Let me outline the main points and the overall structure of the power methods described in [1]. Of course, I will add some points of myself to make some concepts more clear.

After establishing $A^k x$ converges to the eigenvector of A corresponding to the largest eigenvalue, we need some stopping criterion, which is the main concern in numerical analysis because we cannot set max_eiter simply.

Since $x^{(m)}$ cannot be eigenvector identically, there is no real number a satisfying $Ax^{(m)} = ax^{(m)}$. But we can expect the number a which makes

$$\|Ax^{(m)} - ax\|$$

minimizes is a good approximation of the largest eigenvalue. Now we elaborate on how to solve this a.

Rayleigh Quotient. If one see $Av = av$ as an overdetermined equation system, technique about **least square** can be applied to get the solution a which is called Rayleigh quotient.

$$\Lambda_A(v) := \frac{v^* A v}{v^* v} \tag{1}$$

Geometrically, the solution of least square is just to find a scalar a such that $Av - av$ is orthogonal to v.

Shaded Box

One can also deduce the same result (1) algebraically by solving the following optimization problem

$$\min_x \|Av - av\|_2^2, \tag{2}$$

then eq (2) (take real case as example) boils down to finding the critical point of

$$f(a) = v^T va^2 - v^T(A^T + A)va + v^T A^T Av$$

If $v^T A^T v$ is a real number, then $v^T A^T v = v^T Av$, one can use the derivative of $f(a)$ to determine the critical point and the result matches the Least Square. When we consider the complex number field, the expansion of $f(x) = \min_x \|Av - av\|_2^2$ is subtle, saying

$$f(a) \neq v^* va^2 - v^*(A^* + A)va + v^* A^* Av$$

especially when a is complex.

The theoretical support is given by Theorem 4.6 in [1].

Combined with $\|x^{(m)} - v\| \approx 0$ and $|\lambda^{(m)} - \lambda| \approx 0$ where $\lambda, v \in \text{eig}(A)$, we can see that

$$\|Ax^{(m)} - \lambda^{(m)} x^{(m)}\| \to 0$$

where each components can be computed explicitly in the process of iteration. So one option for stopping criterion is to set ϵ and stop the iteration when

$$\|Ax^{(m)} - \lambda^{(m) x^{(m)}}\| < \epsilon$$

References

[1] Steffen Börm and Christian Mehl. *Numerical methods for eigenvalue problems.* Walter de Gruyter, 2012.