



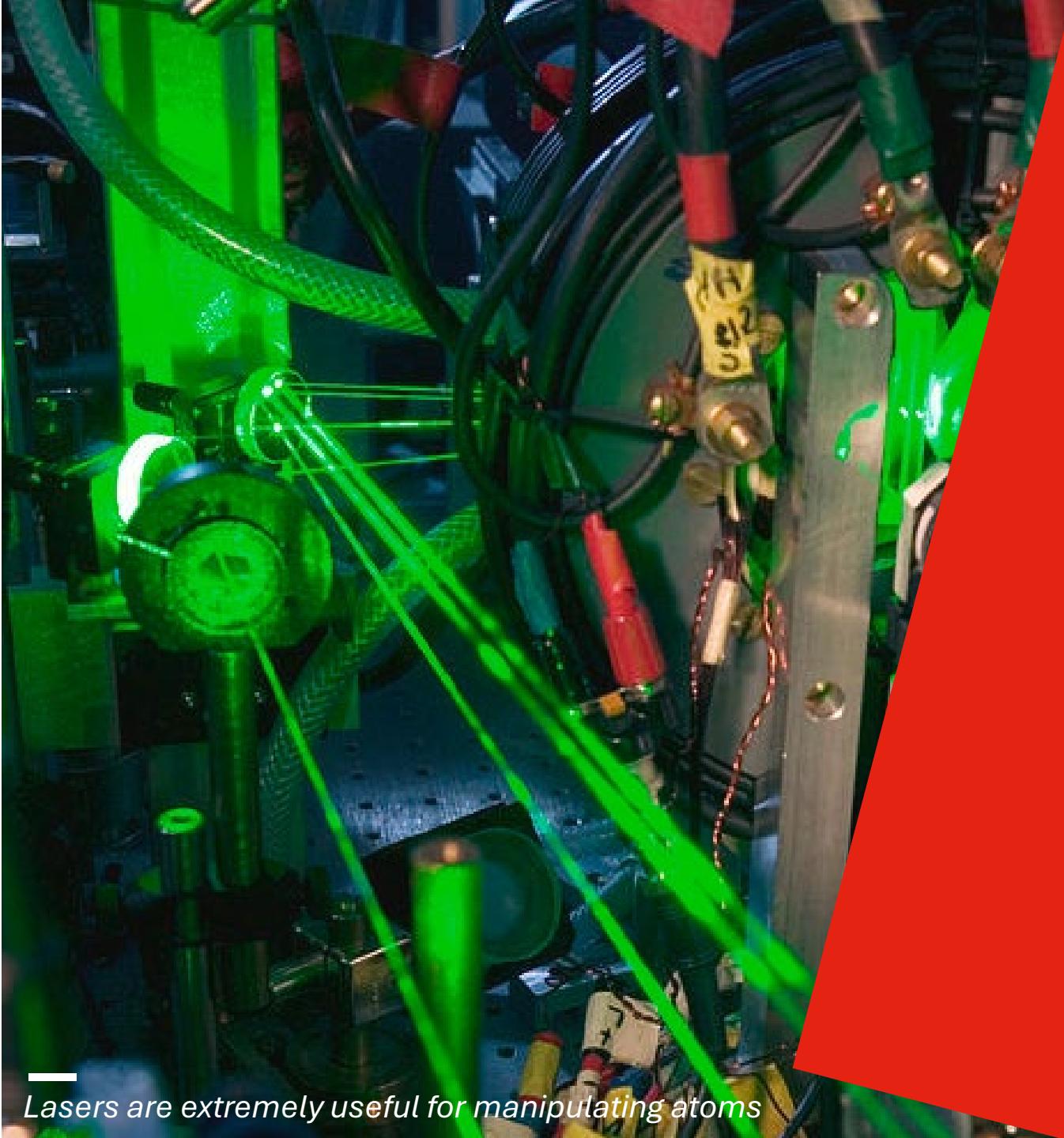
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KYA322

Atomic physics

Andy McCulloch





The quantum state

Foot Ch. 7 // Steck Ch. 4



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Learning outcomes

Week 4, lecture 1

Foot §13.1 – 8.5 // Steck §13.1 – 8.5:
Describing the quantum state

- The density matrix
 - How do quantum states in the wild look
 - Pure and mixed states
 - What is the density operator?
 - The density operator and matrix for a two-level atom

Stocktake

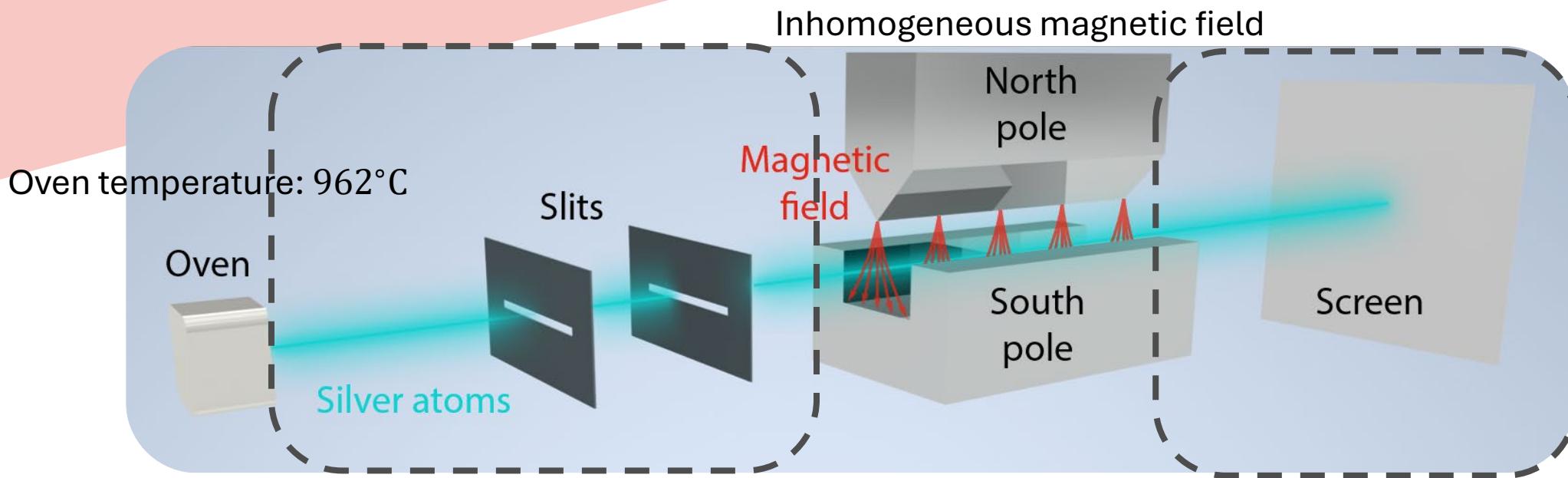
We have studied

1. “Simple” quantum: the hydrogen atom
2. Hyperfine structure and the addition of angular momentum
3. Identical particles and multi-electron atoms
4. Transitions

All these systems are *ideal* systems: they can be fully described by state vector $|\Psi\rangle$

We now want to move from ideal systems to real systems; we have most of the building blocks, but we are missing one important piece

Stern-Gerlach returns



What is the state here?

We do not know how to
describe this!

Final state: either $|s_z; +\rangle$ or $|s_z; -\rangle$

We know how to describe this

When we perform experiments, the systems are (almost) never fully described by a single state vector $|\Psi\rangle$, there is coupling to the environment, and we are working with ensembles. How do we describe these systems?

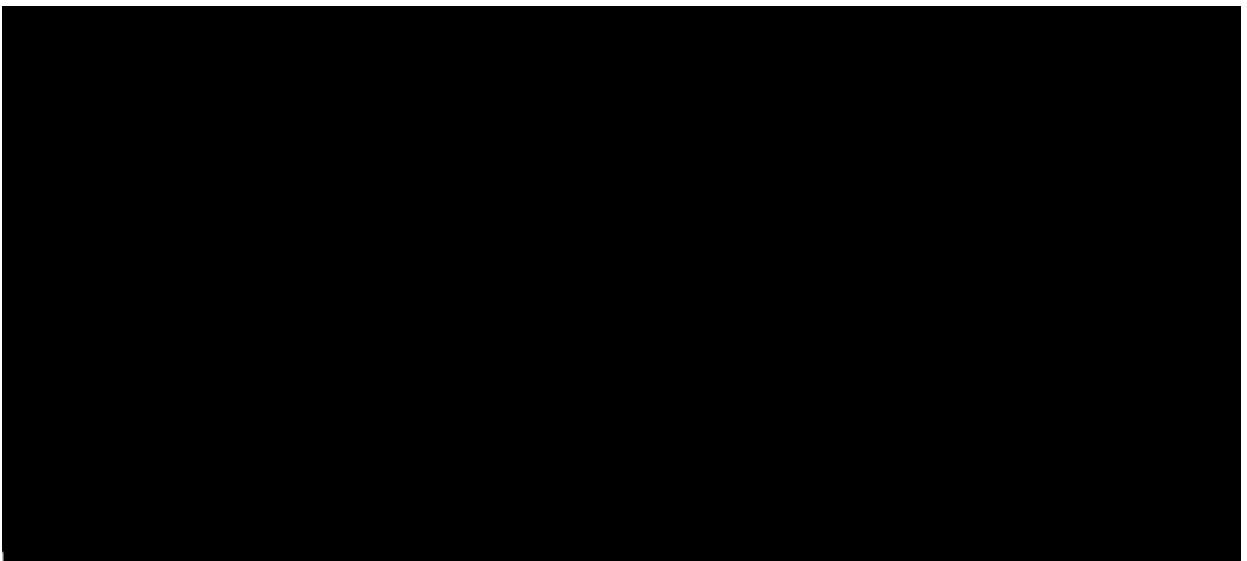
Two-slit interference

Consider Young's classic experiment:

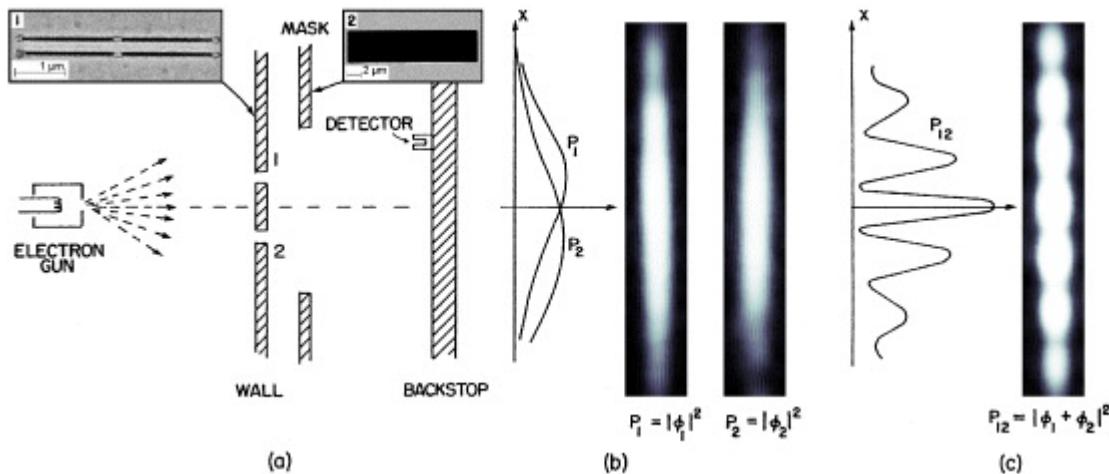
What is the difference between the

state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ and a

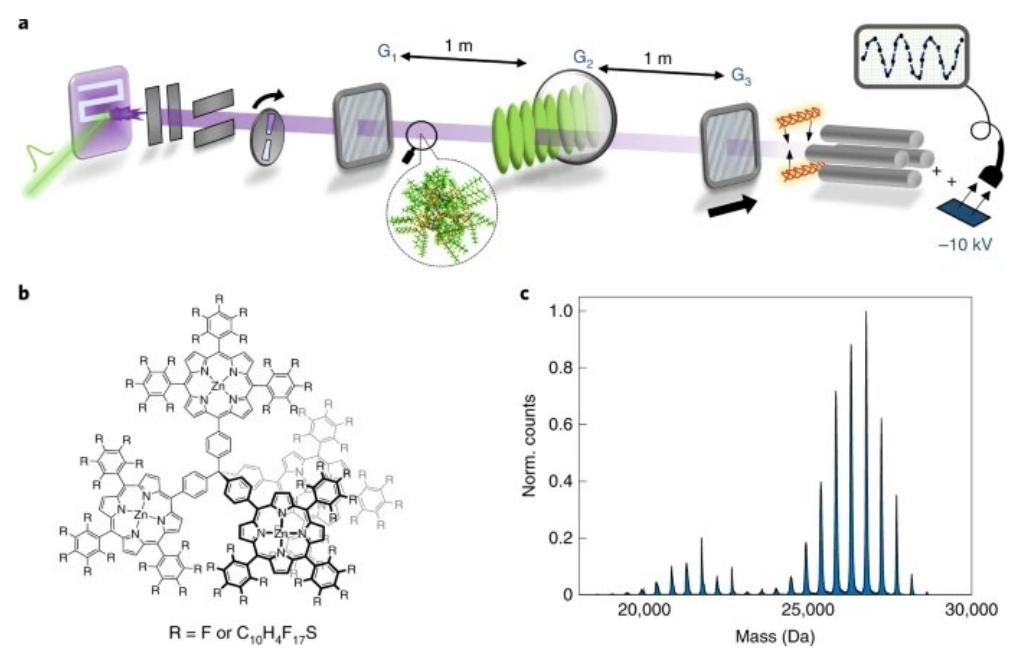
50:50 mixture of states $|\psi_1\rangle$ and $|\psi_2\rangle$?



Buildup of an interference pattern from single electrons



The double-slit experiment using single electrons



Interference of molecules with 25,000 amu

Pure and mixed states

A *pure* ensemble is a collection of physical systems such that every member is characterised by the same state vector $|\Psi\rangle$

A *mixed* ensemble is a statistical mixture of pure states and **cannot** be described by a single state vector $|\Psi\rangle$

Mixed state examples:

- 50% $|+\rangle$, 50% $|-\rangle$
- $w_1|s_x; +\rangle + w_2|s_x; -\rangle + w_3|s_y; +\rangle + \dots$

Mixed states are a collection of state vectors $|\psi_i\rangle$ with fractional populations w_i where $\sum_i w_i = 1$



A mixed state?
 $|g\rangle = 1/\sqrt{2}(|b\rangle + |y\rangle)$

The $|\psi_i\rangle$ are not necessarily orthogonal and can exceed the dimensionality of the state space.

0.3 $|s_z; +\rangle$, 0.3 $|s_z; -\rangle$,
and 0.4 $|s_x; +\rangle$

We can therefore construct systems comprised of mixed constituents



Expectation values

For an operator \hat{A} , the expectation value is

$$\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle$$

for a pure system $|\Psi\rangle$. For a mixed state:

$$\langle \hat{A} \rangle = \sum_i w_i \langle \psi_i | \hat{A} | \psi_i \rangle$$

For an arbitrary operator \hat{B} with orthonormal basis $\{|j\rangle\}$

$$\text{tr}(\hat{B}) = \sum_j \langle j | \hat{B} | j \rangle$$

⇒ the expectation value $\langle \hat{A} \rangle$ can be calculated as

$$\langle \hat{A} \rangle = \text{tr}(\hat{\rho} \hat{A}) \text{ where } \hat{\rho} = \sum_i w_i |\psi_i\rangle \langle \psi_i|$$



Tracing the matrix

This may seem like some mathematical gymnastics, but it turns out that the operator $\hat{\rho}$ is *extremely* useful

The density operator

The density operator is

$$\hat{\rho} = \sum_i w_i |\psi_i\rangle\langle\psi_i|$$

Contains **all** the physically meaningful information we can possibly obtain about the ensemble

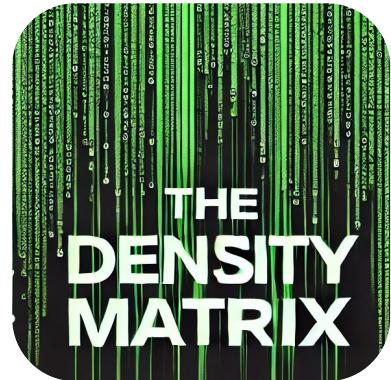
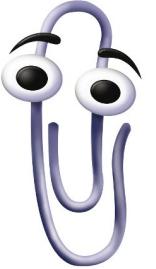
⇒ In the way $|\Psi\rangle$ provides a complete description of a *pure* state, $\hat{\rho}$ provides a complete description of a *mixed* state: it is a “generalised” state vector

We can define $\hat{\rho}$ in terms of a general basis $\{|n\rangle\}$; assuming that the basis is orthonormal and complete* the matrix for $\hat{\rho}$ in the basis $\{|n\rangle\}$ is

$$\langle n|\hat{\rho}|m\rangle = \sum_i w_i \langle n|\psi_i\rangle\langle\psi_i|m\rangle$$

with elements $\rho_{nm} = \langle n|\hat{\rho}|m\rangle$

*We use complete here in the sense that they define all the states $|\psi\rangle$ that we care about



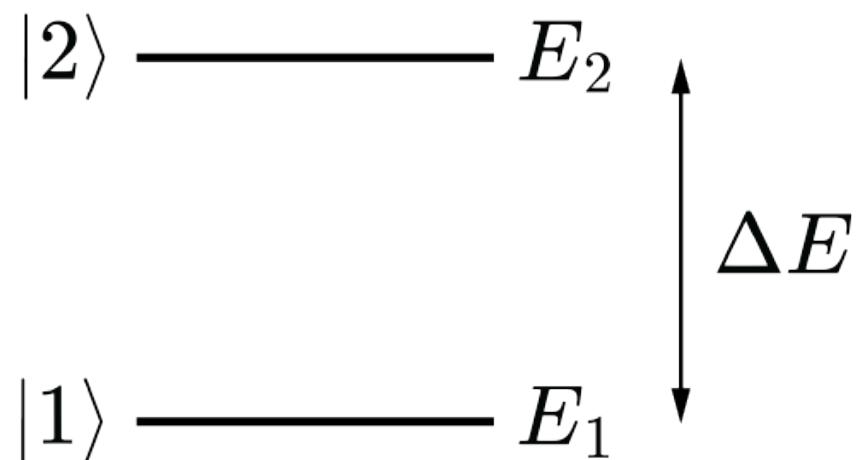
The element ρ_{nm} denotes the matrix element in the n -th row and the m -th column of the density matrix ρ

Computing the density matrix

We are going to talk a lot about two-level atoms, constructed from:

- A ground state $|1\rangle$ with energy E_1
- An excited state $|2\rangle$ with energy E_2
- An energy separation of $\Delta E = E_2 - E_1$

This model is suitable for modelling many complex systems



Compute the density matrix for a two-level atom with $|\Psi\rangle = c_1|1\rangle + c_2|2\rangle$

The two-level atom is the basis for many applications

The density matrix for a two-level atom

For a single atom with $|\Psi\rangle = c_1|1\rangle + c_2|2\rangle$, there are no weightings (or equivalently, $w_1 = 1$) and the density operator is

$$\hat{\rho} = \sum_i w_i |\psi_i\rangle\langle\psi_i| = (c_1|1\rangle + c_2|2\rangle)(c_1^*\langle 1| + c_2^*\langle 2|)$$

The element

$$\begin{aligned}\rho_{11} &= \langle 1|\hat{\rho}|1\rangle = \langle 1|(c_1|1\rangle + c_2|2\rangle)(c_1^*\langle 1| + c_2^*\langle 2|)|1\rangle \\ &= c_1 c_1^* = |c_1|^2\end{aligned}$$

which is the probability of measuring the atom in state $|1\rangle$.

Similarly, we can compute ρ_{12} , ρ_{21} , and ρ_{22} and find

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} c_1 c_1^* & c_1 c_2^* \\ c_2 c_1^* & c_2 c_2^* \end{pmatrix}$$



Dense(evity) examples

Compute the density matrices for the following systems:

1. Pure states:
 - a. A spin-polarised beam $|\Psi\rangle = |S_z; +\rangle$ in the $|S_z; \pm\rangle$ basis
 - b. A spin-polarised beam $|\Psi\rangle = |S_x; -\rangle$ in the $|S_z; \pm\rangle$ basis
2. Mixed states:
 - a. An unpolarised beam in the $|S_z; \pm\rangle$ basis
 - b. A partially polarised beam, with 75% $|S_z; +\rangle$ and 25% $|S_x; -\rangle$
3. Find the expectation value $\langle S_z \rangle$ for the partially polarised beam (2b)
4. Find the expectation values $\langle S_x \rangle$ and $\langle S_y \rangle$ for the same beam

Density matrix properties

Some properties of the density matrix:

- ρ is Hermitian $\Rightarrow \rho_{nm} = \rho_{mn}^*$
- ρ is *normalised*, meaning $\text{tr}(\rho) = 1$
- For a pure ensemble: $\rho^2 = \rho$, $\text{tr}(\rho^2) = 1$
- Diagonal components are real and represent the state populations
- Off-diagonal components are complex and are referred to as *coherences*

Discussion:

What is the concept of coherence? What is the physical origin of coherence? What is an example of a coherent (or incoherent) system?



Coherence



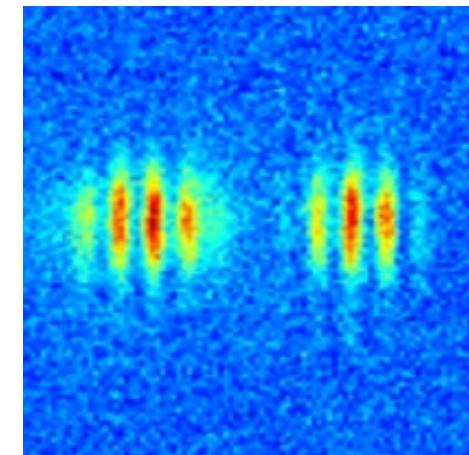
If off-diagonal elements are

- Zero: atoms in the ensemble are in defined states
- Non-zero: the system comprises one or more *superposition* states

Superposition states enforce a phase relationship between states, and we know from wave mechanics, phase plays a critical role in determining the amplitude under time (or spatial) evolution



The Michelson interferometer demonstrates the relationship between phase and intensity



Matter waves interfere like light waves: Rb-87 BEC



The 146 m high shot tower in Bremen gives almost 5 s of freefall

Same, same but different

$$|\psi_1\rangle = 0.866|1\rangle + 0.5|2\rangle$$

$$|\psi_2\rangle = 75\%|1\rangle, 25\%|2\rangle$$

Discussion:

What are the probabilities $\mathcal{P}(|\psi\rangle = |1\rangle)$ and $\mathcal{P}(|\psi\rangle = |2\rangle)$ for both $|\psi_1\rangle$ and $|\psi_2\rangle$? Calculate the density matrices for both systems..

$$\begin{aligned}\text{For } |\psi_1\rangle: \rho &= (0.866|1\rangle + 0.5|2\rangle)(0.866\langle 1| + 0.5\langle 2|) \\ &= 0.866^2|1\rangle\langle 1| + 0.5^2|2\rangle\langle 2| + 0.433|1\rangle\langle 2| + 0.433|2\rangle\langle 1| \\ &= \begin{pmatrix} 0.75 & 0.433 \\ 0.433 & 0.25 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\text{For } |\psi_2\rangle: \rho &= 0.75|1\rangle\langle 1| + 0.25|2\rangle\langle 2| \\ &= \begin{pmatrix} 0.75 & 0 \\ 0 & 0.25 \end{pmatrix}\end{aligned}$$

The 2-slit experiment returns

Consider the state

$$|\Psi\rangle = 1/\sqrt{2} (|\psi_1\rangle + |\psi_2\rangle)$$

Let's change basis; suppose

$$|\Psi\rangle = |\phi_1\rangle$$

then

$$|\psi_1\rangle = 1/\sqrt{2} (|\phi_1\rangle + |\phi_2\rangle)$$

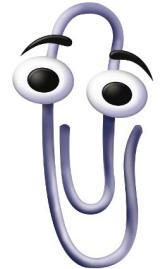
$$|\psi_2\rangle = 1/\sqrt{2} (|\phi_1\rangle - |\phi_2\rangle)$$

If we now make measurements:

50% chance of either $|\psi_1\rangle$ or $|\psi_2\rangle$

100% chance of $|\phi_1\rangle$

*We assume that
the operators
 $[\hat{\psi}, \hat{\phi}] \neq 0$



This can be seen by inspection, as
 $|\psi_1\rangle + |\psi_2\rangle = \sqrt{2}|\phi_1\rangle$ and the state space
must be spanned by two state vectors

Performing the same measurements for the
50:50 mixture of $|\psi_1\rangle$ and $|\psi_2\rangle$:

50% chance of $|\psi_1\rangle$ with 50% chance of $|\phi_1\rangle$

50% chance of $|\psi_2\rangle$ with 50% chance of $|\phi_1\rangle$

⇒ probability of being in $|\phi_1\rangle$

$$= (0.5)(0.5) + (0.5)(0.5) = 0.5$$

What about interference?

In two-slit interference experiments, atoms are approximately in *momentum* eigenstates $|\pm k\rangle$

For the 50: 50 mixture of $|k\rangle$ and $| -k\rangle$:

$$\hat{\rho} = |k\rangle\langle k| + |-k\rangle\langle -k|$$

$$\begin{aligned}\langle \hat{x} \rangle &= \text{tr}(\hat{\rho}\hat{x}) = \langle x|\hat{\rho}|x\rangle \\ \langle x|k\rangle &\propto e^{ik\cdot x}\end{aligned}$$

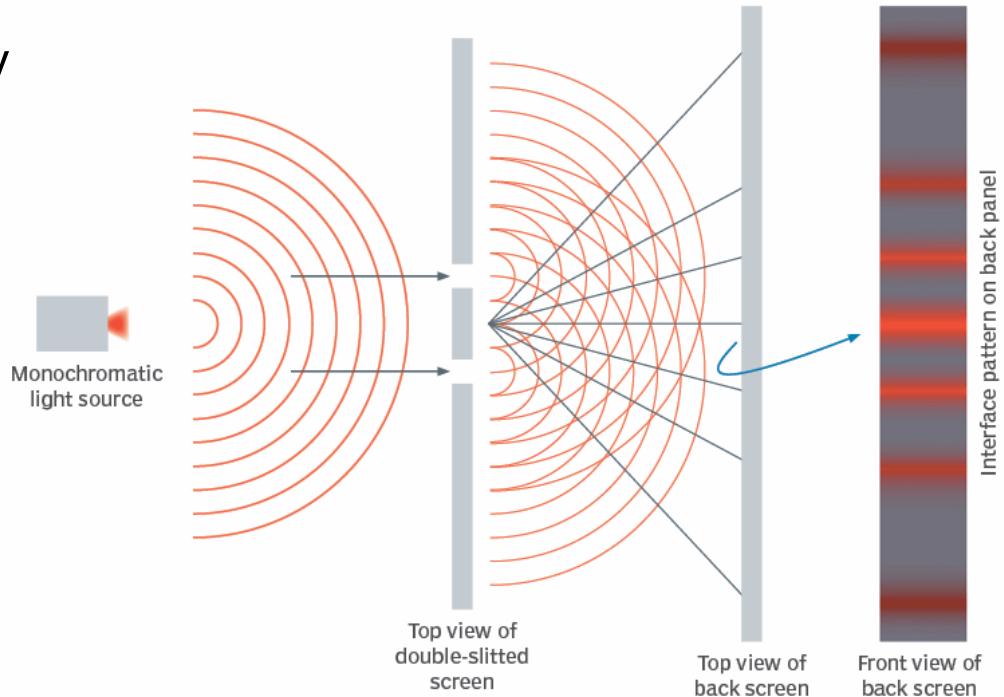
$$\Rightarrow \langle \hat{x} \rangle \propto 1$$

For $|\Psi\rangle = 1/\sqrt{2}(|k\rangle + |-k\rangle)$:

$$\hat{\rho} = 1/2|k\rangle\langle k| + 1/2|k\rangle\langle -k| + 1/2|-k\rangle\langle k| + 1/2|-k\rangle\langle -k|$$

$$\Rightarrow \langle \hat{x} \rangle \propto 1 + \cos(2kx)$$

This Demonstrates that a superposition state is required to observe interference fringes!



Diffraction is best described in terms of momentum states

Interference!



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Summary

Problems

- Quantum states
 - Rarely described by a single state vector $|\Psi\rangle$ and a more sophisticated approach is required
 - Expectation values can be computed as $\langle \hat{A} \rangle = \text{tr}(\hat{\rho}\hat{A})$ where $\hat{\rho} = \sum_i w_i |\psi_i\rangle\langle\psi_i|$
 - The density operator $\hat{\rho}$ describes completely a *mixed* state
- The density matrix
 - Diagonal elements ρ_{nn} represent the probabilities of measuring the system in state $|n\rangle$
 - Off-diagonal elements represent the coherences – a kind of coupling – between states



Time evolution

Foot Ch. 7 // Steck Ch. 5



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Learning outcomes

Week 4, lecture 2

Foot §13.1 – 8.5 // Steck §13.1 – 8.5:
Time evolution of the two-level atom

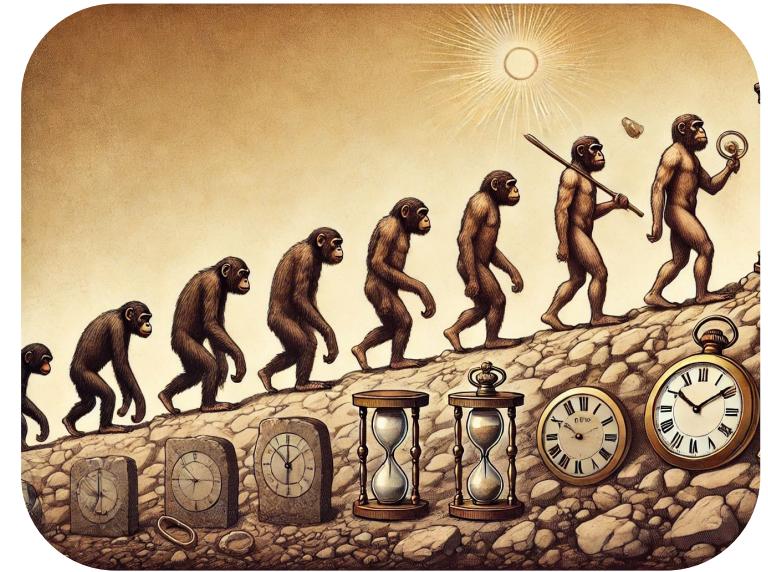
- The two-level atom
 - How do we evolve the density matrix in time?
 - The Rabi frequency
 - Solving for the population of an atomic system coupled to a light field
- The optical Bloch equations
 - How do these relate to the Einstein rate equations
 - Steady-state solution and predictions

Where are we, and where do we want to be?

We have been working on rigorously predicting in the internal structure of atoms, the propensity of a system to transition between states, and recently, how we describe systems which are not simply described (pure states).

Recalling that our goal is to understand modern atomic systems and applications, we need only add a few more tools to our toolkit.

Critically, we need to know how to evolve a system in time, predicting – and ultimately, harnessing - transient effects.



Time evolution... Get it?



Adding to the toolkit

To t' , and beyond!

The density operator $\hat{\rho}$ completely describes the system, and we know how to compute expectation values, but how to change t ?

At some time t_0 , we have

$$\hat{\rho}(t_0) = \sum_i w_i |\psi_i\rangle\langle\psi_i|$$

We are going to assume that the ensemble is left undisturbed (w_i are fixed) and time evolution is contained within the $|\psi_i\rangle$.

State vectors evolve via the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H}(t)|\psi\rangle \quad \text{and} \quad -i\hbar \frac{\partial}{\partial t} \langle\psi| = \langle\psi| \hat{H}(t)$$

$$\Rightarrow i\hbar \frac{\partial \hat{\rho}}{\partial t} = \sum_i w_i (\hat{H}(t)|\psi_i\rangle\langle\psi_i| - |\psi_i\rangle\langle\psi_i| \hat{H}(t))$$

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}(t), \hat{\rho}]$$

which is the *Liouville-von Neumann equation*

*For those familiar with the Heisenberg representation, whilst $\hat{\rho}$ may look like a Heisenberg operator, it is not!



Joseph Liouville



John von Neumann

This is the quantum mechanical analogue of Liouville's theorem in statistical mechanics





The two-level atom

We can put this machinery in action for the two-level atom

Consider a two-level atom with incident monochromatic dipole radiation

$$|\Psi\rangle = c_1|1\rangle + c_2|2\rangle$$
$$\Rightarrow \rho = \begin{pmatrix} c_1 c_1^* & c_1 c_2^* \\ c_2 c_1^* & c_2 c_2^* \end{pmatrix}$$

The time evolution is given by

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}(t), \hat{\rho}]$$

with $H = H_0 + H'$ and for convenience, define $\dot{\rho} = -i/\hbar [\hat{H}, \hat{\rho}]$

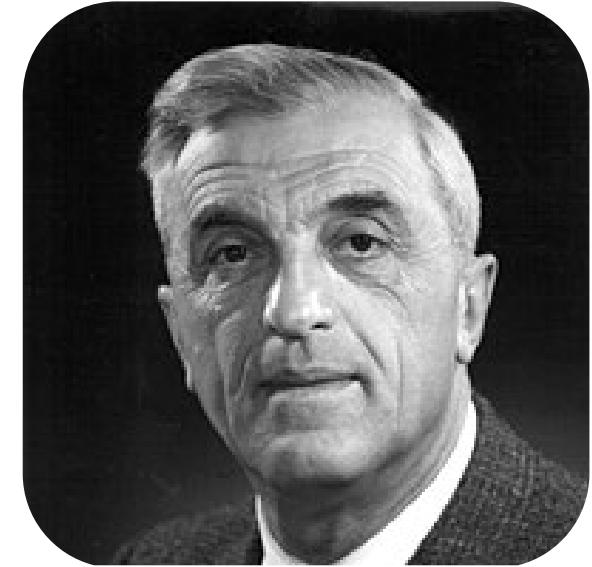
The unperturbed states of the system are

$$\langle 1 | H_0 | 1 \rangle = E_1$$

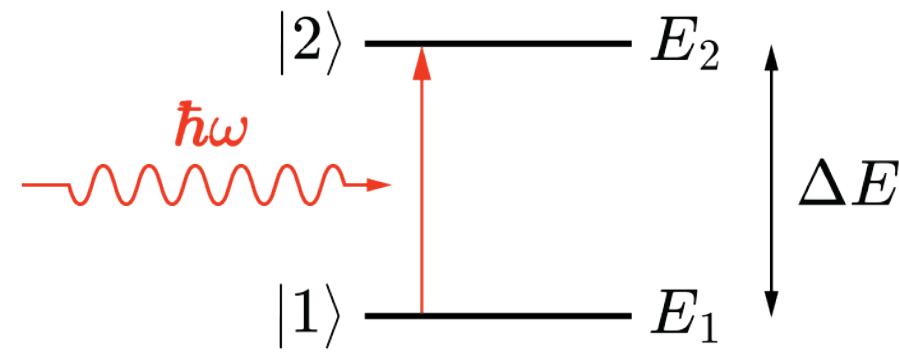
$$\langle 2 | H_0 | 2 \rangle = E_2$$

$$\langle 1 | H_0 | 2 \rangle = \langle 2 | H_0 | 1 \rangle = 0$$

and the perturbing Hamiltonian is $H' = -\mathbf{d} \cdot \mathbf{E}$ (as we have seen previously)



Felix Bloch first studied such systems in “Nuclear induction”



A two-level atom with monochromatic radiation



The perturbing matrix

As we have seen previously, for monochromatic radiation:

$$\mathbf{E}(t) = 2\mathcal{E}_0 \hat{\epsilon} \cos(\omega t) \equiv \mathbf{E}_0 \cos(\omega t)$$

We assume that \mathbf{E}_0 is not affected by the atom, so then for $H' = -\mathbf{d} \cdot \mathbf{E}$, the matrix elements are

$$H'_{11} = -\mathbf{E} \cdot \langle 1 | \mathbf{d} | 1 \rangle = e\mathbf{E} \cdot \langle 1 | \mathbf{r} | 1 \rangle = 0$$

as we have seen previously. The same is true for H'_{22} .

For the off-diagonal elements:

$$H'_{12} = -\mathbf{E} \cdot \langle 1 | \mathbf{d} | 2 \rangle = -\mathbf{E} \cdot \langle 2 | \mathbf{d} | 1 \rangle = H'_{21}$$

which gives

$$H' = \begin{pmatrix} 0 & -\hbar\Omega \cos(\omega t) \\ -\hbar\Omega \cos(\omega t) & 0 \end{pmatrix}$$

where we have defined $\mathbf{d} \cdot \mathbf{E}_0 = \hbar\Omega$

Ω is the *Rabi frequency* and is a measure of the coupling between states



Recall that the dipole operator couples only states with frequency matching the field, and which obey the selection rules.

Intuitively, it makes sense that one cannot induce a dipole moment between the same state

A detour: the Rabi frequency

The Rabi frequency is defined through the relation

$$\mathbf{d} \cdot \mathbf{E}_0 = \hbar\Omega$$

which therefore describes the strength of the interaction between an oscillating field – a laser – and a given transition in an atom. The field is of the form

$$\mathbf{E}(t) = 2\epsilon_0\hat{\epsilon}\cos(\omega t) \equiv \mathbf{E}_0 \cos(\omega t)$$

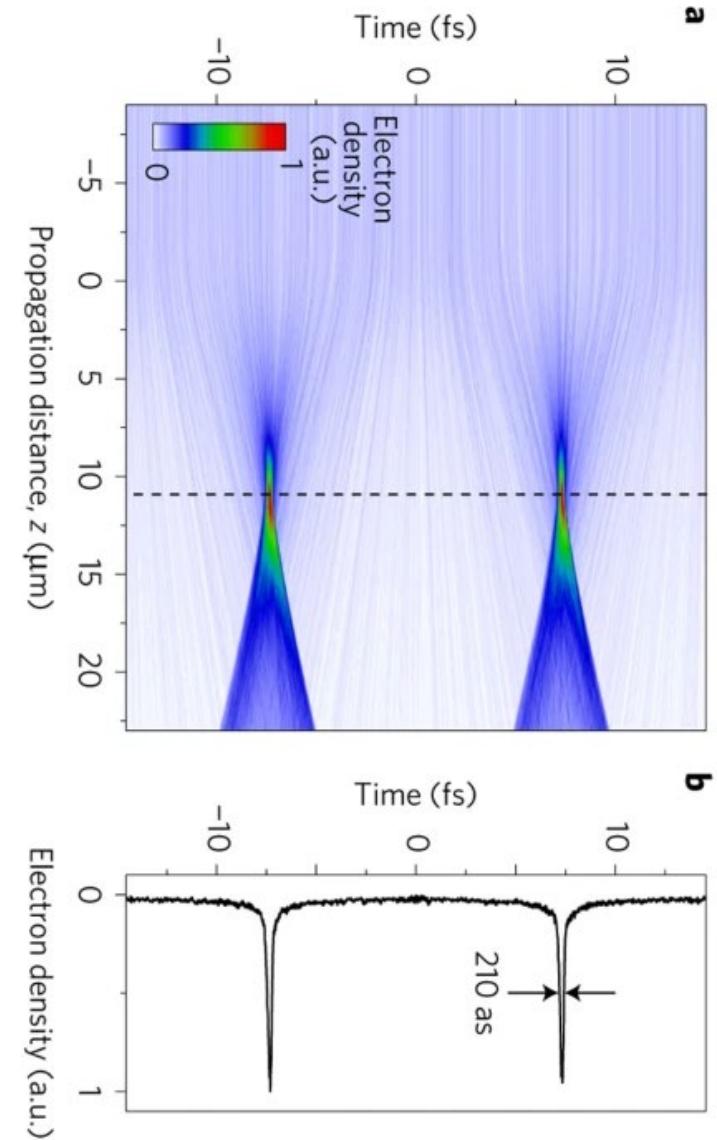
which has the intensity

$$I = \frac{1}{2}\epsilon_0 c E_0^2$$

It should be clear that the Rabi frequency is related to the intensity of the field, since

$$\Omega = \frac{d}{\hbar} \sqrt{\frac{2I}{\epsilon_0 c}}$$

and hopefully this matches your intuition: if I crank up the field, I would expect the system to respond more strongly, and the coupling (Ω) increases with the intensity (I)



*A much blunter instrument:
electron trajectories being altered
by a (very) intense light field*



The Matrix Revolutions

We now have to compute

$$i\hbar \frac{\partial \rho}{\partial t} = [H_0 + H', \rho] = \left[\begin{pmatrix} 0 & -\hbar\Omega \cos(\omega t) \\ -\hbar\Omega \cos(\omega t) & \hbar\omega_0 \end{pmatrix}, \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \right]$$

which (after a grind) gives

$$\begin{aligned}\dot{\rho}_{11} &= i\Omega \cos(\omega t)(\rho_{21} - \rho_{12}) \\ \dot{\rho}_{22} &= -\dot{\rho}_{11} = -i\Omega \cos(\omega t)(\rho_{21} - \rho_{12}) \\ \dot{\rho}_{21} &= -i\omega_0 \rho_{21} - i\Omega \cos(\omega t)(\rho_{22} - \rho_{11}) \\ \dot{\rho}_{12} &= -\dot{\rho}_{21} = i\omega_0 \rho_{21} + i\Omega \cos(\omega t)(\rho_{22} - \rho_{11})\end{aligned}$$

which are the *Optical Bloch Equations (OBEs)*. These allow us to predict the evolution of ρ and thus our quantum system!

But, solving coupled non-linear differential equations is hard...



The rotating wave approximation

We decompose the field into components oscillating at $\pm\omega$ using $\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$ and introduce the *slowly varying* variables σ_{12} and σ_{21} :

$$\rho_{21}(t) = \sigma_{21}(t)e^{-i\omega t}$$

$$\rho_{12}(t) = \rho_{21}^*(t) = \sigma_{12}(t)e^{i\omega t}, \text{ with } \sigma_{21} = \sigma_{12}^*$$

Then

$$\dot{\rho}_{21} = \frac{d}{dt}(\sigma_{21}e^{-i\omega t}) = -i\omega_0(\sigma_{21}e^{-i\omega t}) - \frac{i\Omega}{2}(e^{i\omega t} + e^{-i\omega t})(\rho_{22} - \rho_{11})$$

$$\dot{\sigma}_{21}e^{-i\omega t} = i(\omega - \omega_0)\sigma_{21}e^{-i\omega t} - \frac{i\Omega}{2}(e^{i\omega t} + e^{-i\omega t})(\rho_{22} - \rho_{11})$$

$$\Rightarrow \dot{\sigma}_{21} = i(\omega - \omega_0)\sigma_{21} - \frac{i\Omega}{2}(\rho_{22} - \rho_{11}) - \frac{i\Omega}{2}(\rho_{22} - \rho_{11})e^{2i\omega t}$$

So, in the rotating wave approximation:

$$\dot{\sigma}_{21} = i(\omega - \omega_0)\sigma_{21} - \frac{i\Omega}{2}(\rho_{22} - \rho_{11})$$

This term oscillates at 2ω , and rapidly averages to zero
⇒ we ignore it

Physically, we are effectively moving into the frame which rotates at frequency ω

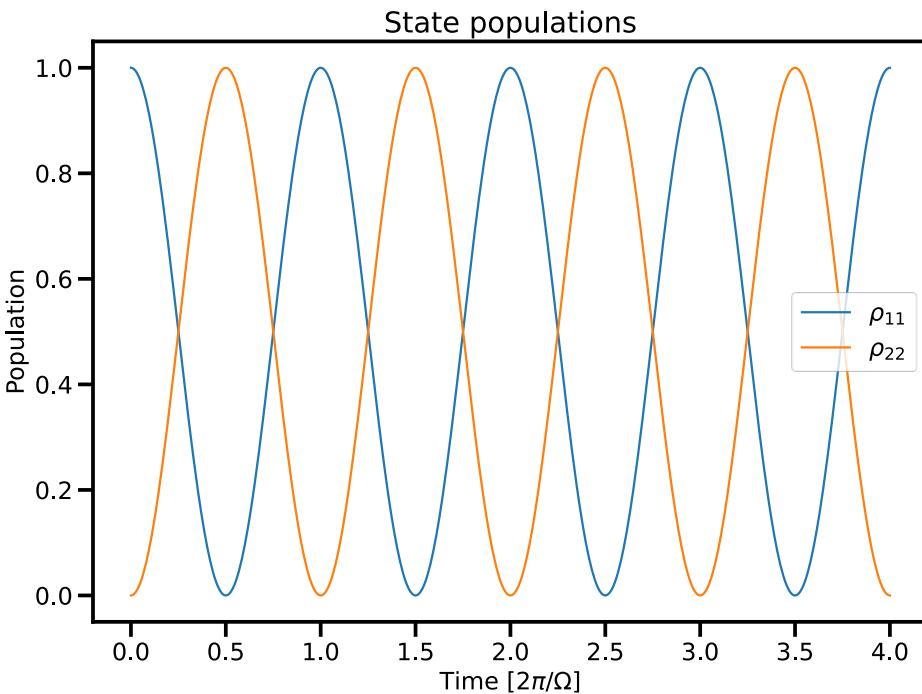


A rotating saucepan lid shows the Bloch-Siegert shift, which the rotating wave approximation ignores

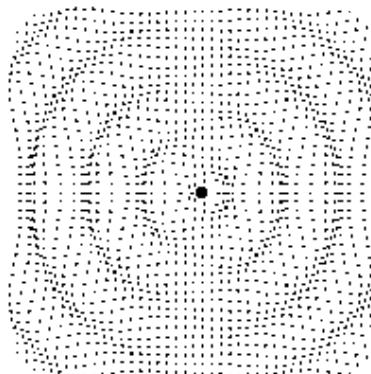
Solutions (manual)

We can numerically solve these equations directly: assuming that the atom is initially in the ground state

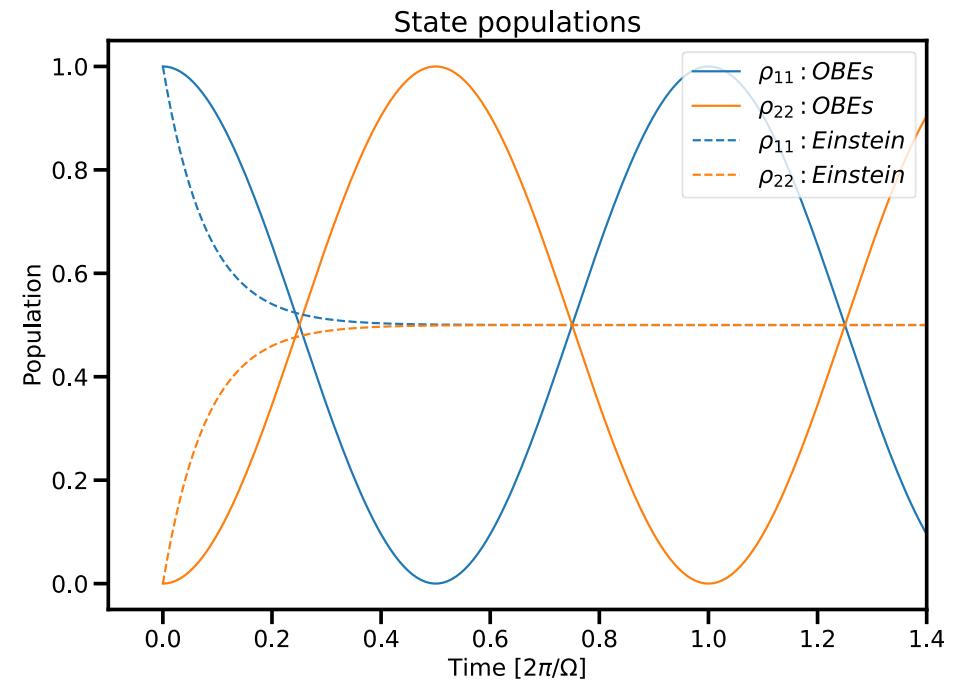
$$\rho_{11}(0) = 1 \quad \rho_{22}(0) = 0$$



Which is
correct?



Under the same conditions, we can predict the time evolution from the Einstein rate equations:



Consider the environment

Neither model includes any *spontaneous decay*.

We have seen how this is incorporated into the Einstein model (the A coefficient)

Again, we phenomenologically introduce *relaxation* into the system via

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}(t), \hat{\rho}] + \hat{\mathcal{L}}_{\text{relax}}$$

Relaxation operator

But what does this look like? How should it act?

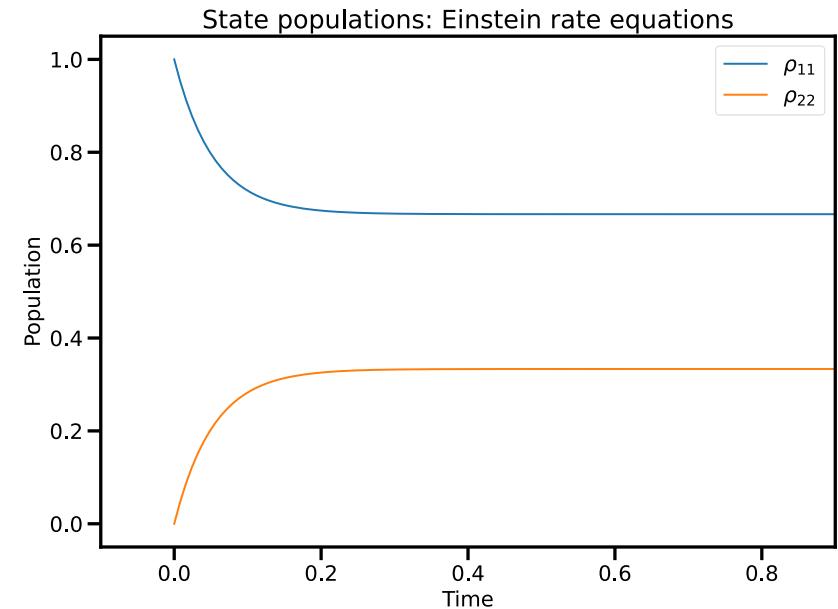
$$\dot{\rho}_{22} = -\Gamma \rho_{22}$$

This is the excited-state decay rate $\Rightarrow \Gamma = A_{21}$

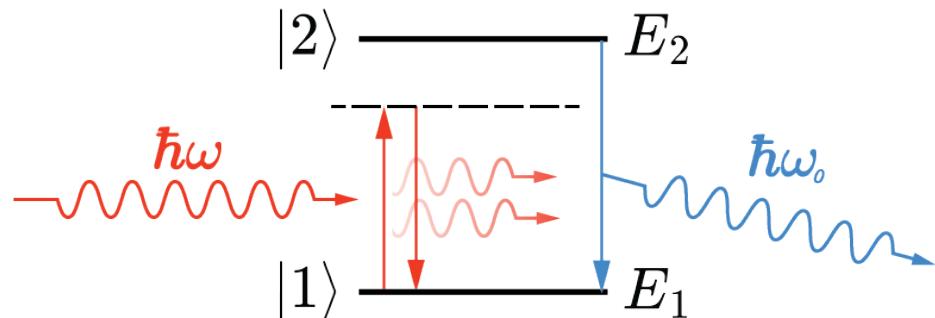
$$\dot{\rho}_{11} = +\Gamma \rho_{22}$$

$$\dot{\sigma}_{12} = -\gamma_{\perp} \sigma_{12}$$

$$\dot{\sigma}_{21} = -\gamma_{\perp} \sigma_{21}$$



Populations in the Einstein model with spontaneous decay



The two-level atom is getting complicated

The optical Bloch equations

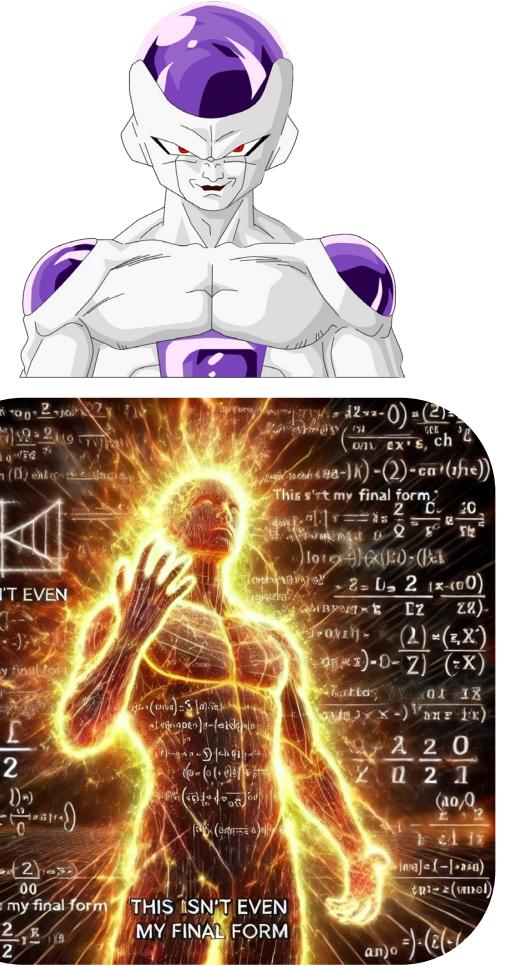
Using the Liouville-von Neumann equation, we arrived at the *Optical Bloch Equations*, which in the rotating wave approximation with baked in relaxation, gives:

$$\begin{aligned}\dot{\rho}_{11} &= \frac{i\Omega}{2} (\sigma_{21} - \sigma_{12}) + \Gamma\rho_{22} \\ \dot{\rho}_{22} &= -\dot{\rho}_{11} = -\frac{i\Omega}{2} (\sigma_{21} - \sigma_{12}) - \Gamma\rho_{22} \\ \dot{\sigma}_{21} &= i(\omega - \omega_0)\sigma_{21} - \frac{i\Omega}{2} (\rho_{22} - \rho_{11}) - \gamma_\perp\sigma_{21} \\ \dot{\sigma}_{12} &= \dot{\sigma}_{21}^* = -i(\omega - \omega_0)\sigma_{12} + \frac{i\Omega}{2} (\rho_{22} - \rho_{11}) - \gamma_\perp\sigma_{12}\end{aligned}$$

where

- $\Gamma = 1/T_1$ is the *relaxation rate* and $T_1 = \tau$ (the excited state lifetime) is the T_1 time
- $\gamma_\perp = 1/T_2$ is the rate at which coherence is lost

These equations are extremely powerful, and can easily be extended to many levels, more light fields, atomic motion, etc.



Both T_1 and T_2 are *homogeneous* broadening mechanisms, that is, they are the same for all atoms

Tsunami warning

It is worth auditing our progress:

- We have developed a tool to describe real quantum systems
- We have a method for computing the time evolution of these systems, and an explicit set of equations for a two-level atom

We now want to look at real systems

- Real systems are full of conventions
- To cover important applications, some mathematical details will be “left as an exercise for the reader”



The storm heralds the arrival of many symbols



Going backwards

Let us first consider the “least quantum case”, i.e. strong damping of coherences (i.e. $\gamma_{\perp} \gg \Omega, \Gamma$), which will make σ_{12} and σ_{21} static, but the populations ρ_{11} and ρ_{22} will still evolve. Then

$$(\gamma_{\perp} - i(\omega - \omega_0))\sigma_{21} = -\frac{i\Omega}{2} (\rho_{22} - \rho_{11})$$

$$(\gamma_{\perp} + i(\omega - \omega_0))\sigma_{12} = \frac{i\Omega}{2} (\rho_{22} - \rho_{11})$$

which gives us the relation

$$\gamma_{\perp} \left(1 + \left(\frac{\Delta}{\gamma_{\perp}} \right)^2 \right) (\sigma_{21} - \sigma_{12}) = -i\Omega(\rho_{22} - \rho_{11})$$

Now

$$\dot{\rho}_{22} = -\frac{i\Omega}{2} (\sigma_{21} - \sigma_{12}) - \Gamma\rho_{22} = -\frac{\Omega^2}{2\gamma_{\perp}(1 + (\Delta/\gamma_{\perp})^2)} (\rho_{22} - \rho_{11}) - \Gamma\rho_{22}$$

$$\frac{d(N_2/N)}{dt} = -\beta(\omega) \left(\frac{N_2}{N} - \frac{N_1}{N} \right) - \Gamma \frac{N_2}{N}$$

and compare this to

$$\frac{dN_2}{dt} = +N_1 B_{12}\rho(\omega_{21}) - N_2 B_{21}\rho(\omega_{21}) - N_2 A_{21}$$

Setting $\dot{\sigma}_{12} = \dot{\sigma}_{21} = 0$ is known as the adiabatic approximation

For the sake of convenience (and sanity), we define the *detuning* from (atomic) resonance as

$$\Delta = \omega - \omega_0$$

We identify $\rho_{11} \sim N_1/N$ and $\rho_{22} \sim N_2/N$, $\Gamma \sim A_{21}$

OBEs: steady-state solutions

Simplest class of applications arise in the steady state.

If $\dot{\sigma}_{21} = 0$:

$$(\sigma_{21} - \sigma_{12}) \Big|_{t \rightarrow \infty} = \frac{2i\Omega\gamma_{\perp}}{\gamma_{\perp}^2 + \Delta^2} \left(\rho_{22} - \frac{1}{2} \right)$$

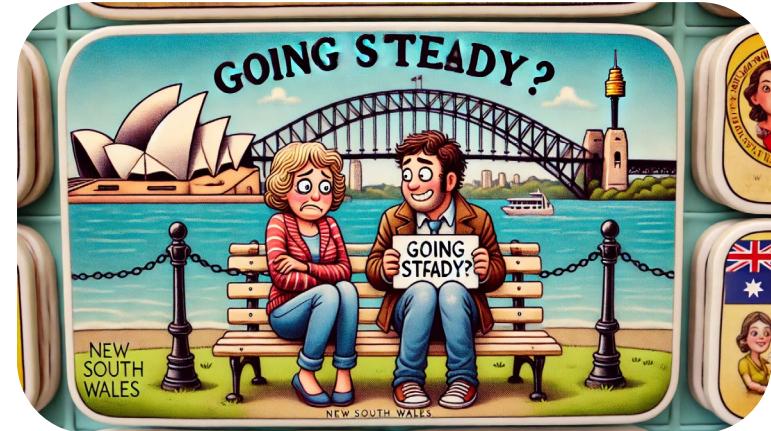
so

$$\rho_{22} \Big|_{t \rightarrow \infty} = \frac{i\Omega}{2\Gamma} (\sigma_{21} - \sigma_{12}) = -\frac{\Omega^2 \left(\frac{\gamma_{\perp}}{\Gamma} \right)}{\gamma_{\perp}^2 + \Delta^2} \left(\rho_{22} - \frac{1}{2} \right)$$

which when solved gives:

$$\rho_{22} \Big|_{t \rightarrow \infty} = \frac{\Omega^2}{2\gamma_{\perp}\Gamma} \frac{1}{1 + \left(\frac{\Delta}{\gamma_{\perp}} \right)^2 + \frac{\Omega^2}{\gamma_{\perp}\Gamma}},$$

$$\sigma_{21} \Big|_{t \rightarrow \infty} = -\frac{i\Omega}{2\gamma_{\perp}} \frac{1 + \frac{i\Delta}{\gamma_{\perp}}}{1 + \left(\frac{\Delta}{\gamma_{\perp}} \right)^2 + \frac{\Omega^2}{\gamma_{\perp}\Gamma}}$$



The steady state – NSW apparently (?)

OBEs: saturation parameter

The steady state expressions are somewhat hideous, but can be simplified by introducing the *saturation parameter*, s , which is defined

$$s = \frac{\Omega^2 / \gamma_{\perp} \Gamma}{1 + \left(\frac{\Delta}{\gamma_{\perp}}\right)^2}$$

which gives the steady state solutions in the form

$$\rho_{22} \Big|_{t \rightarrow \infty} = \frac{s/2}{1 + s}$$

$$\sigma_{21} \Big|_{t \rightarrow \infty} = \frac{\Gamma}{4\gamma_{\perp}} \frac{s}{(1 + s)^2}$$

But what physically is the saturation parameter?

Ω is a measure of the coupling, so s is a measure of the coupling scaled by the relaxation:

e.g. small $s \Rightarrow$ small $\Omega //$ large Γ means small excited state population



Decay rates and line shapes

In the Einstein picture the frequency dependence of the excitation probability is Lorentzian, with a width determined by A_{21} .

In the steady state, by definition, the excitation rate equals the decay rate \Rightarrow the total scattering rate is given by $\Gamma \rho_{22} \Big|_{t \rightarrow \infty}$

$$\rho_{22} \Big|_{t \rightarrow \infty} = \frac{\Omega^2}{2\gamma_{\perp}\Gamma} \frac{1}{1 + \left(\frac{\Delta}{\gamma_{\perp}}\right)^2 + \frac{\Omega^2}{\gamma_{\perp}\Gamma}}$$

Assume atomic decay is homogeneous, which yields the relation $\gamma_{\perp} = \Gamma/2$ and the simplification

$$\rho_{22} \Big|_{t \rightarrow \infty} = \left(\frac{\Omega}{\Gamma}\right)^2 \frac{1}{1 + \left(\frac{2\Delta}{\Gamma}\right)^2 + 2\left(\frac{\Omega}{\Gamma}\right)^2}$$

In the weak field limit (Ω is small) we have

$$\rho_{22} \Big|_{t \rightarrow \infty} = \left(\frac{\Omega}{2}\right)^2 \frac{1}{\Delta^2 + (\Gamma/2)^2}$$

Why $\Gamma/2$? The coherence involves both the ground and excited state, so the overall decay rate is the average of the decay rates of the ground and excited states, i.e. $\Gamma/2$ (assuming a stable ground state)

This is a Lorentzian with a FWHM of Γ , just like Einstein rate equations.

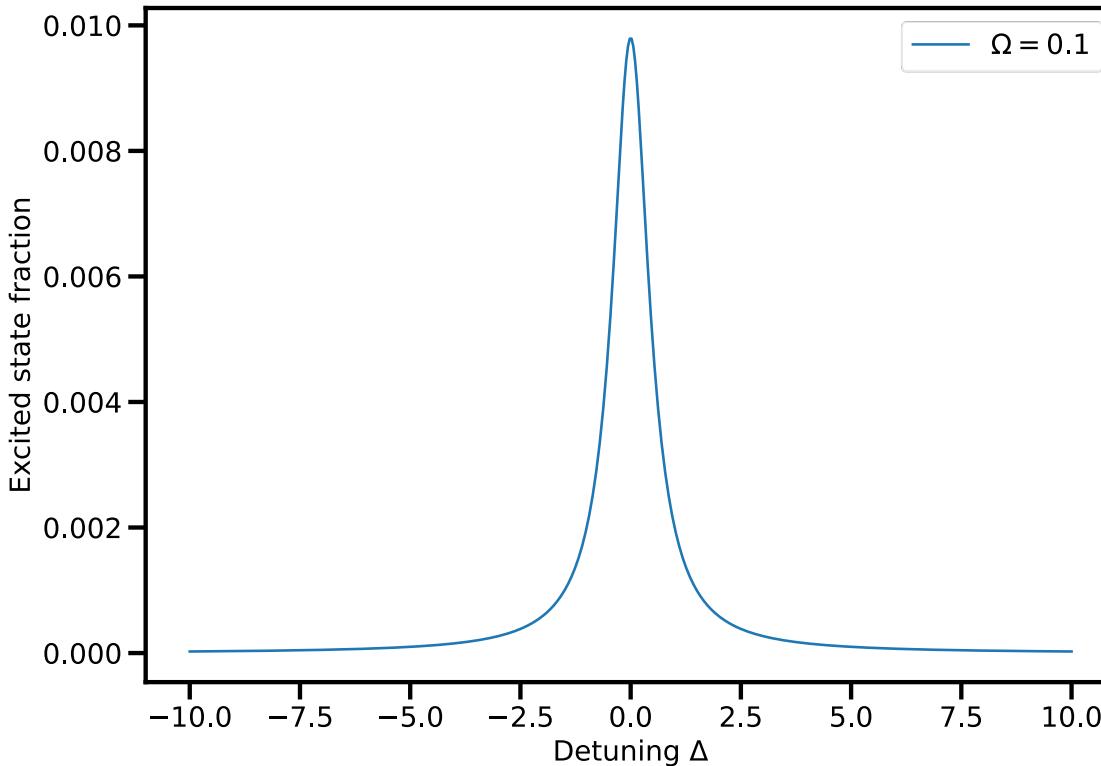
Γ is called the *transition linewidth*

Saturation broadening

That didn't happen last time! As I increase the coupling, the spectral feature broadens.

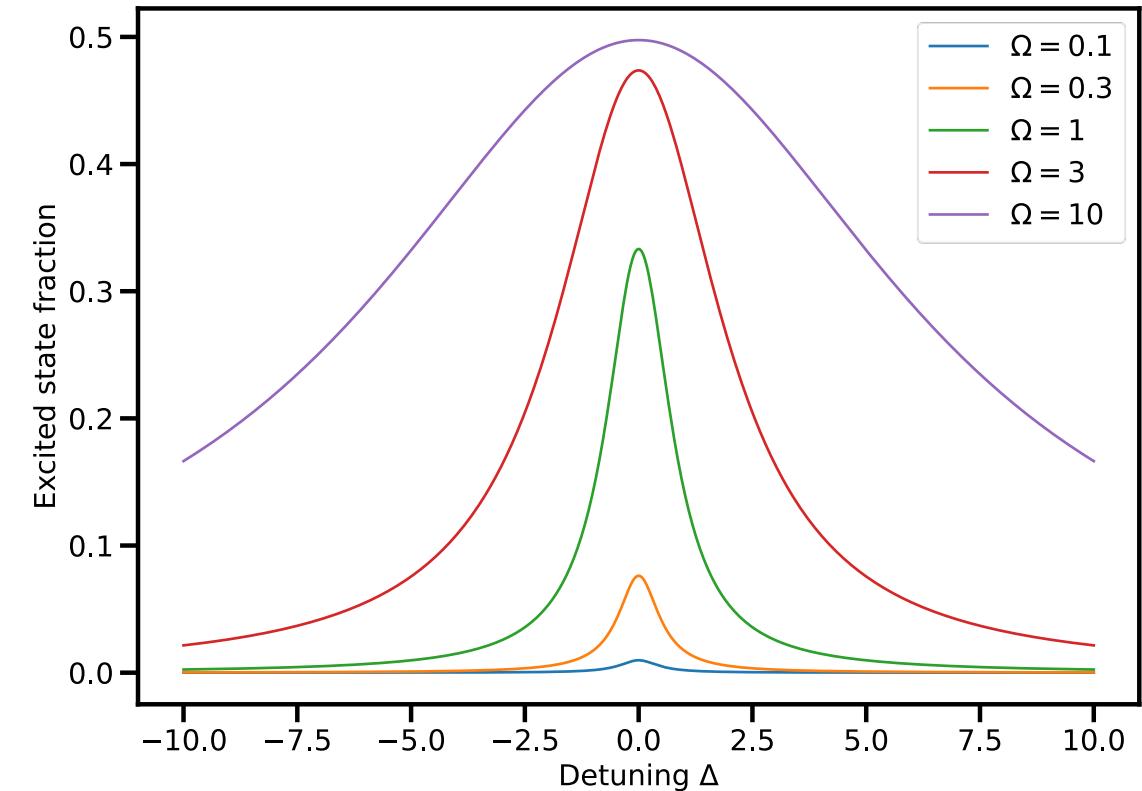
In the weak field (small Ω) we have a Lorentzian with width Γ

$$\rho_{22} \Big|_{t \rightarrow \infty} = \left(\frac{\Omega}{2} \right)^2 \frac{1}{\Delta^2 + (\Gamma/2)^2}$$



In a strong field (large Ω) we have a Lorentzian with width $\sqrt{2}\Omega$

$$\rho_{22} \Big|_{t \rightarrow \infty} = \left(\frac{\Omega}{2} \right)^2 \frac{1}{\Delta^2 + (\Omega/\sqrt{2})^2}$$



The excited-state population as a function of detuning (frequency)

Connecting to the real world

We want to link the microscopic to the macroscopic: how does the quantum behaviour of atoms dictate the absorption of light?

For a plane wave:

$$E = E_0 e^{i(k \cdot x - \omega t)}$$

which is modified in a medium with refractive index n

$$E = E_0 e^{i(nkz - \omega t)}$$

Assuming that the light is propagating in the z direction



*The simplest experiment:
monochromatic light on an atom.
What will happen?*



Likely the most routine and common experiment in modern physics

Polarisation unification

We saw that the dipole operator does not couple to itself, only other states, so with $d_{nm} = \langle n|d|m\rangle$

$$d = \begin{pmatrix} 0 & d_{12} \\ d_{21} & 0 \end{pmatrix}$$

The expectation value is then

$$\begin{aligned}\langle d \rangle &= \text{tr}(d\rho) = \begin{pmatrix} 0 & d_{12} \\ d_{21} & 0 \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \\ &= d_{12}(\rho_{12} + \rho_{21}) \\ &= d_{12}(\sigma_{12}e^{i\omega t} + \sigma_{21}e^{-i\omega t})\end{aligned}$$

and

$$\mathbf{P} = n\langle \mathbf{d} \rangle$$

$$nd_{12}(\sigma_{12}e^{i\omega t} + \sigma_{21}e^{-i\omega t}) = \frac{\varepsilon_0 \hbar \Omega}{2d_{12}} \mathbf{E}_0 (\chi e^{i\omega t} + \chi^* e^{-i\omega t})$$

$$\Rightarrow \chi = \frac{2nd_{12}^2}{\varepsilon_0 \hbar \Omega} \sigma_{12}$$

Recall that

$$\begin{aligned}\mathbf{E}(t) &= \mathbf{E}_0 \cos(\omega t) \\ \mathbf{d} \cdot \mathbf{E}_0 &= \hbar \Omega\end{aligned}$$

The polarisation is also given by

$$\begin{aligned}\mathbf{P}(t) &= \varepsilon_0 \chi(\omega) \mathbf{E}(t) \\ &= \frac{1}{2} \varepsilon_0 \mathbf{E}_0 (\chi(\omega) e^{i\omega t} + \chi(-\omega) e^{-i\omega t}) \\ &= \frac{\varepsilon_0 \hbar \Omega}{2d_{12}} \mathbf{E}_0 (\chi e^{i\omega t} + \chi^* e^{-i\omega t})\end{aligned}$$

We have assumed $d_{12} = d_{21}$, which isn't always true, but is often true

The atomic dipole and susceptibility, and therefore polarisation, are related to the atomic coherences

Susceptible?

The susceptibility tells us what happens when we run our experiment:

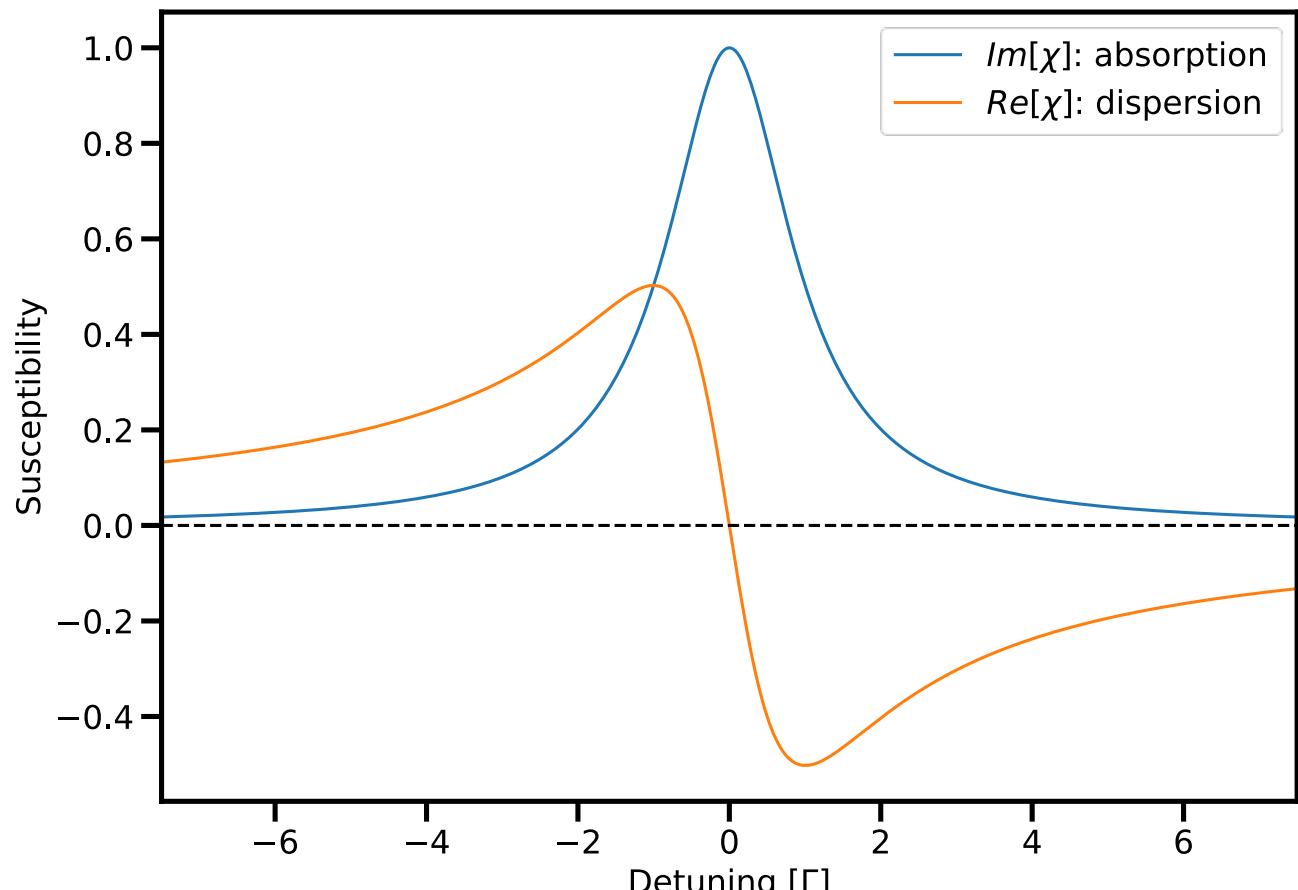


$$\chi = \frac{2nd_{12}^2}{\varepsilon_0 \hbar \Omega} \sigma_{12}$$

Absorption varies as a function of frequency

Rate of phase accumulation varies wildly as a function of frequency

This all depends on the off-diagonal elements of ρ , not the diagonal elements (i.e. the state populations)!



Susceptibility in a two-level atom



Slow light

Recall

$$n = \sqrt{1 + \chi} \approx 1 + \frac{\chi}{2}, \quad Re[n] = 1 + \chi_{Re}/2$$

The *group index of refraction* is

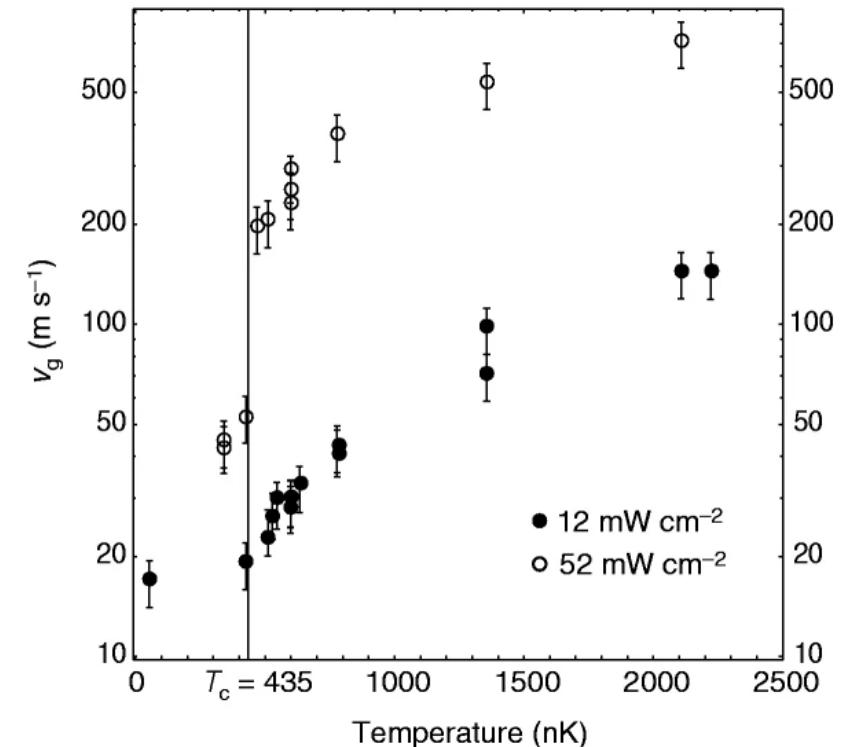
$$n_g = n + \omega \frac{dn}{d\omega}$$

One can show that

$$n_g \approx \frac{2\omega n d_{12}^2}{\epsilon_0 \hbar \Omega^2}$$

which means by controlling the dispersion ($Re[\sigma_{12}]$), we can control the speed of light!

The group index measures the ratio of the speed of light in vacuum to the group velocity of a light pulse



Light speed versus atom temperature

These speeds correspond to optical pulses that are kilometers long being compressed into microns!

Light speed reduction to 17 metres per second in an ultracold atomic gas

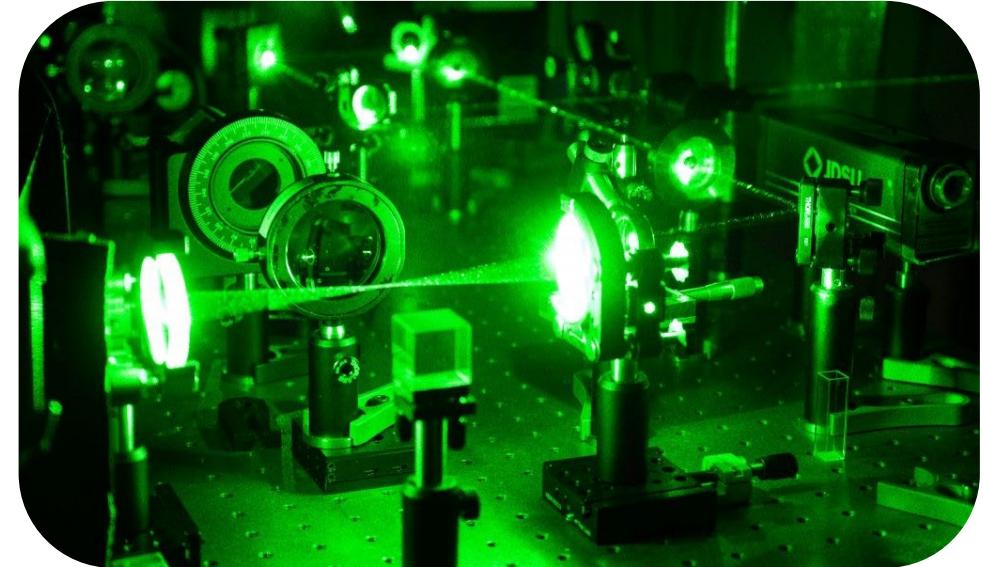
So why do all this?

The Einstein equations neither predict nor explain many phenomena which are explained by the OBEs:

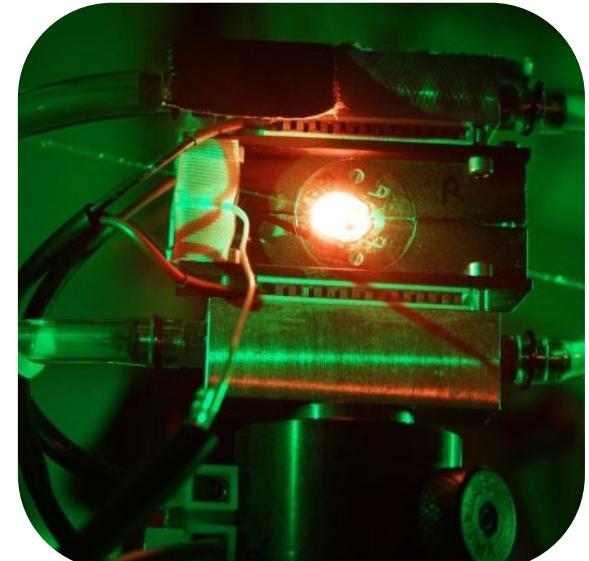
- Coherences
- Dependence on detuning, saturation
- Rabi oscillations
- AC Stark shift
- ...

OBEs have some limitations, notably:

- Raman transitions (far-off resonant light-fields)
- Intense light fields (large Ω)



The OBEs accurately model a huge number of systems



The OBEs do not capture non-linear behaviour



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Summary

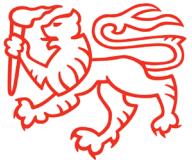
Problems

- Time evolution
 - Evolution of the density operator is given by the Liouville-von Neumann equation:
$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}(t), \hat{\rho}]$$
 - Solving the Liouville-von Neumann equation for a two-level atom give the Optical Bloch Equations (OBEs), which predict the density matrix elements as a function of time
 - The Rabi frequency Ω characterised the coupling between states $\mathbf{d} \cdot \mathbf{E}_0 = \hbar\Omega$
- Steady-state solutions to the OBEs
 - Explain many phenomena not predicted by the Einstein rate equations, including saturation, spectral broadening, dispersion



Transient effects

Foot Ch. 7 // Steck Ch. 4, 5



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Learning outcomes

Week 4, lecture 2

Foot §7.3 – 7.5 // Steck §5.2, 5.4, 5.5:
Time evolution of the two-level atom

- Time dependent solutions of the optical Bloch equations
 - Rabi oscillations
- Bloch sphere
 - Geometric interpretation of states

The optical Bloch equations (slide 1 of 1)

The optical Bloch equations in the rotating wave approximation are

$$\begin{aligned}\dot{\rho}_{11} &= \frac{i\Omega}{2} (\sigma_{21} - \sigma_{12}) + \Gamma\rho_{22} \\ \dot{\rho}_{22} &= -\dot{\rho}_{11} = -\frac{i\Omega}{2} (\sigma_{21} - \sigma_{12}) - \Gamma\rho_{22} \\ \dot{\sigma}_{21} &= i(\omega - \omega_0)\sigma_{21} - \frac{i\Omega}{2} (\rho_{22} - \rho_{11}) - \gamma_\perp\sigma_{21} \\ \dot{\sigma}_{12} &= \dot{\sigma}_{21}^* = -i(\omega - \omega_0)\sigma_{12} + \frac{i\Omega}{2} (\rho_{22} - \rho_{11}) - \gamma_\perp\sigma_{12}\end{aligned}$$

where the density matrix elements are related to the slowly-rotating components

$$\begin{aligned}\rho_{21}(t) &= \sigma_{21}(t)e^{-i\omega t} \\ \rho_{12}(t) &= \rho_{21}^*(t) = \sigma_{12}(t)e^{i\omega t}, \quad \text{with } \sigma_{21} = \sigma_{12}^*\end{aligned}$$

and the Rabi frequency is

$$\mathbf{d} \cdot \mathbf{E}_0 = \hbar\Omega$$

Transient transience

All effects have thus far have been *steady-state effects*, that is, with no time dependence. We now turn to *transient* effects.

From the OBEs, we have:

$$\dot{\rho}_{22} = -\dot{\rho}_{11} = -\frac{i\Omega}{2}(\sigma_{21} - \sigma_{12})$$

$$\dot{\sigma}_{12} = \dot{\sigma}_{21}^* = -i\Delta\sigma_{21} + \frac{i\Omega}{2}(\rho_{22} + \rho_{11})$$

How to solve these equations?

Ultimately looks a bit like the differential equation for an exponential \Rightarrow look for a solution of the form

$$\sigma_{ij}(t) = \sigma_{ij}(0)e^{\lambda t}$$



We are using a notational convenience: here we are simply defining

$$\begin{aligned}\sigma_{11} &\equiv \rho_{11} \\ \sigma_{22} &\equiv \rho_{22}\end{aligned}$$



Crank the handle

Putting the trial solution into the OBEs yields

$$\begin{pmatrix} -\lambda & 0 & -\frac{i\Omega}{2} & \frac{i\Omega}{2} \\ 0 & -\lambda & \frac{i\Omega}{2} & -\frac{i\Omega}{2} \\ -\frac{i\Omega}{2} & \frac{i\Omega}{2} & -\lambda & -\Delta \\ \frac{i\Omega}{2} & -\frac{i\Omega}{2} & \Delta & -\lambda \end{pmatrix} \begin{pmatrix} \rho_{11}(0) \\ \rho_{22}(0) \\ \sigma_{12}(0) \\ \sigma_{21}(0) \end{pmatrix} = 0$$

which in turn yields

$$\lambda^2(\lambda^2 + \Delta^2 + \Omega^2) = 0$$

$$\Rightarrow \lambda = 0, \pm i\Omega' \text{ where } \Omega'^2 \equiv \Omega^2 + \Delta^2$$

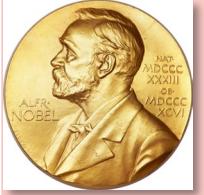
The solution to the density matrix elements will then be a sum of these solutions

$$\sigma_{ij}(t) = \sigma_{ij}^{(0)} + \sigma_{ij}^{(i\Omega')} e^{i\Omega' t} + \sigma_{ij}^{(-i\Omega')} e^{-i\Omega' t}$$



Time to get cranking

Ω' is the *generalised Rabi frequency* and is used as the dynamics of a system with detuning Δ are the same as a system with no detuning and a Rabi frequency $\Omega = \Omega'$



Rabi oscillations

Assuming that the excited state population is initially zero, the transient excited state population is given by

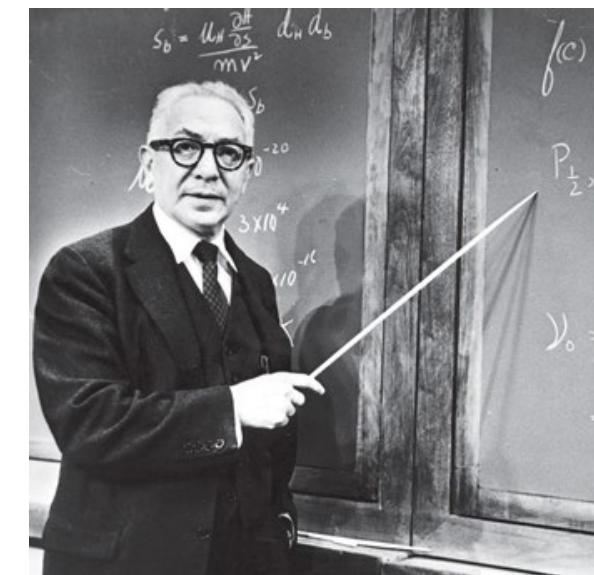
$$\rho_{22}(t) = \left(\frac{\Omega}{\Omega'}\right)^2 \sin^2\left(\frac{\Omega't}{2}\right)$$

Looking at case of resonance ($\Delta = 0$):

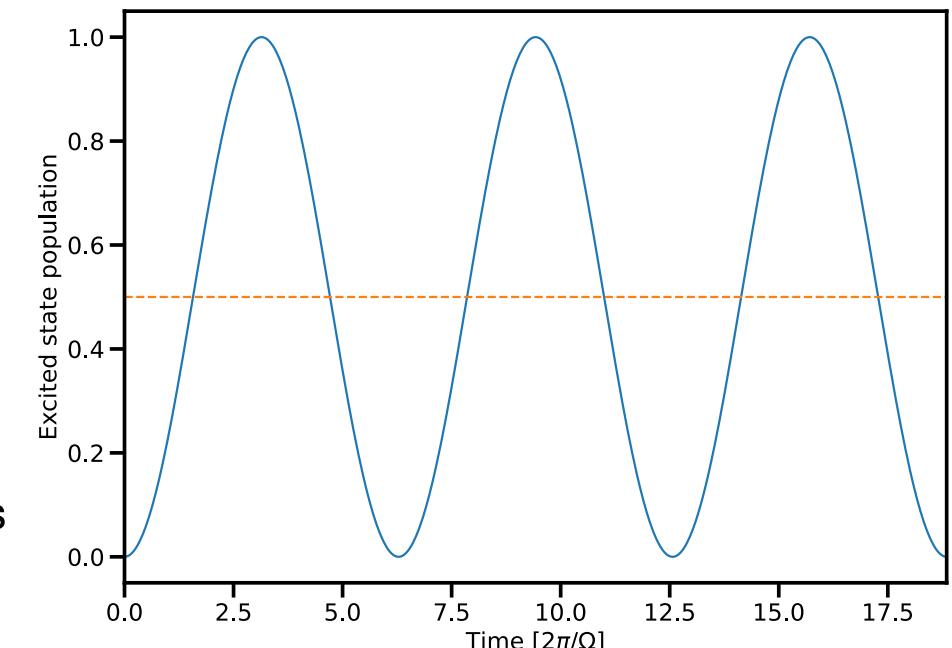
$$\rho_{22}(t) = \sin^2\left(\frac{\Omega t}{2}\right)$$

This should strike you as odd for a number of reasons:

- A coupling should drive atomic excitation, shouldn't it?
- This suggests that I can put a system into a state whereby I can guarantee that excited-state population is 100%
- I don't recall seeing atomic systems undergoing oscillations indefinitely



Isidor Isaac Rabi ...



... and his oscillations: Rabi oscillations



Rabi oscillations

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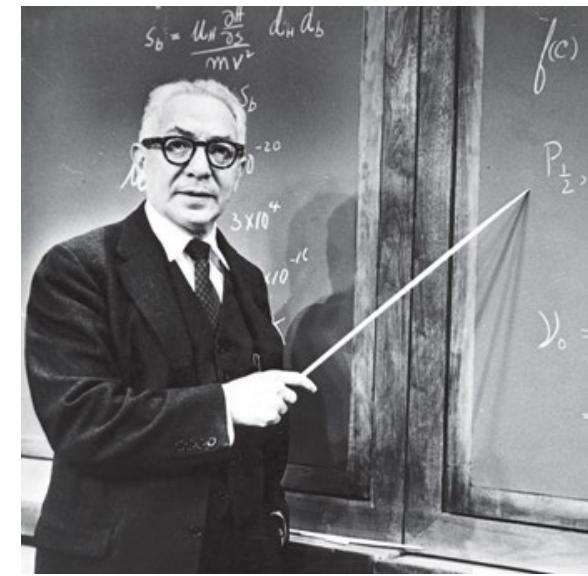
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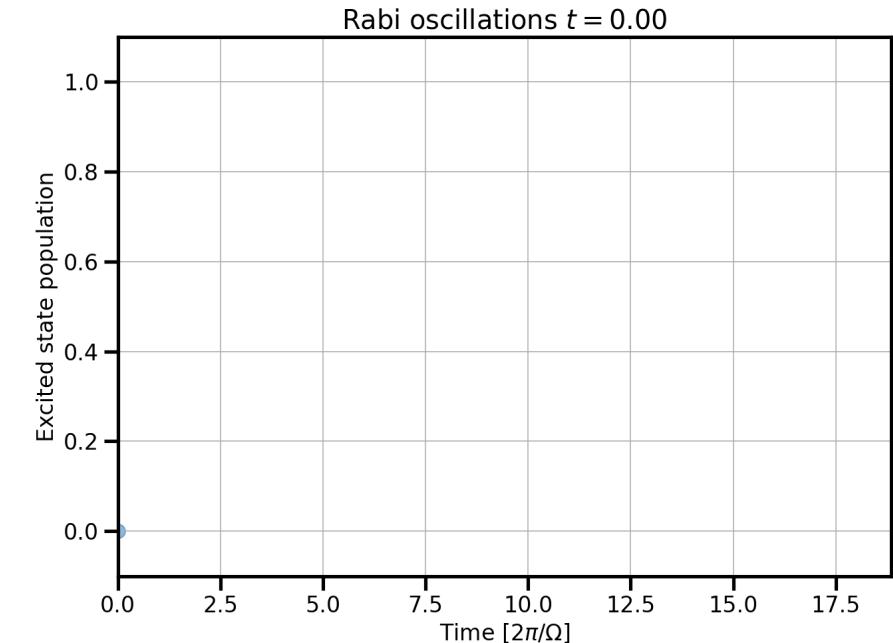
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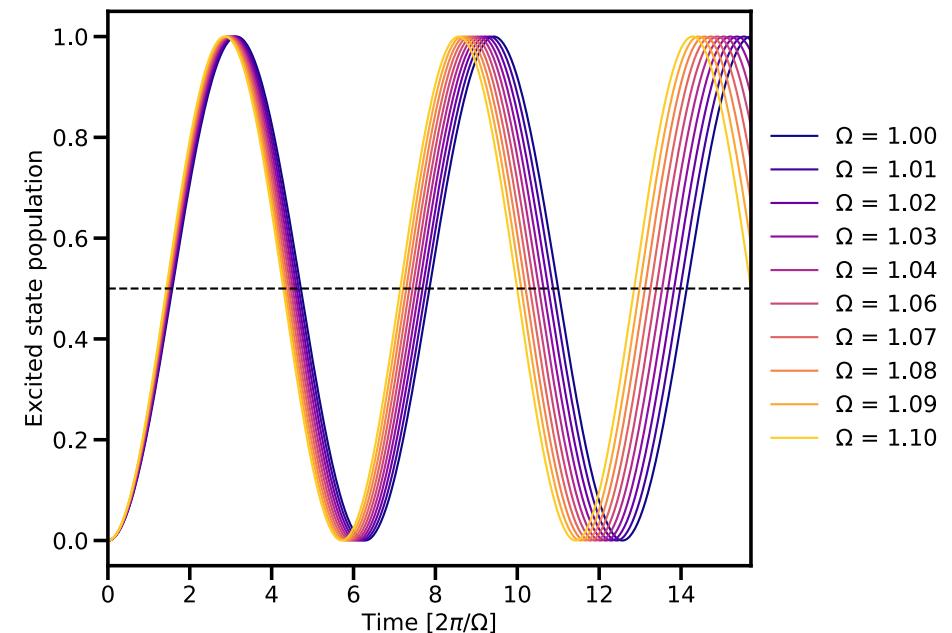
Tuning the oscillations

Implications of the relation

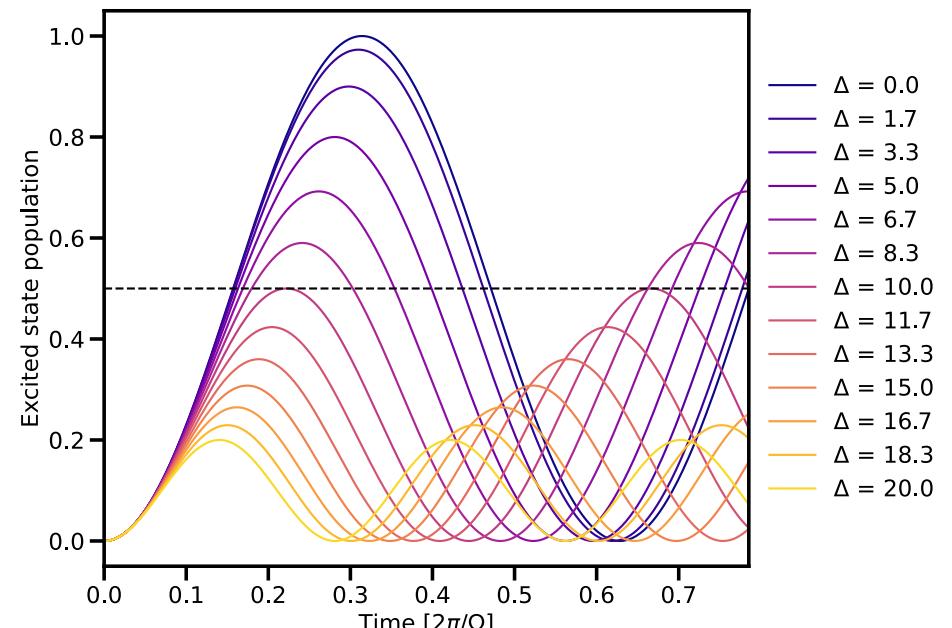
$$\rho_{22}(t) = \left(\frac{\Omega}{\Omega'}\right)^2 \sin^2\left(\frac{\Omega't}{2}\right)$$

We can control the excited state population, and its evolution!

- Changing the coupling between the atoms and field (Ω) changes the rate at which oscillations occur
- Introducing a detuning (Δ) reduces the efficiency of excitation and induces faster oscillations

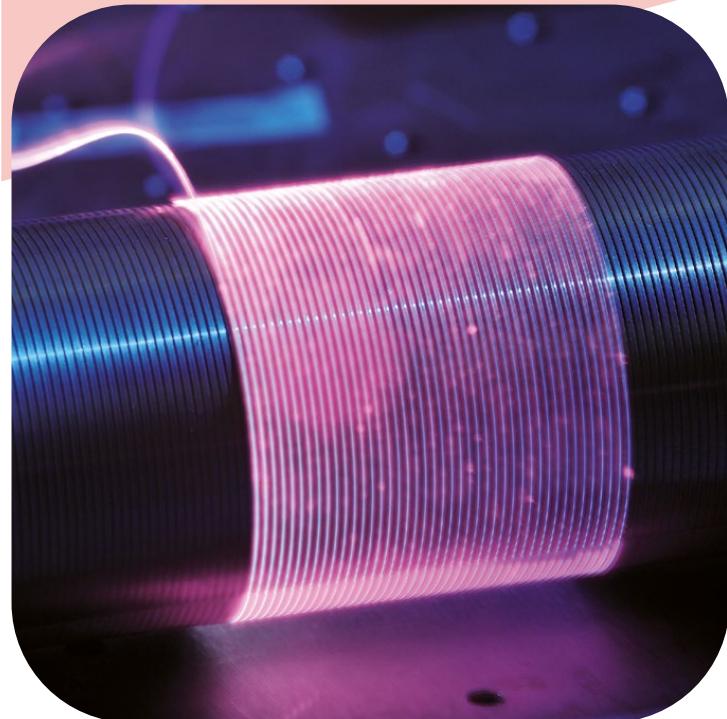


Rabi oscillations for different coupling strengths

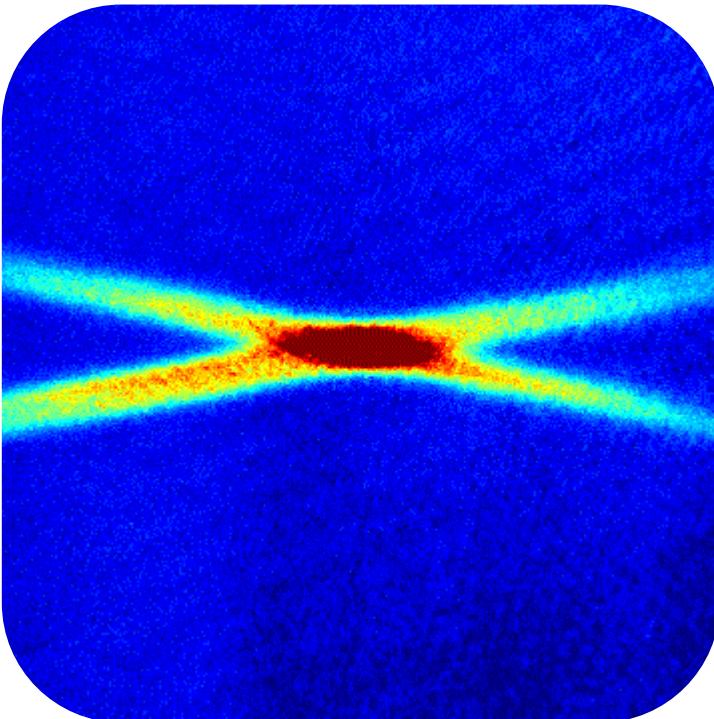


Rabi oscillations for different detunings

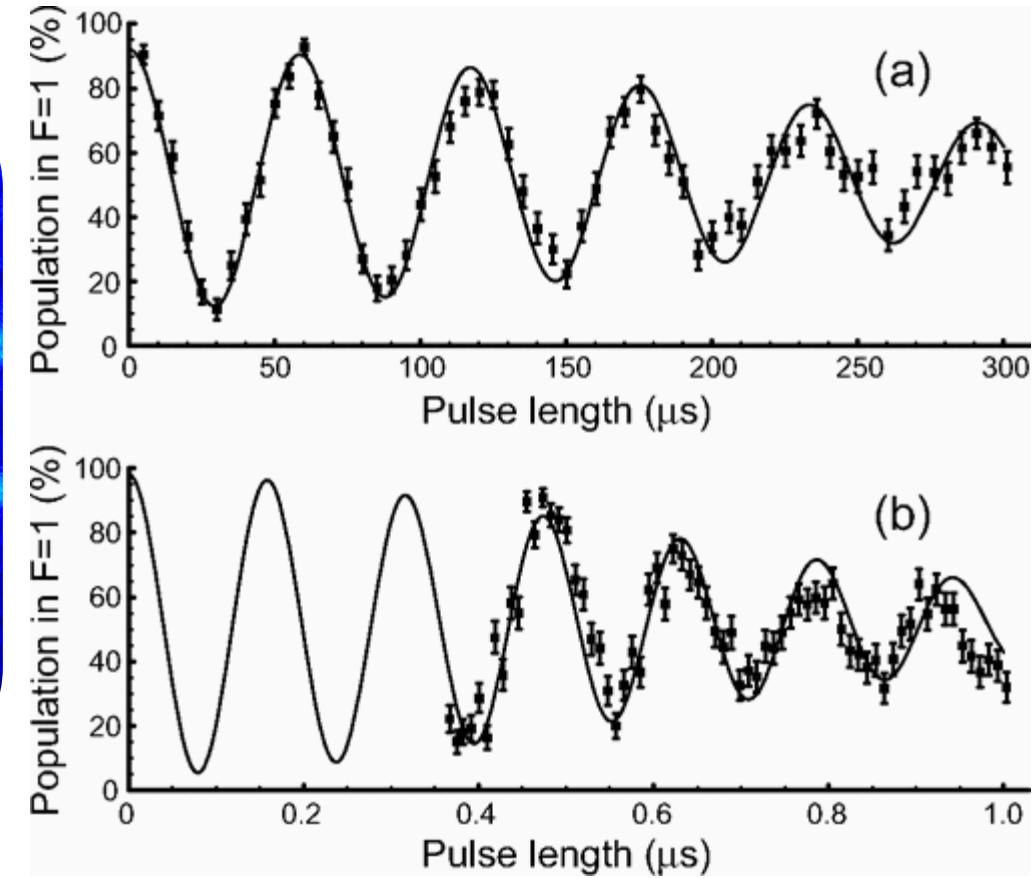
Oscillations of the population don't actually occur, right?



Fibre lasers can be extremely powerful



One can trap (single) atoms in an optical trap



Rabi oscillations of a single atom

The cycle continues, well, at least for some time

Rabi oscillations are an inherently quantum phenomenon, and it stands to reason they would not continue indefinitely.

In constructing the optical Bloch equations, we phenomenologically baked in relaxation via

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}(t), \hat{\rho}] + \hat{\mathcal{L}}_{\text{relax}}$$

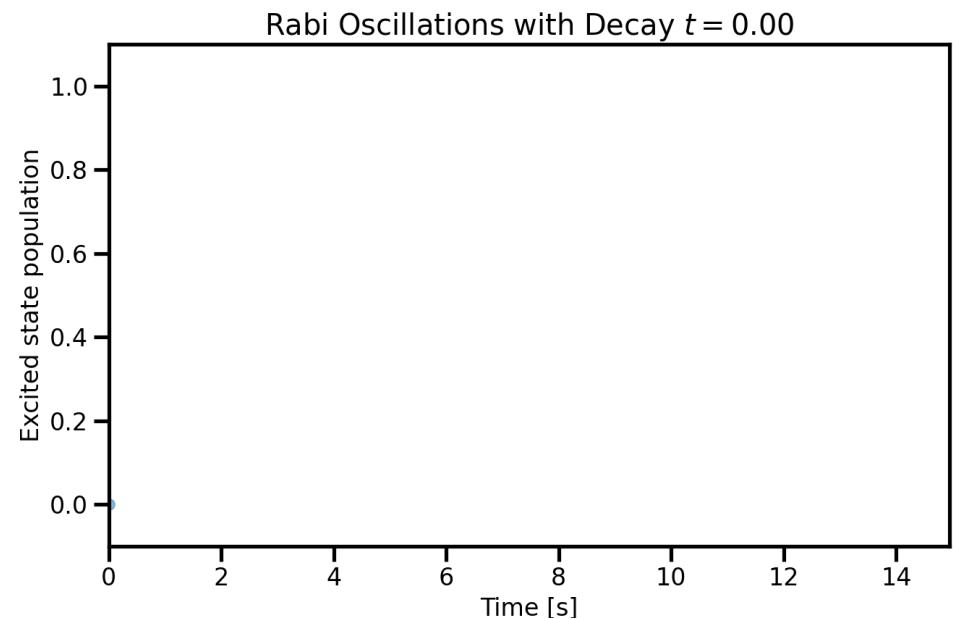
Explicitly for a two-level atom, we have

$$\dot{\rho}_{22} = -\Gamma \rho_{22}$$

$$\dot{\sigma}_{12} = -\gamma_{\perp} \sigma_{12}$$

The question is how does this work for a single atom?

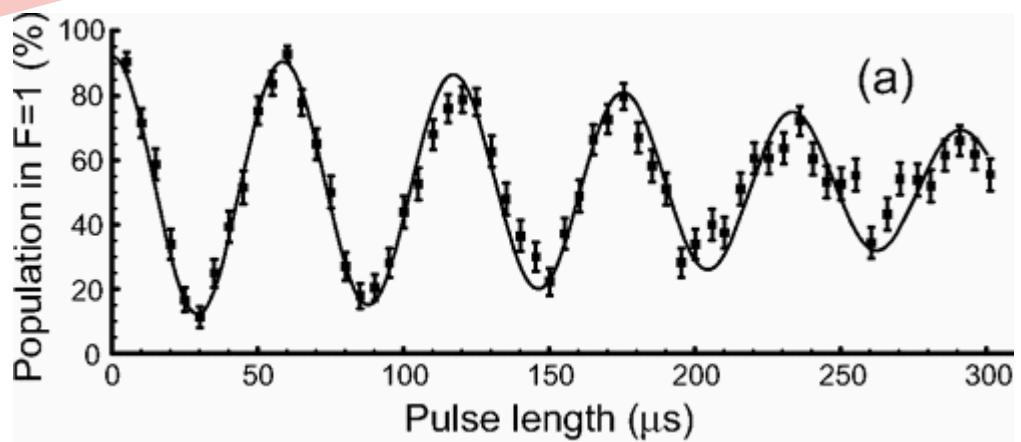
- Introduce Poisson decay process
- Exponential decay with rate parameter Γ



The evolution of a single atom with decay

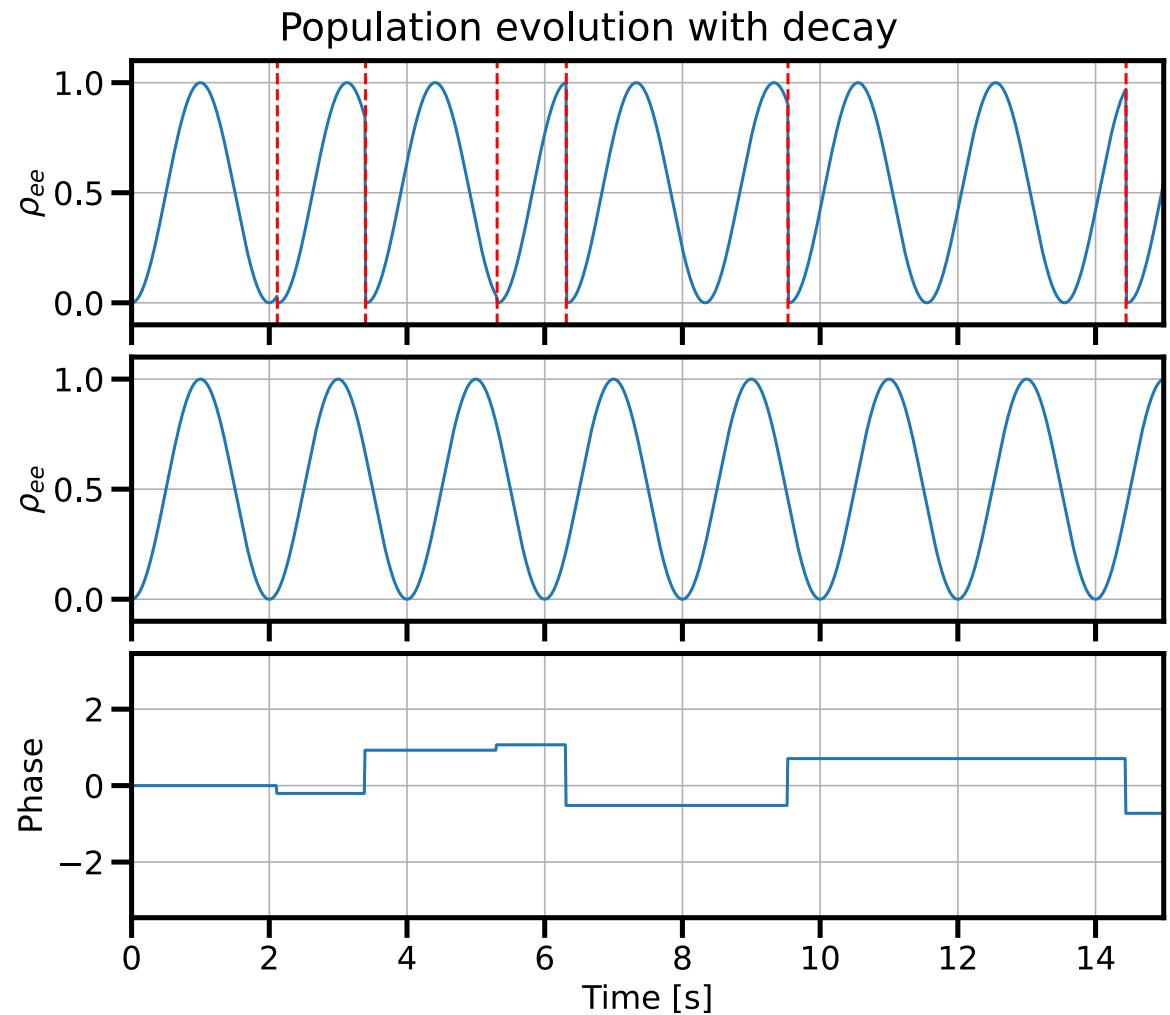
The ensemble evolves

But why do the oscillations look smooth?



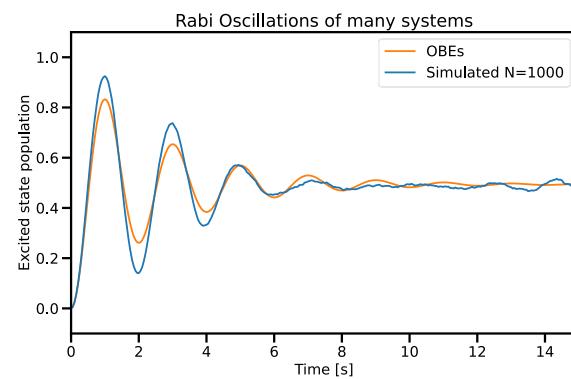
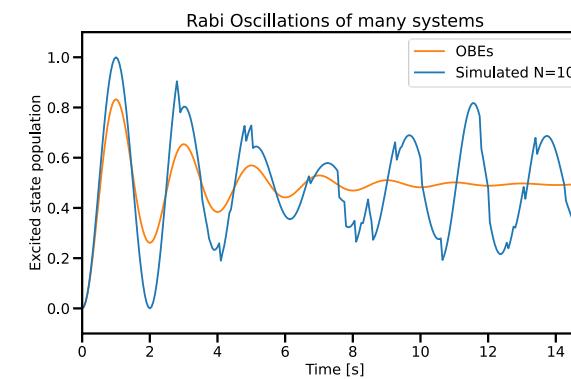
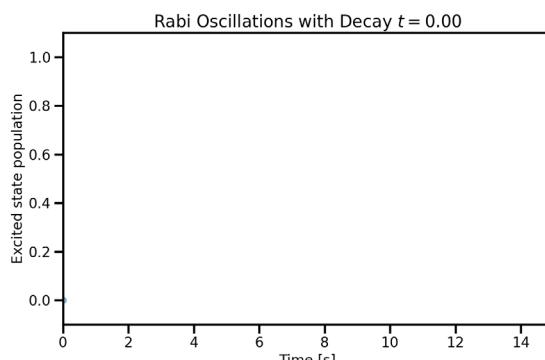
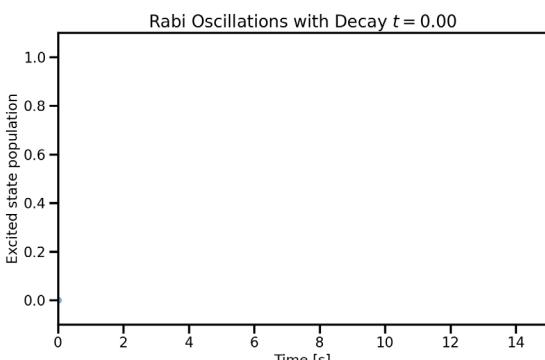
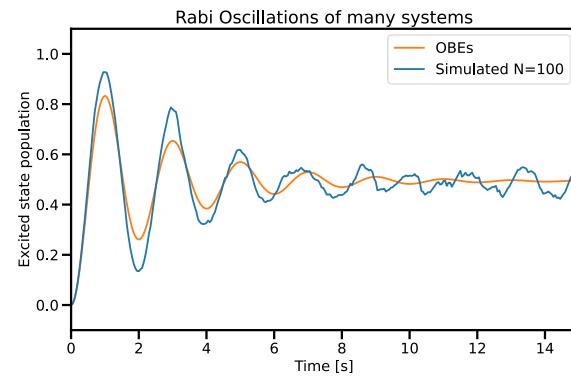
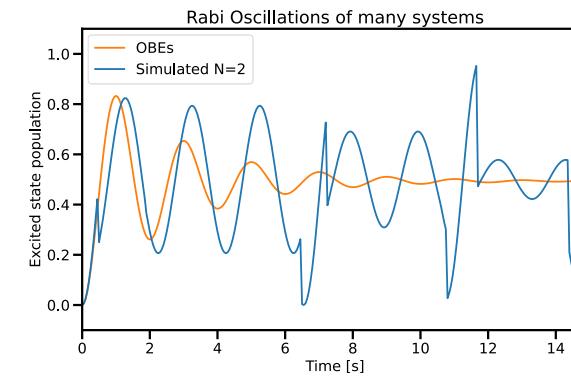
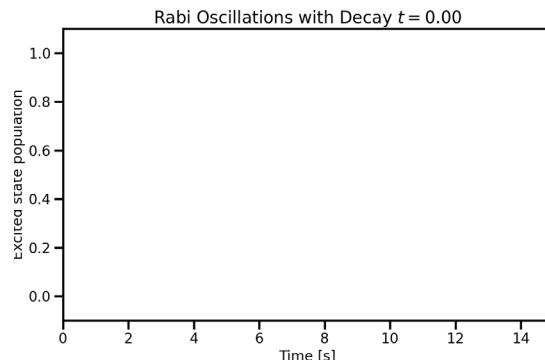
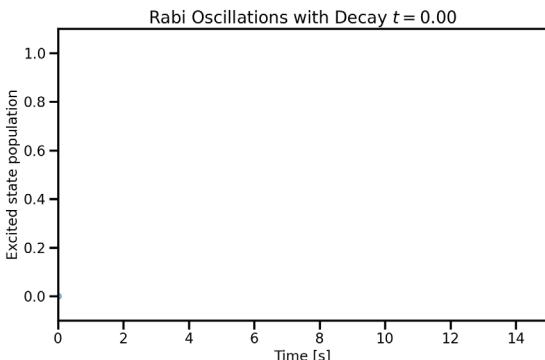
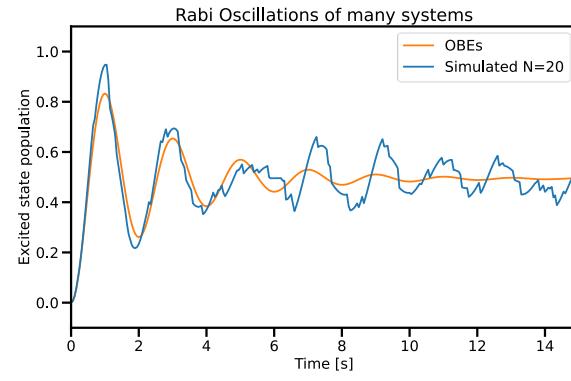
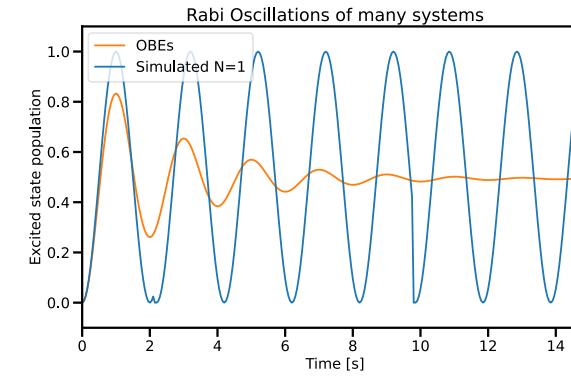
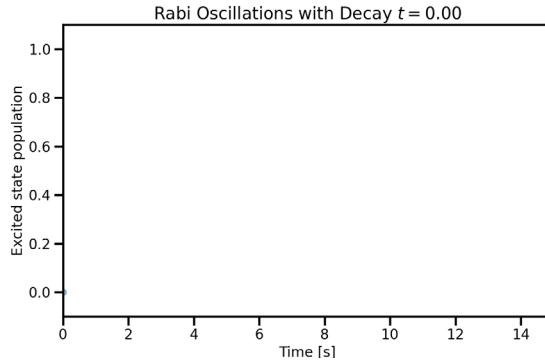
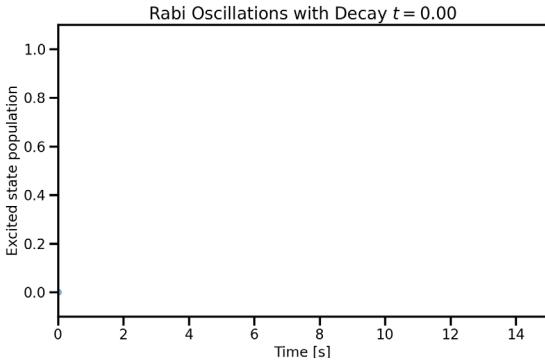
Rabi oscillations of Rubidium

The evolution of a single atom may have discontinuities, but in an ensemble, a decay is equivalent to dephasing relative to the initial Rabi oscillation



Atoms with and without decay, and associated phase shift

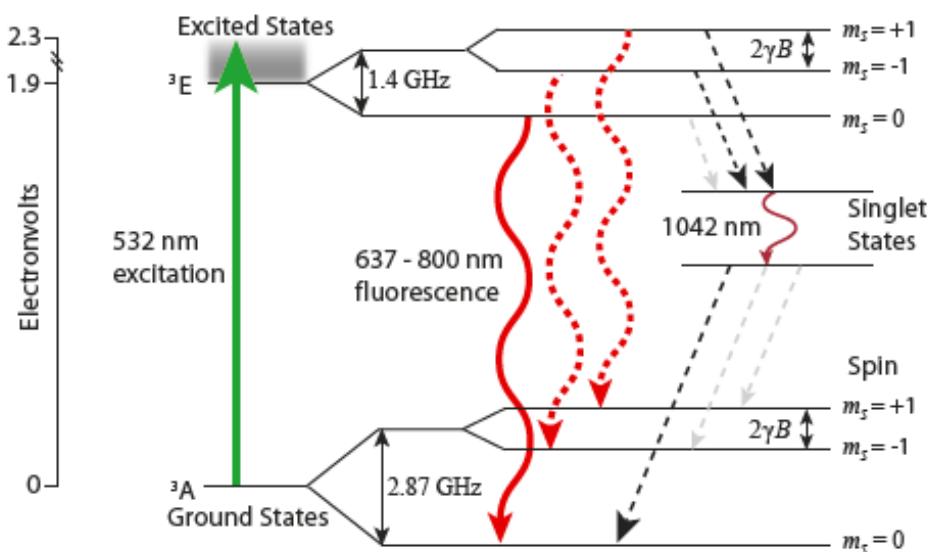
Compare the pair



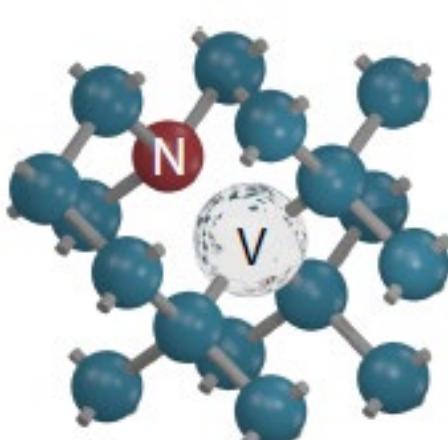
Does this mean my Rabi cycles are doomed?

Decay makes the quantum go away: what can we do?

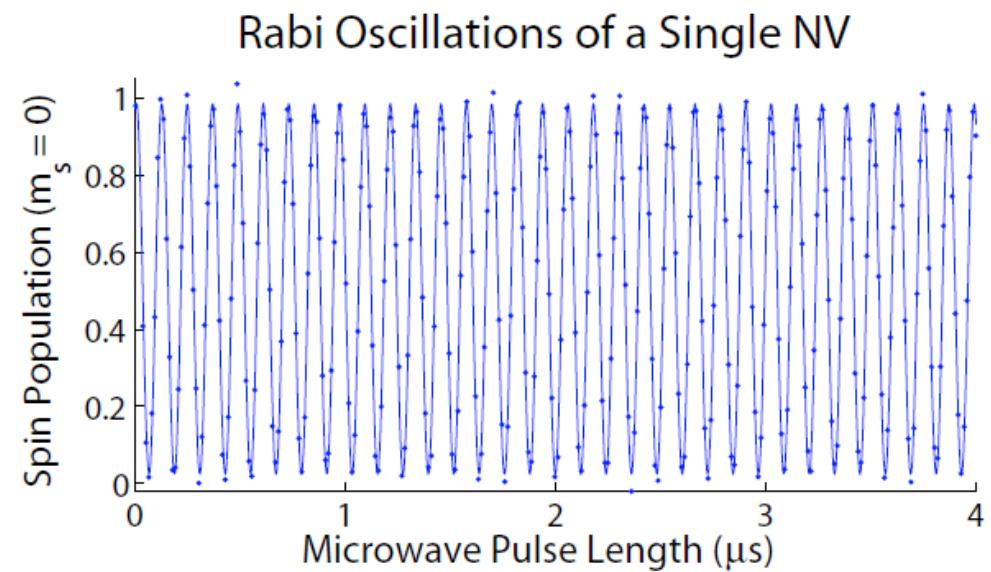
- Small linewidth/long lifetime
- Good environment



Energy-level diagram of the nitrogen vacancy centre in diamond



The structure of nitrogen vacancy centres



Rabi oscillations of a nitrogen vacancy centre in diamond: very little decay

Dephasing

The time to *dephase* – lose coherence between states – is characterised by $T_2 = 1/\gamma_{\perp}$.

Whereas $T_1 = 1/\Gamma$ is a property of the atom, T_2 is greatly influenced by the environment

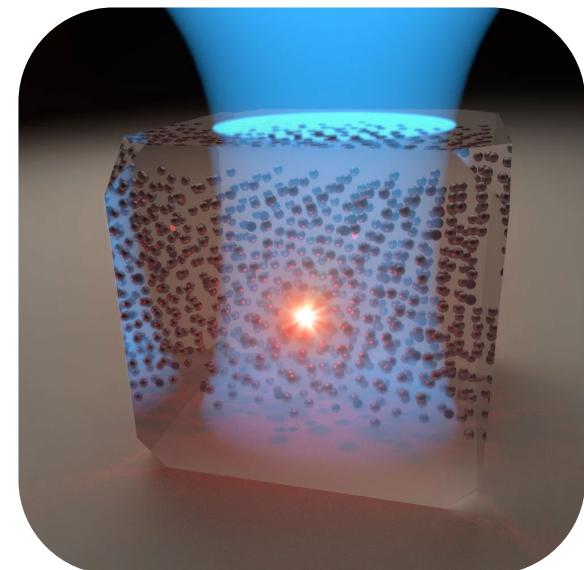
- Magnetic field fluctuations and inhomogeneities
- Temperature changes
- Interactions with neighbouring spins

⇒ Make stuff small, use big fields, cool stuff down, find “magic” materials

E.g. Praseodymium-doped Yttrium Orthosilicate ($\text{Pr}^{3+}:\text{Y}_2\text{SiO}_5$) has coherence times on the order of many hours!



Yttrium orthosilicate

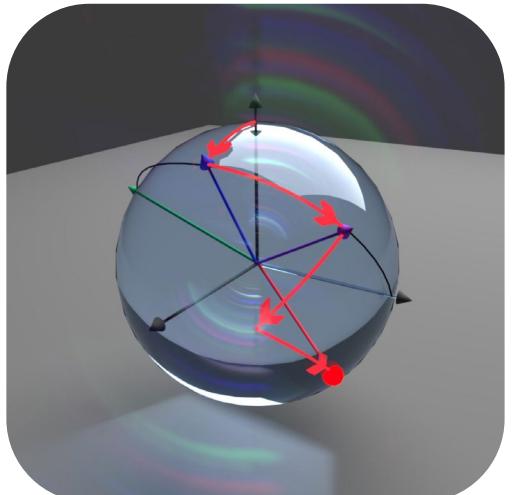


Single praseodymium ions in a yttrium orthosilicate have extremely long T_2 times

The problem

To this point, we have really only looked at following the excited- (and ground-) state population.

We have (complex-valued) coherences which determine properties of the system, and we want to visualise how the system evolves. How can we possibly do this?



The answer:
The Bloch sphere



The symbol to summon Bloch



Felix Bloch



To be expected

$$\sigma = |2\rangle\langle 1|, \quad \rho = \begin{pmatrix} \rho_{11} & \rho_{21} \\ \rho_{12} & \rho_{22} \end{pmatrix}$$

If we compute expectation values:

$$\langle \sigma \rangle = \text{tr}(\sigma \rho) = \sigma_{21}$$

$$\langle \sigma^\dagger \rangle = \sigma_{12}$$

$$\langle \sigma^\dagger \sigma \rangle = \rho_{22}$$

$$\langle \sigma \sigma^\dagger \rangle = \rho_{11}$$

Optical Bloch equations:

$$\dot{\rho}_{22} = -\dot{\rho}_{11} = -\frac{i\Omega}{2}(\sigma_{21} - \sigma_{12})$$

$$\dot{\sigma}_{12} = \dot{\sigma}_{21}^* = -i\Delta\sigma_{21} + \frac{i\Omega}{2}(\rho_{22} + \rho_{11})$$

$$\frac{d\langle \sigma_x \rangle}{dt} = \dot{\sigma}_{21} + \dot{\sigma}_{12}$$

$$= i\Delta(\sigma_{21} - \sigma_{12}) - i\Omega(\rho_{22} - \rho_{11})$$

$$= \Delta\langle \sigma_y \rangle - i\Omega\langle \sigma_z \rangle$$

$$\frac{d\langle \sigma_y \rangle}{dt} = -\Delta\langle \sigma_x \rangle$$

$$\frac{d\langle \sigma_z \rangle}{dt} = -\Omega\langle \sigma_y \rangle$$

Then

$$\langle \sigma_x \rangle = \langle \sigma \rangle + \langle \sigma^\dagger \rangle = \sigma_{21} + \sigma_{12}$$

$$\langle \sigma_y \rangle = i(\sigma_{21} - \sigma_{12})$$

$$\langle \sigma_z \rangle = (\rho_{22} - \rho_{11})$$

Measuring up

We are close... All that remains is to look at the length of the expectation value of σ :

$$\begin{aligned} |\langle \sigma \rangle^2| &= \langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 \\ &= (\sigma_{21} + \sigma_{12})^2 - (\sigma_{21} - \sigma_{12})^2 + (\rho_{22} - \rho_{11})^2 \\ &= 4\sigma_{21}\sigma_{12} + \rho_{22}^2 + \rho_{11}^2 - 2\rho_{22}\rho_{11} \\ &= (\rho_{22} + \rho_{11})^2 = 1 \end{aligned}$$

We assume here that we have a pure state; in this case:

$$\rho_{11}\rho_{22} = \sigma_{21}\sigma_{12}$$

In the case of a mixed state, this value will be less than one, and for a completely mixed state, it will be zero.

Z is for...

So, we have defined an operator, with associated vector

$$(\langle \sigma_x \rangle, \langle \sigma_y \rangle, \langle \sigma_z \rangle)$$

Why!?

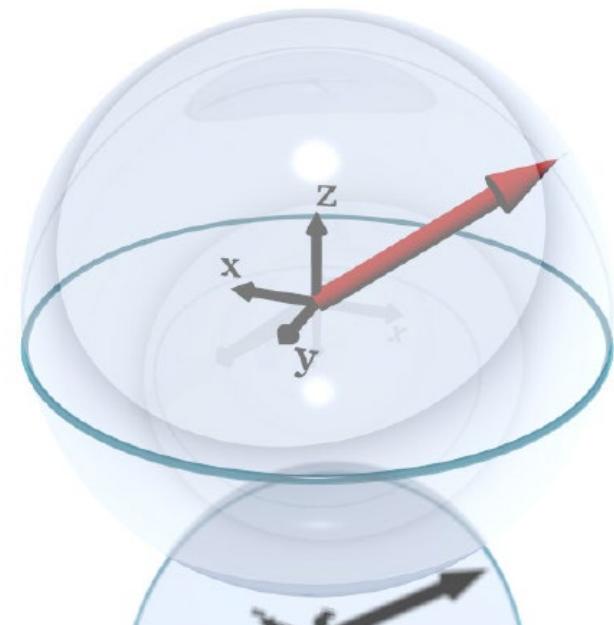
The component $\langle \sigma_z \rangle = (\rho_{22} - \rho_{11})$ quantifies the excited state population

Poles indicate system is either in state $|1\rangle$ (south) or $|2\rangle$ (north)

Note: in other conventions, the poles are reversed

$$\sigma = |2\rangle\langle 1|$$

$$\begin{aligned}\langle \sigma_x \rangle &= \sigma_{21} + \sigma_{12} \\ \langle \sigma_y \rangle &= i(\sigma_{21} - \sigma_{12}) \\ \langle \sigma_z \rangle &= (\rho_{22} - \rho_{11})\end{aligned}$$



Enter the Bloch sphere: visualising states

X and Y are for...

Recall the dipole operator, which couples states $|1\rangle$ and $|2\rangle$, is

$$d = \begin{pmatrix} 0 & d_{12} \\ d_{21} & 0 \end{pmatrix}$$

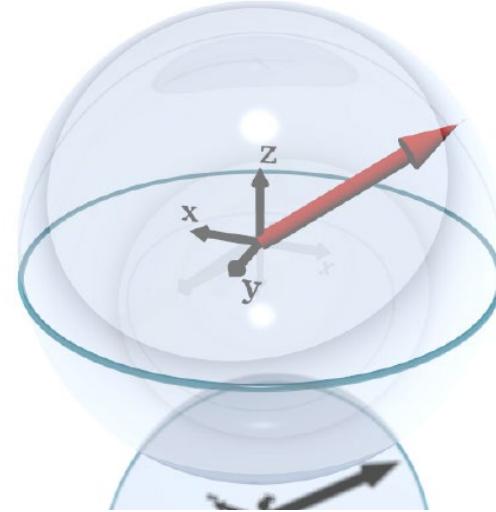
The expectation value $\langle d \rangle$ is then

$$\begin{aligned} \text{tr}(d\rho) &= d_{12}(\rho_{12} + \rho_{21}) \\ &= d_{12} (\sigma_{12}e^{i\omega t} + \sigma_{21}e^{-i\omega t}) \\ &= d_{12}[(\sigma_{12} + \sigma_{21}) \cos(\omega t) + i(\sigma_{12} - \sigma_{21}) \sin(\omega t)] \\ &= d_{12}(\langle \sigma_x \rangle \cos(\omega t) - \langle \sigma_y \rangle \sin(\omega t)) \end{aligned}$$

In phase In quadrature

We can decompose a sinusoid into two components: one *in phase* with $\sin(x)$, and one *in quadrature*, that is, exactly out of phase with $\sin(x)$, or in phase with $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$

$$\sin(x + \phi) = \sin(x) \cos(\phi) + \sin\left(x + \frac{\pi}{2}\right) \sin(\phi)$$



The Bloch sphere contains much information

Meaning of the Bloch vector

The Bloch vector is defined as $\mathbf{r} = (\langle \sigma_x \rangle, \langle \sigma_y \rangle, \langle \sigma_z \rangle)$, where the components have the physical interpretations:

$\langle \sigma_z \rangle \equiv w$ is the degree of atomic excitation

$\begin{pmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \end{pmatrix} \equiv \begin{pmatrix} u \\ v \end{pmatrix}$ are the components of $\langle \hat{d} \rangle$ which oscillate in phase and quadrature with the coupling field.

What does that mean?

That there will only be a dipole moment when the atom is a superposition of $|1\rangle$ and $|2\rangle$

$$\chi = \frac{2nd_{12}^2}{\varepsilon_0 \hbar \Omega} \sigma_{12}$$

We have seen this before in the case of the susceptibility

The density matrix can be constructed from the Bloch vector via

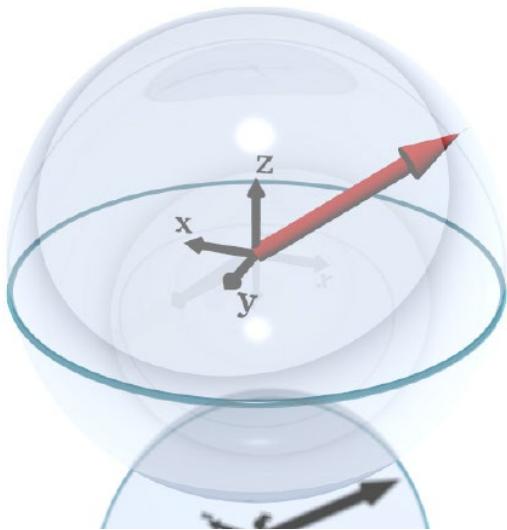
$$\rho = \frac{1}{2}(\mathbf{1} + \mathbf{r} \cdot \boldsymbol{\sigma})$$

u component: oscillates in phase with the coupling field, represents one part of the coherence between states $|1\rangle$ and $|2\rangle$

v component: oscillates in quadrature with the coupling field, represents the other part of the coherence.

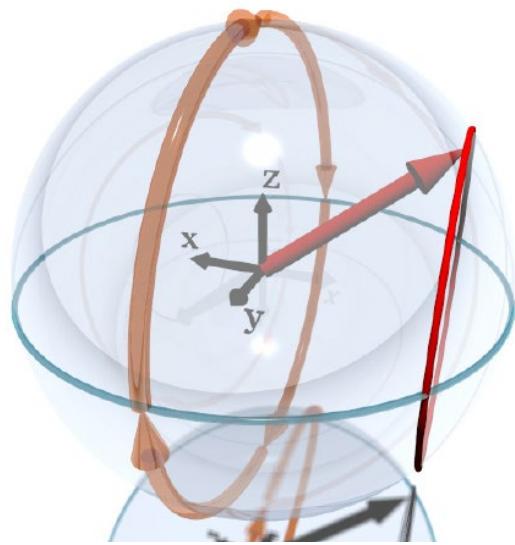
Implication: there is a dipole moment when the atom is in a superposition of states $|1\rangle$ and $|2\rangle$ because the coherence (u and v) allows the atom to interact with the coupling field and exhibit a dipole moment.

States and operations on the Bloch sphere

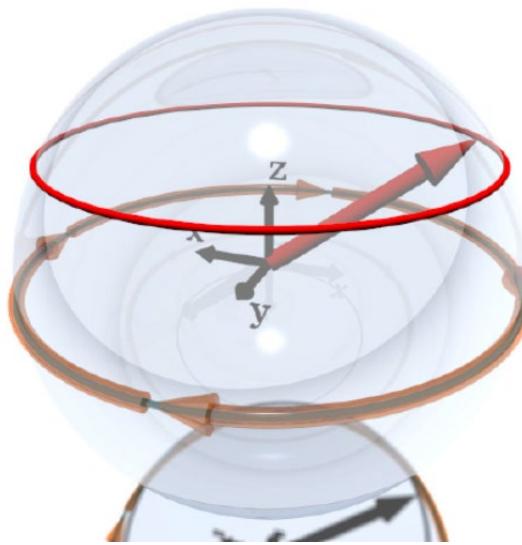


General state

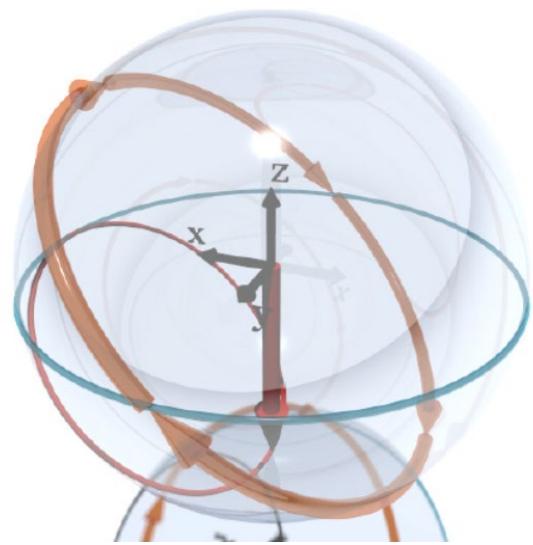
Point not on the
surface of the sphere?
Mixed



On-resonant excitation



Free evolution



Off-resonant excitation

Bloch says, relax

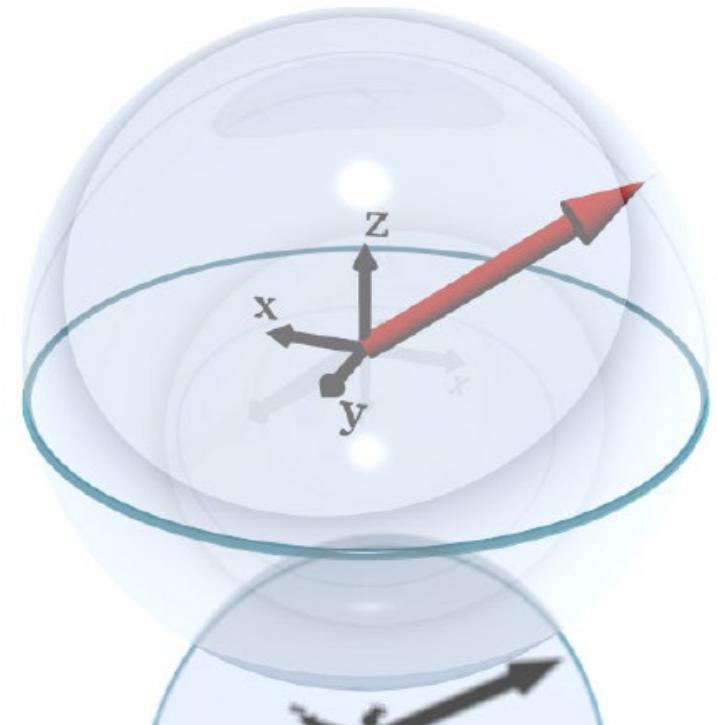
Recall that in the absence of a coupling field:

$$\dot{\rho}_{22} = -\Gamma \rho_{22}$$

$$\dot{\sigma}_{12} = -\gamma_{\perp} \sigma_{12}$$

$$\Gamma = \frac{1}{T_1}, \gamma_{\perp} = \frac{1}{T_2}$$

What does this look like on the Bloch sphere?



Cheat sheet

Fundamental quantities

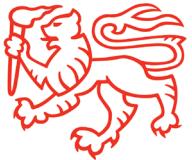
| Name | Symbol | Unit |
|----------------------------|------------------|-------|
| Linewidth | Γ | Hertz |
| Transverse relaxation rate | γ_{\perp} | Hertz |
| Resonant frequency | ω_0 | rad/s |

Derived quantities

| Name | Symbol | Definition |
|----------------------------|-----------|---|
| Detuning | Δ | $\omega - \omega_0$ |
| Rabi frequency | Ω | $\hbar\Omega = \mathbf{d} \cdot \mathbf{E}_0$ |
| Generalised Rabi frequency | Ω' | $\Omega'^2 = \Omega^2 + \Delta^2$ |
| Saturation parameter | s | $\frac{\Omega^2/\gamma_{\perp}\Gamma}{1 + (\Delta/\gamma_{\perp})^2}$ |
| T_1 time | T_1 | $1/\Gamma$ |
| T_2 time | T_2 | $1/\gamma_{\perp}$ |



The Greek alphabet attacks!



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Summary

Problems

- Time-dependent solutions to the optical Bloch equations
 - Probabilities evolve in time sinusoidally in the absence of decay, e.g.
$$\rho_{22}(t) = \left(\frac{\Omega}{\Omega'}\right)^2 \sin^2\left(\frac{\Omega' t}{2}\right)$$
 - Rate and excitation probability altered by coupling strength and detuning from resonance
 - Pulsing the coupling can deterministically shift atomic populations
- Decay and dephasing
 - Need to understand the link between individual atomic behaviour and ensemble behaviour
 - Decay and interaction leads to a loss of coherence, quantum behaviour dissipates
- The Bloch sphere
 - Provides a way to visualise states and their evolution