Given $f, g \in L^p(\mu)$, and we investigate

$$\phi(t) = \int_X |f + tg|^p \, d\mu.$$

One can expect that $\Phi(t)$ is differentiable becasuse the integrand is continuous. First, let me give an example

$$\phi(t) = \int_0^{2\pi} |\sin(x) + t\cos(x)|^2 \, d\mu,$$

which is illustrated in the following figure.

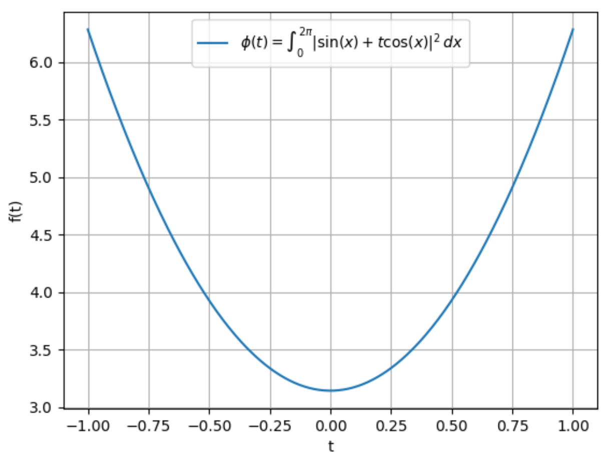

Figure 1:

To prove this theoretically, we take $t = 0$ as example

$$\begin{aligned} \phi'(0) &= \lim_{t\to 0} \frac{\int_X |f + tg|^p - |f|^p}{t} \\ &= \int_X \lim_{t\to 0} \frac{|f + tg|^p - |f|^p}{t} \, d\mu. \end{aligned}$$

Notice that we need some convergence theorem to change the order of $\lim_{t\to 0}$ and $\int_X$.

It's the convexity of $|a+bt|^p, p \geq 1$ that make it available. For $t \in (-0.5, 0.5)$, we have

$$\frac{|f + tg|^p - |f|^p}{t} \leq \frac{|f + g|^p - |f|^p}{1 - 0} = |f + g|^p - |f|^p.$$

and it's easy to see that $|f+g|^p - |f|^p \in L^1(\mu)$ by the simple inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$.

Hence we apply L Hospital's rule to get

$$\phi'(0) = p \int_X |f|^{p-2} fg \, d\mu.$$

Convex function. Here I wanna write something about convex function. The simplest convex function which is not differentiable is $|x|$.

More generally, a convex function can consist of connected line segments, also known as a piecewise linear function. Interestingly, we can obtain such a function by connecting the tangent lines of a convex function or connecting the points $(x_i, f(x_i))$ where f is convex.