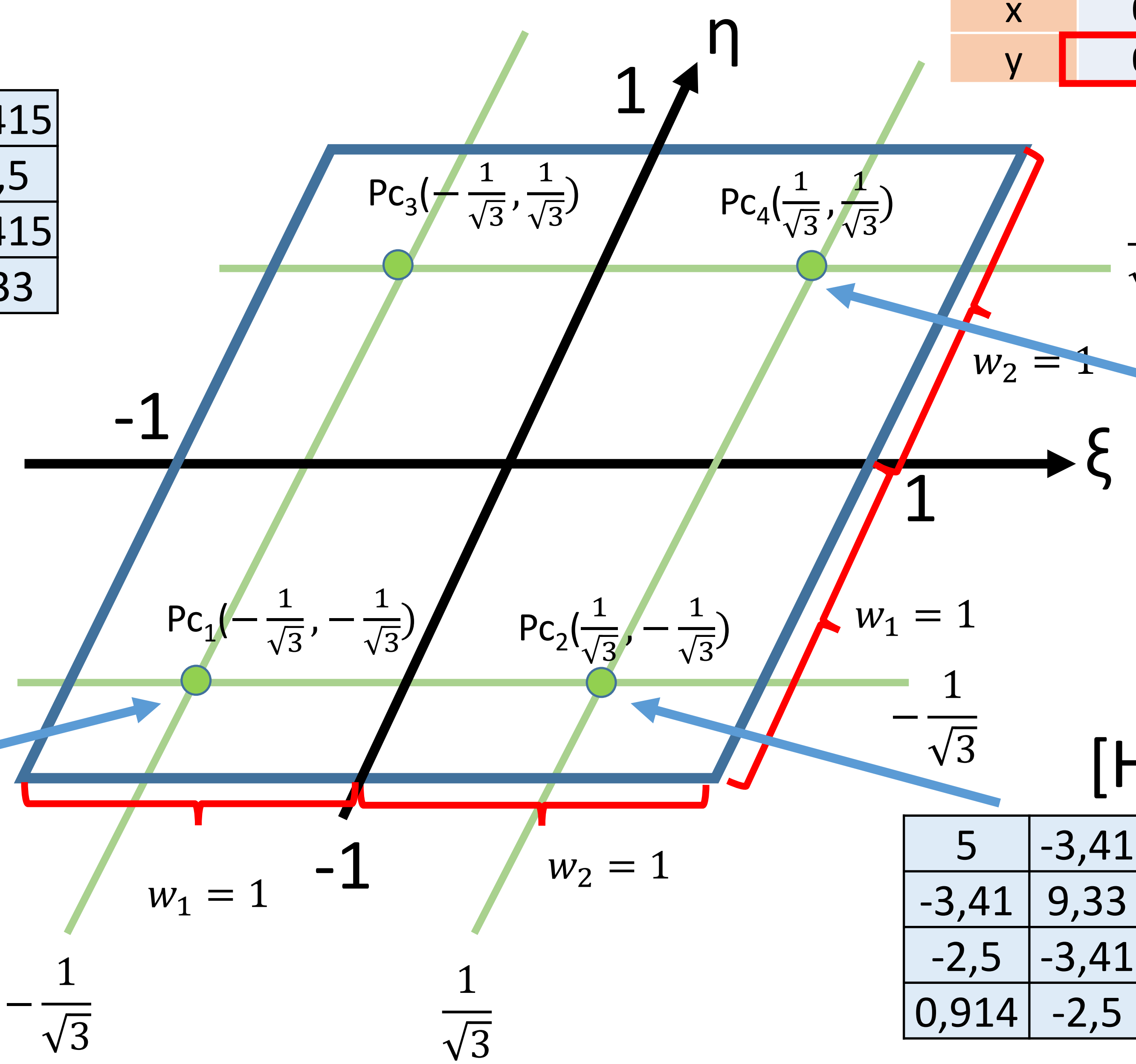
dV realizujemy poprzez przemnożenie wyniku przez Jakobian przekształcenia tego punktu całkowania

$$[H] = 30 * \left(\begin{matrix} 995,21 & -995,21 & -266,66 & 266,66 \\ -995,21 & 995,21 & 266,66 & -266,66 \\ -266,66 & 266,66 & 71,45 & -71,45 \\ 266,66 & -266,66 & -71,45 & 71,45 \end{matrix} + \begin{matrix} 995,21 & 266,66 & -266,66 & -995,21 \\ 266,66 & 71,45 & -71,45 & -266,66 \\ -266,66 & -71,45 & 71,45 & 266,66 \\ -995,21 & -266,667 & 266,66 & 995,21 \end{matrix}\right) * \ 0,000156$$

ID	1	2	3	4
x	0	0,025	0,025	0
y	0	0	0,025	0,025

[H]pc3

5	0,915	-2,5	-3,415
0,915	0,67	0,915	-2,5
-2,5	0,915	5	-3,415
-3,415	-2,5	-3,415	9,33



[H]pc4

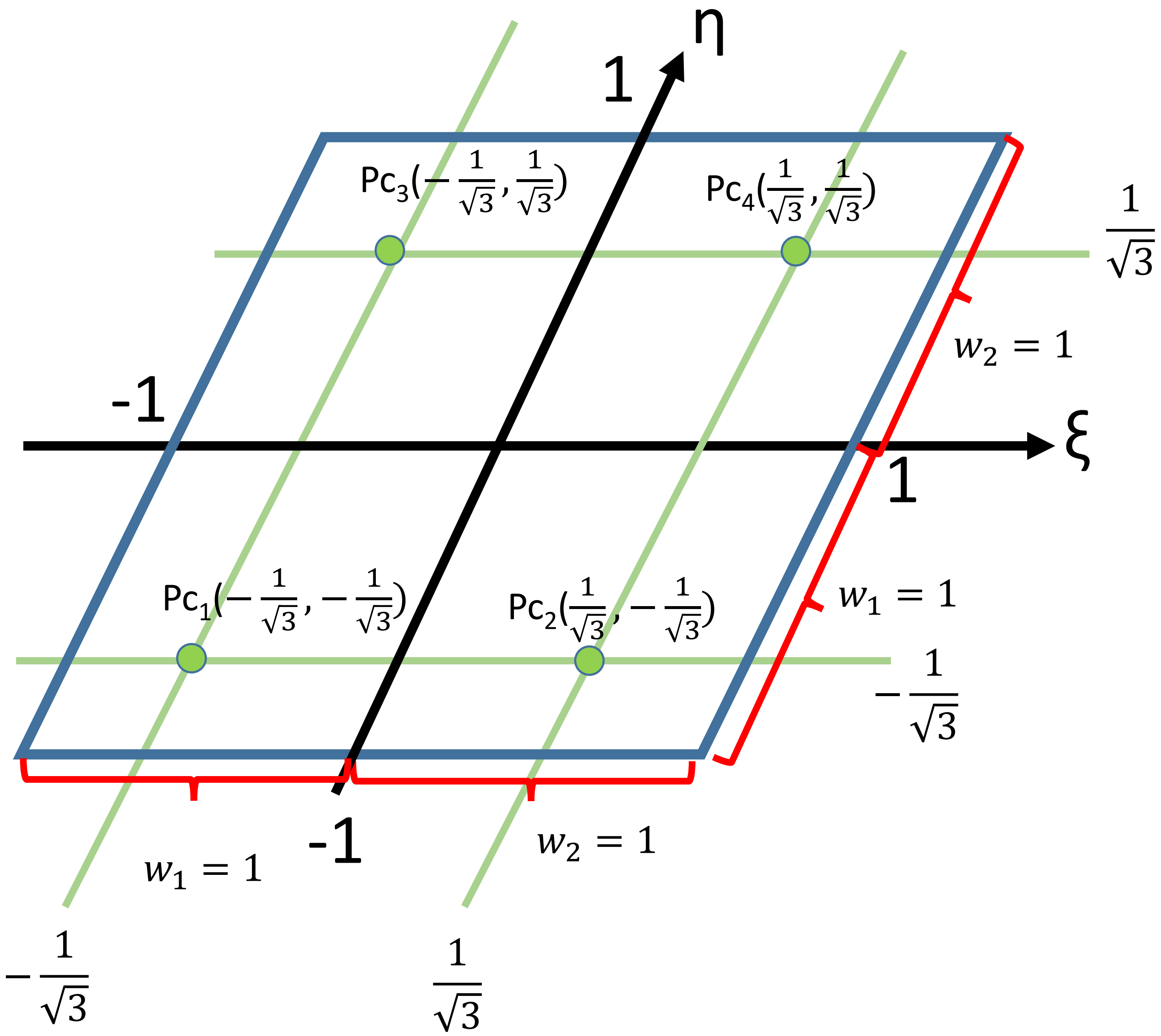
0,66	0,91	-2,5	0,916
0,91	5	-3,41	-2,5
-2,5	-3,41	9,33	-3,4
0,916	-2,5	-3,41	5

[H]pc1

9,33	-3,41	-2,5	-3,41
-3,41	5	0,91	-2,5
-2,5	0,91	0,66	0,91
-3,41	-2,5	0,91	5

[H]pc2

5	-3,41	-2,5	0,91
-3,41	9,33	-3,41	-2,5
-2,5	-3,41	5	0,91
0,914	-2,5	0,914	0,66



ID	1	2	3	4
x	0	0,025	0,025	0
y	0	0	0,025	0,025

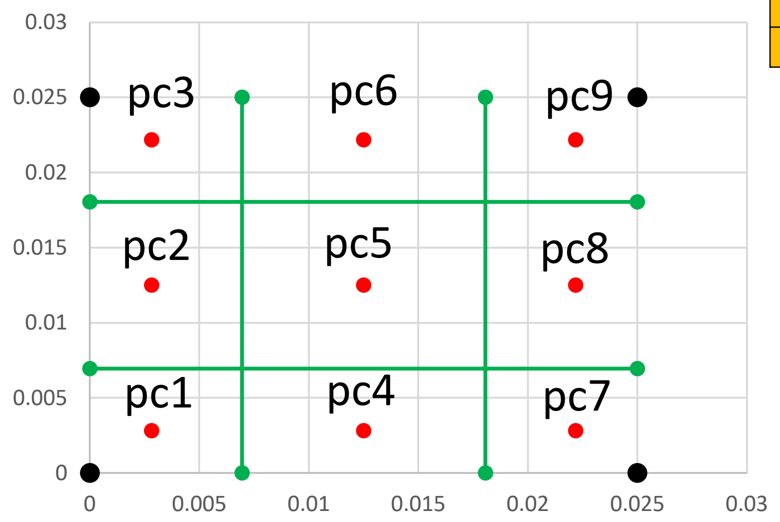
Poszczególne macierze H to wartości funkcji w punktach całkowania. Celem obliczenia całki należy przemnożyć je przez odpowiednie wagi.

20	-5	-10	-5
-5	20	-5	-10
-10	-5	20	-5
-5	-10	-5	20

H= $H_{pc1} * w_1 * w_1 + H_{pc2} * w_2 * w_1 + H_{pc3} * w_1 * w_2 + H_{pc4} * w_2 * w_2 =$

Przykład testowy - 1

ID	1	2	3	4
x	0	0.025	0.025	0
y	0	0	0.025	0.025



conductivity	
k	**30**

pc	dN1/dKsi	dN2/dKsi	dN3/dKsi	dN4/dKsi
1	-0.44365	0.44365	0.05635	-0.0564
2	-0.25	0.25	0.25	-0.25
3	-0.05635	0.05635	0.44365	-0.4436
4	-0.44365	0.44365	0.05635	-0.0564
5	-0.25	0.25	0.25	-0.25
6	-0.05635	0.05635	0.44365	-0.4436
7	-0.44365	0.44365	0.05635	-0.0564
8	-0.25	0.25	0.25	-0.25
9	-0.05635	0.05635	0.44365	-0.4436

pc	dN1/dEta	dN2/dEta	dN3/dEta	dN4/dEta
1	-0.444	-0.056	0.056	0.443649
2	-0.444	-0.056	0.056	0.443649
3	-0.444	-0.056	0.056	0.443649
4	-0.25	-0.25	0.25	0.25
5	-0.25	-0.25	0.25	0.25
6	-0.25	-0.25	0.25	0.25
7	-0.056	-0.444	0.444	0.056351
8	-0.056	-0.444	0.444	0.056351
9	-0.056	-0.444	0.444	0.056351

pc	dN1/dx	dN2/dx	dN3/dx	dN4/dx
1	-35.4919	35.4919	4.50807	-4.5081
2	-20	20	20	-20
3	-4.50807	4.50807	35.4919	-35.492
4	-35.4919	35.4919	4.50807	-4.5081
5	-20	20	20	-20
6	-4.50807	4.50807	35.4919	-35.492
7	-35.4919	35.4919	4.50807	-4.5081
8	-20	20	20	-20
9	-4.50807	4.50807	35.4919	-35.492

1pc	3.6449	-1.59097	-0.46296	-1.590968
	-1.59097	1.85185	0.202079	-0.462963
	-0.46296	0.20208	0.058804	0.202079
	-1.59097	-0.46296	0.202079	1.851852

2pc	3.841846	-0.55556	-1.296296	-1.98999
	-0.555556	0.972969	0.878883	-1.2963
	-1.296296	0.878883	0.972969	-0.55556
	-1.989994	-1.2963	-0.555556	3.841846

3pc	1.85185	0.20208	-0.463	-1.591
	0.20208	0.0588	0.20208	-0.463
	-0.463	0.20208	1.85185	-1.591
	-1.591	-0.463	-1.591	3.6449

4pc	3.842	-1.99	-1.296	-0.556
	-1.99	3.842	-0.556	-1.296
	-1.296	-0.556	0.973	0.879
	-0.556	-1.296	0.879	0.973

8pc	0.972969	-0.5555	-1.2963	0.878883
	-0.55556	3.8418	-1.9899	-1.2963
	-1.2963	-1.9899	3.8418	-0.55556
	0.878883	-1.296	-0.5555	0.972969

9pc	0.058804	0.202079	-0.46296	0.202079
	0.202079	1.851852	-1.59097	-0.46296
	-0.46296	-1.59097	3.6449	-1.59097
	0.202079	-0.46296	-1.59097	1.851852

pkt całk	1	2	3	4	5	6	7	8	9
J_1_1	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125
J_1_2	0.00	0	0	0	0	0	0	0	0
J_2_1	0	0	0	0	0	0	0	0	0
J_2_2	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125

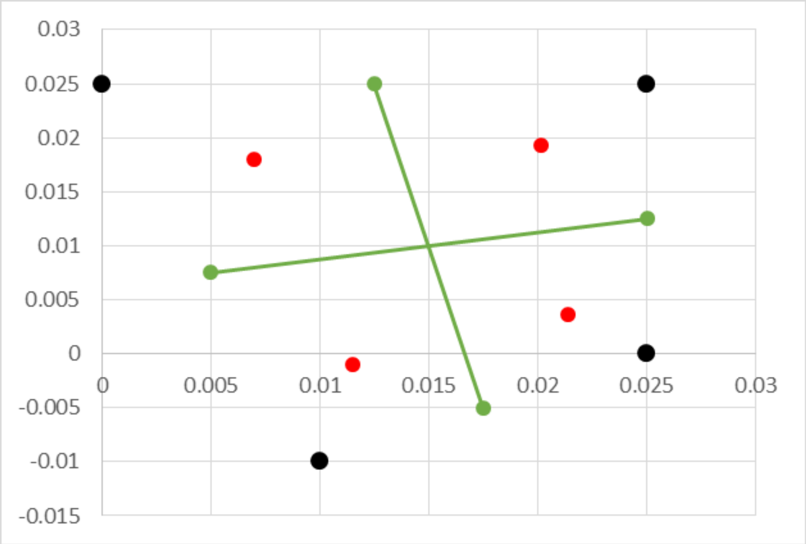
pkt całk	1	2	3	4	5	6	7	8	9
Det J	0.000156	0.000156	0.000156	0.000156	0.000156	0.0001563	0.00015625	0.000156	0.000156

Macierz H - po całkowaniu			
20	-5	-10	-5
-5	20	-5	-10
-10	-5	20	-5
-5	-10	-5	20

Przykład testowy - 2

conductivity	
k	30

ID	1	2	3	4
x	0.01	0.025	0.025	0
y	-0.01	0	0.025	0.025



pkt całk	1	2	3	4
J_1_1	8.56E-03	0.008556624	0.011443376	0.0114434
J_1_2	0.0039434	0.003943376	0.001056624	0.0010566
J_2_1	-0.00394338	-0.001056624	-0.00105662	-0.003943
J_2_2	1.64E-02	0.013556624	0.013556624	0.0164434

pkt całk	1	2	3	4
Det J	0.00015625	0.000120166	0.00015625	0.0001923

pc	dN1/dKsi	dN2/dKsi	dN3/dKsi	dN4/dKsi	ksi	eta
1	-0.394338	0.394338	0.105662	-0.10566	-0.5774	-0.57735
2	-0.394338	0.394338	0.105662	-0.10566	0.57735	-0.57735
3	-0.105662	0.105662	0.394338	-0.39434	0.57735	0.57735
4	-0.105662	0.105662	0.394338	-0.39434	-0.5774	0.57735

pc	dN1/dEta	dN2/dEta	dN3/dEta	dN4/dEta	ksi	eta
1	-0.394338	-0.10566	0.105662	0.394338	-0.5774	-0.57735
2	-0.105662	-0.39434	0.394338	0.105662	0.57735	-0.57735
3	-0.105662	-0.39434	0.394338	0.105662	0.57735	0.57735
4	-0.394338	-0.10566	0.105662	0.394338	-0.5774	0.57735

1pc	9.330	-7.147	-2.5	0.316
	-7.147	9.224	1.915	-3.992
	-2.5	1.915	0.669	-0.084
	0.3169	-3.992	-0.084	3.760

2pc	6.501	-7.517	-0.998	2.014
	-7.517	14.072	-2.785	-3.770
	-0.998	-2.785	3.037	0.746
	2.014	-3.770	0.746	1.010

3pc	0.669	0.647	-2.5	1.183
	0.647	4.375	-2.415	-2.607
	-2.5	-2.415	9.330	-4.415
	1.183	-2.607	-4.415	5.839

4pc	4.062	0.228	-3.438	-0.852
	0.228	0.631	1.496	-2.356
	-3.438	1.496	7.526	-5.584
	-0.852	-2.356	-5.584	8.792

Macierz H				
	20.563	-13.788	-9.436	2.6619
	-13.788	28.304	-1.788	-12.726
	-9.436	-1.788	20.563	-9.338
	2.6619	-12.726	-9.338	19.402

Zadanie domowe – zrealizuj całkowanie macierzy H

$$[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right)dV$$

Czytanie siatki z pliku - > tworzenie struktur danych – global data, element, node, element uniwersalny,

Implementacja pętli po elementach e:

Pobieranie wartości *x* oraz *y* węzłów elementu skończonego e,

Pętla po punktach całkowania *pc* *(dla 2d pc = 4, 9, 16…)*

Obliczanie macierzy Jakobiego J, Jakobianu i macierzy odwrotnej J^{-1} dla punktu całkowania *pc*

Obliczamy dN/dx oraz dN/dy -> Macierz H w dla punktu całkowania *pc*

Sumujemy macierze H z punktów całkowania *pc_1 – pc_n (dla 2d pc_n = 4, 9, 16…)*

H= $H_{pc1} * w_1 * w_1 * detJpc_1 + H_{pc2} * w_2 * w_1 * detJpc_2 + H_{pc3} * w_1 * w_2 * detJpc_3 + H_{pc4} * w_2 * w_2 * detJpc_4$