Algebraic Geometry

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Affine varieties and coordinate rings. Projective space, projective varieties and homogenous coordinates. Rational and regular maps. [4]

Discussion of basic commutative algebra. Dimension, singularities and smoothness. [4]

Conics and plane cubics. Quadric surfaces and their lines. Segre and Veronese embeddings. [4]

Curves, differentials, genus. Divisors, linear systems and maps to projective space. The canonical class. [8]

Statement of the Riemann-Roch theorem, with applications. [4]

Algebraic Geometry is a 24-lectures course in part II. It is focusing on studying the geometry and toplogy of the solution sets by analysing the algebraic structures of their polynomials equations.

Contents

1 Affine Varieties

First, let's have a short introduction of the course. The whole course of the algebraic geometry is a story of duality. That is,

$$\{\text{Systems of polynomials equations (non-linear)}\} \\ \longleftrightarrow \{\text{Geometry or topology of the solution sets}\}$$

If we have a polynomial system $f_1, \cdots, f_r \in k[X_1, \cdots, X_n]$, where k a field. Then we will have a solution space associated to these polynomial equations, which is,

$$V = \{p \in k^n : f_1(p) = f_2(p) = \cdots = f_r(p) = 0\} \subset k^n.$$

We can have a natural ideal in $k[\mathbf{X}]$ ($\mathbf{X} = (X_1, \cdots, X_n)$), namely $I = (f_1, \cdots f_r)$.

The duality we are mainly interested in are the follows,

$$R := k[\mathbf{X}]/I \longleftrightarrow \text{Geometry of } V.$$

The left hand side is the algebra and the right hand side is the geometry.

There is a **note** about the field k. We may have the following possible assumption about k,

- k is *algebraically closed*, which is the weakest condition we can accept, since we would like the solutoin set $V \in k^n$.
- A stronger assumption is $k = \overline{k}$ (k has no proper *algebraic extension*) and $\mathrm{char}(k) = 0$.
- Our assumption throughout the course is much stronger, which is:

$$k = \mathbb{C}.$$

Hence, with our assumption, we are studying,

$$R = \mathbb{C}[\mathbf{X}]/I \longleftrightarrow V \subset \mathbb{C}^n.$$

There are possible questions of the course we might want to study:

(i) To what extend do $\mathbb{R}$ and V determine each other?

(ii) Can we obtain wether $V \subset \mathbb{C}^n$ is a manifold based on $\mathbb{R}$?

(iii) What is the right notion of dimension of V (in terms of algebra)?

(iv) Is V compact as a topological space? If not is there a natural compactification $\overline{V}$ that is in some sense algebraic?

Now let's start our course.

...under construction...

1.1 Affine Space and Affine Varieties

Definition (Affine Space)**.** The *affine space* of dimension n (over $\mathbb{C}$) is the set $\mathbb{A}^n = \mathbb{C}^n$ (not vector space).

Its elements are called *points* denoted $p = (\mathbf{a}) = (a_1, \cdots, a_n)$ where $a_i \in \mathbb{C}$.

Definition (Affine Subspace)**.** An *affine subspace* of $\mathbb{A}^n$ is any subset of the form $v + U$ where $v \in \mathbb{C}^n$ and $U \subset \mathbb{C}^n$ is a linear subspace.

Note $\mathbb{A}^n$ is the natural set on which $\mathbb{C}[X_1, \cdots X_n]$ is a ring of functions (evaluating functions). Given $f \in \mathbb{C}[\mathbf{X}]$, we get a function,

$$f : \mathbb{A}^n \longrightarrow \mathbb{C}.$$

The subset $\mathbb{C} \in \mathbb{C}[\mathbf{X}]$ are called *constant functions*.

Now we have the following proposition.

Proposition 1.1.1. The polynomial ring $\mathbb{C}[\mathbf{X}]$ satisfy:

(i) $\mathbb{C}[\mathbf{X}]$ is a unique factorisation domain.

(ii) Every ideal $I \lhd \mathbb{C}[\mathbf{X}]$ is finitely generated.

The first statement is the Gauss lemma and the second statement is the Hilbert basis theorem (see Part IB GRM).

Note for $\mathbf{X} = X$, $\mathbb{C}[X]$ is a ED with the Euclidean function $\phi(f) = \deg(f)$. And, $\mathbb{C}[X_1, \cdots, X_n]$ can be considered inductively as $\mathbb{C}[X_1, \cdots, X_{n-1}][X_n]$, which is a polynomial ring of a UFD, so a UFD as well by Gauss lemma. Also, since $\mathbb{C}$ is a PID, so a Noetherian ring. By Hilbert basis theorem, we have $\mathbb{C}[\mathbf{X}]$ is a Noetherian ring inductively.

Definition (Vanishing Locus and Affine Varieties)**.** Let $S \subset \mathbb{C}[\mathbf{X}]$ be any subset. The *vanishing locus* of S is defined as,

$$\mathbb{V}(S) = \{p \in \mathbb{A}^n : f(p) = 0, \forall f \in S\}.$$

An *affine (algebraic) variety* in $\mathbb{A}^n$ is any set of the form $\mathbb{V}(S)$ for some S.

Warning: it is a n inconsistent terminology in literature.

Example. If $n = 1$ and $f \in \mathbb{C}[X]$ then $\mathbb{V}(f) = \{\text{zeros of } f\}$.

Conversly, if $V \subset \mathbb{A}$ is finite. Then $V = \mathbb{V}(f)$ where $f = \prod_{a \in V}(X - a)$. However, it is not a bijection, since we can have different power of $(X - a)$, such as $(X - a)^2$.

Example (Hypersurface)**.** A *hypersurface* in $\mathbb{A}^n$ is a variaty of the form $\mathbb{V}(f)$ for some $f \in \mathbb{C}[\mathbf{X}]$.

Example. It is often convient to represent variables parametrically. For example, the *affine twisted cubic*,

$$C = \left\{(t, t^2, t^3) : t \in C\right\} \in \mathbb{A}^3.$$

C is the variety set,

$$C = \left\{X_1^2 - X_2 = X_1^3 - X_3 = 0\right\} \in \mathbb{A}^3.$$

Now let us introduce our first theorem.

Theorem 1.1.2. Let $V = \mathbb{V}(S)$ where $S \in \mathbb{C}[\mathbf{X}]$. Then we have the follwoing statement be true:

(i) Let $I \subset \mathbb{C}[\mathbf{X}]$ be the ideal generated by S. Then $\mathbb{V}(S) = \mathbb{V}(I)$.

(ii) There exists a *finite* subset $\{f_i\} \subset S$ s.t. $\mathbb{V}(S) = \mathbb{V}(\{f_i\})$.

Proof. The general idea is first prove (i), then use the finite generating property to prove (ii).

For (i), suppose $p \in \mathbb{A}^n$. Then $f(p) = 0$ for all $f \in S$ iff $f(p) = 0$ for all $f \in I$.

For (ii) by statement (i) we know that $\mathbb{V}(S) = \mathbb{V}(I)$. Also there exist $h_1, \cdots, h_r \in I$ that generates I because I is finitely generated. Hence, use statement (i) reversely, since I is the ideal generated by $\{h_i\}$, we have $\mathbb{V}(S) = \mathbb{V}(I) = \mathbb{V}(\{h_i\})$.

Then what we need to do is to reduce $\{h_i\}$ into some $\{f_i\} \subset S$. Note $\{h_i\}$ and S generate the same ideal. So we can find $g_{ij} \in \mathbb{C}[\mathbf{X}]$ s.t.,

$$h_i = \sum_{j=1}^{m} g_{ij} f_j,$$

where $\{f_1, \cdots, f_m\}$ is some *finite* subset of S, since this is the definition of ideal (finite linear combination).

Hence, $I \subset (f_1, \cdots, f_m)$. Also, since $\{f_i\} \subset S$, so $(f_1, \cdots, f_m) \subset I$. Thus, $(f_1, \cdots, f_m) = I$. HEnce, by (i) again, we have,

$$\mathbb{V}(S) = \mathbb{V}(I) = \mathbb{V}(\{h_i\}) = \mathbb{V}(\{f_i\}).$$

□

Proposition 1.1.3. Let $S, T \subset \mathbb{C}[\mathbf{X}]$. Then we have:

(i) If $S \subset T$, then we have $\mathbb{V}(T) \subset \mathbb{V}(S)$.

(ii) $\mathbb{V}(0) = \mathbb{A}^n$ and $\mathbb{V}(\mathbb{C}(\mathbf{X})) = \emptyset = \mathbb{V}(\lambda)$, where $\lambda \in \mathbb{C} \setminus \{0\}$.

...under construction...

1.2 Morphisms

...under construction...

People might be interested in what happens in the neighbourhood of a point p in some affine variety. In order to study that, we need to define the *local ring*. We have the definition as follows.

Definition (Local Rings)**.** Let V be an irreducible variety and $p \in V$. Then the *local ring of V at p* is,

$$\mathcal{O}_{V,p} = \{f \in \mathbb{C}(V) : f \text{ is regular at } p\}.$$

Additionally, we say a ring R is *local* or is *a local ring* if it contains a *unique* maximal ideal.

Remark: The definition of regular tells us, f here shoulde be written in the reduced form, since if f *can* have non-zero dominator at p, then it is regualer at p. Hence, the definition of the local ring of V at p is well-defined.

We first introduce a lemma about local rings.

Lemma 1.2.1. A ring R is local iff $R \setminus R^\times$ is an ideal. If so the unique maximal ideal is $\mathfrak{m} = R \setminus R^\times$.

Proof. ($\Leftarrow$): Suppose $A \subset R$ is an ideal. Then A is a proper ideal iff $A \subset R \setminus R^\times$. So if $\mathfrak{m} = R \setminus R^\times$ is an ideal, it is automatically the unique maximal ideal.

($\Rightarrow$): Conversly, let R be a local ring with the maxiaml ideal $\mathfrak{m}$. Then $\mathfrak{m} \subset R \setminus R^\times$. If $x \in R \setminus R^\times$, then $(x) \neq R$. By the maximality of $\mathfrak{m}$, we have $(x) \subset \mathfrak{m}$. In particular, $x \in \mathfrak{m}$. Hence, $R \setminus R^\times \subset \mathfrak{m}$. Hence, $\mathfrak{m} = R \setminus R^\times$. $\square$

One might think is the definition of the local ring and a long ring consistent? We have the following proposition.

Proposition 1.2.2. Let V be an irreducible variety and $p \in V$. The ring $\mathcal{O}_{V,p}$ is local, with *unique* maximal given by,

$$\mathfrak{m}_{V,p} = \{f \in \mathcal{O}_{V,p} : f(p) = 0\} = \operatorname{Ker} \theta,$$

where $\theta : \mathcal{O}_{V,p} \longrightarrow \mathbb{C}$ is the homomorphism defined by $f \longmapsto f(p)$.

Furthermore, the invertible elements of $\mathcal{O}_{V,p}$ are those f s.t. $f(p) \neq 0$.

Understanding. Evaluate map on a polynomial ring is always a homomorphsim. Particularly, the kernel of the map is an ideal. Hence, this quickly tells us why $\mathfrak{m}_{V,p}$ is an ideal.

Proof. If $\mathcal{O}_{V,p} \subset \mathbb{C}(V)$ and $\dfrac{f}{g} \in \mathcal{O}_{V,p}$. It is invertible if and only if $(\dfrac{f}{g})(p) \neq 0$, since $\mathbb{C}(V)$ is a field.

Define $\mathfrak{m}_{V,p}$ as the proposition. Then, the ideal $\mathfrak{m}_{V,p}$ contains all the non-units elements in $\mathcal{O}_{V,p}$. Hence, by the previous lemma, we have $\mathcal{O}_{V,p}$ is local and $\mathfrak{m}_{V,p}$ is the unique maximal ideal. $\square$

The ring structure of local rings are a bit different from what we know, like polynomials rings. Let's see two exmaples of local rings to help us build our intuitions.

Example. Define,

$$R = \left\{ \frac{f}{g} \in \mathbb{C}(t) : \text{ in the lowest terms } g(0) \neq 0 \right\}.$$

That is, in the lowest terms, g has a non-zero constant.

Note this is a local ring, since it is atually, $\mathcal{O}_{\mathbb{A}^1,0}$. The maximal ideal is (t) and $R/(t) = \mathbb{C}$.

Example. Take S be the ring of *formal power series*, i.e.,

$$S = \mathbb{C}[\![t]\!] = \left\{ \sum_{i=1}^{\infty} a_i t^i : a_i \in \mathbb{C} \right\}.$$

This is also a local ring. Its maximal ideal is (t) by Lemma XXX. Indeed, $f \in \mathbb{C}[\![t]\!]$ is invertible iff f has non-zero constant term. Note the infinity ∞ here doesn't mean the limit, but means infinite terms.

...under construction...

In fact, $R \subset S$, which we can prove.

2 Projective Geometry

After setting up all the things about affine variety, we are now at the point similar where we are when we just finish studying linear algebra. What we then went to do is to study more complex concept, the manifolds, which are locally governed by linear algebra. What we are now going to do is similar, that is, studying geometries spaces that locally look like affine varieties.

...under construction...

The projective space is one of the examples.

2.1 Projective Spaces

Projective space $\mathbb{P}^n$ is a replacement or an upgrade over $\mathbb{A}^n$. The following statement serve as the motivations.

(i) Every pair of lines in $\mathbb{P}^2$ are distinct intersect at a unique point, which is not true in $\mathbb{A}^n$.

(ii) If V is a *projective variety* defined by a degree d polynomial in $\mathbb{P}^2$. If V is a manifold, then V is homeomorphic (Euclidean) to a closed orrientable topological surface of genus $\binom{d-1}{2}$.

(iii) More general, we have two projective varieties homeomorphic if they are defined by the polynomials with the same degree and they are both manifolds.

(iv) Most importantly, $\mathbb{P}^n$ is compact in the Euclidean topology, but $\mathbb{A}^n$ is not.

Definition (Projectivization)**.** Let U be a finite dimension complex vector space. The *projectivization* of U is,

$$\mathbb{P}(U) = \{\text{lines in } U \text{ through } \mathbf{0} \in U\} .$$

Define the n-dimensional *projective space* $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$.

Notation: we index coordinates on $\mathbb{C}^{n+1}$ by $(0, \cdots, n)$. A line is given by,

$$L_{(a_0,\cdots,a_n)} = \{(a_0 t, \cdots, a_n t) | t \in \mathbb{C}\} ,$$

where $(a_0, \cdots, a_n)$ are fixed and not *all zero*. Note the lines in $\mathbb{C}^{n+1}$ are points in $\mathbb{P}^n$. We write $(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n$ for the corresponding point. Thus, we have,

$$\mathbb{P}^n = \{(a_0 : \cdots : a_n) | a_i \in \mathbb{C} \text{ not all } a_i = 0\} / \sim = \left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right) / \sim,$$

where $\sim$ is the equivalence relation defined by scaling by non-zero elements of $\mathbb{C}$. E.g. $(2:1:-2) = (4:2:-4) \in \mathbb{P}^n$.

Now we can have some elementary **observations**: We can decompose,

$$\begin{aligned}\mathbb{P}^1 &= \left\{(a_0 : a_1) \in \mathbb{P}^1 | a_0 \neq 0\right\} \cup \left\{(a_0 : a_1) \in \mathbb{P}^1 | a_0 = 0\right\} \\ &= \left\{(1 : \frac{a_1}{a_0}) \in \mathbb{P}^1 | a_0 \neq 0\right\} \cup \{(0,1)\} \\ &= \left\{(1 : z) \in \mathbb{P}^1 | z \in \mathbb{C}\right\} \cup \{(0,1)\} \\ &= \mathbb{A}^1 \cup \{\text{a point (at infinity)}\}.\end{aligned}$$

Hence, we can think of $\mathbb{P}^1$ as $\mathbb{A}^1$ with *infinity.*

More generally, we have inductively,

$$\begin{aligned}\mathbb{P}^n &= \{(a_0 : \cdots : a_n) | a_0 \neq 0\} \cup \{(a_0 : \cdots : a_n) | a_0 = 0\} \\ &= \mathbb{A}^n \cup \mathbb{P}^{n-1} = \mathbb{A}^n \amalg \mathbb{A}^{n-1} \amalg \cdots \amalg \mathbb{A}^1 \amalg \{\text{a point}\}.\end{aligned}$$

In this case, we think of $\mathbb{A}^{n-1} \amalg \cdots \amalg \mathbb{A}^1 \amalg \{\text{a point}\}$ at infinity. Note the $\amalg$ means independent, in order to emphasize that they are all distinct affine spaces.

Definition (Zariski and Euclidean Topology on $\mathbb{P}^n$)**.**
The Zariski (Euclidean) topology on $\mathbb{P}^n$ is the quotient topology for the subspace topology of the Zariski (Euclidean) topology on $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$, where,

$$\mathbb{P}^n = \left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right)/\sim \text{ and } \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \subset \mathbb{C}^{n+1}.$$

We use quotient topology for the first equality and subspace topology for the second equality.

We also can write $\mathbb{P}^n = \left(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}\right)/\mathbb{C}^\times$ where $\mathbb{C}^\times$ is the equivalence relation we defined previously.

Corollary 2.1.1. The projective space $\mathbb{P}^n$ is compact with Euclidean topology.

Proof. We have the following commutative diagram. Hence, the quotient map q restricted in $S^{2n+1} \subset \mathbb{C}^{n+1}$ is still continuous and surjective. Hence, $\mathbb{P}^n$ is a continous image of a compact space, thus compact. □

Definition (Hyperplanes)**.** For $0 \leqslant j \leqslant n$, the *jth coordinate hyperplane* is defined as,

$$H_j = \{(a_0 : \cdots : a_n) : a_j = 0\} \subset \mathbb{P}^n.$$

Note we have $H_j \cong \mathbb{P}^{n-1}$.

Note if we have $a_j = c \neq 0$, then it is not well-defined, since two tuples differed by scaling will not be regarded as the same. But we have $a_j = 0$ here so everthing is fine.

Definition (Standard Affine Patches)**.** The *jth standard affine patch* is defined as,

$$U_j = \mathbb{P} = \{(a_1 : \cdots : a_n) : a_j \neq 0\}.$$

The reason we call it jth standard affine patch is that we have a natual bijection,

$$\begin{aligned} \mathbb{A}^n &\longleftrightarrow U_j \\ (\frac{a_0}{a_j}, \cdots, 1, \cdots, \frac{a_n}{a_j}) &\longleftarrow\!\shortmid (a_0 : \cdots : a_n) \\ (b_1, \cdots, b_n) &\longmapsto (b_1 : \cdots : b_{j-1} : 1 : b_{j+1} : \cdots : b_n). \end{aligned}$$

We **obeserve** $\{U_j\}_{0 \leqslant j \leqslant n}$ is an open cover of $\mathbb{P}^n$ in Zariski topology. It is a cover since one of a_j has to be non-zero for every point in $\mathbb{P}^n$. It is open since H_j is a closed set by the definition. Indeed, the preimage of H_j is $\mathbb{V}(X_j) \subset \mathbb{A}^{n+1} \setminus \{\mathbf{0}\}$ which is closed.

We have mentioned that an important property of projective spaces is that every two line intersect each other at one point. This is because compared to affine spaces, we add things at infinity to make it into a compact space (we can think of Riemann sphere of complex plane).

2.2 Projective Varieties

Now let us introduce the *projective varieties* on the playground we have built up. We expect it to be some vanishing locus of some polynomials. However, there is something to be careful.

Consider the polynomial $X_0 + 1 \in \mathbb{C}[X_0, X_1]$. Then,

- $X_0 + 1$ does *not* define a function on $\mathbb{P}^1$ since $(1 : 1) \sim (2, 2)$ but $(X_0 + 1)(1 : 1) \neq (X_0 + 1)(2 : 2)$, so not well-defined.
- The vanishing locus of $X_0 + 1$ is also undefined directly, since on $\mathbb{P}^1$, we have $(-1 : 0) \sim (1 : 0)$ but $X_0 + 1$ vanishes on the first but not the second, so not well-defined.

This means we can't use the normal polynomial. Instead we should use a special class of polynomials, *homogeneous polynomials*.

Definition (Monomial and Homogeneous Polynomials)**.** A *monomial* in $\mathbb{C}[\mathbf{X}] = \mathbb{C}[X_0, \cdots, X_n]$ is an element of the form $X_0^{d_0} X_1^{d_1} \cdots X_n^{d_n}$ for $d \geqslant 0$.

A *term* is a non-zero multiple of a monomial. The *degree* of a term $cX_0^{d_0} \cdots X_n^{d_n}$ is $\sum_{i=0}^n d_i$. A *homogenous polynomial* of degree d is a *finite* sum of degree d terms.

Notation: we can always write a polynomial f in the form of,

$$f = \sum_{i=0}^{\infty} f_{[i]},$$

where $f_{[i]}$ is the degree i homogeneous polynomial. Note the sum here is finite.

What is the advantage of working with homogeneous polynomials?

Lemma 2.2.1. Let $f \in \mathbb{C}[\mathbf{X}]$ be homogeneous and for $(a_0, \cdots, a_n) \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ we have if $f(\mathbf{a}) = 0$ then $f(\lambda a_0, \cdots, \lambda a_n) = 0 \; \forall \, \lambda \in \mathbb{C}^{\times}$.

Proof. Suppose that f has m terms,

$$f = \sum_{i=1}^{m} X_0^{d_{i0}} X_1^{d_{i1}} \cdots X_n^{d_{in}},$$

where $\sum\limits_{j=1}^{n} d_{ij} = d \geqslant 0$ for any i. Then, we have,

$$f(\lambda \mathbf{a}) = \sum_{i=1}^{m} (\lambda a_0)^{d_{i0}} (\lambda a_1)^{d_{i1}} \cdots (\lambda a_n)^{d_{in}} = \lambda^d f(\mathbf{a}) = 0.$$

□

Hence, there is some pay to work on projective spaces, that is, we can't work with arbitrary polynomials, but only homogeneous polynomials.

Corollary 2.2.2. Let $f \in \mathbb{C}[\mathbf{X}]$ be homogeneous. Then the set,

$$\mathbb{V}(f) = \left\{ p \in \mathbb{P}^n : f(\mathbf{a}) = 0 \text{ for any } \mathbf{a} \in \mathbb{C}^{n+1} \text{ representing } p \right\}$$

is well-defined.

Proof. Immediately follows the previous lemma. □

We also can extend the definition of homogeneous to ideals.

Definition (Homogeneous Ideals). An ideal $I \in \mathbb{C}[\mathbf{X}]$ is *homogeneous* if it is generated by homogeneous polynomials (potentially of different degrees).

Lemma 2.2.3. Let $I \subset \mathbb{C}[\mathbf{X}]$ an ideal. The following statements are equivalent:

(i) The ideal I is homogeneous.

(ii) If $f \in I$, then the homogeneous polynomials $f_{[r]} \in I$ for all r.

Proof. (i)$\Rightarrow$(ii): Let g_j be homogeneous generator for I of degree d_j. If $f = \sum\limits_j h_j g_j$ for $h_j \in \mathbb{C}[\mathbf{X}]$, we can split h_j into pieces $h_{j[r]}$. Now $h_{j[r]}g_j \in I$ and is homogeneous.

Now we try to build up f_r by $h_{j[r]}g_j \in I$. Indeed, we have,

$$f_{[r]} = \sum_j h_{j[r-d_j]}g_j \in I,$$

as required.

(ii)$\Leftarrow$(i): We can always decompose generators of I into homogeneous pieces that still in I. And, those homogeneous pieces actually generates I. $\square$

Remark: (i) tells us $\mathbb{V}(I) = \mathbb{V}(\{g_i\})$ is well-defined where g_i are generators. (ii) is what we usually use to determine a homogeneous ideal.

Now we are sort of confident to define *projective varieties.*

Definition (Vanishing Locus and Projective Varieties). Let $I \subset \mathbb{C}[\mathbf{X}]$ be homogeneous. Define the *vanishing locus* on $\mathbb{P}^n$ as,

$$\mathbb{V}(I) = \{p = (a_i) \in \mathbb{P}^n | f((a_i)) = 0 \ \forall \, f \in I\}.$$

It is well-defined by the definition of homogeneous ideal. A *projective variety* in $\mathbb{P}^n$ is any set of the form.

Let us see some examples.

Example. Let $U \subset \mathbb{C}^{n+1}$ be a vector subspace. Then $\mathbb{P}(U) \subset \mathbb{P}^n$ is a projective variety. We can write U as a kernel of a linear map (naturally map to its quotient space). Then we have,

$$U = \left\{ v = (v_0, \cdots, v_n) \in \mathbb{C}^{n+1} : \sum_{i=0}^{n} a_i^{(j)} v_i = 0 \ \forall \, j \right\},$$

where $\left\{\mathbf{a}^{(j)} = (a_0^j, \cdots, a_n^{(j)})\right\}$ is a subset of $\mathbb{C}^{n+1}$. Then we can see easily $\mathbb{P}(U) = \mathbb{V}(I)$, where I is generated by $F_j = \sum\limits_{i=0}^{n} a_i^{(j)} X_i \in \mathbb{C}[\mathbf{X}]$ which are homogeneous.

We call $\mathbb{P}(U) \subset \mathbb{P}^n$ the *projective linear subsapce.*

Understanding. For subset $U \in \mathbb{C}^{n+1} = \mathbb{A}^{n+1}$, if $\mathbb{V}(I) = U$ on $\mathbb{A}^{n+1}$ where I is homogeneous. Then $\mathbb{V}(I) = \mathbb{P}(U)$ on $\mathbb{P}^n$.

Example. We can easily see that,

$$\left\{ \text{projective linear spaces in } \mathbb{P}^n \longleftrightarrow \text{ linear spaces in } \mathbb{C}^{n+1} \right\}.$$

We can have $\mathrm{GL}(n+1, \mathbb{C})$ acts on $\mathbb{P}^n$ in the natural way (via its action on $\mathbb{C}^{n+1}$). Then, the normal subgroup (scaling matrices) $\mathbb{C}^\times \subset \mathrm{GL}(n+1, \mathbb{C})$ acts trivially. The quotient of actions (permutation representation) is,

$$\mathrm{PGL}(n+1, \mathbb{C}) \cong \mathrm{GL}(n+1, \mathbb{C})/\mathbb{C}^\times.$$

$\mathrm{PGL}(n+1, \mathbb{C})$ acts extremely transitivilly on $\mathbb{P}^n$. That is transitive and faithful, can map any $x \in \mathbb{P}^n$ to $y \in \mathbb{P}^n$. Moreover, any $G \in \mathrm{PGL}(n+1, \mathbb{C})$ maps one to another (see IA Groups).

A more interesting example is as follows.

Example. The *Segre surface* is a (hypersurface) defined by,

$$S_{11} = \mathbb{V}(X_0X_3 - X_1X_2) \subset \mathbb{P}^3.$$

To understand, consdier,

$$\begin{aligned} \sigma_{11} : \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ ((a_0 : a_1), (b_0 : b_1)) &\longmapsto (a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1). \end{aligned}$$

Note this map is well-defined. Moreover, we prove that the image of σ_{11} is exactly S_{11}.

Indeed, consider first,

$$\begin{aligned} \mathbb{C}^2 \times \mathbb{C}^2 &\longrightarrow \mathbb{C}^4 = \mathbb{C}^{2\times 2} \\ (v, w) &\longmapsto vw^\intercal \end{aligned}$$

where the element in $\mathbb{C}^{2\times 2}$ is 2×2 matrices. The image is precisely the set of rank less than 2 matrices because of our construction. This is said, $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ is the determinant 0 matrices in $\mathbb{P}^3$ under equivalence relations, which is exactly $\mathbb{V}(X_0X_3 - X_1X_2)$.

Understanding. Since the definition of varieties repsects the equivalence relation defining projective space. We can always operate things in affine spaces then transfer to proejctive spaces by quotient map.
For this example, the map $\sigma_{11} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$, we can consider,
Then, we can study the property of σ_{11} by $\tilde{\sigma}_{11}$.

We have seen that $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$ ($\{U_i\}$ is an open cover of $\mathbb{P}^n$) when we defining the standard affine patches.

We have three ways to think of Zariski topology on $\mathbb{P}^n$:

(i) Subquotient of the Zariski topology on $\mathbb{C}^{n+1}$ which is what we defined previously.

(ii) A set Z in $\mathbb{P}^n$ is Zariski closed iff $G = \mathbb{V}(I)$ for some $I \lhd \mathbb{C}[\mathbf{X}]$ homogeneous.

(iii) Gluing topology: a set Z in $\mathbb{P}^n$ is Zariski closed iff $Z \cap U_i$ is closed in U_i for all i where we regard $U_i = \mathbb{A}^n$.

Fact: There three definitions of Zariski topology on $\mathbb{P}^n$ conincides (see EX sheet 2).

Remark: Note the projective varieties are automatically closed by (i).

Consider the (iii) more carefully. Suppose $V \subset \mathbb{P}^n$ and $V = \mathbb{V}(I)$ for some $I \lhd \mathbb{C}[\mathbf{X}]$ homogeneous. Consider $V \cap U_0 \subset U_0 \subset \mathbb{P}^n$. We have,

$$\mathbb{V}(I) \cap \{a_0 : \cdots : a_n | a_0 \neq 0\} = \mathbb{V}\left(\left\{f = F(1, \frac{X_1}{X_0}, \cdots, \frac{X_n}{X_0}) : F \in I\right\}\right),$$

where X_i are coordinate functions. We can do this because $X_0(p) \neq 0$ for $p \in \mathbb{P}^n$. Defining $Y_i = X_i/X_0$, we can have $V \cap U_0 = \mathbb{V}(I_0)$, where,

$$I_0 = \{f = F(1, Y_1, \cdots, Y_n) : F \in I\} \lhd \mathbb{C}[Y_1, \cdots, Y_n].$$

Here we still consider $\mathbb{V}(I_0) \subset \mathbb{P}^n$ (but $\mathbb{V}(I_0)$ is not a projective variety since Y_i are not coordinate functions). However, we can also think $V \cap U_0 = \mathbb{V}(I_0) \subset U_0 \cong \mathbb{A}^n$. Map $\mathbb{V}(I_0)$ by the isomorphism $U_0 = \{(a_0 : \cdots : a_n) : a_0 \neq 0\} \leftrightarrow \{(1, \frac{a_1}{a_0}, \cdots, \frac{a_n}{a_0})\}$, then the coordinate functions of $U_0 = \mathbb{A}^n$ are exactly Y_i. Hence, $V \cap U_0 = \mathbb{V}(I_0)$ are naturally an affine variety in $\mathbb{A}^n = U_0$.

Likewise, setting $X_j = 1$ defines an ideal I_j whose associated affine variety is $U_j \cap V \subset U_j = \mathbb{A}^n$. Notice, if we consider $U_0 \cap V \subset \mathbb{P}^n$, we have,

$$U_0 \cap V = \{(1 : \mathbf{a}) | \mathbf{a} \in \mathbb{V}(I_0) \subset \mathbb{A}^n\} \subset \mathbb{P}^n.$$

Remark: This gives a proof of one direction of (iii) of the Zariski topology.

Understanding. Geometrically, we should think $\mathbb{V}(I_j) = U_j \cap V \in \mathbb{A}^n$ is the affine variety defined by $V \in \mathbb{A}^{n+1}$ cutted by the plane $X_j = 1$. And, it is bijective with $U_j \cap V \subset \mathbb{P}^n$.

Also note, $U_j \cap V$ is closed in $\mathbb{A}^n$ is not equivalent to $U_j \cap V$ is closed in $\mathbb{P}^n$ (which is equivalent to $U_j \cap V$ is closed in $\mathbb{A}^{n+1}$).

Conversely, suppose we have have a variety $W \subset \mathbb{A}^n$. If we identify $\mathbb{A}^n = U_0$. We can view $W \subset \mathbb{P}^n$, which is almost certainly not a projective variety, since it is even not closed in $\mathbb{A}^{n+1}$. Then, we can have the Zariski closure $\overline{W} \subset \mathbb{P}^n$ of W, which is a projective variety (see IB AT). We are going to find the explicit expresiion of $\overline{W}$. It is like we are trying to find what is the thing in the infinity by previous insight of regarding $\mathbb{P}^n = \mathbb{A}^n \amalg \cdots \amalg \{\text{a point}\}$.

Definition (Homogenization)**.** Fix $f \in \mathbb{C}[Y_1, \cdots, Y_n]$ of total degree d. The *homogenization* of f is,

$$F = f^h = X_0^d f(\frac{X_1}{X_0}, \cdots, \frac{X_n}{X_0}) \in \mathbb{C}[X_0, \cdots, X_n].$$

This is homogeneous of degree d.

If $I \lhd \mathbb{C}[Y_1, \cdots, Y_n]$. Its homogenization $I^* = I^h$ is the ideal generated by f^h for all $f \in I$. I^* is then a homogeneous ideal in $\mathbb{C}[X_0, \cdots, X_n]$.

Definition (Porjective Closure)**.** Given a variety $V \subset \mathbb{A}^n$. The *projective closure* is $V^* = \mathbb{V}(I(V)^h) \subset \mathbb{P}^n$.

Let us see some examples for homogeneouzation.

Example. Let $f(Y_1, Y_2) = 1+Y_1^2+Y_1Y_2^2 \subset \mathbb{C}[Y_1, Y_2]$. Then, $f^h = X_0^3+X_1^2X_0+X_1X_2^2$. We just simply multiply every term by X_0 until obtain the highest degree.

Proposition 2.2.4. Let $V \subset \mathbb{A}^n$ be an affine variety. The Zariski closure $\overline{V} \subset \mathbb{P}^n$ (via $\mathbb{A}^n = U_0$) conincides with $\mathbb{V}(I(V)^h) \subset \mathbb{P}^n$.

Proof. Suppose $V \subset \mathbb{A}^n$ is an affine variety and $I = I(V)$. Let $\overline{V}$ denote the Zariski closure, which is the smallest closed set contaning $V \subset \mathbb{P}^n$. Let I^h be the homogeneozation. Then $\mathbb{V}(I^h)$ is Zariski closed.

Now what we are going to show is $\mathbb{V}(I^h)$ is the smallest. Suppose $Y \supset V$ closed with $Y = \mathbb{V}(I')$ where I' homogeneous. Then, any homogeneous element in I' can always be written as $X_0^d f^h$ where $f \in \mathbb{C}[Y_1, \cdots, Y_n]$ and $d \in \mathbb{Z}_{\geqslant 0}$ (Note X_0^d is necessary or X_0 can't be written in the form of some homogenenization).

Then, for $X_0^d f^h \in I'$, we have,

$$X_0^d f^h = 0 \text{ on } V \subset \mathbb{P}^n \implies f = 0 \text{ on } V \subset \mathbb{A}^n,$$

since the way we map V from $\mathbb{A}^n$ to $\mathbb{P}^n$ is $(b_1, \cdots, b_n) \longmapsto (1 : b_1 : \cdots : b_n)$. Then, we have, $f \in I(V) = I$, so $f^h \in I^h$. It implies that $X_0^d f^h \in I^h$.

Hence, we are done since it implies $I' \subset I^h$, which means $Y = \mathbb{V}(I') \supset \mathbb{V}(I^h) = V^h$. Hence, $V^h = \mathbb{V}(I^h) = \overline{V}$. □

There is a corollary about the form of projective closure. Actually, we can think it in a more natural way.

Corollary 2.2.5. Let $V = \mathbb{V}(I) \subset \mathbb{A}^n$. Then, $\mathbb{V}(I(V)^h) = \mathbb{V}(I^h)$.

Proof. We know $\mathbb{V}(I(V)^h) \subset \mathbb{V}(I^h)$. By Strong Nullstellensatz, We need to show $I(V)^h = \sqrt{I}^h \subset \sqrt{I^h}$. Suppose $f \in \sqrt{I}^h$, then there exists $g \in \sqrt{I}$ s.t. $X_0^d g^h = f$. Then,

$$\begin{aligned} & g^m \in I \text{ for some positive integer } m \\ \Longrightarrow\ & (g^m)^h = (g^h)^m \in I^h \\ \Longrightarrow\ & g^h \in \sqrt{I^h} \implies f = X_0^d g^h \in \sqrt{I^h}. \end{aligned}$$

Hence, we get $I(V)^h \subset \sqrt{I^h}$, which implies $\mathbb{V}(\sqrt{I^h}) = \mathbb{V}(I^h) \subset \mathbb{V}(I(V)^h)$. Hence, $V(I(V)^h) = \mathbb{V}(I^h)$. □

Warning:

(i) Let $V \subset \mathbb{P}^n$ and $W = V \cap U_0 \subset \mathbb{A}^n = U_0$. Then $V \neq \overline{W}$. This is because we has no way to recover the information at the infinity. We illustrate by the figure below.

There is also a trivial example. Let $V = \mathbb{V}(X_0)$, then $W = V \cap U_0 = \phi$, so $\overline{W} = \phi \neq V$. The reason is exactly the ambiguity when we represent element in $I(V)$ by $X_0^d f^h$, we will lose the information about X_0.

(ii) Suppose $I = (f_1, \cdots, f_r) \in \mathbb{C}[Y_1, \cdots, Y_n]$. Now let $J := (f_1^h, \cdots, f_r^h)$. We have $J \neq I^h$ (see EX sheet 2). (If I is principal then is not a problem i.e. $I = (f)$ and $I^h = (f^h)$).

Let us do an example.

Example. Let $V \subset \mathbb{P}^2$ be $\mathbb{V}(X_0X_1 - X_2^2)$. We have 3 natural affine varieties:

- For $V_0 \subset U_0$ by setting $X_0 = 1$, we have $V_0 = \mathbb{V}(Y_1 - Y_2^2)$, a parabla.
- For $V_1 \subset U_1$ by setting $X_1 = 1$, we have $V_1 = \mathbb{V}(Y_0 - Y_2^2)$, also a parabla.
- For $V_2 \subset U_2$ by wetting $X_2 = 1$, we have $V_2 = \mathbb{V}(Y_0Y_1 - 1)$, a rectangular hyperbola.

Theorem 2.2.6. Let $Q \subset \mathbb{P}^n$ be given by $\mathbb{V}(f)$ where f is a homogeneous quadratic polynomial. After a $\mathrm{PGL}(n+1, \mathbb{C})$ change of coordinates, Q will have the form (isomorphic to) $\mathbb{V}(X_0^2 + \cdots + X_r^2)$ where r is the rank of quadratic-form f.

Proof. See IB Linear Algebra. □

Now we can also have the Nullstellensatz for the projective space.

Theorem 2.2.7 (Projective Nullstellensatz Theorem)**.**

(i) If $\mathbb{V}(I) = \emptyset \subset \mathbb{P}^n$, then $I \supset (X_0^{m_1}, \cdots, X_n^{m_n})$ for some $m_i \in \mathbb{N}$.

(ii) If $V = \mathbb{V}(I) \neq \emptyset$, then $I^h(V) = \sqrt{I}$, where $I^h(V)$ is the ideal generated by all homogeneous polynomial vanishing on V.

Proof. We try to prove it by the reduction to the affine Nullstellensatz.

...under construction... □

Definition (Subvariety)**.** Let V be a projective variety in $\mathbb{P}^n$. Then if $W \subset V$ closed in V, we say W is a *closed subvariety* of V. Then complement $V \setminus W$ is called an *open subvariety.*

Remark: The closed (open) subvarieties of V is defined to satisfy the axioms of closed (open) subsets of a topology.

Definition (Irreducible)**.** We say V is *irreducible* if $V \neq V_1 \cup V_2$ for proper closed $V_1, V_2 \in V$.

Now we can have a similar proposition to affine spaces. All the intuitions in affine spaces work here.

Proposition 2.2.8.

(i) Every projective variety is a finite union of irreducibles.

(ii) $V \subset \mathbb{P}^n$ is irreducible iff $I^h(V)$ is prime.

Proof. (i) Identical to the affine case, since it is basically a topological statement by using the fact in Notheorian rings.

(ii) ...under construction... □

Recall the definition of dense (see IB AT).

Definition (Dense)**.** Let X be a topological space. Let $S \subset X$ be a subset of X. We say S is *dense* in X if $\overline{S} = X$.

We often say that the open sets in $\mathbb{A}^n$ and $\mathbb{P}^n$ are very large. Why do we say that? Let $S \subset V$ be any subset. S is Zariski dense iff every homogeneous polynomial vanishing on S also vanishes on V.

...under construction...

2.3 Functions on Projective Varieties

We have already seen that the norma polynomials in $\mathbb{C}[\mathbf{X}]$ has no well-defined zero sets, so we define the homogeneous polynomials. However, even homogeneous polynomials which have well-defined zero sets, are not able to have a well-defined value on a point of $\mathbb{P}^n$.

Luckily, there is a main **observation**: the ratio of homogeneous polynomials of the *same degree* has a well-defined value away from the vanishing of the denominator.

Definition (Function Field)**.** Let $V \subset \mathbb{P}^n$ be irreducible projective variety. The *function field* (or *field of valid functions*) are defined as,

$$\mathbb{C}(V) = \left\{ \frac{F}{G} : F, G \in \mathbb{C}[\mathbf{X}] \text{ homogeneous with same degree, } G \notin I^h(V) \right\} / \sim,$$

where $\dfrac{F_1}{G_1} \sim \dfrac{F_2}{G_2}$ iff $F_1G_2 - F_2G_1 \in I^h(V)$.

Remark: This looks weird (difficult to identify two functions) sice we can't define coordinate rings on $\mathbb{P}^n$, so need to define $\mathbb{C}(V)$ without $\mathbb{C}[V]$. Otherwise, it is basically the same case in $\mathbb{A}^n$.

We now have not yet set up all the things. We need to check $\sim$ is indeed a equivalence relation and $\mathbb{C}(V)$ is indeed a field.

Lemma 2.3.1. The relation $\sim$ is an equivalence relation.

Proof. The reflexivive and symmetric is clear.

For the transitive, suppose $\dfrac{F_1}{G_1} \sim \dfrac{F_2}{G_2}$ and $\dfrac{F_2}{G_2} \sim \dfrac{F_3}{G_3}$. Then, we have,

$$F_1G_2 - F_2G_1 \text{ and } F_2G_3 - F_3G_2 \in I^h(V).$$

Consider $F_1G_3 - F_3G_1$. Multiply by G_2 get $F_1G_2G_3 - F_3G_1G_2$. Since $G_2 \notin I^h(V)$ and the primality of $I^h(V)$, it is sufficient to show $F_1G_2G_3 - F_3G_1G_2 = 0$ in $\mathbb{C}[\mathbf{X}]/I^h(V)$.

In this ring $\mathbb{C}[\mathbf{X}]/I^h(V)$ we have relations $F_1G_2 = F_2G_1$ and $F_2G_3 = F_3G_2$. Plugging in, we get $F_1G_2G_3 - F_3G_1G_2 = 0$. □

Remark:

(i) Note here we need the primarity, i.e. irreducible of V. Just like we need irreducible in $\mathbb{A}^n$ to keep $\mathbb{C}[V]$ to be an integral domain, then we can define the field of fraction.

(ii) It is easy to show that $\mathbb{C}(V)$ is a field.

(iii) (Non-examinable) There is more structure lurking here than one initially sees. The definition above examines homogeneous rational functions of total degree 0. For any integer d, one could examine a set $\mathbb{C}(V, d)$ of homogeneous rational functions of degree d. An object of this form would not be a field though, as it doesn't make sense to multiply two such, since the closure of the set. However, it does make sense to multiply a rational function of degree d by a rational function of degree 0, i.e. an element of $\mathbb{C}(V)$. In other words, the set $\mathbb{C}(V, d)$ of degree d is a vector space (or module) over $\mathbb{C}(V)$!

Let's look more carefully at the field.

Definition (Finitely Generated Field and Field Extension)**.** Let F be a field. We say F is *finitely generated* if there exists $f_1, \cdots, f_r \in F$ s.t. the smallest subfield containing them $(f_1, \cdots, f_r) = F$.

Let K be another field. If F is a subfield of K, we say K is a *field extension* of K, denoted K/F. Moreover, K is a *finitely generated field extension* of F if there exists $k_1, \cdots, k_n \in K$ s.t. the smallest subfield containing both k_i and F, denoted by $F(k_1, \cdots, k_n) = K$.

Proposition 2.3.2. The field $\mathbb{C}(V)$ is a *finitely generated* field extension of $\mathbb{C}$.

Proof. Assuming V nonempty, then there is some coordinate function X_i that is non-zero on V. Reoder to let $X_i = X_1$.

We **claim** that $\dfrac{X_1}{X_0}, \cdots \dfrac{X_n}{X_0}$ (elements in $\mathbb{C}(V)$) generate the filed extension $\mathbb{C}(V)$.

Explicitly, this means if $\dfrac{F}{G}$ is a degree 0 ratio, it can be written interms of $\dfrac{X_j}{X_0}$ and $\mathbb{C}$ and field expansion.

Now we do reduction, in order to get $\dfrac{F}{G}$ from what we have,

- Sufficient to get the form $\dfrac{\text{monomial}}{G}$.
- Sufficient to get the form $\dfrac{G}{\text{monomial}}$.
- Sufficient to get the form $\dfrac{\text{monomial}}{\text{monomial}}$, which is clear to get by algebra.

Hence, we are done. $\square$

Remark: The proposition means the functions $\dfrac{X_i}{X_0}$ play the role of coordinate functions X_i play in $\mathbb{C}[V]$ of $\mathbb{A}^n$.

Corollary 2.3.3. Let $V \subset \mathbb{P}^n$ be irrducible projective variety not contained in $\{x_0 = 0\}$. Let $V_0 = V \cap U_0$ where $U_0 \cong \mathbb{A}^n$ is the first affine patch. Then, we have,

$$\mathbb{C}(V_0) = \mathrm{FF}(\mathbb{C}[V_0]) = \mathbb{C}(V).$$

Proof. V_0 has corrdinate ring,

$$\mathbb{C}[V_0] = \mathbb{C}\left[\frac{X_1}{X_0}, \cdots \frac{X_n}{X_0}\right] / I(V_0).$$

So $\mathbb{C}(V_0) = \mathrm{FF}(\mathbb{C}[V_0])$ is the field generated by $\dfrac{X_1}{X_0}, \cdots, \dfrac{X_n}{X_0}$, which means $\mathbb{C}(V)$ and $\mathbb{C}[V]$ are made up of the same element.

Then, we need to check the equivalence relation.

...under construction... $\square$

Remark: This provides us with a good method to look for $\mathbb{C}(V)$.

> *Understanding.* We can use affine patches decomposition of a variety to convert a projective variety into several affine varieties which we are more familiar with. That's the reason why we said projective space is locally a affine space.

We now are going to redefine the things we have on affine space.

Definition (Regular and Rational Functions)**.** Let $\varphi \in \mathbb{C}(V)$ and $p \in V$. Then we say φ is *regular* (or *well-defined*) at p if φ can be expressed (up to the equivalence relation) as $\dfrac{F}{G}$, homogeneous of same degree s.t. $G(p) \neq 0$.

In this case, φ is a partially defined function from V to $\mathbb{C}$.

$$\varphi : V \setminus \{p : \varphi \text{ is not regular at } p\} \longrightarrow \mathbb{C}.$$

We say (φ, U) is a *rational function*, where U is the domain of φ.

Definition (Local Rings)**.** The *local ring* of V at p is a subring of $\mathbb{C}(V)$, which is defined as,

$$\mathcal{O}_{V,p} = \{\varphi \in \mathbb{C}(V) : \varphi \text{ is regular at } p\} \subset \mathbb{C}(V).$$

Again, the local ring is a *local ring* in the sense of ring theory. We had better the following picture in our mind.

Proposition 2.3.4. Let $V \subset \mathbb{P}^n$ is irreducible projective variety and not contained in $\{x_0 = 0\}$. Let $V_0 = V \cap U_0$. Fix $p \in V_0$, then there is a natural isomorphism,

$$\mathcal{O}_{V,p} \cong \mathcal{O}_{V_0,p},$$

respecting the isomorphism $\mathbb{C}(V) \cong \mathbb{C}(V_0)$.

Proof. ...under construction... □

2.4 Rational Maps on Projective Varieties

In this subsection, we are going to consider the *rational maps* between two varieties. First we introduce a **notation**: we will define a notion of *rational map*, e.g. $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$.

Let $F_0, \cdots, F_m \in \mathbb{C}[X_0, \cdots, X_n]$ homogeneous of the same degree d. Define,

$$\mathbf{F} = (F_0, \cdots, F_m) : \mathbb{C}^{m+1} \longrightarrow \mathbb{C}^{n+1}.$$

Proposition 2.4.1. The map $\mathbf{F}$ descends to a well-defined map,

$$\varphi : \mathbb{P}^n \setminus \left(\bigcap_j \mathbb{V}(F_j) \right) \longrightarrow \mathbb{P}^n.$$

Moreover, if p is *represented* $\mathbf{a} = (a_0, \cdots, a_n)$, then $\varphi(p)$ is represented by,

$$(F_0(\mathbf{a}), \cdots, F_m(\mathbf{a})).$$

Proof. ...under construction... □

Things are interesting, since when we define the function field on $\mathbb{P}^n$, there is some ambiguity, so we use ratios. On the other hand, for now, we cancel out the ambiguity by just changing our codomain to $\mathbb{P}^n$. But things are related.

Notation: denote such maps $\mathbf{F} = (F_1, \cdots, F_m)$ by $\mathbf{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ for convenient.

Observe that let G be non-zero, homogeneous in $X_0, \cdots, X_n$. Given $\mathbf{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. We can also consider,

$$G\mathbf{F} = (GF_1, \cdots, GF_m) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n.$$

These maps $\mathbf{F}$ and $G\mathbf{F}$ have different domains since there is *ambiguity* under "$\dashrightarrow$" (partially defined). The domain potentially gets slightly smaller since we multiply $\mathbf{F}$ by G. But on the common domain of definition, i.e. $\bigcap_j \mathbb{V}(GF_j)$ the maps are set theoretically the same.

We want to consider these two maps $G\mathbf{F}$ and $\mathbf{F}$ are the same.

Definition (Rational Maps)**.** Let $V \subset \mathbb{P}^n$ be irreducible projective variety. Let $F_0, \cdots, F_m$ be homogeneous and same degree d elements in $\mathbb{C}[X_1, \cdots X_n]$ not all contained in $I^h(V)$. They determine a set theoretic map,

$$V \setminus \bigcap_j \mathbb{V}(F_j) \longrightarrow \mathbb{P}^m,$$

by previous construction. Two such pairs $(F_0, \cdots, F_m)$ and $(G_0, \cdots, G_m)$ are said to *determine the same map* if $F_iG_j - F_jG_i \in I^h(V)$ for all i, j.

A *rational map* from V to $\mathbb{P}^m$ is an equivalence class of tuples $(F_0, \cdots, F_m)$ as above. Two tuples are *equivalent* iff they determine the same map.

Understanding. The definition of rational functions enables us to use the functions in the same equivalence class to cover as many points as we can. Also, these functions at least agree on the points they share.

Definition (Regular Points and Domains)**.** A point $p \in V$ is called a *regular point* of a rational map $\varphi : V \dashrightarrow \mathbb{P}^m$ if there exists a representative $(F_0, \cdots, F_m)$ of φ s.t. $F_j(p) \neq 0$ for some j.

The *domain* of φ is the set of regular points, denoted $\mathrm{dom}(\varphi)$.

Definition (Morphisms)**.** A rational map φ is called a *morphism* if teh domain of φ is V, denoted as $\varphi : V \longrightarrow \mathbb{P}^n$.

Because of this a bit messy definition of rational functions, we need to change lots of times the representatives of a rational map, to study every point in our potential domain. Let see some examples.

Example (Linear maps)**.** A *linear map* $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is given by an $(m+1) \times (n+1)$ matrix (a_{ij}). Concretely, $\varphi = (F_0, \cdots, F_m)$ where $F_j = \sum_i a_{ij} X_i$.

The bad locus is where $F_0, \cdots, F_m$ all vanishes, that is the kernel of the matrix. If the matrix has rank $n + 1 \leqslant m + 1$, then φ is a morphism. In particular, there is no linear morphism from higher dimension to lower dimension projective spaces.

Example (Projection from a Point)**.** Let $P = (0 : 0 : 1)$ in $\mathbb{P}^2$. The *projection from* p,

$$\begin{aligned} \pi : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ (a_0 : a_1 : a_2) &\longmapsto (a_0 : a_1). \end{aligned}$$

π is not regular at P. Let $C = \mathbb{V}(f_d)$ where f_d is a degree d homogeneous polynoial. Assume that $P \notin C$.

Note we can get a morphism by restriction $\varpi : C \longrightarrow \mathbb{P}^1$.

...under construction...

Example. Let $C = \mathbb{V}(X_0X_2 - X_1^2) \subset \mathbb{P}^2$. Consider the proejction from $(0 : 0 : 1)$ and restrict to C to get $\pi : C \dashrightarrow \mathbb{P}^1$. The map π is determined by (X_0, X_1).

Observe that $(0 : 0 : 1) \in C$, so we have a *potentially* non-regular point on C for π. But we must look for other (F_0, F_1) that determine the same rational map as (X_0, X_1). I.e. $F_1X_0 - F_0X_1 \in I^h(C) = (X_0X_2 - X_1^2)$. Hence, we can try $(F_0, F_1) = (X_1, X_2)$. We get,

$$F_1X_0 - F_0X_1 = X_2X_0 - X_1X_1 \in (X_0X_2 - X_1^2).$$

Hence in fact, π is regular at $(0 : 0 : 1)$ so π is a morphism on C.

Further **observation**: if we consider $\pi : C \longrightarrow \mathbb{P}^1$, for every $p \in \mathbb{P}^1$, $\pi^{-1}(q)$ is a single point. With a little more work, we can show π is also surjective. Hence, we might guess π is an "isomorphism". See the Propositin XXX.

More terminologies.

Definition (Rational Maps to Varieties)**.** Let $W \subset \mathbb{P}^n$ be a projective variety, then a *rational map* (or *morphism*) $V \dashrightarrow W$ is a rational map (morphism) $V \dashrightarrow \mathbb{P}^m$ such that the image of domain is contained in W.

Definition (Isomorphism)**.** A morphism $\varphi : V \longrightarrow W$ is an *isomorphism* if it has a two sided inverse also a morphism.

Lemma 2.4.2. Suppose $f \in \mathbb{C}[X_0, \cdots, X_n]$ is an irreducible quadratic form. Then, after coordinate changing, it is still irreducible.

Proof. ...under construction... □

Proposition 2.4.3. Let C be the vanishing locus of $f \in \mathbb{C}[X_0, X_1, X_2]$ homogeneous of degree 2 on $\mathbb{P}^2$. Then, if f is irreducible, then $C \cong \mathbb{P}^1$.

Proof. Notice by changing the coordinate, we can write all quadratic form in the form of X_0^2, $X_0^2 + X_1^2$ or $X_0^2 + X_1^2 + X_2^2$. Note the first two are not irreducible, by our previous lemma, f is conjugate to $X_0^2 + X_1^2 + X_2^2$ which is conjugate to any rank 3 polynomial, in particular, $X_0X_2 - X_1^2$. (see IB LA).

By previous example, we have a morphism $\pi : C \longrightarrow \mathbb{P}^1$ by projection from $(0 : 0 : 1)$. Then, notice we almost have $x_2 = x_1^2/x_0$. Hence, we define the inverse $\mu : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ by $(Y_0^2 : Y_0Y_1 : Y_1^2)$ which has image in C. It is easy to check this works as an inverse at $C \setminus (0:0:1)$ and also works for $(0:0:1)$ where π determined by $(X_1 : X_2)$. Also, μ is regular on $\mathbb{P}^1$ so a morphism.

Hence, π is a isomorphism and $C \cong \mathbb{P}^1$. □

Example (Cremona Transformation)**.** ...under construction...

Example (Veronese Embeddings)**.** Let $F_0, \cdots, F_m$ be the list of degree d monomials in variables $X_0, \cdots, X_n$, where $m = \binom{n+d}{d} - 1$. We get a morphism,

$$\begin{aligned} \nu_d : \mathbb{P}^n &\longrightarrow \mathbb{P}^m \\ (\mathbf{a}) &\longmapsto (F_0(\mathbf{a}), \cdots, F_m(\mathbf{a})). \end{aligned}$$

It is a morphism, since if $\nu_d(\mathbf{a}) = \mathbf{0}$, then $\mathbf{a} = \mathbf{0}$ as $X_i^d \in \{F_j\}$ for all i.

In fact, ν_d is injective and image of ν_d is a projective variety isomorphic to $\mathbb{P}^n$. (This is a straight forward but tedious result.) ...under construction...

Before the next example, we should **notice** that $\mathbb{P}^n \times \mathbb{P}^m \neq \mathbb{P}^{n+m}$ even in the sense of two sets (not same elements). For affine spaces, although $\mathbb{A}^n \times \mathbb{A}^m \neq \mathbb{A}^{n+m}$ in topological sense, they are the same the sense of sets.

Example (Segre Embeddings)**.** The *Segre embedding* is the map,

$$\begin{aligned} \sigma_{mn} : \mathbb{P}^m \times \mathbb{P}^n &\longrightarrow \mathbb{P}^{mn+m+n} \\ ((x_i), (y_j)) &\longmapsto (x_i y_j). \end{aligned}$$

We will label the coordinates of $\mathbb{P}^{mn+m+n}$ as z_{ij} with $o \leqslant i \leqslant m$ and $o \leqslant j \leqslant n$. To understand this, we can think without equivalence relation, it is a tensor product $\mathbb{C}^m \times \mathbb{C}^n \longrightarrow \mathbb{C}^m \otimes \mathbb{C}^n$.

But we still don't know the structure of $\mathbb{P}^m \times \mathbb{P}^n$ until we have the following theorem.

Theorem 2.4.4. The map σ_{mn} we defined above is a bijection between $\mathbb{P}^m \times \mathbb{P}^n$ and the projective variety $\mathbb{V}(I) \subset \mathbb{P}^{mn+m+n}$ where I is generated by $Z_{ij}Z_{pq} - Z_{iq}Z_{pj}$ with $i, p \in \{0, \cdots, m\}$ and $j, q \in \{0, \cdots, n\}$, also $i \neq p$ and $j \neq q$.

Proof. ...under construction...

□

Remark: This theorem told us we can think of $\mathbb{P}^m \times \mathbb{P}^n$ as the projective variety $\mathbb{V}(I) \subset \mathbb{P}^{mn+m+n}$ which is irreducible. ...under construction...

Understanding. ...under construction...

It follows from this that if V and W are projective varieties that $V \times W$ also "naturally" a projective variety, which we will see later.

Suppose we have $\varphi : V \dashrightarrow W$ and $\psi : W \dashrightarrow Z$ are rational maps between irreducible varieties. The composition of $\psi \circ \varphi$ need not be defined, because the image of the domain of φ might be entirely contained in the locus of indeterminacy of ψ.

Definition. A rational map $\varphi : V \dashrightarrow W$ is *dominant* if $\varphi(\mathrm{dom}(\varphi))$ is dense in W (in Zariski topology).

We have a lemma here.

Lemma 2.4.5. Let X be a topological space. Let S and T be two *open dense* subset of X. We have $S \cap T$ also an open dense subset of X.

Proof. (non-examinable) See IB Analysis and Topology. □

Then, we can have the folloing prposition.

Proposition 2.4.6. Let $\varphi : V \dashrightarrow W$ be a dominant rational map. Let $\psi : W \dashrightarrow Z$ be a rational map. Then, $\psi \circ \varphi$ is a well-defined (partially defined) on a open dense subset of V.

Proof. Note the the map $\psi \circ \varphi$ can only be defined on $\mathrm{dom}(\varphi) \cap \varphi^{-1}(\mathrm{dom}\, \psi)$, which might be empty in general.

However, suppose φ is dominant. We have $\bigcap_j F_j$ is closed on V ($\varphi = (F_j)$). Then, because V is irreducible, $\mathrm{dom}(\varphi) = V \setminus \bigcap_j F_j$ is open. Similarly, $\varphi^{-1}(\mathrm{dom}\, \psi)$ is also open, since φ is continuous (see XXX). Hence, $\mathrm{dom}(\varphi) \cap \varphi^{-1}(\mathrm{dom}\, \psi)$ is also open on V.

In order to show it is non-empty, consider,

$$\varphi(\mathrm{dom}(\varphi)) \cap \varphi^{-1}(\mathrm{dom}\, \psi) = \varphi(\mathrm{dom}\, \varphi) \cap \mathrm{dom}\, \psi,$$

which is open and dense, so non-empty by our previous lemma and φ is dominant. Hence, $\psi \circ \varphi$ is defined on an open dense subset. And, the fact it is a rational map follows that composition of polynomials are still polynomials □

Understanding. Note let $f : U \longrightarrow V$ and $S \subset V$. Then, $f(f^{-1}(S)) = f(U) \cap S$ in general.

(non-examinable) Algebraically, we can think ...under construction... (prove open dense with the fact $\psi \circ \varphi$ is a rational map and something in other books.)

Hence, no matter geometrically or algebraically, we can see **once** $\psi \circ \varphi$ is well-defined, then it is defined on an open dense subset.

For morphism, if we have a inverse which is also a morphism, it is an isomorphism. For more general rational maps, we have the following definition replaces isomorphism.

Definition (Birational Maps)**.** Let $\varphi : V \dashrightarrow W$ and $\psi : W \dashrightarrow V$ be two rational maps. If $\varphi \circ \psi$ and $\psi \circ \varphi$ are both equivalent to the identity map (respectively on W and on V), then we say that V and W are *birational* and φ and ψ are *birational maps.*

Understanding. We can roughly think that a birational map is an isomorphism between two open dense subsets.

Proposition 2.4.7. Let $\varphi : V \dashrightarrow W$ be a birational map. Then φ is dominant. Moreover, φ defines an isomorphism between two open dense subsets of V and W.

Proof. (non-examinable) ...under construction... □

Example. Any isomorphism is a birational map.

Example (Cremona Transformation)**.** Take the map,

$$\begin{aligned} \kappa : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^2 \\ (x_0 : x_1 : x_2) &\longmapsto (x_1x_2 : x_0x_2 : x_0x_1). \end{aligned}$$

which is mainly $(x_0 : x_1 : x_2) \longmapsto (\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2})$. Then κ is self-inverse as a rational map (check on where it is defined), so birational but not isomorphism since it even is not a morphism.

(non-examinable) can build an interesting group $\mathrm{Bir}(\mathbb{P}^2)$, the birational automorphism group. ...under construction...

Remember for affine varieties we have V and W are isomorphic iff there coordinate rings $\mathbb{C}[V]$ and $\mathbb{C}[W]$ are isomorphic as $\mathbb{C}$-algebra. Now, we have a similar theorem for projective varieties.

Theorem 2.4.8. Let V and W be irreducible. Then V is birational to W iff the field $\mathbb{C}(V)$ and $\mathbb{C}(W)$ isomorphic as fields.

Proof. ...under construction... $\square$

3 Tangent Spaces and Nonsingularity

...under construction...

3.1 Tangent Spaces

We first consider the geometric setting of tangent space. Let $V \subset \mathbb{A}^n$ be an affine hypersurface, i.e. $V = \mathbb{V}(f)$ for some $f \in \mathbb{C}[\mathbf{X}]$. Assume f is irreducible so V is also irreducible.

Pick a point $p = (a_1, \cdots, a_n) \in V$. In order to figure out the condition of a tangent, we first consider an affine line through p. It has the form of,

$$L = \{(a_1 + b_1 t, \cdots, a_n + b_n t) : t \in \mathbb{C}\},$$

with $(\mathbf{b}) \in \mathbb{C} \setminus \{\mathbf{0}\}$ is fixed.

Now consider the intersection $V \cap L$. It is the set of points on L where f vanishes. That is, we have,

$$0 = f(a_1 + b_1 t, \cdots, a_n + b_n t) = g(t) = \sum_r c_r t^r.$$

Since $p \in V \cap L$, we know that $c_0 = g(0) = f(\mathbf{a}) = 0$. We can also compute the linear term,

$$c_1 = \frac{\mathrm{d}f}{\mathrm{d}t}(0) = \sum_{i=1}^n b_i \frac{\partial f}{\partial X_i}(\mathbf{a}),$$

where the (partial) differentiation here is the formal differentiation, i.e. a map on $\mathbb{C}[\mathbf{X}]$ (which can be proved also respecting the chain rule).

Since $g(t)$ measure the value of f at along the line through p, a line at p tangent to V should have no linear term (a higher order zero at 0), i.e. $c_1 = 0$. It means $(\mathbf{b}) \cdot (\partial_i f(p)) = 0$, so $(\partial_i f(p))$ is the tangent vector. Recall the equation of plane in Euclidean space ($\mathbf{n} \cdot \mathbf{x} = 0$), we have the following definition.

Definition (Tangent Space)**.** Let $V = \mathbb{V}(f)$ be an irreducible affine hypersurface in $\mathbb{A}^n$. A line L through $p = (\mathbf{a}) \in V$ is *tangent* to V at p iff it contained in an affine plane,

$$\mathcal{T}_{V,p} = \mathbb{V}(g) = \mathbb{V}\left(\sum_{i=1}^n \left(\frac{\partial f}{\partial X_i}\right)(p)(X_i - a_i)\right).$$

The affine linear subspace $\mathcal{T}_{V,p}$ is called the *tangent space.*

Remark: We should get the same result if we apply the Euclidean topology on $\mathbb{A}^n$ and $\mathbb{V}(f)$. If we define $\mathbb{V}(f) = \{\mathbf{x} : f(\mathbf{x}) = 0\}$, then ∇f is the normal, and we get the same result.

...under construction...

3.2 Function Field Geometry

Recall we have defined the *field extension* in XXX. We can have the following definition in field theory.

There is an important fact that if K/F is a field extension, then the multiplication in K always makes K a *vector field* over F. Then, we can have the following definition.

Definition (Degree of Field Extensions)**.** The ***degree*** (***index***) of a field extension K/F, denoted $[K : F]$, is the dimension of K as a vector space over F (i.e. $[K : F] = \dim_F K$). The extension is said to be *finite* if $[K : F]$ is finite and is said to be *infinite* otherwise.

Definition (Alegbraic and Transcendental)**.** Let L/K be a field extension. We say $\alpha \in L$ is ***transcendental*** over K if it is *not* a solution to a non-trivial polynomial equation $f(t) = 0$ with $f \in K[t]$. If $\alpha \in L$ is not transcendental, then it is ***algebraic*** over K.

Note that if α is algebraic over a field K, then it is algebraic over all extensions of K.

Definition (Algebraically Independent)**.** Let L/K be a field extension and $S \subset L$ be a subset. We say S is ***algebraically independent*** over K if they do not satisfy a non-trivial polynomial relation over K. That is, for any finite elements $\alpha_1, \cdots, \alpha_n \in S$, there is no polynomial $f(t_1, \cdots, t_n) \in K[t_1, \cdots, t_n]$ s.t. $f(\alpha_1, \cdots, \alpha_n) = 0$. The elements of S are called ***independent transcendentals*** over K.

The reason we only check finite elements in S is because the definition of polynomial is finite.

Definition (Pure Transcendental Extension)**.** A field extension $K/\mathbb{C}$ is a ***pure transcendental extension*** if there exists $x_i \in K$ s.t. $K = \mathbb{C}(x_1, \cdots, x_n)$ (field generated by $\mathbb{C}$ and $x_1, \cdots, x_n$) and $\{x_i\}$ are algebraically independent over $\mathbb{C}$.

This definition can be extendet to any field extension L/K.