2 Finite Difference Method

2.1 Finite Difference Methods for One-Dimensional Problem

Numerical scheme (constant coefficient):

$$-\frac{d^2u}{dx^2} + \frac{du}{dx} + u = f(x), x \in [a, b]$$

Discrete difference scheme:

$$-\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} + \frac{u(x_{i+1}) - u(x_{i-1})}{h} + u(x_i) = f(x_i), i = 1, 2 \cdots, N-1.$$

Example 2.1

$$\begin{cases} -\dfrac{d^2u}{dx^2} + \dfrac{du}{dx} = \pi^2 \sin(\pi x) + \pi \cos(\pi x), \quad x \in [0, 1] \\ u(0) = 0, u(1) = 0. \end{cases}$$

Exact solution: $u(x) = \sin(\pi x)$.

```
% fdm1d1.m
% finite difference method for 1D problem
% -u''+u'=pi^2*sin(pi*x)+pi*cos(pi*x)  in [0,1]
% u(0)=0, u(1)=0 ;
% exact solution : u=sin(pi*x)
clear all
h=0.05;
x=0:h:1;
N=length(x)-1;
A=diag((2/h^2)*ones(N-1,1))...
    +diag((1/(2*h)-1/h^2)*ones(N-2,1),1)...
    +diag((-1/(2*h)-1/h^2)*ones(N-2,1),-1);
b=pi^2*sin(pi*x(2:N))+pi*cos(pi*x(2:N));
u=A\b';
u=[0;u;0];
ue=sin(pi*x)';
plot(x,ue,'b-',x,u,'r+','LineWidth',1.5)
Error=max(abs(u-ue))
xlabel('x','fontsize', 16), ylabel('y','fontsize',16,'Rotation',0)
legend('Exact ','Numerical','location','North')
title('Finite Difference Method','fontsize',14)
set(gca,'fontsize',14)
```

Numerical scheme (variable coefficient):

$$-\frac{d}{dx}(p\frac{du}{dx}) + r\frac{du}{dx} + qu = f(x), x \in (a, b)$$

Discrete difference scheme:

$$-\frac{2}{h_i + h_{i+1}}\left[p_{i+\frac{1}{2}}\frac{u(x_{i+1}) - u(x_i)}{h_{i+1}} + p_{i-\frac{1}{2}}\frac{u(x_i) - u(x_{i-1})}{h_i}\right] +$$
$$\frac{r_i}{h_i + h_{i+1}}(u(x_{i+1}) - u(x_{i-1})) + q_i u(x_i) = f(x_i), i = 1, \cdots, N-1.$$

Example 2.2

$$\begin{cases} -\dfrac{d}{dx}\left(x\dfrac{du}{dx}\right) + x\dfrac{du}{dx} = \pi^2 x\sin(\pi x) + \pi(x-1)\cos(\pi x), x \in (0,1) \\ u(0) = 0, u(1) = 0. \end{cases}$$

Exact solution: $u(x) = \sin(\pi x)$.

```
% fdm1d2.m
% finite difference method for 1D problem
% -(xu')'+x*u'=pi^2*x*sin(pi*x)-pi*cos(pi*x)+pi*x*cos(pi*x) in [0,1]
% u(0)=0, u(1)=0 ;
% exact solution : u=sin(pi*x)
clear all
h=0.05;
x=0:h:1;
N=length(x)-1;
A=diag(2*x(2:N)./h^2)+diag(x(2:N-1)./(2*h)-(x(2:N-1)+0.5*h)./h^2,1)...
    +diag(-x(3:N)./(2*h)-(x(3:N)-0.5*h)./h^2,-1);
b=pi^2*x(2:N).*sin(pi*x(2:N))+pi*(x(2:N)-1).*cos(pi*x(2:N));
u=A\b';
u=[0;u;0];
ue=sin(pi*x');
plot(x,ue,'b-',x,u,'r+','LineWidth',1.5)
Error=max(abs(u-ue))
xlabel('x','fontsize', 16), ylabel('y','fontsize',16,'Rotation',0)
legend('Exact ','Numerical','location','North')
title('Finite Difference Method','fontsize',14)
set(gca,'fontsize',14)
```

2.2 Finite Difference Methods for Two-Dimensional Problem

Consider the two-dimensional Poisson problem:

$$\begin{cases} -\Delta u = f(x,y), & (x,y)\in\Omega, \\ u|_{\partial\Omega} = \phi(x,y), & (x,y)\in\partial\Omega. \end{cases}$$

Discrete difference scheme:

$$-\frac{1}{h_2^2}u_{i,j-1}-\frac{1}{h_1^2}u_{i-1,j}+2\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right)u_{i,j}-\frac{1}{h_1^2}u_{i+1,j}-\frac{1}{h_2^2}u_{i,j+1}=f\left(x_i,y_j\right),$$
$$1\leqslant i\leqslant N-1,\quad 1\leqslant j\leqslant M-1.$$

Define the vector: $\boldsymbol{u}_j=\left(u_{1j},u_{2j},\ldots,u_{N-1,j}\right)^{\mathrm{T}},\quad 0\leqslant j\leqslant M.$

The discrete scheme to matrix form:

$$\boldsymbol{D}\boldsymbol{u}_{j-1}+\boldsymbol{C}\boldsymbol{u}_j+\boldsymbol{D}\boldsymbol{u}_{j+1}=\boldsymbol{f}_j,\quad 1\leqslant j\leqslant M-1.$$

$$\boldsymbol{C}=\begin{pmatrix} 2\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) & -\frac{1}{h_1^2} & & & \\ -\frac{1}{h_1^2} & 2\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) & -\frac{1}{h_1^2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h_1^2} & 2\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) & -\frac{1}{h_1^2} \\ & & & -\frac{1}{h_1^2} & 2\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right) \end{pmatrix}$$

$$\boldsymbol{D}=\begin{pmatrix} -\frac{1}{h_2^2} & & & & \\ & -\frac{1}{h_2^2} & & & \\ & & \vdots & & \\ & & & -\frac{1}{h_2^2} & \\ & & & & -\frac{1}{h_2^2} \end{pmatrix}\qquad \boldsymbol{f}_j=\begin{pmatrix} f(x_1,y_j)+\frac{1}{h_1^2}\phi(x_0,y_j) \\ f(x_2,y_j) \\ \vdots \\ f(x_{N-2},y_j) \\ f(x_{N-1},y_j)+\frac{1}{h_1^2}\phi(x_N,y_j) \end{pmatrix}$$

Next, above can be written in the following matrix form

$$\begin{pmatrix} \boldsymbol{C} & \boldsymbol{D} & & & \\ \boldsymbol{D} & \boldsymbol{C} & \boldsymbol{D} & & \\ & \ddots & \ddots & \ddots & \\ & & \boldsymbol{D} & \boldsymbol{C} & \boldsymbol{D} \\ & & & \boldsymbol{D} & \boldsymbol{C} \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_{M-2} \\ \boldsymbol{u}_{M-1} \end{pmatrix}=\begin{pmatrix} \boldsymbol{f}_1-\boldsymbol{D}\boldsymbol{u}_0 \\ \boldsymbol{f}_1 \\ \vdots \\ \boldsymbol{f}_{M-2} \\ \boldsymbol{f}_{M-1}-\boldsymbol{D}\boldsymbol{u}_N \end{pmatrix}$$

Example 2.3

$$\begin{cases} -\Delta u = f(x,y), (x,y) \in \Omega = (0,1)\times(0,1) \\ u = 0, (x,y) \in \partial\Omega. \end{cases}$$

where $f(x,y) = -2\pi^2 e^{\pi(x+y)}(\sin \pi x \cos \pi y + \cos \pi x \sin \pi y)$.

Exact solution: $u(x,y) = e^{\pi(x+y)} \sin \pi x \sin \pi y$, $(x,y) \in \Omega = (0,1)\times(0,1)$.

```
% fdm2d1.m
% finite difference method for 2D problem
% -d^2u/dx^2-d^2u/dy^2=f(x,y)
% f(x,y)=-2*pi^2*exp(pi*(x+y))*(sin(pi*x)*cos(pi*y)+cos(pi*x)*sin(pi*y))
% exact solution: ue=exp(pi*x+pi*y)*sin(pi*x)*sin(pi*y)
clear all
h=0.01;
x=[0:h:1]';
y=[0:h:1]';
N=length(x)-1;
M=length(y)-1;
[X,Y]=meshgrid(x,y);
X=X(2:M,2:N);
Y=Y(2:M,2:N);
% generate the matrix of RHS
f=-2*pi^2*exp(pi*X+pi*Y).*(sin(pi*X).*cos(pi*Y)+cos(pi*X).*sin(pi*Y));
% constructing the coefficient matrix
C=4/h^2*eye(N-1)-1/h^2*diag(ones(N-2,1),1)-1/h^2*diag(ones(N-2,1),-1);
D=-1/h^2*eye(N-1);
A=kron(eye(M-1),C)+kron(diag(ones(M-2,1),1)+diag(ones(M-2,1),-1),D);
% solving the linear system
f=f';
u=zeros(M+1,N+1);
u(2:M,2:N)=reshape(A\f(:),N-1,M-1)';
u(:,1)=0;
u(:,end)=0;
ue=zeros(M+1,N+1);
ue(2:M,2:N)=exp(pi*X+pi*Y).*sin(pi*X).*sin(pi*Y);
% compute maximum error
Error=max(max(abs(u-ue)))
mesh(x,y,u)
xlabel('x','fontsize', 16), ylabel('y','fontsize',16), ...
    zlabel('u','fontsize',16,'Rotation',0)
title('Finite Difference Method','fontsize',14)
set(gca,'fontsize',14)
```

```
% fdm2d1_error.m
% finite difference method for 2D problem
% -d^2u/dx^2-d^2u/dy^2=f(x,y)
% f(x,y)=-2*pi^2*exp(pi*(x+y))*(sin(pi*x)*cos(pi*y)+cos(pi*x)*sin(pi*y))
% exact solution: ue=exp(pi*x+pi*y)*sin(pi*x)*sin(pi*y)
clear all
Nvec=2.^[3:10];  Err=[];
for n=Nvec
   h=1/n;
   x=[0:h:1]';      y=[0:h:1]';
   N=length(x)-1;  M=length(y)-1;
   [X,Y]=meshgrid(x,y);
   X=X(2:M,2:N);
   Y=Y(2:M,2:N);
   % generate the matrix of RHS
   f=-2*pi^2*exp(pi*X+pi*Y).*(sin(pi*X).*cos(pi*Y)+cos(pi*X).*sin(pi*Y));
   % constructing the coefficient matrix
   e=ones(N-1,1);
   C=1/h^2*spdiags([-e 4*e -e],[-1 0 1],N-1,N-1);
   D=-1/h^2*eye(N-1);
   e=ones(M-1,1);
   A=kron(eye(M-1),C)+kron(spdiags([e e],[-1 1],M-1,M-1),D);
   % solving the linear system
   f=f';
   u=zeros(M+1,N+1);
   u(2:M,2:N)=reshape(A\f(:),N-1,M-1)';
   u(:,1)=0;
   u(:,end)=0;
   ue=zeros(M+1,N+1);         % numerical solution
   ue(2:M,2:N)=exp(pi*X+pi*Y).*sin(pi*X).*sin(pi*Y);
   err=max(max(abs(u-ue)));  % maximum error
   Err=[Err,err];
end
plot(log10(Nvec),log10(Err),'ro-','MarkerFaceColor','w','LineWidth',1.5)
grid on,hold on, plot(log10(Nvec), log10(Nvec.^(-2)), '--')
xlabel('log_{10}N','fontsize', 16), ...
    ylabel('log_{10}Error','fontsize',16),
title('Convergence of Finite Difference Method','fontsize',14)
set(gca,'fontsize',14)
for i=1:length(Nvec)-1     % computating convergence order
   order(i)=-log(Err(i)/Err(i+1))/(log(Nvec(i)/Nvec(i+1)));
end
Err
order
```

Example 2.4

$$\begin{cases} -\Delta u = \cos 3x \sin \pi y, & (x, y) \in G = (0, \pi) \times (0, 1), \\ u(x, 0) = u(x, 1) = 0, & 0 \leqslant x \leqslant \pi, \\ u_x(0, y) = u_x(\pi, y) = 0, & 0 \leqslant y \leqslant 1\,. \end{cases}$$

Exact solution: $u = (9 + \pi^2)^{-1} \cos 3x \sin \pi y$.

Rectangular division: $h_1 = \frac{\pi}{N}$, $h_2 = \frac{1}{N}$, grid node $x_i = ih_1$, $y_j = jh_2$, $i, j = 0, 1, \cdots, N$.

Discrete difference scheme:

$$-\left(\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h_1^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h_2^2} \right) = \cos 3x_i \sin \pi y_j$$

$$i, j = 1, 2, \cdots, N-1.$$

Boundary conditions:

$$\begin{aligned} &u_{i0} = u_{iN} = 0, i = 0, \cdots, N \\ &u_{0j} = u_{1j}\,, j = 1, \cdots, N-1 \\ &u_{Nj} = u_{N-1,j}\,, j = 1, \cdots, N-1 \end{aligned}$$

Discrete scheme:

$$\boldsymbol{D}\boldsymbol{u}_{j-1} + \boldsymbol{C}\boldsymbol{u}_j + \boldsymbol{D}\boldsymbol{u}_{j+1} = \boldsymbol{f}_j, \quad 1 \leqslant j \leqslant M-1\,.$$

$$\boldsymbol{C} = \begin{pmatrix} \left(\frac{1}{h_1^2} + \frac{2}{h_2^2}\right) & -\frac{1}{h_1^2} & & & \\ -\frac{1}{h_1^2} & 2\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right) & -\frac{1}{h_1^2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h_1^2} & 2\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right) & -\frac{1}{h_1^2} \\ & & & -\frac{1}{h_1^2} & \left(\frac{1}{h_1^2} + \frac{2}{h_2^2}\right) \end{pmatrix}$$

$$\boldsymbol{D} = \begin{pmatrix} -\frac{1}{h_2^2} & & & & \\ & -\frac{1}{h_2^2} & & & \\ & & \vdots & & \\ & & & -\frac{1}{h_2^2} & \\ & & & & -\frac{1}{h_2^2} \end{pmatrix} \qquad \boldsymbol{f}_j = \begin{pmatrix} f(x_1, y_j) \\ f(x_2, y_j) \\ \vdots \\ f(x_{N-2}, y_j) \\ f(x_{N-1}, y_j) \end{pmatrix}$$

Matrix form:

$$\begin{pmatrix} \boldsymbol{C} & \boldsymbol{D} & & & \\ \boldsymbol{D} & \boldsymbol{C} & \boldsymbol{D} & & \\ & \ddots & \ddots & \ddots & \\ & & \boldsymbol{D} & \boldsymbol{C} & \boldsymbol{D} \\ & & & \boldsymbol{D} & \boldsymbol{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_{M-2} \\ \boldsymbol{u}_{M-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_1 \\ \vdots \\ \boldsymbol{f}_{M-2} \\ \boldsymbol{f}_{M-1} \end{pmatrix}$$

```
% fdm2d2_error.m
% finite difference method for 2D problem
% -\Delta u=cos(3*x)*sin(pi*y)  in (0,pi)x(0,1)
% u(x,0)=u(x,1)=0  in [0,pi]
% u_x(0,y)=u_x(pi,y)=0 in [0,1]
% exact solution: ue=(9+pi^2)^(-1)*cos(3*x)*sin(pi*y)
clear all; close all;
Nvec=2.^[2:7];  MErr=[];
for N=Nvec
    h1=pi/N;  h2=1/N;
    x=[0:h1:pi]';   y=[0:h2:1]';
    [X,Y]=meshgrid(x,y);
    X1=X(2:N,2:N);  Y1=Y(2:N,2:N);
    % generate the matrix of RHS
    f=cos(3*X1).*sin(pi*Y1);
    % constructing the coefficient matrix
    e=ones(N-1,1);
    C=diag([1/h1^2+2/h2^2, (2/h1^2+2/h2^2)*ones(1,N-3), ...
        1/h1^2+2/h2^2])...
        -1/h1^2*diag(ones(N-2,1),1)-1/h1^2*diag(ones(N-2,1),-1);
    D=-1/h2^2*eye(N-1);
    %A=kron(eye(N-1),C)+kron(diag(ones(N-2,1),1)+diag(ones(N-2,1),-1),D);
    A=kron(eye(N-1),C)+kron(spdiags([e e],[-1 1],N-1,N-1),D);
    % solving the linear system
    f=f';
    u=zeros(N+1,N+1);
    u(2:N,2:N)=reshape(A\f(:),N-1,N-1)';
    % Neumann boundary condition
    u(:,1)=u(:,2);
    u(:,end)=u(:,end-1);
    ue=1/(9+pi^2)*(cos(3*X)).*(sin(pi*Y));
    Merr=max(max(abs(u-ue)));   %maximum error
    MErr=[MErr,Merr];
end
plot(log10(Nvec),log10(MErr),'ro-','MarkerFaceColor','w','LineWidth',1.5)
grid on,hold on, plot(log10(Nvec),log10(Nvec.^(-1)),'--')
xlabel('log_{10}N','fontsize', ...
    16),ylabel('log_{10}Error','fontsize',16),
title('Convergence of Finite Difference Method','fontsize',14)
set(gca,'fontsize',14)
for i=1:length(Nvec)-1     % computating convergence order
    order(i)=log(MErr(i)/MErr(i+1))/(log(Nvec(i)/Nvec(i+1)));
end
order
```