1 Introduction

1.1 Limit

Let's first show the process of limit by approximating a real number by $\{k/2^m\}$.

Example 1.1. *Given $x = \sqrt{2}/2$, we search the nearest dyadic number from $m = 1$ to ∞. Notice that we just need consider $1/2^m$ because $2/2^m = 1/2^{m-1}$ is supposed to be added before.*

1. *$m = 1$, there is $1/2 < x$ and we denote $x - 1/2$ by x_1.*
2. *$m = 2$, $1/4 > x_1$ and we enter next m.*
3. *$m = 3$, $1/8 < x_1$ and we denote $x_1 - 1/8$ by x_2,*
4. *continue...*

Thanks to the countable property of $k/2^m$, we can continue this procedure and have

$$\frac{\sqrt{2}}{2} = 1/2 + 1/8 + \cdots$$

One can view this as decomposition of open interval $(0, x)$ or any interval with length x by shifting for need. More general, we have

Theorem 1.2. *Every open set U in $\mathbb{R}^n$ can be written as a countable union of almost disjoint cubes.*

Proof. Consider the collection of all cubes in $\mathbb{R}^n$ whose vertices have coordinates of the form $k/2^m$, where $k \in \mathbb{Z}, m \in \mathbb{N}^*$. Denote these dyadic cubes by

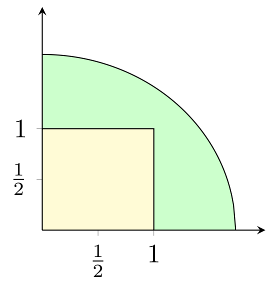
$$A_m = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, \frac{k_i}{2^m} \leq x_i < \frac{k_i + 1}{2^m} \text{ for } i = 1, 2, \cdots, n\}$$

For fixed m, A_m is countable since it's determined by $(k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n$. Therefore $\mathcal{A} := \cup_m A_m$ is also countable and we create an enumeration $D_1, D_2, \cdots$.

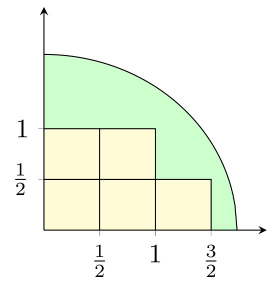
Last, we show how to select a sub-collection from $\mathcal{A}$ to satisfy $U = \cup D_j$. Let U be an open set in $\mathbb{R}^n$. For each $x \in U$, since U is open, there exists a dyadic cube containing x and fully contained within U.

1. We start with an empty collection, and then iterate through the dyadic cubes contained in U
2. For each cube, if it is almost disjoint from every cube already in our collection, we add it.

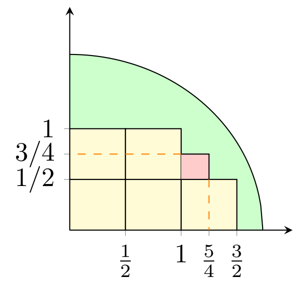
Since $\mathcal{A}$ is countable, the sub-collection we get is also countable. $\square$



(a) collection C_1



(b) collection C_2



(c) collection C_3

Figure 1: We use dyadic square to approximate the quarter. Note that the sub-collection is not unique since we can choose 4 smaller quads in place of 1-by-1 quad. This process can be viewed as 2D-version of determining series $\{a_n\} \in \mathbb{Z}^n$ such that $x = \sum a_n/2^n$ for any real number x.

1.2 Limit of sets

One advantage of Lebesgue measure is to deal with some special set which is a limit of a collection of sets. For example, $(0,1) \cap \mathbb{Q}$ can be viewed as an infinite union of set

$$E_k = \left\{ \frac{m}{k} : \gcd(m,k) = 1, m < k \right\}.$$

More interesting and important sets are

$$\bigcap_{i=1} \bigcup_{k=i}^{\infty} A_k$$

representing elements appeared in A_k for infinite k and

$$\bigcup_{i=1} \bigcap_{k=i}^{\infty} A_k \tag{1.1}$$

representing elements appeared in A_k for all but finite k.

Application. We can reinterpret the set of points where $f_k(x)$ converges to $f(x)$ pointwise into

$$\bigcap_{n=1} \boxed{\bigcup_{i=1} \bigcap_{k \geq i} \left\{ x : |f_k(x) - f(x)| < \frac{1}{n} \right\}} \tag{1.2}$$

Notice that the boxed part is exactly interpretation of "for all $k \geq K(n,x)$, we have..." when n (or ε) is given. In other words, it's those points that can appear in all but finite $A_{k,n}$, where n can be seen as a additional parameter to eq (1.1)

And the set whose point is uniform convergent is

$$\bigcap_{n=1} \boxed{\bigcap_{k\geq K(n)} \left\{ x : |f_k(x) - f(x)| < \frac{1}{n} \right\}}. \tag{1.3}$$

The essence of Egoroff theorem lies in

$$\lim_{K\to\infty} \mu \left(\bigcup_{k\geq K} \left\{ x : |f_k(x) - f(x)| \geq \frac{1}{n} \right\} \right) = 0 \tag{1.4}$$

due to (1.2) and **finite measure** of X. In detail, since

$$\mu \left(\bigcap_{i\geq 1} \bigcup_{k\geq i} \left\{ x : |f_k(x) - f(x)| \geq \frac{1}{n} \right\} \right) = 0, \tag{1.5}$$

which implies $\bigcup_{k\geq i} \left\{ x : |f_k(x) - f(x)| \geq \frac{1}{n} \right\}$ are shrinking to a null set and we can control it to a small size saying $\varepsilon/2^n$ by terminating at $K_{n,\varepsilon}$-th step.

Remark 1.3. *We must have $\mu(X) < \infty$ to deduce eq(1.4) from (1.5), i.e. we can apply*

$$\lim_{n\to\infty} \mu(A_i) = \mu(\cap A_i) = 0$$

to a descending set A_i only if $\mu(X) < \infty$. For instance,

$$\boxed{f_k(x) = \chi_{[k,k+1]} \textit{ and } f_k \to 0 \textit{ pointwisely in } \mathbb{R}.}$$

Theorem 1.4. *If $\{f_k\}$ converges to f in measure, one can find a subsequence f_{k_j} such that $f_{k_j} \to f$, a.e.*

Proof. We give a construction about subsequence by diagonal principle. We aim to prove there exists a subsequence $\{f_{k_j}\}$ satisfying

$$\mu \left(\bigcap_{i=1}^{\infty} \bigcup_{j\geq i} \left\{ x : \boxed{|f_{k_j}(x) - f(x)| \geq \frac{1}{n}} \right\} \right) = 0,\ \forall n, \tag{1.6}$$

Claim: $\{f_{k_j}\}$ satisfying

$$\mu \left(\bigcup_{j\geq i} \left\{ x : \boxed{|f_{k_j}(x) - f(x)| \geq \frac{1}{i}} \right\} \right) = \frac{1}{i},\ \forall i$$

is eligible for eq(1.6).

View $1/i$ as ε, we decompose ε as $\sum \varepsilon/2^j$ and determine $f_{i,k_1}, f_{i,k_2}, \cdots, f_{i,k_n}$ such that

$$\mu\left(\{x : |f(x) - f_{i,k_j}(x)| \geq 1/i\}\right) \leq \frac{\varepsilon}{2^j}. \tag{1.7}$$

If we use $f_{i,n}$ in place of f_{i,k_n} for simplicity, we can pick the diagonal element as desired subsequence.

$$\begin{array}{cccccc}
\boxed{f_{1,1}} & f_{1,2} & f_{1,3} & \cdots & f_{1,n} & \cdots \\
f_{2,1} & \boxed{f_{2,2}} & f_{2,3} & \cdots & f_{2,n} & \cdots \\
f_{3,1} & f_{3,2} & \boxed{f_{3,3}} & \cdots & f_{3,n} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \\
f_{n,1} & f_{n,2} & f_{n,3} & \cdots & \boxed{f_{n,n}} & \cdots
\end{array}$$

□

Proof of Claim. Note that

$$\begin{aligned}
\bigcap_{i=1}^{\infty}\bigcup_{j\geq i}\left\{x : \boxed{|f_{k_j}(x) - f(x)| \geq \frac{1}{n}}\right\} &\subset \bigcap_{i=n}^{\infty}\bigcup_{j\geq i}\left\{x : \boxed{|f_{k_j}(x) - f(x)| \geq \frac{1}{n}}\right\} \\
&\subset \bigcap_{i=n}^{\infty}\bigcup_{j\geq i}\left\{x : \boxed{|f_{k_j}(x) - f(x)| \geq \frac{1}{i}}\right\}
\end{aligned}$$

and

$$B_i := \bigcup_{j\geq i}\left\{x : \boxed{|f_{k_j}(x) - f(x)| \geq \frac{1}{i}}\right\}$$

is descending sequence with measure less than $1/i$. To ensure the validity of

$$\mu\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_i \mu(B_i) = \lim_i 1/i = 0$$

it suffices to show that $\mu(B_1) < \infty$. Actually, $\mu(B_1) \leq 1$ due to our condition (1.7).