ASSESSMENT

Atomic physics

UNIVERSITY of TASMANIA

Assignment 1: Hydrogen, calculation methods, angular momenta

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Exercise 1 *Tritium* (4 points)

Tritium is an isotope of hydrogen, with a nucleus comprising one proton and two neutrons. The tritium nucleus (triton) is radioactive, decaying via β emission to a helium-3 nucleus which comprises two protons and one neutron. An electron is initially in the ground state of a tritium atom. After the (instantaneous) β decay, what is the probability that the electron is in the ground state of the new atom?

Exercise 2 *Muonic hydrogen* (11 points)

Muons are effectively heavy electrons. They rapidly decay into electrons with a lifetime of approximately 2.2 μs - see the *lifetime of the muon experiment* in third year labs - but through some technical wizardry, it is possible to produce muonic hydrogen and perform precision spectroscopy on these atoms before they decay. Indeed, said spectroscopy has led to a still-unanswered problem, the so-called *proton radius problem*.

As muons are much heavier than electrons, muonic hydrogen is a much smaller atom the vanilla hydrogen. As such, the finite size of the nucleus plays an more significant role in the energy-level structure of the system. The effective Coulomb potential can be approximated as

$$V(r) = \begin{cases} -\frac{Ze^2}{r} & r \geq R \\ -\frac{Ze^2}{r}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right) & r \leq R \end{cases}$$

where R is the region over which the nuclear charge is distributed.

1. (1 mark) Express the Hamiltonian for muonic hydrogen as a perturbation of a hydrogenic system.
2. (3 marks) Posit how the energies for all the states with $n = 1, 2$, and 3 will be shifted. How will the levels be shifted relative to absolutely, and relatively? Draw an energy-level diagram, indicating the unperturbed states, the perturbed states, and discuss the physical origins of any differences between states.
3. (2 marks) Calculate the first-order change in energy for the ground state of muonic hydrogen. You can make the approximation that $R \ll a_\mu$, where a_μ is the Bohr radius for the muon.

   Hint: make this approximation early!
4. (3 marks) The charge radius R is often measured by probing the $2s \to 2p$ transition. Find an expression for the (angular) frequency of this transition, thus showing one can indeed measure R from measurement of this transition.
5. (2 marks) Can you suggest reasons that the $2s \to 2p$ transition is used to measure R instead of directly measuring the ground state energy?

Exercise 3 *Variational method* (12 points)

Electrons in many-electron atoms generally experience a *screened* potential, that is, a potential that is smaller in magnitude than that due solely to its interaction with the nucleus due to other electrons shielding the nuclear charge.

In atomic units, we can model this screened potential using

$$V(r) = -\frac{Z}{r} + \frac{1 - e^{-\mu r}}{r}$$

where μ is a constant determined by atomic properties[a]. We are going to use our intuition and propose a trial wavefunction for the ground state of the form

$$|\psi_{trial}\rangle \propto e^{-\lambda Z r} Y_0^0(\theta, \phi)$$

You are going to use the variational method to find a bound on the ground-state energy.

1. (1 Mark) Determined the normalisation constant for the trail wavefunction $|\psi_{trial}\rangle$
2. (2 marks) Calculate the expectation value $\langle\psi_{trial}|T|\psi_{trial}\rangle$, where T is the kinetic energy operator
3. (2 marks) Calculate the expectation value $\langle\psi_{trial}|V_0|\psi_{trial}\rangle$, where V_0 is the normal Coulomb potential
4. (2 marks) Calculate the expectation value $\langle\psi_{trial}|V'|\psi_{trial}\rangle$, where V' is the screening potential
5. (5 marks) Using the above, calculate a bound on the ground state energy of the system, assuming a screening potential of $\mu = 1 \times 10\,\mathrm{m}^{-1}$. You may solve non-trivial polynomials computationally (indeed, it is encouraged) but ensure to include executable code in response.

[a] The symbol μ here is convention, and is unfortunate given the question above, but the two μ are very much unrelated!

Exercise 4 *Angular momentum* (4 points)

In class, we used the fact that the electron and proton spin observables $\mathbf{S}^2$ and $\mathbf{I}^2$ commute with the hyperfine Hamiltonian $H'_{hf} = A\mathbf{S} \cdot \mathbf{I}/\hbar^2$, and stated that the component observables S_z and I_z do not.

1. (2 marks) Explicitly show that $\mathbf{S}^2$ and $\mathbf{I}^2$ commute with the hyperfine Hamiltonian
2. (2 marks) Explicitly show that S_z and I_z do not commute with the hyperfine Hamiltonian

Exercise 5 *Positronium* (12 points)

In February 2024, positronium was laser cooled, heralding the era for precision measurements involving antimater. Positronium atom is a hydrogen-like atom with a positron ($m = m_e$, $q = +e$, spin 1/2) as the nucleus and a bound electron. The hyperfine structure in the ground state of positronium is described by a perturbation Hamiltonian $H' = A\mathbf{S}_1 \cdot \mathbf{S}_2/\hbar^2$ where $\mathbf{S}_i$ are the spins of the electron and positron.

1. (2 marks) What is the Bohr energy of the ground state of positronium (you can ignore the hyperfine structure for this one)?
2. (3 marks) The electron and positron spins can be coupled to form the total spin $\mathbf{S}$ of the atom. Write down the spin states of the coupled and uncoupled bases and how they relate to each other.
3. (5 marks) Express the hyperfine Hamiltonian in the ground state as a matrix in both the coupled and uncoupled spin bases.
4. (2 marks) Determine the effect of the hyperfine perturbation interaction on the ground state of positronium. Draw an energy level diagram to illustrate your results.