De Rham's Theorem

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Abstract

Some of the most important concepts from algebraic topology are homology and cohomology. The concept of both is to count holes of some topological manifold by examining cycles and boundaries as we decrease dimension or increase dimension, respectively. By applying a homeomorphic function to a manifold, said manifold can be continuously transformed into a complex that can further be generalized into a vector field. By considering the exact and closed differential forms of some smooth manifold and applying the exterior derivative, a cohomology group can be formed known as the De Rham cohomology group. Georges De Rham then proved that this group is isomorphic to the singular cohomology group of the smooth manifold, thus making the De Rham cohomology group a topological invariant. This paper investigates preliminary knowledge, offers a proof for De Rham's Theorem, and discusses some potential uses for research.

1 Introduction

Topology is best described as the abstraction or generalization of geometry [1]. The reason for studying topology is the observation that when metrics such as angles or side lengths are removed from a geometric setting, there are still properties that can be observed. Despite this abstraction, there are some restrictions when working with topology. For instance, topological objects still have boundaries that must be observed [2], as proved by the Jordan Curve Theorem. Because of these boundaries, we are able to distinguish differences between topological objects. For example, consider a circle and a square that has been partitioned into two triangles. Since we have to respect the boundaries of a topological object, we cannot continuously morph this partitioned square into a circle. The notion of continuously morphing one topological object into another is called homeomorphism [3]. By using homeomorphism, we are able continuously morph topological objects in vector spaces while still respecting the boundaries. From here, we are able to observe which closed cycles are not boundaries for the entire topological object, thus describing the number of holes the object has. This method is called homology when we describe cycles in a decreasing dimension and cohomology when we describe cycles in an increasing dimension.

When we think about calculus, our immediate intuition is to take the problem that is being worked and create a geometric structure to help draw conclusions. After Poincaré's paper famous paper, Analysis Situs [3], work began to generalize the geometric ideas behind calculus. This lead to the establishment of the differentiable (smooth) manifold, a topological object in which calculus calculus could be performed on [4]. In differential topology, we can perform calculus in terms of abstract algebraic functions such as differential forms and the exterior derivative in order to describe aspects about circulation and area, without the need for metrics. The problem with this is that there was a lack of geometric intuition associated with this abstract notation. De Rham's Theorem shows that by examining the exterior derivative on sets of differential forms, a cohomology space can be created from the closed differential forms modulo the exact differential forms. According to De Rham's Theorem, this cohomology is isomorphic to the singular cohomology and shows that these notions of differential forms on a smooth manifold is directly related to the number of holes in the smooth manifold.

Both differential topology and De Rham's Theorem are relatively new subjects in mathematics, only dating back to the early 1900s. In the late 1920s, a mathematician by the name of Georges de Rham studied topology in Paris under famous mathematician Henri Lebesgue [5]. He would then begin his PhD thesis, Sur l'Analysis Situs des Variétés à n dimensions [6], in which he established singular cohomology as a reverse of homology described by Poincar' in Analysis Situs [3]. In this thesis, De Rham used the lessons in analysis and topology he learned from Lebesgue and created the De Rham cohomology. Later in the thesis, he would further prove that this new cohomology described the same things that the regular cohomology on manifolds would.

Although the history of De Rham's Theorem is short, it has found itself to be a very useful theorem. The abstraction of geometry in a calculus setting has been useful for the fields of vector analysis in physics, specifically relating to quantum mechanics. Since the theorem relates the ideas of calculus on a general manifold to the geometric shape of the manifold, it has been useful for studying the relation between solution in general spaces to their geometric structure, such as in the Hodge Conjecture [12] and is related to the Fundamental Theorem of Calculus on higher dimensional, smooth manifolds [7].

2 Definitions and Preliminaries

Before being able to prove De Rham's Theorem, there is a number of introductory definitions and theorems that will be needed. We'll start by introducing some the basic notations of algebraic topology and working towards defining the singular cohomology group De Rham cohomology group.

2.1 Topology

When removing metrics from some geometric structures there are still properties that can be observed. The following theorem claims that boundaries are an important topological invariant. Although you will see this paper use $\mathbb{R}$, keep in mind we are removing metrics. We refer to the real numbers since they are a complete set.

Definition 1. Let $X \subseteq \mathbb{R}$. The subset X is an open set if for every $x \in X$ there is an interval (a, b) that contains x and is contained in X.

Theorem 2. *(Jordan Curve) A simple closed curve C in the Euclidean plane separates the plane into two open connected sets with C as their common boundary. Exactly one of these open connected sets is bounded.*

The intuition behind this theorem seems obvious, but the theorem would not have a proper proof until 50 years after the theorem was conceived.

Proof. The complexity of this proof warrants a paper on its own. For proof, refer to [9]. The main idea is to create a sequence of open disks for any point inside the curve and use analysis to show that the sequence is bounded. $\square$

The Jordan Curve theorem allows us to distinguish different topological shapes based on these boundaries. We can consider the interior of the simple closed curve to be a hole in some partitioned original topological space. We can further classify these spaces.

Definition 3. Let X be a set and T be a collection of subsets of X. Then T is a *topological space* if it satisfies the following properties

1. Both X, $\emptyset \in T$
2. The union of any collection of sets in T is also in T
3. The intersection of any finite collection of sets in T is also in T

For simplicity, we can consider a topological space to merely be a collection of open sets. Since we have removed metrics, we can define a way to continuously morph one topological space into another

Definition 4. Let X and Y be topological spaces. We claim that X and Y are *homotopic equivalent* if there exists a continuous function $H : X \times [0, 1] \to Y$.

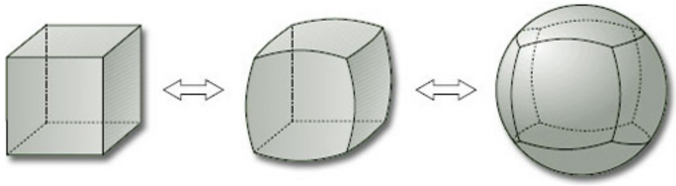


Figure 1: The cube is a manifold that is homeomorphic to a sphere [17]. Notice that we can morph the cube into a sphere without mashing any of the points together. This function is not only continuous, but bijective as we are not losing any of the points through the transformation

This function allows us to continuously morph one topological space into another, but we need to create another function if we want to go the opposite way. To avoid creating a brand new function, we define the homeomorphism.

Definition 5. Topological spaces X and Y are *homeomorphic* if the function $F : X \to Y$ satisfying the following,

1. F is continuous
2. F is bijective
3. The inverse, F^{-1} is also continuous

Since we will be performing calculus on manifolds, we need to further specify morphing a topological space into another while keeping its differentiability.

Definition 6. A *diffeomorphism* is a homeomorphism that is also differentiable.

The notion of having a collection of open sets is vague. To compute (co)homology, we will need a stricter version of a topological space, specifically one that will allow us to relate back to euclidean geomtry and to apply algebraic techniques to describe.

Definition 7. *Manifolds*, M, are topological spaces such that for any $m \in M$, the open ball $B_r(m)$ is homeomorphic to a Euclidean n-dimensional sphere.

Example 1. Figure 1 [15] shows a cube manifold being morphed into a sphere

We like to think of manifolds as topological spaces that are 'nice' to work on in that they are similar to some n-dimensional euclidean space. We can further relate manifolds by using a homeomorphism to morph the manifold into a simple euclidean object, or collection of simple euclidean objects, known as simplexes, or complexes.

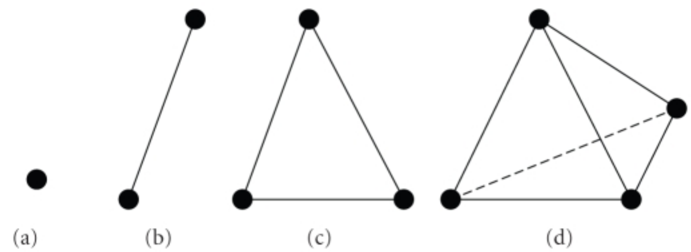


Figure 2: The simplexes a, b, c, d are 0, 1, 2, 3-simplexes respectively [16]

Definition 8. The *simplex* is the simplest polygon that can be formed in a k-dimensional euclidean space. A *complex* is a collection of $0 \leq k$-dimensional simplexes.

Example 2. Figure 2 [14] has examples of the 0, 1, 2, 3-simplexes

Example 1 states that we can continuously morph the circle into a triangle or a 2-simplex. This implies we can continuously morph any manifold into some complex and vice versa.

2.2 Homology

By morphing a manifold into a complex, we can consider the complex to be some vector space defined by a linearly independent basis. For example, let M be a manifold that is homeomorphic to n-complex, K for $n \geq 0$. Then for each k-dimensional simplex $\sigma_k = (a_1 ... a_k) \in K$,

$$C_k(M) = span\{\sigma_{k_1}, \sigma_{k_2}, ..., \sigma_{k_m}\}$$

is a vector space. We will refer to each linear combination in the span as a k-chain. We then can define a homomorphism, $\partial(\sigma_k) = \sum_i^k (-1)^i (a_1 a_2 ... a_{i-1} a_{i+1} ... a_k)$, that describes the boundary of each k-chain in terms of the $(k-1)$-chain that make up the boundary.

Definition 9. The *chain complex* is defined as

$$... \xrightarrow{\partial_{k+1}} C_k(M) \xrightarrow{\partial_k} C_{k-1}(M) \xrightarrow{\partial_{k-1}} ... \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} C_0(M) \xrightarrow{\partial_0} 0$$

such that $\partial_k \circ \partial_{k+1} = 0$.

To define the homology vector space, we need to observe which chains are cycles and which are boundaries. We can refer to the cycle subspace of $C_k(M)$ as $Z_k = \ker(\partial_k)$ and the boundary subspace of $C_k(M)$ as $B_{k+1} = Im(\partial_{k+1})$. Note that every boundary is itself a cycle. Thus B_{k+1} is a subspace of Z_k.

Definition 10. The *homology space* is the quotient space cycles modulo boundaries,

$$H_k(M) = Z_k/B_{k+1}$$

This quotient space tells us which cycles in the complex fail to bound the entire complex. By taking the order of this space, we can use this information to count the number of holes of a manifold or complex as we decrease dimension.

2.3 Cohomology

Cohomology can be considered to be the reverse of homology. Instead of examining k-chains as k decreases, we study them as k increases. We refer to each vector space of k-chains as $C^k(M)$ to separate notation between homology and cohomology.

Definition 11. The *cochain complex* is defined as

$$0 \xrightarrow{\partial^0} C^0(M) \xrightarrow{\partial^1} ... \xrightarrow{\partial^{k-2}} C^{k-1}(M) \xrightarrow{\partial^{k-1}} C^k(M) \xrightarrow{\partial^k} ...$$

where the coboundary homomorphism is

$$\partial^{k+1}(a_1...a_{k+1}) = a_1(a_2...a_{k+1}) + \sum_i^k (-1)^i(a_1...a_{i-1}a_ia_{i+1}...a_{k+1})(-1)^{i+1}(a_1...a_k)$$

such that $(\partial^k \circ \partial^{k-1}) = 0$.

The subspace $Z^k = \ker(\partial^k)$ is the subspace of cocycles and $B^{k-1} = Im(\partial^{k-1})$ is the subspace of coboundaries.

Definition 12. The *cohomology space* is the quotient space

$$H^k(M) = Z^k/B^{k-1}$$

Similar to the homology space, the order of the cohomology space describes the number of holes in a manifold, except now instead of describing cycles and boundaries by decreasing dimension, we are moving in the opposite direction by increasing dimension.

2.4 De Rham Cohomology

From this point forward, we will be observing smooth manifolds, M, or topological manifolds that calculus can be performed on. Since we are working in topology, we want to create a method of doing vector calculus without the use of a coordinate system. If given a finite amount of vectors on a smooth manifold, we can observe the exterior product to describe a k-dimensional parallelopiped. For example, for vectors u and v, the exterior product denoted $u \wedge v$ describes a 2-dimensional parallelogram. The intuitive idea is similar to a determinant without metric values to compute it.

Definition 13. A *differential k-form* is the smooth alternating tensors of a manifold

Example 3. All smooth functions f are 0-forms. For a smooth function f, fdx is a 1-form.

This definition is a bit vague. Differential forms share a relation to the tangent spaces of smooth manifolds. For the purpose of this paper, we can consider differential forms to describe a notion of area produced by the exterior product of vectors on a manifold. The exterior product only gives information about a cycle related to some k-dimensional parallelopiped. By taking the exterior product of differential forms, the notion of area can be defined. There are no good geometrically intuitive ways to think about differential forms, which is one of the reasons for the De Rham Theorem.

If we consider the Fundamental Theorem of Calculus, notice that $\int_a^b f dx = F(b) - F(a)$ for the antiderivative of f, F over $[a, b]$. Similarly for some 2-form, $f dx \wedge dy$, the double integral will give the area of the function. Notice that integrating gets rid of a differential form. We can define the exterior derivative to add a new differential form.

Definition 14. Let M be a smooth manifold. The set $\Omega^k(M)$ is the set of differential k-forms.

Definition 15. The *exterior derivative* of a smooth function $f(x)$ is

$$d(f dx_1 \wedge ... \wedge dx_k) = \sum_i \frac{\partial f}{\partial x} dx_{k+1} \wedge ... \wedge dx_1$$

This definition is similar to the (co)boundary homomorphism. Since the exterior derivative is a linear operator, it is commutative between functions. Similarly to the (co)boundary homomorphism, we can set up a complex.

Definition 16. For the exterior derivative, d, the De Rham complex is defined as

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} ... \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} ...$$

such that $d \circ d = 0$

Just like in (co)homology, we need some type of kernal and image to work with. To do this, we will define the closed and exact forms.

Definition 17. For some $[w] \in \Omega^k(M)$, w is a *closed form* if $dw = 0$. Also, $[w]$ is a *exact form* if $w = d[c]$ for some $[c] \in \Omega^{k-1}(M)$.

Definition 18. The De Rham cohomology space is defined as the set of all closed forms (Z^k_{dR}) modulo exact forms (B^{k-1}_{dR}), denoted

$$H^k_{dR}(M) = Z^k_{dR}/B^{k-1}_{dR}$$

The intuition behind understanding the De Rham cohomology space is that we are counting the closed forms, or intuitively closed curves, that do not come from a smaller form. This cohomology is extremely difficult to think about intuitively. By proving De Rham's Theorem, a more intuitive way to think about how closed and exact forms relate to cocycles and coboundaries. For now, consider the Fundamental Theorem of Calculus. Closed forms mean that the derivative of some form is 0 which implies that form is a constant. By the Fundamental Theorem of Calculus, the constant form should have an antiderivative. But by examing the closed forms modulo the exact forms, whenever the De Rham cohomology set is not empty, there are constant forms that do not have antiderivatives. This implies that the De Rham cohomology set is a measure of how much the fundamental theorem fails on general smooth manifolds. With De Rham's Theorem, we will show that the cohomology set shows the failure of the fundamental theorem of calculus on general smooth manifolds of varying dimension.

One last tool before proving De Rham's Theorem will allow us to describe a function between the cohomology spaces of two manifolds. The following definition defines this function over the De Rham cohomology since this is the only instance we will need it, but it can be applied to any (co)homology space.

3 De Rham's Theorem

3.1 Preliminary Information

We'll start this section by defining the De Rham homomorphism and what it means for a smooth manifold to be a De Rham manifold. Before that, we need to define integration over a smooth k-simplex and a generalization of Stokes' Theorem.

Definition 19. Let $F : M \to N$ be a smooth diffeomorphism between two smooth manifolds M and N. The *pullback* of F, denoted F^*, is a function between the cohomology of the manifolds, $F^* : H^k_{dR}(N) \to H^k_{dR}(M)$.

This definition can be applied to other (co)homology rather than just to the De Rham cohomology, however, this will be a result of the De Rham Theorem.

Definition 20. Let $w \in \Omega^k(M)$ and σ be a k-simplex, $\sigma : \Delta_k \to M$. The *integral w over σ* is

$$\int_\sigma w = \int_{\Delta_k} \sigma^* w$$

After defining integration of a form over a simplex, we can utilize Stokes' Theorem in order to define the De Rham homomorphism.

Theorem 21. *(Stokes' Theorem) Let c be a smooth k-cycle in a smooth manifold M and let w be a $(k-1)$-form on M. Then*

$$\int_{\partial(c)} w = \int_c dw$$

Proof. We will prove the statement for the case of simplexes. Let σ be a k-simplex and w be a k-form. Then

$$\int_{\sigma} dw = \int_{\Delta_k} \sigma^* dw \tag{1}$$

$$= \int_{\Delta_k} d\sigma^* w \tag{2}$$

$$= \int_{\partial\Delta_k} \sigma^* w \tag{3}$$

The boundary homomorphism is the same as we defined in section 2.2, but we can redefine it in function terms, $\partial(\Delta_k) = \sum_i^k (-1)^i (a_1 a_2 ... a_{i-1} a_{i+1} ... a_k) = \sum_i^k (-1)^i \Delta_k \circ F_{i,k}$. Then

$$\int_{\partial\Delta_k} \sigma^* w = \int_{\sum_i^k (-1)^i \Delta_k \circ F_{i,k}} \sigma^* w \tag{4}$$

$$= \sum_i^k (-1)^i \int_{\Delta_k \circ F_{i,k}} \sigma^* w \tag{5}$$

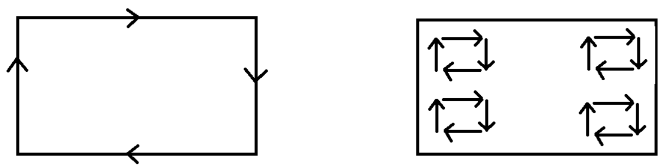
$$= \sum_i^k (-1)^i \int_{\Delta_{k-1}} F_{i,k}^* \sigma^* w \tag{6}$$

$$= \sum_i^k (-1)^i \int_{\Delta_{k-1}} (\sigma \circ F_{i,k})^* w \tag{7}$$

$$= \sum_i^k (-1)^i \int_{\sigma \circ F_{i,k}} w \tag{8}$$

$$= \int_{\partial\sigma} w \tag{9}$$

$\square$



$$\int_{\partial M} f \quad = \quad \int_M df$$

Figure 3: On the left, we can take the integral of some function, f, over the oriented boundary of the manifold by considering it to be a curve. Stoke's Theorem states that we can evaluate this integral by examining the changes that are taking place on the manifold itself, which is what the exterior derivative is describing

Stoke's theorem is sometimes considered to be a generalization of the Fundamental Theorem of Calculus to smooth manifolds of n-dimension. This geometric intuition will be proved by De Rham's Theorem, but for now we can imagine the integral of some function f over a smooth manifold. In vector calculus, when we integrate a function over a region, we take a line integral over some curve. If we want to integrate a function over a smooth manifold, we consider the boundary of that manifold to be our curve. Since the exterior derivative describes the changes cycles, or curvature, Stoke's Theorem states that we can integrate a function (or form) over the boundary by examining the smaller changes in curvature found on that manifold. For a full analytical proof of Stokes Theorem, refer to Bredon's Topology and Geoemtry [2] theorem 4.1.

Definition 22. Let $[w] \in H^k_{dR}(M)$ and $\bar{c}$ be a smooth k-cycle on M, or $\bar{c} \in [c] \in H^\infty_k(M)$. The *De Rham homomorphism* $I : H^k_{dR}(M) \to H^k(M)$ is

$$I[w][c] = \int_{\bar{c}} w$$

This homomorphism allows us to describe the abstract nature of closed k-forms in terms of geometric dimension. The goal of this paper is to prove that this homomorphism is an isomorphism for any smooth manifold. When we think about calculus, or real/complex analysis, our immediate intuition is to relate the ideas to some geometric structure. By proving that the De Rham homomorphism is isomorphic to the cohomology of any smooth manifold, we can convert many of these abstract ideas in differential topology and relate them to the geometric properties. Furthermore, we can make inferences about calculus on smooth manifolds of any dimension.

Definition 23. A smooth manifold is a *De Rham manifold* if the De Rham homomorphism is an isomorphism.

Definition 24. A set is considered *convex* if for any 2 points, the line connecting those 2 points is contained in the set.

We are now ready to begin proving De Rham's Theorem. Proving De Rham's Theorem will come down to 3 steps.

1. Proving that I is isomorphic for any subset of M diffeomorphic to a convex open subset of $\mathbb{R}^n$

2. Proving that if I is isomorphic for subsets U, V, $U \cap V \in M$, the I is isomorphic for $U \cup V \in M$

3. Proving that if I is isomorphic for each disjoint partition U_α, then I is isomorphic for $\bigcup U_\alpha$

The following lemma from Bredon's Topology and Geometry [2] will prove that these are the only properties that need to be satisfied.

Lemma 25. *Let M be an smooth manifold. Suppose that $P(U)$ is a statement about open subsets of M satisfying the following three properties*

1. *$P(U)$ is true for U diffeomorhphic to a convex open subset of $\mathbb{R}^n$*

2. *$P(U)$, $P(V)$, $P(U \cap V)$ imply $P(U \cup V)$*

3. *For disjoint $\{U_\alpha\}$ and $P(U_\alpha)$, $P(\bigcup U_\alpha)$ for every α*

Then $P(M)$ is true

Proof. Since M is a smooth manifold, any open subset of M is diffeomorphic to some subset of $\mathbb{R}^n$. Since U is open in a euclidean space, it can be written as a countable union of open sets. Since U is an open subset of M, each of these are diffeomorphic to some open ball in $\mathbb{R}^n$. For any open ball W, we have $P(W)$ since any open ball is convex and the intersection of any 2 open balls is also convex. By (2), the union of all of these open balls is convex. Therefore $P(U \cup V)$ and $P(\bigcup U_\alpha)$ are true and $P(M)$ is also true. □

3.2 Step 1: Every Open Convex Subset U that is diffeomorphic to $\mathbb{R}$ is De Rham

Before proving the first step we first need the next proposition that will help us compute the De Rham homomorphism on an open convex set.

Proposition 26. *Let M be a smooth manifold that is convex. Then $H^0_{dR}(M) = \mathbb{R}$ and $H^k_{dR}(M) = 0$ for any $k > 0$*

Proof. Since we define $\Omega^k(M)$ to be the set of all k-forms on M, the set $\Omega^0(M)$ is the set of functions whose derivatives are 0. We define $\Omega^c(M) = 0$ for any $c < 0$, thus meaning that there are 0 exact forms in $\Omega^0(M)$. Since $\Omega^0(M)$ is the set of all constant functions, $H^0_{dR}(M) = \mathbb{R}$.
Since M is convex, any line between 2 points on M is also in M meaning that M is homotopic to any point in M (contractable). This implies that each constant function in $\Omega^0(M)$ is mapped to a form under the exterior derivative. Since M will always remain contractable, this can be generalized for any $\Omega^{k-1}(M) \to \Omega^k(M)$. This means that every closed form is exact and $H^k_{dR}(M) = 0$ $\square$

Not every smooth manifold will be convex, but this lemma will be useful when proving De Rham's Theorem later on to prove that open convex subsets of $\mathbb{R}^n$ are De Rham.

Lemma 27. *Every open convex subset U of $\mathbb{R}^n$ is De Rham.*

Proof. Since each open convex subset of $\mathbb{R}$ is contractable, we can use Proposition 25 to show that $I : H^k_{dR}(U) \to H^k(U)$ is an isomorphism. Since $H^k_{dR} = 0$ for $k > 0$, the mapping I is trivially an isomorphism. But $H^0_{dR}(U) = \mathbb{R}$ as well as $H^0(U) = \mathbb{R}$. If we take $\sigma : \Delta_0 \to U$ where $\Delta_0 = 0$ and $f : U \to \mathbb{R}$ we get the following.

$$I[f][\sigma] = \int_{\Delta_0} \sigma^\star f \tag{10}$$

$$= (f \circ \sigma)(0) \tag{11}$$

$$= f \tag{12}$$

Since f is a constant function that exists in $\mathbb{R}$ the mapping I maps f over the point Δ_0 to itself making it an isomorphism. $\square$

3.3 Step 2: If U, V, $U \cap V$ are De Rham, then $U \cup V$ is De Rham

The next lemma will set up a commutative diagram between 2 Mayer-Vietoris sequences of the De Rham cohomology and the singular cohomology. Note that this lemma uses the famous Five Lemma [10] from category theory. The proof for the Five Lemma can be shown by examining the properties of each mapping individually, but this is beyond the scope of this paper so we will refer to Michael's A Note on the Five Lemma [10] for further details. Before getting to the lemma, we need to establish the commutative property of the De Rham homomorphism.

Proposition 28. *Let M and N be smooth manifolds and $F : M \to N$ be a smooth function. Then following diagram is commutative.*

$$\begin{tikzcd} H^k_{dR}(N) \arrow[r, "F^*"] \arrow[d, "I"] & H^k_{dR}(M) \arrow[d, "I"] \\ H^k(N) \arrow[r, "F^*"] & H^k(M) \end{tikzcd}$$

Proof. The proof for this lemma simply comes down to showing the commutative property holds for I. By applying the definition,

$$\begin{aligned} I(F^*[w])[\sigma] &= \int_{\Delta_k} \sigma^* F^* w && (13) \\ &= \int_{\Delta_k} (F \circ \sigma)^* w && (14) \\ &= I[w][F \circ \sigma] && (15) \\ &= I[w]F^*[\sigma] && (16) \end{aligned}$$

□

Definition 29. The following sequence for some smooth manifold M such that $U \cup V = M$ is a *Mayer-Vietoris* sequence.

$$H^{k-1}_{dR}(U) \bigoplus H^{k-1}_{dR}(V) \longrightarrow H^{k-1}_{dR}(U \cap V) \longrightarrow H^k_{dR}(U \cup V) \longrightarrow H^k_{dR}(U) \bigoplus H^k_{dR}(V) \longrightarrow H^k_{dR}(U \cap V)$$

such that the sequence is exact, or each Kernal is equal to its respective Image.

We can now continue proving De Rham's Theorem.

Lemma 30. *Let U and V be open subsets of some smooth manifold M such that $U \cup V = M$. If U, V, and $U \cap V$ are De Rham, then $U \cup V$ is also De Rham.*

Proof. We will construct the following commutative diagram based on the Mayer-Vietoris sequences for the De Rham cohomology group and singular cohomology groups respectively.

$$\begin{tikzcd} H^{k-1}_{dR}(U) \bigoplus H^{k-1}_{dR}(V) \arrow[r] \arrow[d, "I_a"] & H^{k-1}_{dR}(U \cap V) \arrow[r] \arrow[d, "I_b"] & H^k_{dR}(M) \arrow[r] \arrow[d, "I_c"] & H^k_{dR}(U) \bigoplus H^k_{dR}(V) \arrow[r] \arrow[d, "I_d"] & H^k_{dR}(U \cap V) \arrow[d, "I_e"] \\ H^{k-1}(U) \bigoplus H^{k-1}(V) \arrow[r] & H^{k-1}(U \cap V) \arrow[r] & H^k(M) \arrow[r] & H^k(U) \bigoplus H^k(V) \arrow[r] & H^k(U \cap V) \end{tikzcd}$$

Since I_a, I_b, I_d, I_e, are isomorphisms by assumption, and the Mayer-Vietoris sequences are exact, by the Five Lemma, I_c is also isomorphic. □

3.4 Step 3: If each disjoint U_α is De Rham, then $\bigcup U_\alpha$ is De Rham

Lemma 31. *For disjoint subsets of M, U_α, if U_α is De Rham for each alpha, then $\bigcup U_\alpha$ is also De Rham.*

Proof. In the previous lemma, we proved the case where U and V were disjoint subset partitions of M. By taking an inductive step and letting $U = U_1 \cup U_2 \cup ... \cup U_k$ and $V = U_{k+1}$, the previous lemma's proof sufficiently proves this lemma. □

3.5 The De Rham Theorem Proof

Now that we have proved the each of the properties of lemma 24 are satisfied, the proof for De Rham's Theorem simply boils down to compiling each lemma we have proven.

Theorem 32. *(De Rham) $I : H^k_{dR}(M) \to H^k(M)$ is an isomorphism.*

Proof. By lemma 24, let $P(U)$ be a statement that U is De Rham. Then by lemmas 27, 29 and 30, all three properties of lemma 24 are satisfied. Therefore any smooth manifold M is De Rham. □

4 Application

The goal of homology and cohomology is to count holes as a topological invariance as the dimension decreases or increases respectively. Each have their purposes, but cohomology has been used far more than homology partly due to the De Rham Theorem. The relation between the De Rham cohomology and singular cohomology allow us to describe cohomology as the failure of local solutions to become global solutions.

Methods of differential topology as most often used in the physics setting. From Stokes' Theorem in classical calculus, $\int_{\partial S} F \cdot dr = \int\int_S curlF \cdot dS$, for a vector field, F, and a surface, S, relates the rotation of a vector field on some surface to the integral the surfaces boundary. This version of Stokes' Theorem implies a relation between 1-forms and 2-forms in $\mathbb{R}^3$. Further, it implies the relation between the boundary of a surface and the rotation of the surface. De Rham's Theorem implies that this is directly related to k-dimensional holes as we increase dimension. Because of this relationship, we can say that De Rham cohomology describes the the failure of a local solution to become a global solution. For this reason, cohomology has been a useful tool with studying various topics in physics, dynamic systems, and differential equations.

5 Discussion

Although cohomology has been extremely useful in studying physics, its application has found more use in pure mathematic subjects such as number theory and algebraic geometry. One of the most famous statements involving the De Rham cohomology is the Hodge Conjecture. The Great Mathematical Problems by Ian Stewart [14] claims that there is no possible way to fully explain the Hodge Conjecture in layman terms. Many mathematicians have claimed there is no possible way that our modern mathematics will be able to prove that the Hodge Conjecture true, although the same was, likely, said for Fermat's Last Theorem which was proved by Andrew Wiles [15].

The Hodge Conjecture creates a relationship between the subject of algebraic geometry and algebraic topology. In algebraic geometry, the goal is to observe solutions various curves created by polynomials for some dimension n. This is often extremely complicated for some cases. This is why algebraic topology is used. Each of these curves, as strange as they may be, are continuous and must have some other curve they are homeomorphic to. The aim of the Hodge Conjecture is to define a relation between these two concepts, allowing us to study the solutions to polynomials based on cohomology cocycles. This is where De Rham's Theorem comes into play. In algebraic geometry, these curves are generated by polynomials and from any introductory calculus course, polynomials are easily differentiable. Instead of observing the cohomology of n-dimensional curves directly, we can examine the De Rham cohomolgy instead which will give us information about the cohomology of curves, via the De Rham Theorem, by performing calculus on these polynomials.

Currently the Hodge Conjecture is unproven with some skeptical if it will ever be proven. The Clay Institute has included the conjection on their list of 7 millennium problems and are offering a \$1,000,000 dollar reward for its proof [13].

Acknowledgments

This paper would not have been possible without my instructor, Dr. Yu-Min Chung, for teaching me homology in his Intuitive Concepts of Topology course at U.N.C. Greensboro. Without rigorously studying how homology worked, I would not have been able to describe the difficult theorems, lemmas, and proofs needed to explain De Rham Cohomology. This is the most difficult topic I have encountered and without having some experience with the rigor of algebraic topology, I would have not been able to complete the paper.

I also would like to thank Dr. Austin Lawson. His work on my research project for Yu-Min Chung's course recently caused for the our paper to get recognized by the International Conference on Pattern Recognition. The experience I gained from this research has helped tremendously with

my experience in writing mathematics.

Lastly I owe a debt of gratitude to all the friends and family who have listened to my lectures on this subject. Even with my friend's mathematics degrees, on top of my family's lack of degrees, my practice explaining these difficult topics have made a significant impact on my analysis skills.

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