2D Geometric Transformations

(Chapter 5 in FVD)

2D geometric transformation

- Translation
- Scaling
- Rotation
- Shear
- Matrix notation
- Compositions
- Homogeneous coordinates

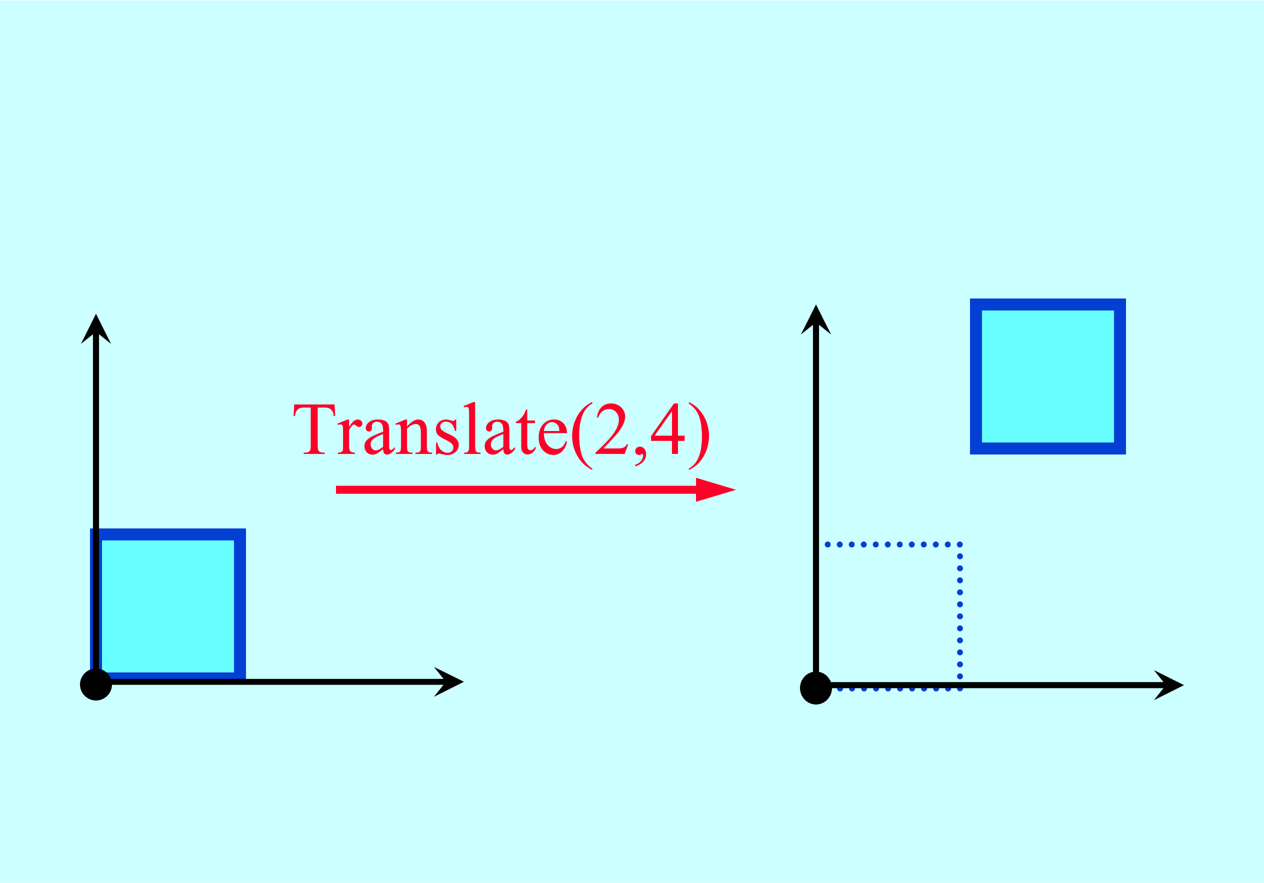
2D Geometric Transformations

- **Question**: How do we represent a geometric object in the plane?
- **Answer**: For now, assume that objects consist of points and lines. A point is represented by its Cartesian coordinates: (x,y).

- **Question**: How do we transform a geometric object in the plane?
- **Answer**: Let (A,B) be a straight line segment and T a general 2D transformation: T transforms (A,B) into another straight line segment (A',B'), where $A'=TA$ and $B'=TB$.

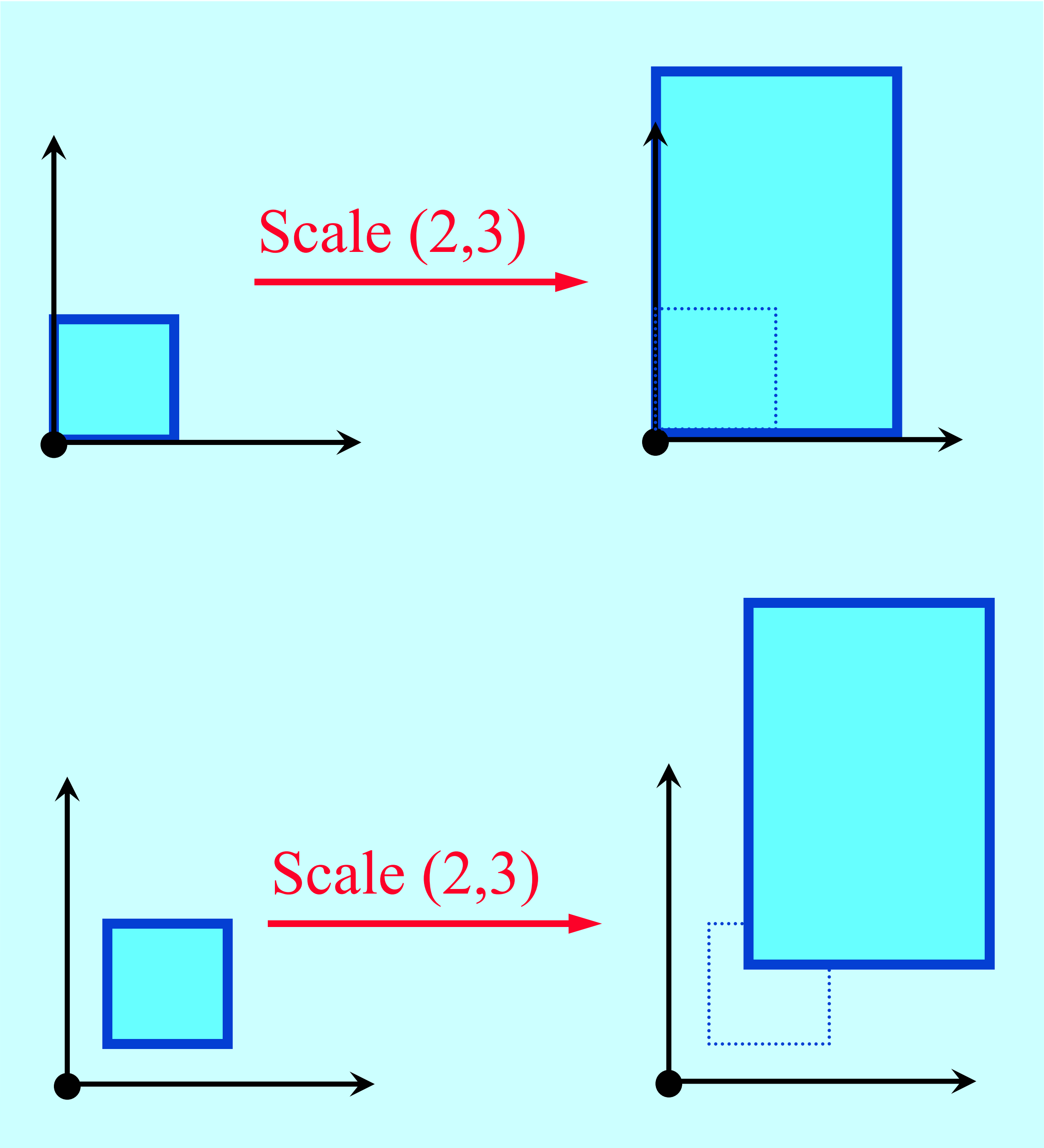
Translation

- Translate (a,b): $(x,y) \longrightarrow (x+a,y+b)$

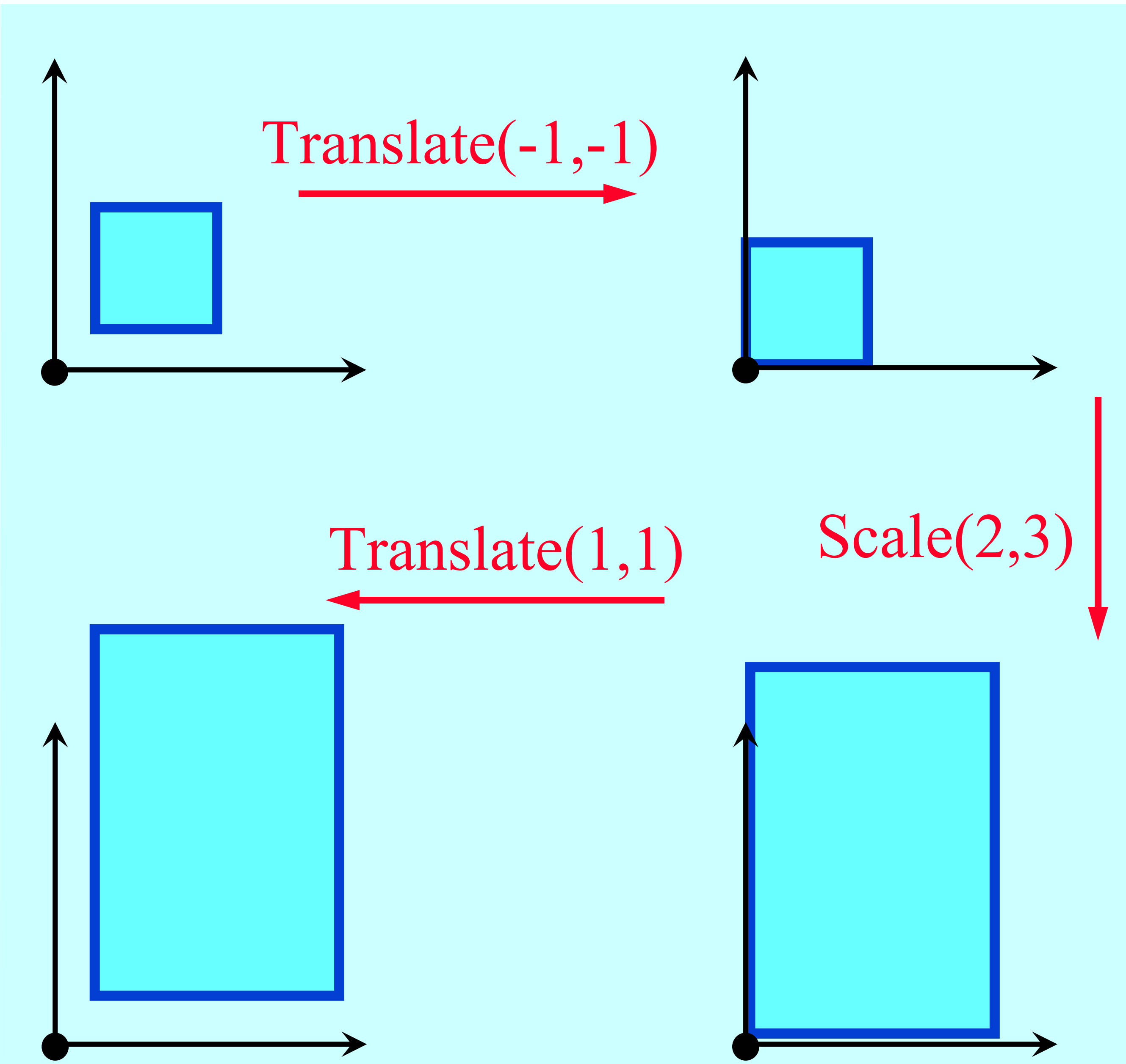


Scale

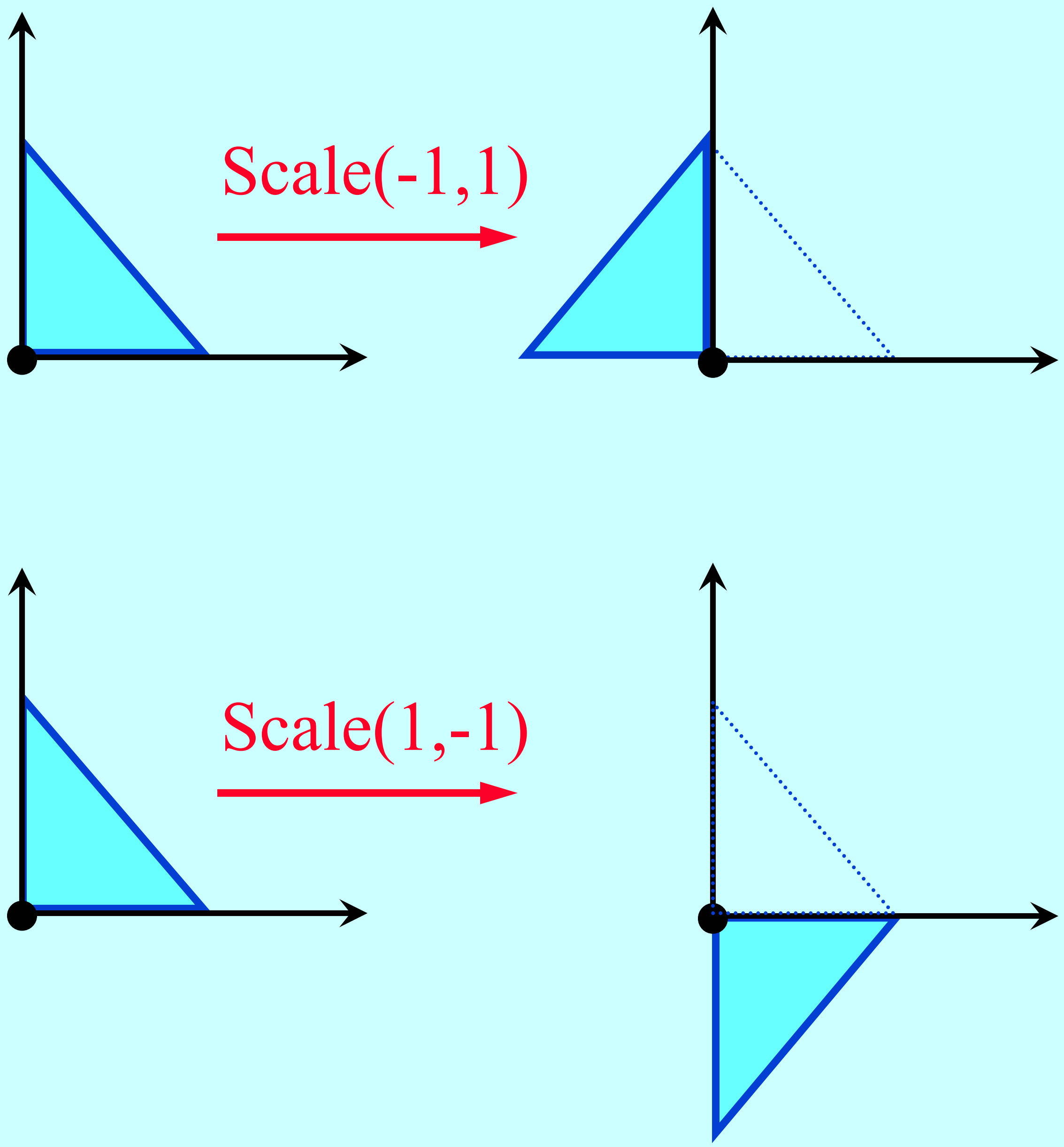
- Scale (a,b): $(x,y) \longrightarrow (ax,by)$



- How can we scale an object without moving its origin (lower left corner)?



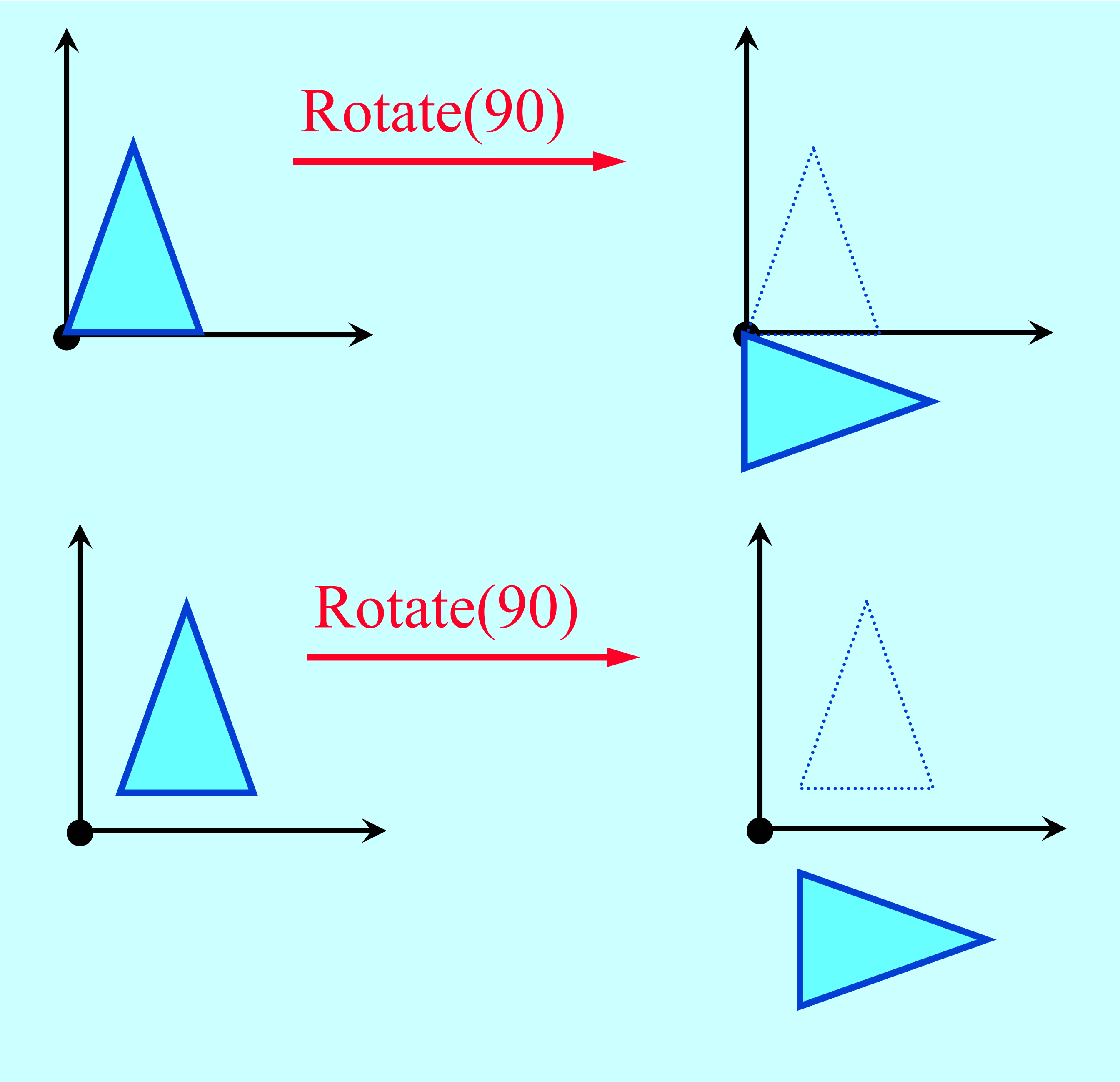
Reflection



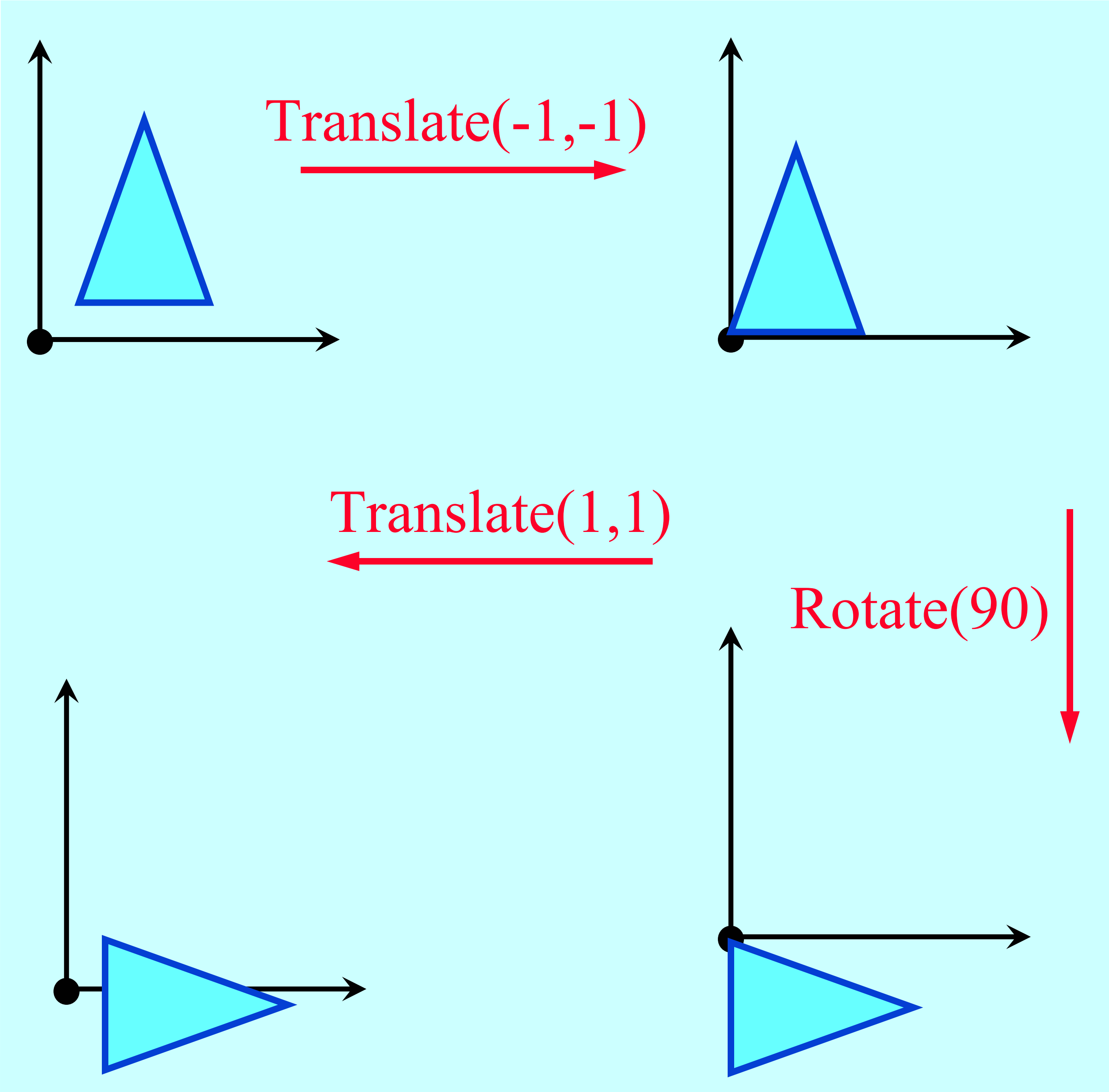
Rotation

- Rotate(θ):

(x,y) ⟶ (x cos(θ)+y sin(θ), -x sin(θ)+y cos(θ))

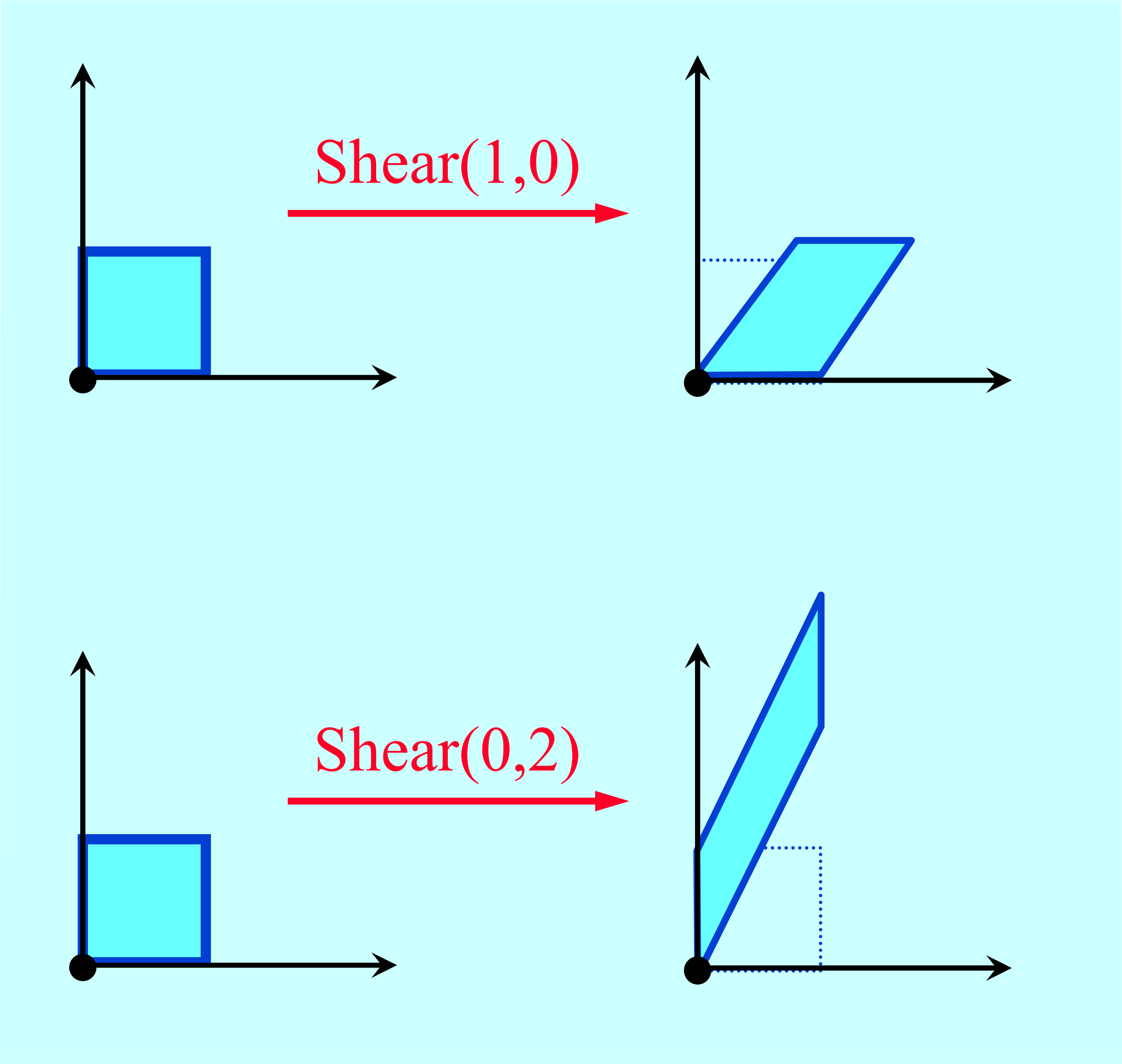


- How can we rotate an object without moving its origin (lower left corner)?



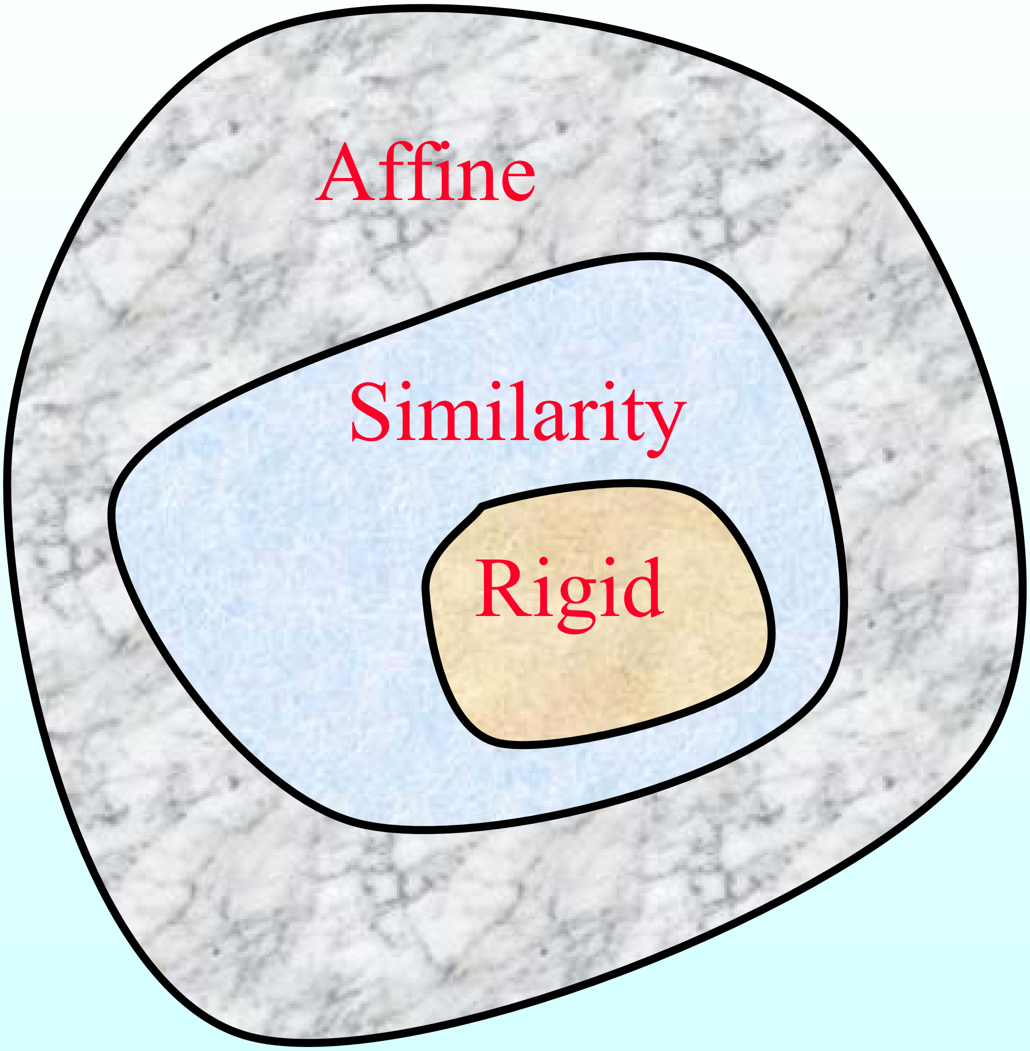
Shear

- Shear (a,b): $(x,y) \longrightarrow (x+ay, y+bx)$



Composition of Transformations

- Rigid transformation:
  - Translation + Rotation (distance preserving).
- Similarity transformation:
  - Translation + Rotation + uniform Scale (angle preserving).
- Affine transformation:
  - Translation + Rotation + Scale + Shear (parallelism preserving).
- All above transformations are groups where Rigid $\subset$ Similarity $\subset$ Affine.


Affine
Similarity
Rigid

Matrix Notation

- Let's treat a point (x,y) as a 2x1 matrix (a column vector):

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

- What happens when this vector is multiplied by a 2x2 matrix?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

2D Transformations

- 2D object is represented by points and lines that join them.
- Transformations can be applied only to the the points defining the lines.
- A point (x,y) is represented by a 2x1 column vector, and we can represent 2D transformations using 2x2 matrices:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scale

- Scale(a,b): $(x,y) \longrightarrow (ax,by)$

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

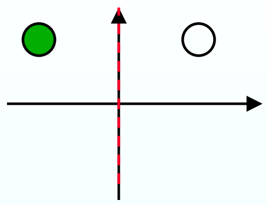
- If *a* or *b* are negative, we get reflection.

- Inverse: $S^{-1}(a,b)=S(1/a,1/b)$

Reflection

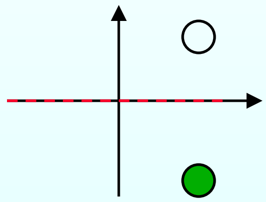
- Reflection through the y axis:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



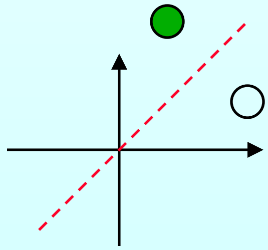
- Reflection through the x axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



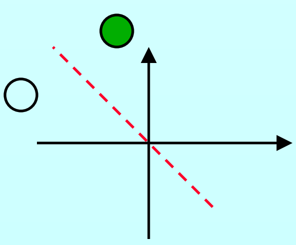
- Reflection through $y=x$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



- Reflection through $y=-x$:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



Shear, Rotation

- Shear(a,b): (x,y) $\longrightarrow$ (x+ay,y+bx)

$$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ay \\ y+bx \end{bmatrix}$$

- Rotate(θ):

(x,y) $\longrightarrow$ ($x$cosθ+$y$sinθ , -$x$sinθ + $y$cosθ)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}$$

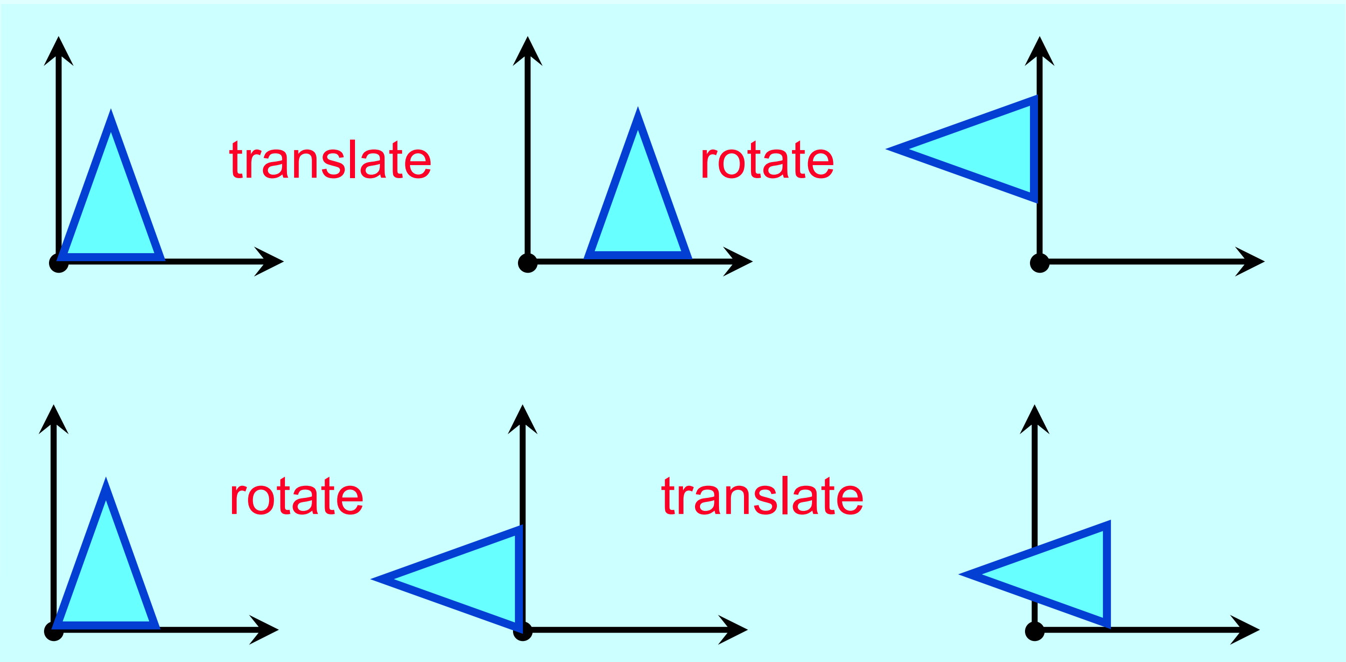
- Inverse: $R^{-1}(\theta)=R^{T}(\theta)=R(-\theta)$

Composition of Transformations

- A sequence of transformations can be collapsed into a single matrix:

$$[A][B][C]\begin{bmatrix} x \\ y \end{bmatrix} = [D]\begin{bmatrix} x \\ y \end{bmatrix}$$

- Note: order of transformations is important! (otherwise - commutative groups)



Composition of Transformations (Cont.)

$$D = A\ B\ C$$

$$D^{-1} = C^{-1}B^{-1}A^{-1}$$

Proof:

$$D * D\text{-}1 = ABC * C\text{-}1B\text{-}1A\text{-}1 =$$
$$AB*I*B\text{-}1*A\text{-}1=$$
$$A*I*A = I$$

Translation

- Translation(a,b): $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x+a \\ y+b \end{bmatrix}$

- **Problem**: Cannot represent translation using 2x2 matrices.

- **Solution**:

  Homogeneous Coordinates

Homogeneous Coordinates

- Homogeneous Coordinates is a mapping from R^n to R^{n+1}:

$$\boxed{(x, y) \rightarrow (X, Y, W) = (tx, ty, t)}$$

- Note: (tx, ty, t) all correspond to the same non-homogeneous point (x, y). E.g. $(2,3,1) \equiv (6,9,3)$.

- Inverse mapping:

$$(X, Y, W) \rightarrow \left(\frac{X}{W}, \frac{Y}{W}\right)$$

Translation

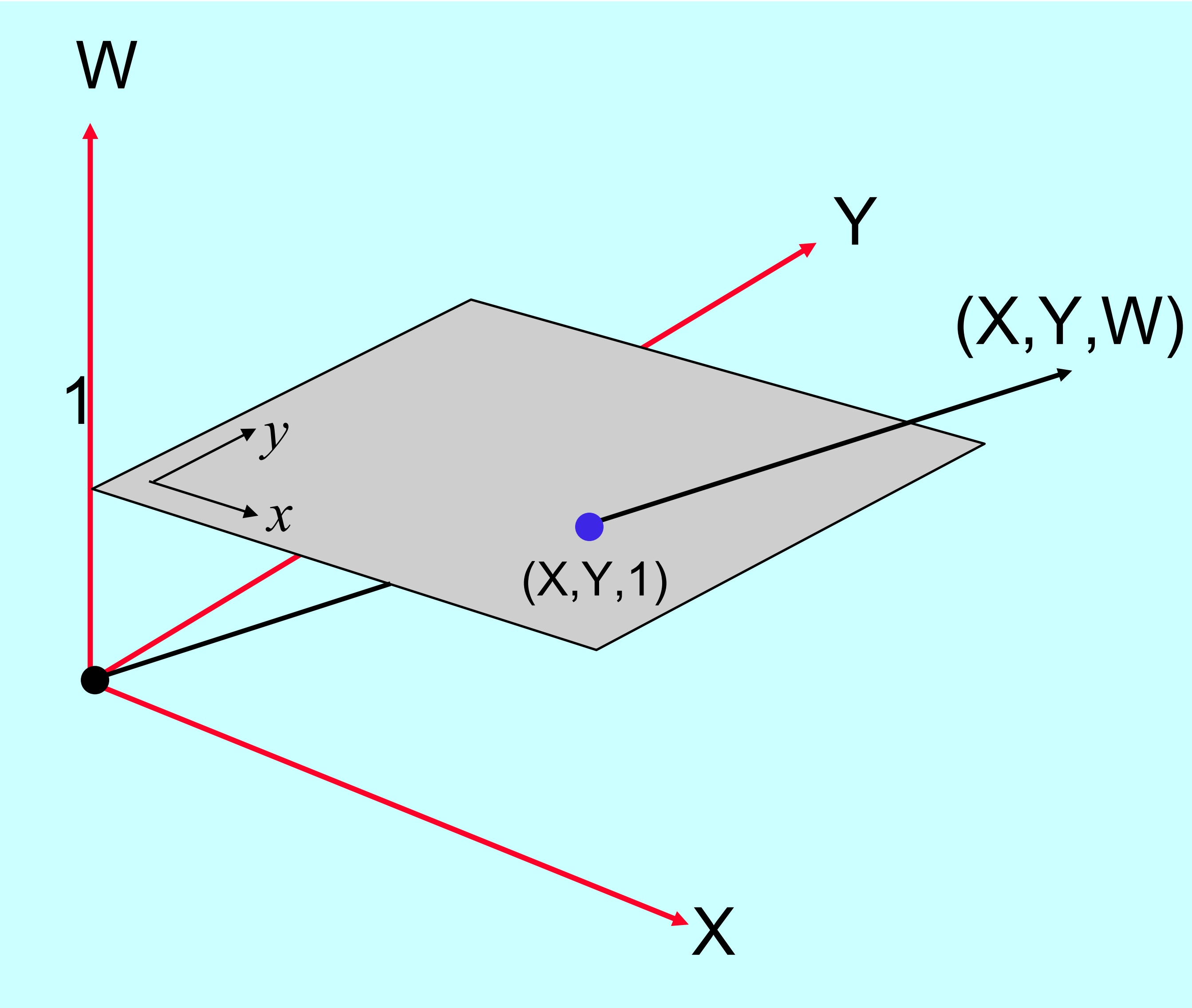
- Translate(a,b):

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

- Inverse: $T^{-1}(a,b)=T(-a,-b)$
- Affine transformation now have the following form:

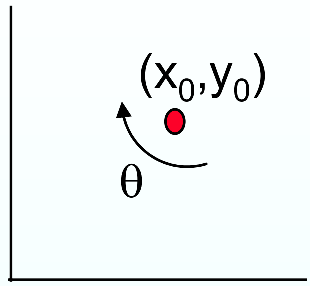
$$\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix}$$

Geometric Interpretation



- A 2D point is mapped to a line (ray) in 3D. The non-homogeneous points are obtained by projecting the rays onto the plane Z=1.

- Example: Rotation about an arbitrary point:

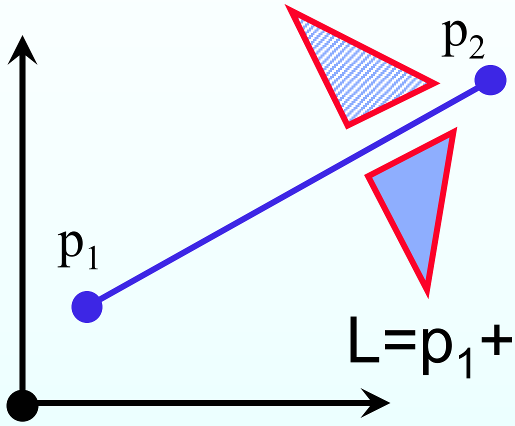


- Actions:
  - Translate the coordinates so that the origin is at (x_0,y_0).
  - Rotate by θ.
  - Translate back.

$$\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_0(1-\cos\theta)+y_0\sin\theta \\ \sin\theta & \cos\theta & y_0(1-\cos\theta)-x_0\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Another example: Reflection about an Arbitrary Line:

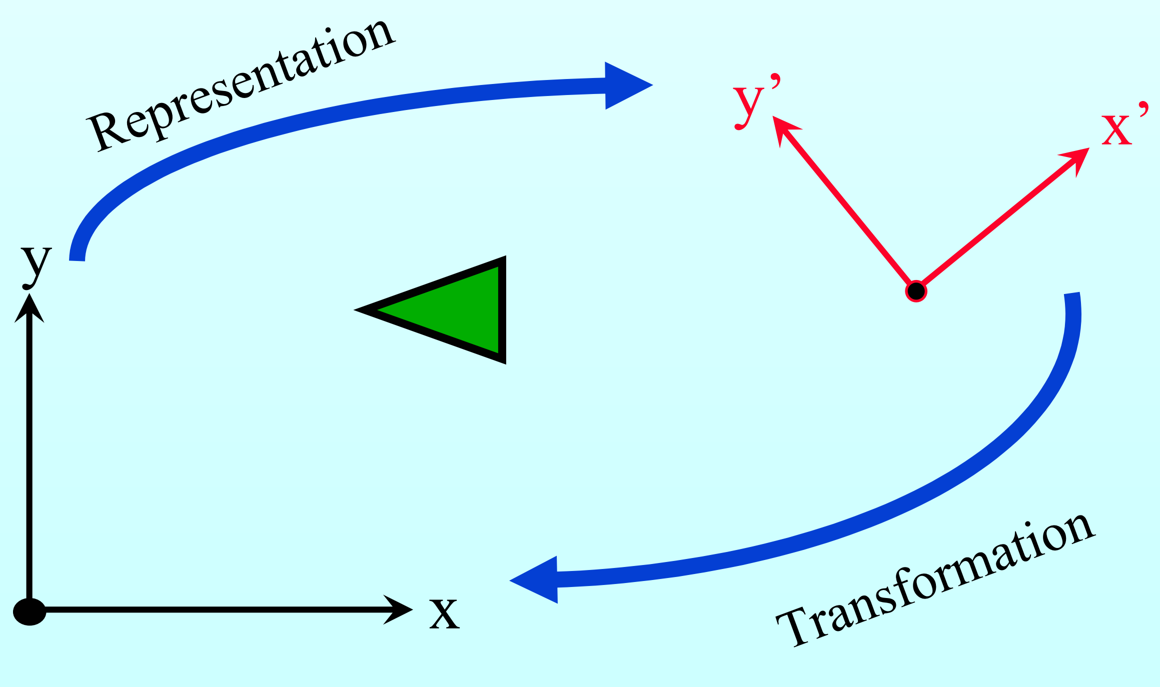


$L=p_1+t\ (p_2-p_1)=t\ p_2+(1-t)\ p_1$

- Actions:
  - Translate the coordinates so that P_1 is at the origin.
  - Rotate so that L aligns with the x-axis.
  - Reflect about the x-axis.
  - Rotate back.
  - Translate back.

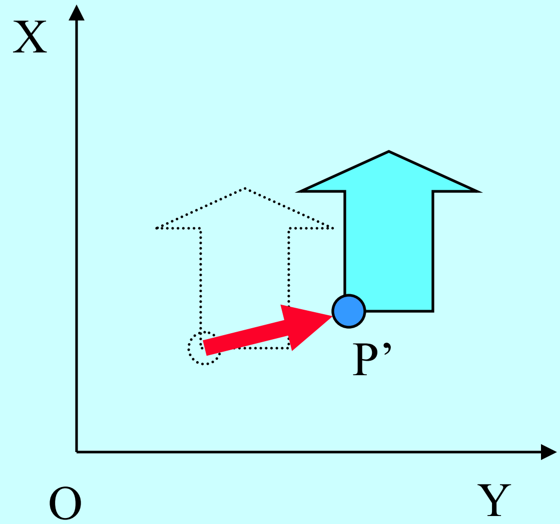
Change of Coordinates

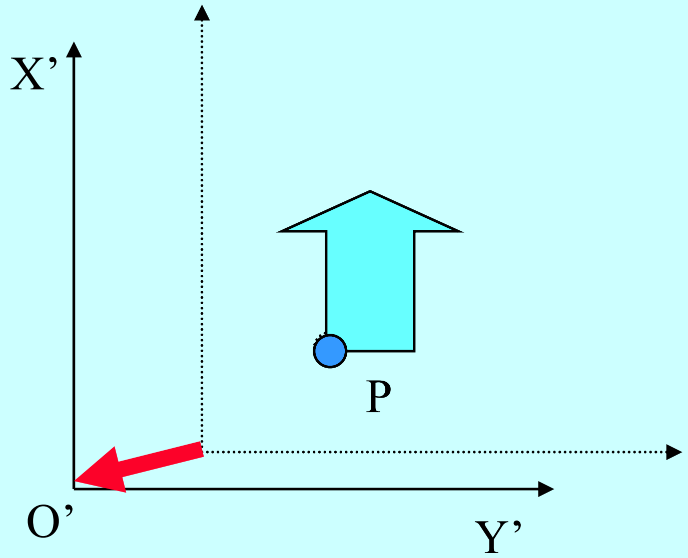
- It is often requires the transformation of object description from one coordinate system to another.
- How do we transform between two Cartesian coordinate systems?
- **Rule**: Transform one coordinate frames towards the other in the opposite direction of the representation change.



Change of Coordinates (Cont.)

Example:



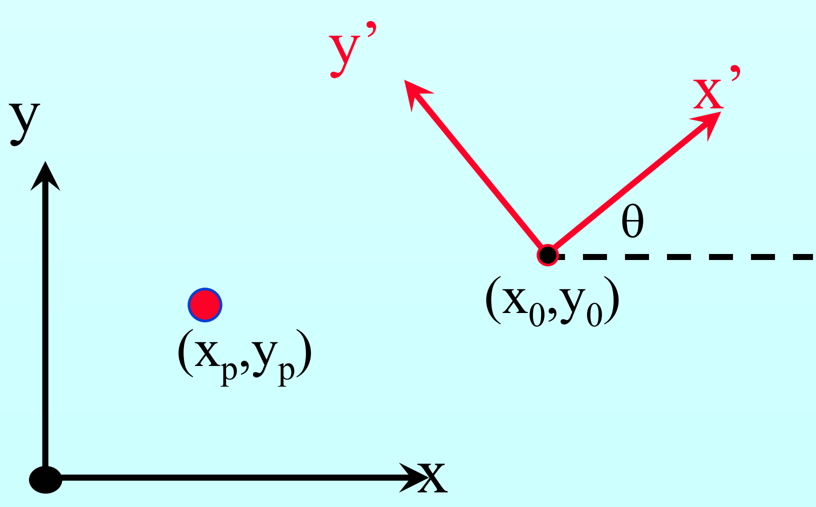


- Example:
  - Represent the point $P=(x_p,y_p,1)$ in the (x',y') coordinate system.

$$P' = MP$$

where

$$M = R^{-1}T^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

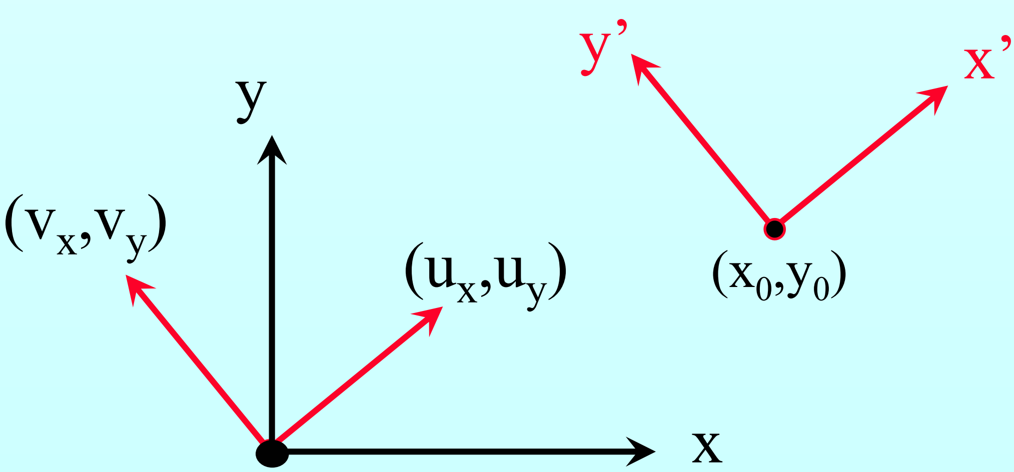


- **Alternative method**:
  - Assume x'=(u_x,u_y) and y'=(v_x,v_y) in the (x,y) coordinate system .

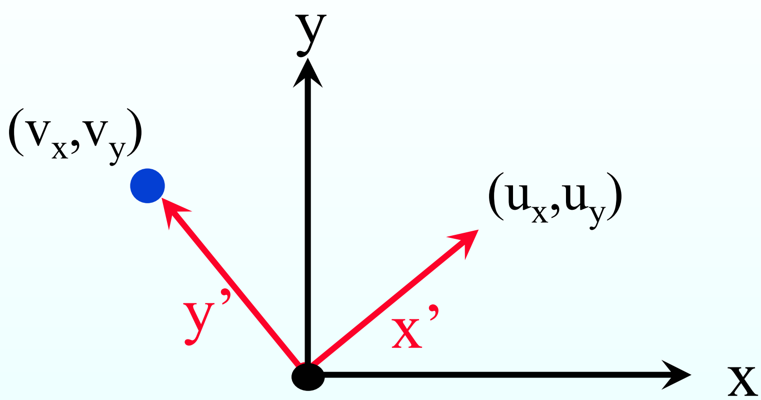
$$P' = MP$$

where

$$M = \begin{pmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix}$$



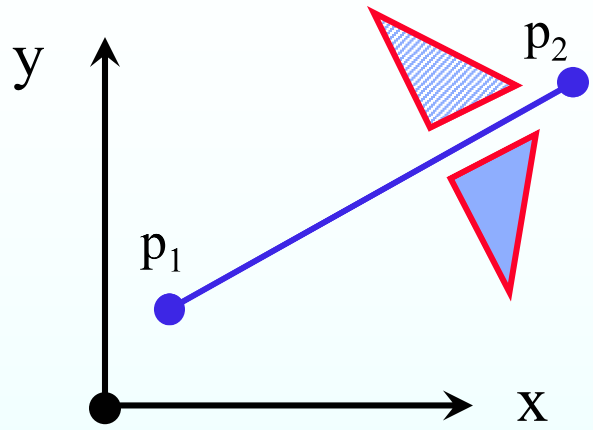
- **Example:**
  - P is at the y' axis $\mathbf{P=(v_x,v_y)}$:



$$P' = MP = \begin{pmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

  - What is the inverse?

- Another example: Reflection about an Arbitrary Line:



- Define a coordinate systems (u,v) parallel to P_1P_2:

$$\mathbf{u} = \frac{p_2 - p_1}{|p_2 - p_1|} \equiv \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} -u_y \\ u_x \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 & p_{1_x} \\ 0 & 1 & p_{1_y} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_x & v_x & 0 \\ u_y & v_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -p_{1_x} \\ 0 & 1 & -p_{1_y} \\ 0 & 0 & 1 \end{pmatrix}$$