Definition 0.1 (Isometric matrix)**.** *Let $n, k \in \mathbb{N}$, and let $Q \in \mathbb{F}^{n\times k}$. If $Q^*Q = I$ holds, Q is called isometric. A square isometric matrix is called **unitary**.*

Remark 0.2. *Given isometric Q, one cannot have $Q^*Q = QQ^*$ and QQ^* is just a orthogonal projection in $\mathbb{R}^n$. For example,*

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We will use this technique in invariant subspace by simultaneous iterations.

Let me give a general example about **orthogonal projection**. Given

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix}$$

One can check $Q^TQ = I_2$ which justifies the isometric of Q and

$$QQ^T = \begin{bmatrix} 5/6 & -1/6 & 1/3 \\ -1/6 & 5/6 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

which is usually denoted by P.

One can see the action of QQ^T to v by

$$\begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} q_1^* \\ q_2^* \end{bmatrix} v = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} q_1^* v \\ q_2^* v \end{bmatrix} = C_1 q_1 + C_2 q_2 \tag{1}$$

which means QQ^T sends v to the subspace spanned by q_1 and q_2. Of course, one can guess that it is similar to

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$$

because we have $Pq_i = q_i, i = 1, 2$, i.e. two eigenvectors with eigenvalue 1. It's easy to use eq(1) to check that QQ^T has the diagonal form under basis

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$$

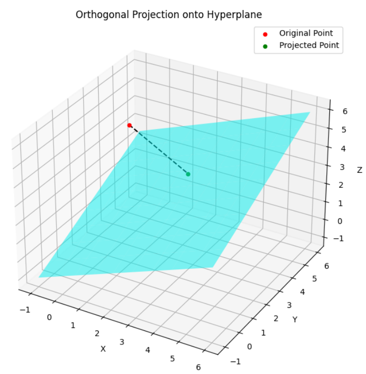

Figure 1: QQ^T projects vector v to the hyperplane generated by the column vector of Q, i.e. $q_1 = [1/\sqrt{3}, 1\sqrt{3}, 1\sqrt{3}]$ and $q_2 = [1/\sqrt{2}, -1/\sqrt{2}, 0]$. For instance, point $[1\ 2\ 6]^T$ in red is projected to $[1.5 \quad 2.5 \quad 3]^T$

where q_3 is the extended normalized vector. The illustration is given in the following figure.

We need prepare some lemmas to be at bottom of simultaneous iteration, assume V is matrix consisting of linearly independent columns $[v_1, v_2, \cdots, v_k]$ and we have the following proposition

Lemma 0.3. *1. V is injective is equivalent to $\widehat{V}$ is injective.*

2. PV is injective is equivalent to $\widehat{P}\widehat{V}$ is injective.

3. PV is injective implies that V is injective.

Proof. The first can be easily proven by $v_i = QQ^* v_i = Q\widehat{v}_i$.

We can view PV as new matrix consisting of w_i and find that $\widehat{PV} = \widehat{P}\widehat{V}$. □