Communication Laboratory: Fourier Transforms

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Outline

References

- A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, Signals & Systems (2nd Ed.). Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1996.
- A. V. Oppenheim and R. W. Schafer, Discrete-Time Signal Processing (3rd Ed.). Upper Saddle River, NJ, USA: Prentice Hall Press, 2009.

Outline

The Fourier Transform Family

Frequency \ Time	Continuous	Discrete
Continuous	Continuous-Time Fourier Transform (CTFT) $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t$	Discrete-Time Fourier Transform (DTFT) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
Discrete	Fourier Series (FS) $a_k = \frac{1}{T}\int_{T} x(t)e^{-j\frac{2\pi kt}{T}}\mathrm{d}t$	Discrete Fourier Transform (DFT) $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi nk}{N}}$

Outline

Continuous-Time Fourier Transforms

Convert a continuous signal $x(t)$ in the time domain into a continuous signal $X(j\omega)$ in the frequency domain

Continuous-Time Fourier Transforms in ω (Radians)

- Analysis equation (Time domain $x(t)$ to frequency domain $X(j\omega)$)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \mathrm{d}t.$$

- Synthesis equation (Frequency domain $X(j\omega)$ to time domain $x(t)$)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \mathrm{d}\omega.$$

[1] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, Signals & Systems (2nd Ed.). Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1996.

Continuous-Time Fourier Transform in f (Hertz)

- Analysis equation (Time domain $x(t)$ to frequency domain $X(f)$)

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \mathrm{d}t.$$

- Synthesis equation (Frequency domain $X(f)$ to time domain $x(t)$)

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \mathrm{d}f.$$

1. Change of variables: $\omega = 2\pi f$.
2. Symmetry of the analysis and synthesis equations
3. $X(j\omega)$ and $X(f)$ are interchangeable

[1] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, Signals & Systems (2nd Ed.). Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1996.

Properties of CTFT

Consider the following Fourier transform pairs (Times $\leftrightarrow$Frequency):

$$x(t) \leftrightarrow X(j\omega), \qquad\qquad y(t) \leftrightarrow Y(j\omega).$$

Then

$$\begin{aligned}
&\text{Linearity:} & \alpha x(t) + \beta y(t) &\leftrightarrow \alpha X(j\omega) + \beta Y(j\omega), \\
&\text{Time-shifting:} & x(t - t_0) &\leftrightarrow e^{-j\omega t_0} X(j\omega), \\
&\text{Frequency-shifting:} & e^{j\omega_0 t} x(t) &\leftrightarrow X(j(\omega - \omega_0)), \\
&\text{Conjugation:} & x^*(t) &\leftrightarrow X^*(-j\omega), \\
&\text{Time and frequency scaling:} & x(at) &\leftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right), \\
&\text{Convolution:} & x(t) * y(t) &\leftrightarrow X(j\omega)Y(j\omega), \\
& & &\vdots
\end{aligned}$$

Fourier Transforms as Inner Products

- Define the inner product between $x(t)$ and $y(t)$:

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) \mathrm{d}t$$

- Three axioms of inner products:
  - Linearity[1]: $\langle \alpha x_1(t) + \beta x_2(t), y(t) \rangle = \alpha \langle x_1(t), y(t) \rangle + \beta \langle x_2(t), y(t) \rangle$.
  - Conjugate symmetry: $\langle x(t), y(t) \rangle = \langle y(t), x(t) \rangle^*$.
  - Positive-definiteness: $\langle x(t), x(t) \rangle \geq 0$. The equality holds if and only if $x(t) = 0$.
- The CTFT can be interpreted as[2]

$$X(j\omega) = \left\langle x(t), e^{j\omega t} \right\rangle.$$

[1] In some references, the inner product is defined over the linearity with respect to the second argument.

[2] As a result, the CTFT is closely related to inner product spaces in linear algebra. The relations will be more clear in the context of discrete Fourier transforms.

Outline

Discrete-Time Fourier Transforms

Convert a <u>discrete</u> signal $x[n]$ in the time domain
into a <u>continuous</u> signal $X(e^{j\omega})$ in the frequency domain

Discrete-Time Fourier Transforms (DTFT) in ω (Radians)

- Analysis equation ($x[n]$ to $X(e^{j\omega})$)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \tag{1}$$

- Synthesis equation ($X(e^{j\omega})$ to $x[n]$)

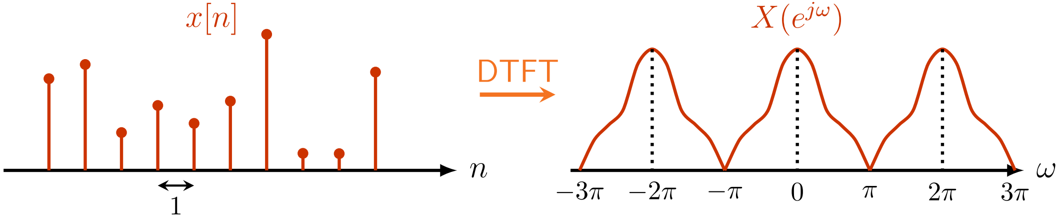
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega. \tag{2}$$

- Properties of DTFT are very similar to those of CTFT.

[1] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, Signals & Systems (2nd Ed.). Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1996.

$X(e^{j\omega})$ has period 2π

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \underbrace{e^{-j2\pi n}}_{1} = X(e^{j\omega}).$$



Linear Convolution and DTFT

$$x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$y[n] \xrightarrow{\text{DTFT}} Y(e^{j\omega})$$

$$\boxed{x[n] * y[n] \triangleq \sum_{\ell=-\infty}^{\infty} x[\ell]y[n-\ell]} \xrightarrow{\text{DTFT}} \boxed{X(e^{j\omega})Y(e^{j\omega})}$$

Linear convolution | Multiplication

Computing CTFT

- Computing the spectrum (or equivalently CTFT) is important in many engineering problems.
- Difficulties in obtaining $X(j\omega)$ (the CTFT of $x(t)$)
  1. Integration is involved
  2. t is defined over the real axis $\mathbb{R}$, from $-\infty$ to ∞.
- These difficulties can be handled by
  1. DTFT (for discrete-time sequences)
  2. DFT (for finite-duration discrete-time sequences)

Computing CTFT from DTFT (1/2)

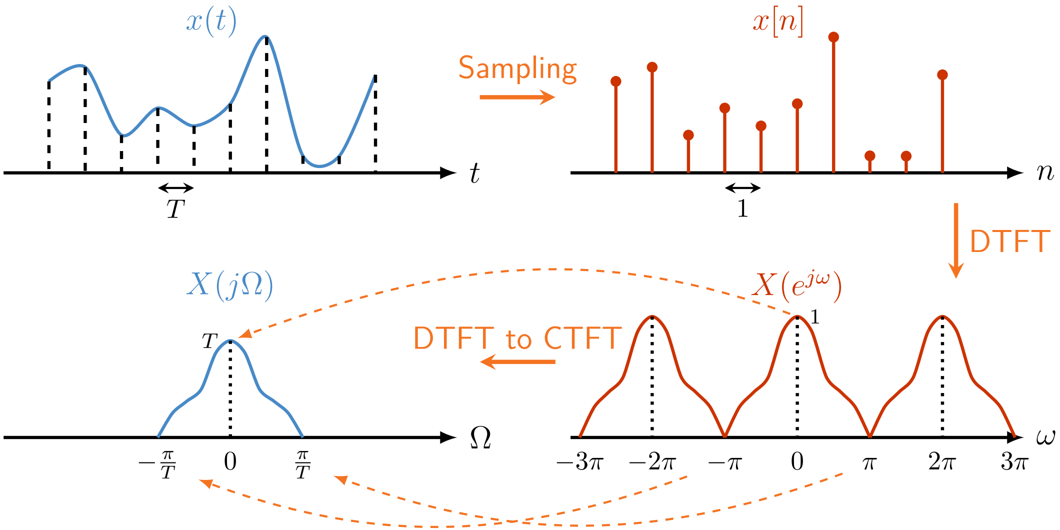
- $X(j\Omega)$: The CTFT of a bandlimited signal $x(t)$. ($X(j\Omega) = 0$ for $|\Omega| \geq \pi/T$)
- $x[n] = x(nT)$.
- $X(e^{j\omega})$: The DTFT of $x[n]$.

For $|\Omega| < \pi/T$,

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\mathrm{d}t \\ &\approx \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega nT} \times T \qquad \text{(Riemann sums, } t = nT\text{)} \\ &= T\sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega T)n} \\ &= TX(e^{j\Omega T}). \end{aligned}$$

[1] The argument of Riemann sums is only for demonstration. In fact, it can be shown rigorously that $X(j\Omega) = TX(e^{j\Omega T})$ for $|\Omega| < \pi/T$.

Computing CTFT from DTFT (2/2) ($X(j\Omega) \approx TX(e^{j\Omega T})$)



Outline

Discrete Fourier Transforms

Convert a discrete signal $x[n]$ in the time domain
into a discrete signal $X[k]$ in the frequency domain

Discrete Fourier Transforms

- N is the number of samples
- The input signal $x[n]$, $n = 0, 1, 2, \ldots, N-1$.
- The discrete Fourier transform (DFT) of $x[n]$ is denoted by $X[k]$:

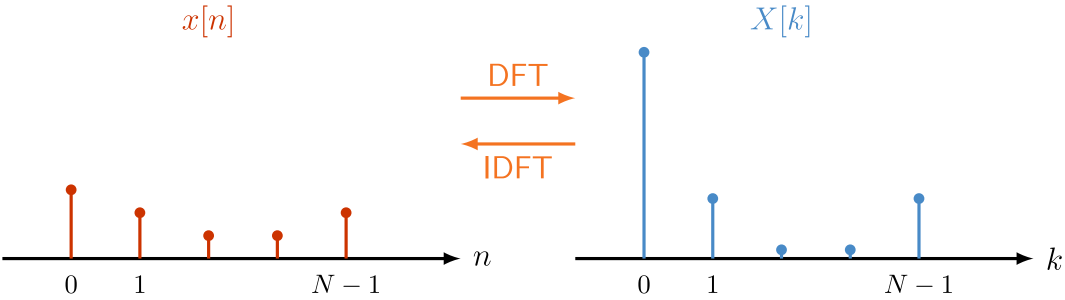
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}}, \qquad k = 0, 1, \ldots, N-1.$$

- The inverse discrete Fourier transform (IDFT) is given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi nk}{N}}, \qquad n = 0, 1, \ldots, N-1.$$

[1] In some books, the definition of the DFT has an $1/\sqrt{N}$ factor.

Example



Properties of DFT

- Many properties are very similar to those of DTFT
- In this lecture, we will focus on
  - Circular shift
  - Circular time-reversal
  - Circular convolution
  - Hermitian symmetry

Circular Shift and Circular Time-Reversal

- Define $((m))_N$ as the remainder of m divided by N,
  e.g., $((3))_5 = 3$, and $((2))_5 = 3$.
- Assume that $N = 10$

$$
\begin{array}{rcllllllllll}
n & = & 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9 \\
x[n] & = & x[0], & x[1], & x[2], & x[3], & x[4], & x[5], & x[6], & x[7], & x[8], & x[9] \\
x[((n-1))_N] & = & x[9], & x[0], & x[1], & x[2], & x[3], & x[4], & x[5], & x[6], & x[7], & x[8] \\
x[((n-2))_N] & = & x[8], & x[9], & x[0], & x[1], & x[2], & x[3], & x[4], & x[5], & x[6], & x[7] \\
x[((-n))_N] & = & x[0], & x[9], & x[8], & x[7], & x[6], & x[5], & x[4], & x[3], & x[2], & x[1]
\end{array}
$$

Circular Convolution and DFT

$$x[n] \xrightarrow{\text{DFT}} X[k]$$

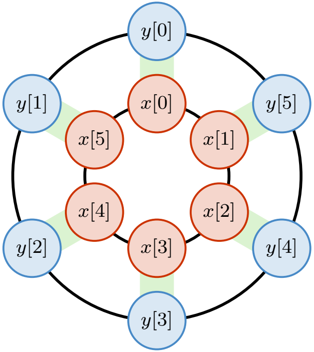
$$y[n] \xrightarrow{\text{DFT}} Y[k]$$

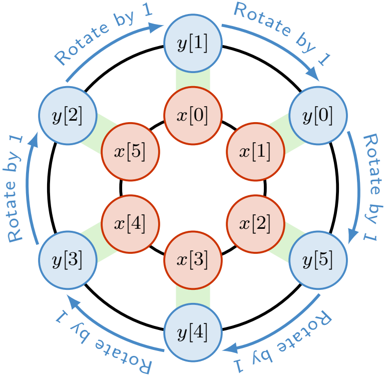
$$\boxed{x[n] \circledast_N y[n] \triangleq \sum_{\ell=0}^{N-1} x[\ell]y[((n-\ell))_N]} \xrightarrow{\text{DFT}} \boxed{X[k]Y[k]}$$

Circular convolution | Multiplication

- $n, k = 0, 1, \ldots, N-1$.
- $((m))_N$: The remainder of m divided by N, e.g., $((3))_5 = 3$, and $((-2))_5 = 3$.
- $y[((n-\ell))_N]$: a circular shift of $y[n]$.

An Example of Circular Convolution ($z[n] = x[n] \circledast_6 y[n]$)





$$z[0] = x[0]\,y[0] + x[1]\,y[5] + x[2]\,y[4] + x[3]\,y[3] + x[4]\,y[2] + x[5]\,y[1]$$

$$z[1] = x[0]\,y[1] + x[1]\,y[0] + x[2]\,y[5] + x[3]\,y[4] + x[4]\,y[3] + x[5]\,y[2]$$

Hermitian Symmetry

$x[n]$ is real if and only if $X[k] = X^*[((-k))_N]$ for all $k = 0, 1, \ldots, N-1$.

Example: $N = 6$

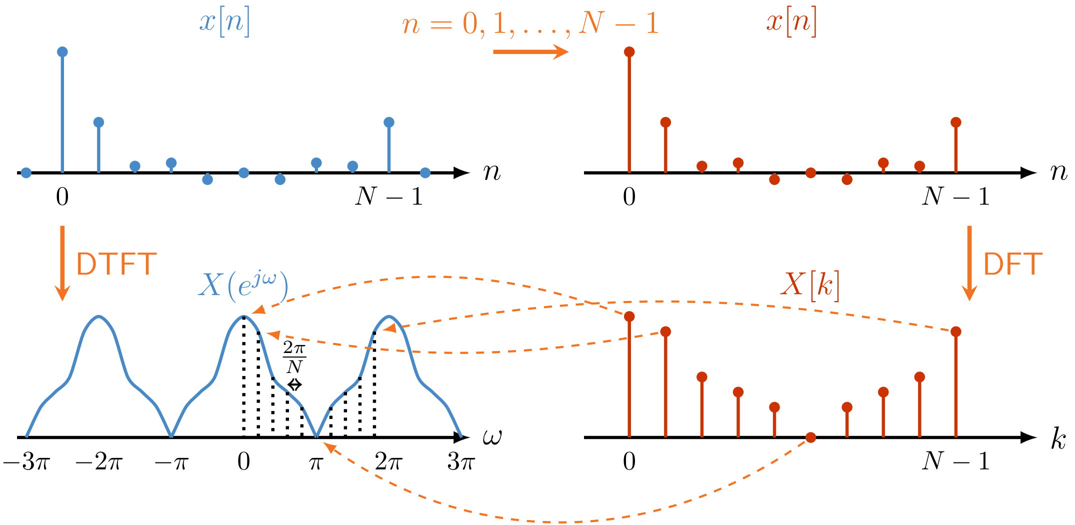
n	$=$	$0,$	$1,$	$2,$	$3,$	$4,$	$5,$
$x[n]$	$=$	$1,$	$1,$	$3,$	$0,$	$1,$	$8,$
k	$=$	$0,$	$1,$	$2,$	$3,$	$4,$	$5,$
$X[k]$	$=$	$14,$	$3.5 + 4.3301j,$	$-5.5 + 7.7942j,$	$-4,$	$-5.5 - 7.7942j,$	$3.5 - 4.3301j$

$$X[0] = X^*[((-0))_6] = X^*[0], \qquad X[1] = X^*[((-1))_6] = X^*[5],$$
$$X[2] = X^*[((-2))_6] = X^*[4], \qquad X[3] = X^*[((-3))_6] = X^*[3].$$

Computing CTFT

- Computing the spectrum (or equivalently CTFT) is important in many engineering problems.
- Difficulties in obtaining $X(j\omega)$ (the CTFT of $x(t)$)
  1. Integration is involved
  2. t is defined over the real axis $\mathbb{R}$, from $-\infty$ to ∞.
- These difficulties can be handled by
  1. DTFT (for discrete-time sequences)
  2. DFT (for finite-duration discrete-time sequences)

Computing DTFT from DFT ($X(e^{j2\pi k/N}) = X[k]$)



Advanced Topics on DFT (Potential Topics for Survey)

- Fast Fourier Transform (FFT): A divide-and-conquer algorithm for DFT
  - DFT has complexity $\mathcal{O}(N^2)$, where N is the number of sample points.
  - FFT has complexity $\mathcal{O}(N \log_2 N)$.
- Discrete transforms related to DFT
  - Discrete cosine transforms (DCT)
  - Discrete sine transforms (DST)
  - DIscrete Hartley transforms
- Orthogonal Frequency Division Multiplexing (OFDM)
  - OFDM uses the inverse discrete Fourier transform (IDFT)
  - OFDM will be introduced later in this course

Outline

Signals and Systems from the Linear Algebra Perspective

- Signals and systems:

A signal: $x[n]$

Sample values: $x[0], x[1], \ldots$

- Linear algebra:

A column vector: $$\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \in \mathbb{C}^N.$$

  - $\mathbf{x}$ is analogous to $x[n]$.
  - In this lecture, we focus on discrete-time signals with N points.

[1] *Conceptually*, discrete-time sequences with infinite duration are associated with infinite-dimensional vectors. However, this case requires special attention since some of the results in linear algebra might not hold.

Linear Combinations of Signals

- Assume the signals $x[n]$ and $y[n]$ correspond to the vectors $\mathbf{x}$ and $\mathbf{y}$, defined as

$$\mathbf{x} \triangleq \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}, \qquad \mathbf{y} \triangleq \begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix}.$$

- Linear combination of two signals

$$z[n] = \alpha x[n] + \beta y[n], \qquad \mathbf{z} \triangleq \begin{bmatrix} \alpha x[0] + \beta y[0] \\ \alpha x[1] + \beta y[1] \\ \vdots \\ \alpha x[N-1] + \beta y[N-1] \end{bmatrix} = \alpha \mathbf{x} + \beta \mathbf{y}.$$

Linear Systems as Matrix Multiplication

- Matrix representation of linear transformations in linear algebra
- Let $\mathcal{S}$ be a linear system satisfying

$$y[n] = \mathcal{S}\left(x[n]\right).$$

- Then $\mathcal{S}$ can be characterized by

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} s_{0,0} & s_{0,1} & \cdots & s_{0,N-1} \\ s_{1,0} & s_{1,1} & \cdots & s_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{N-1,0} & s_{N-1,1} & \cdots & s_{N-1,N-1} \end{bmatrix}}_{\mathbf{S}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\mathbf{x}}.$$

Main Concept

Signals and Systems	Linear Algebra
A Discrete-Time Signal $x[n]$ with Finite Duration	A Column Vector $\mathbf{x}$
A Linear System $\mathcal{S}$	A Matrix $\mathbf{S}$

Next some examples will be demonstrated

Time-Reversal of N-point Sequences

- The time-reversal of a discrete-time sequence with N points:

$$y[n] = x[((-n))_N], \qquad \text{for } n = 0, 1, \ldots, N-1.$$

- A numerical example with $N = 6$.

$$\begin{cases} y[0] = x[((0))_6] = x[0], \\ y[1] = x[((-1))_6] = x[5], \\ y[2] = x[((-2))_6] = x[4], \\ y[3] = x[((-3))_6] = x[3], \\ y[4] = x[((-4))_6] = x[2], \\ y[5] = x[((-5))_6] = x[1], \end{cases} \qquad \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{A permutation matrix}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \end{bmatrix}.$$

Delay

- The system of unit delay is defined as

$$y_1[n] = \mathcal{D}\left(x[n]\right) = x[((n-1))_N] \qquad \text{for } n = 0, 1, \ldots, N-1.$$

- We can define *the system of delay by M units* as

$$y_M[n] = \underbrace{\left(\mathcal{D} \circ \mathcal{D} \circ \cdots \circ \mathcal{D}\right)}_{M \text{ times}} \left(x[n]\right) = \mathcal{D}^M\left(x[n]\right) \qquad \text{for } n = 0, 1, \ldots, N-1.$$

- Numerical examples with $N = 4$

$$\underbrace{\begin{bmatrix} y_1[0] \\ y_1[1] \\ y_1[2] \\ y_1[3] \end{bmatrix}}_{\text{Delay by 1}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\substack{\text{The matrix } \mathbf{D} \\ \text{w.r.t the system } \mathcal{D}(\cdot)}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}, \quad \underbrace{\begin{bmatrix} y_2[0] \\ y_2[1] \\ y_2[2] \\ y_2[3] \end{bmatrix}}_{\text{Delay by 2}} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\substack{\text{The matrix} \\ \text{w.r.t the system } \mathcal{D}^2(\cdot) \\ \text{(Exactly } \mathbf{D}^2\text{)}}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Circular Convolution (1/2)

- The circular convolution of $h[n]$ and $x[n]$ is defined as

$$y[n] = h[n] \circledast_N x[n] = \sum_{\ell=0}^{N-1} h[\ell] x[((n-\ell))_N].$$

- An example with $N = 4$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \underbrace{\begin{bmatrix} h[0] & h[3] & h[2] & h[1] \\ h[1] & h[0] & h[3] & h[2] \\ h[2] & h[1] & h[0] & h[3] \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix}}_{\text{A circulant matrix } \mathbf{H}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

Circular Convolution (2/2)

$$y[n] = h[n] \circledast_N x[n], \quad \mathbf{y} = \underbrace{\begin{bmatrix} h[0] & h[N-1] & \dots & h[2] & h[1] \\ h[1] & h[0] & \dots & h[3] & h[2] \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h[N-2] & h[N-3] & \dots & h[0] & h[N-1] \\ h[N-1] & h[N-2] & \dots & h[1] & h[0] \end{bmatrix}}_{\mathbf{H}} \mathbf{x}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{3}$$

Properties of the matrix $\mathbf{H}$

- $\mathbf{H}$ is a circulant matrix
- $\mathbf{H}$ can be decomposed as $\mathbf{H} = \sum_{\ell=0}^{N-1} h[\ell]\mathbf{D}^{\ell}$, where $\mathbf{D}$ is defined in (3).
- The columns of the DFT matrix $\mathbf{W}_N$ are the eigenvectors of $\mathbf{H}$.
- If $\mathbf{H}_1, \mathbf{H}_1 \in \mathbb{C}^{N\times N}$ are both circulant matrices, then $\mathbf{H}_1\mathbf{H}_2 = \mathbf{H}_2\mathbf{H}_1$.

Modulation

- The modulation of $x[n]$ is defined as

$$y[n] = x[n]e^{j2\pi f_c n}, \qquad \text{for } n = 0, 1, \ldots, N-1.$$

- A numerical example with $N = 5$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{j2\pi f_c} & 0 & 0 & 0 \\ 0 & 0 & e^{j4\pi f_c} & 0 & 0 \\ 0 & 0 & 0 & e^{j6\pi f_c} & 0 \\ 0 & 0 & 0 & 0 & e^{j8\pi f_c} \end{bmatrix}}_{\text{A diagonal matrix}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}.$$

Modeling DFT Using Matrices and Vectors

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}} = \sum_{n=0}^{N-1} x[n] W_N^{nk},$$

Twiddle factor: $W_N = e^{-j\frac{2\pi}{N}}$,

DFT matrix $\mathbf{W}_N$

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-2] \\ X[N-1] \end{bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & W_N & W_N^2 & \dots & W_N^{N-2} & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-2)} & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & W_N^{N-2} & W_N^{2(N-2)} & \dots & W_N^{(N-2)(N-2)} & W_N^{(N-2)(N-1)} \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-2)(N-1)} & W_N^{(N-1)(N-1)} \end{bmatrix}}_{\mathbf{W}_N} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-2] \\ x[N-1] \end{bmatrix}}_{\mathbf{x}}.$$

Properties of the DFT Matrix

- The DFT matrix is an orthogonal matrix. In particular,

$$\mathbf{W}_N^H \mathbf{W}_N = \begin{bmatrix} N & 0 & 0 & \dots & 0 \\ 0 & N & 0 & \dots & 0 \\ 0 & 0 & N & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & N \end{bmatrix} = N\mathbf{I}_N, \tag{4}$$

  where $\mathbf{I}_N$ is the identity matrix of size $N \times N$.
- $\mathbf{W}_N$ is a Vandermonde matrix,
  i.e., columns of $\mathbf{W}_N$ are of the form $\begin{bmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{N-1} \end{bmatrix}^T$ for some α.

Question

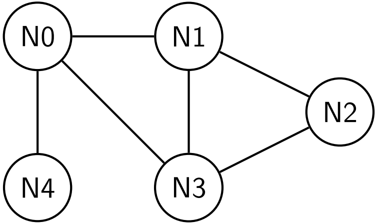
Consider discrete-time sequences $x[n]$ and $y[n]$. We focus on the range from $n = -4$ to $n = 4$ and define the columns vectors:

$$\mathbf{x} = \begin{bmatrix} x[-4] & x[-3] & x[-2] & x[-1] & x[0] & x[1] & x[2] & x[3] & x[4] \end{bmatrix}^T,$$
$$\mathbf{y} = \begin{bmatrix} y[-4] & y[-3] & y[-2] & y[-1] & y[0] & y[1] & y[2] & y[3] & y[4] \end{bmatrix}^T.$$

1. Assume that $y[n] = \mathcal{S}_{\downarrow M}(x[n])$ (downsampling by M). What is the expression of the matrix $\mathbf{S}_{\downarrow M}$?
   Hint: $\mathbf{y} = \mathbf{S}_{\downarrow M}\mathbf{x}$. Undefined quantities are assumed to be zero.
2. What are the rank, the nullity, the range space, and the null space of $\mathbf{S}_{\downarrow M}$?
3. What about the case of upsampling by L, for the previous two questions?

Advanced Topics (Potential Topics for Survey)

- Graph signal processing (Hot research topic in signal processing):
  - DFT is based on a uniform grid of data ($n = 0, 1, 2, \ldots$).
  - What if the data are defined over a nonuniform grid? (e.g, $n = 0, 5, 6, 7, 10, 15, 20$).
  - Moreover, what if the data are defined over a graph?



[1] D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega and P. Vandergheynst, "The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains, " in *IEEE Signal Processing Magazine*, vol. 30, no. 3, pp. 83-98, May 2013; A. Ortega, P. Frossard, J. Kovačević, J. M. F. Moura and P. Vandergheynst, "Graph Signal Processing: Overview, Challenges, and Applications," in *Proceedings of the IEEE*, vol. 106, no. 5, pp. 808-828, May 2018.
References are not limited to these two papers.

Outline

The Fourier Transform Family

Frequency \ Time	Continuous	Discrete
Continuous	Continuous-Time Fourier Transform (CTFT) $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \mathrm{d}t$	Discrete-Time Fourier Transform (DTFT) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$
Discrete	Fourier Series (FS) $a_k = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi k t}{T}} \mathrm{d}t$	Discrete Fourier Transform (DFT) $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi n k}{N}}$

Signals and Systems / Linear Algebra

Signals and Systems	Linear Algebra
A Discrete-Time Signal $x[n]$ with Finite Duration	A Column Vector $\mathbf{x}$
A Linear System $\mathcal{S}$	A Matrix $\mathbf{S}$