Day 2
Predicates, quantifiers, relations

Lecturer: Msc. Minh Tan Le

Outline

I. Predicates, quantifiers
II. Sets, tuples, Cartesian products
III. Relations and their properties
IV. Matrix of relation

I. Predicates, quantifiers

- In calculus, we can define y using a function with parameter(s) and variable(s):

$$y = x^2 + 1$$

- There are multiple y that satisfy by setting values for x.
- Is there any method to define a group of propositions?

- **Predicate** is a part of **propositional function**.

$$P(x) = x \textit{ is greater than 3}$$

- x is the **variable** (subject of statement).
- *is greater than 3* is the **predicate** (property).
- The **function** matches these **propositions** and **truth values**:
  - 5 is greater than 3. → 1.
  - 5.5 is greater than 3. → 1.
  - 1 is greater than 3. → 0.

- What if we don't want to set specific x? How about *all, none, few,...* as x?
- **Quantifier** comes to the rescue!

$\exists x P(x)$ = There exists an x *that is greater than 3*

- In case of using domain to narrow the possibility of x:

$\exists x \in \mathbb{R}, P(x)$ = There exists a **real number** x *that is greater than 3*

$\forall x > 4, P(x)$ = For every $x > 4$, x *is greater than 3*

- Propositional functions **do not have** truth value(s).
- Propositions, quantifiers **have** truth value(s).
- Quantifiers can be:
  - Universal quantification: $\forall x P(x)$.
  - Existential quantification: $\exists x P(x)$.
  - Uniqueness quantification (one & only one): $\exists !\, P(x)$ or $\exists_1 P(x)$.

Example:

Not all the students finished the exercise. (1)

There is at least 1 student who didn't finish the exercise. (2)

1. With x as student and $P(x)$ as propositional function for (1), rewrite (1) and (2) as expressions.
2. Are (1) and (2) equivalent?

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

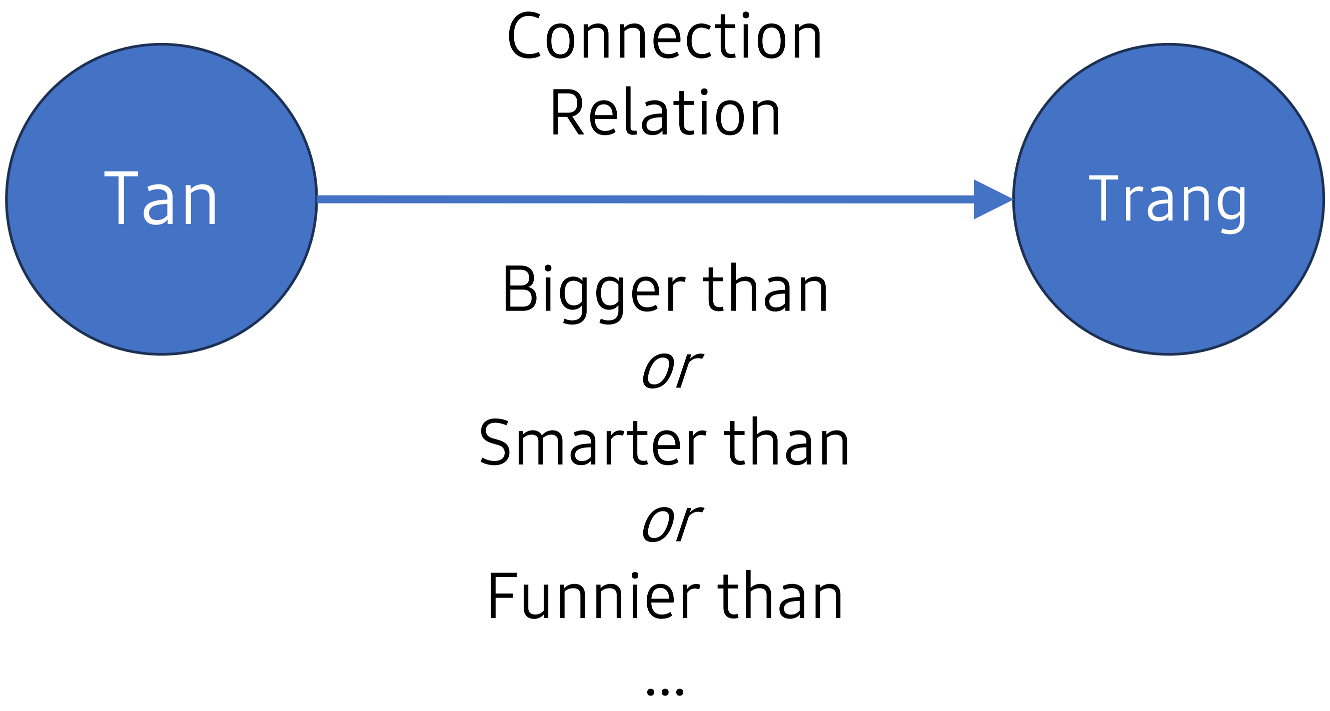
Negating quantified expressions

II. Sets, tuples, Cartesian products

- A set is an **unordered** collection of objects/members/elements.
- There can be:
  - … multiple same elements in a set.
  - … infinite number of elements.
  - … number, set or anything in set.
- Examples:
  - $O = \{1, -2, 3, 2.5, 7\}$
  - $\mathrm{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is a set of integers.
  - $P = \{1,3,3,3\}$ is the same as $Q = \{1,3\}$, or $P = Q$.

Name	Description	Expression
Equal sets	Two sets are equal if and only if they have the same element.	$A = B$ $\forall x(x \in A \leftrightarrow x \in B)$ $A \subseteq B \wedge B \subseteq A$
Subset	A is a subset of B if and only if every element of A is also an element of B.	$A \subseteq B$ $\forall x(x \in A \rightarrow x \in B)$
Proper subset	A is a subset of B, but not equal to B.	$A \subset B$ $\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$
Size	If there are exactly n distinct elements in S then S is a finit set and n is the cardinality.	$n = \|S\|$
Power set	The power set of S contains all the possible subsets of it.	$\mathcal{P}(S)$

Tuples



Connection
Relation
Tan
Trang
Bigger than
or
Smarter than
or
Funnier than
...

- An **ordered** collection of n objects is a $\boldsymbol{n}$**-tuple**.

$$(a_1, a_2, \ldots, a_n)$$

- Two tuples $(a_1, a_2, \ldots a_n)$, $(b_1, b_2, \ldots b_m)$ are unequal if:
  - $n \neq m$
  - Otherwise, $\exists a_i (a_i \neq b_i)$ is true.

Cartesian products

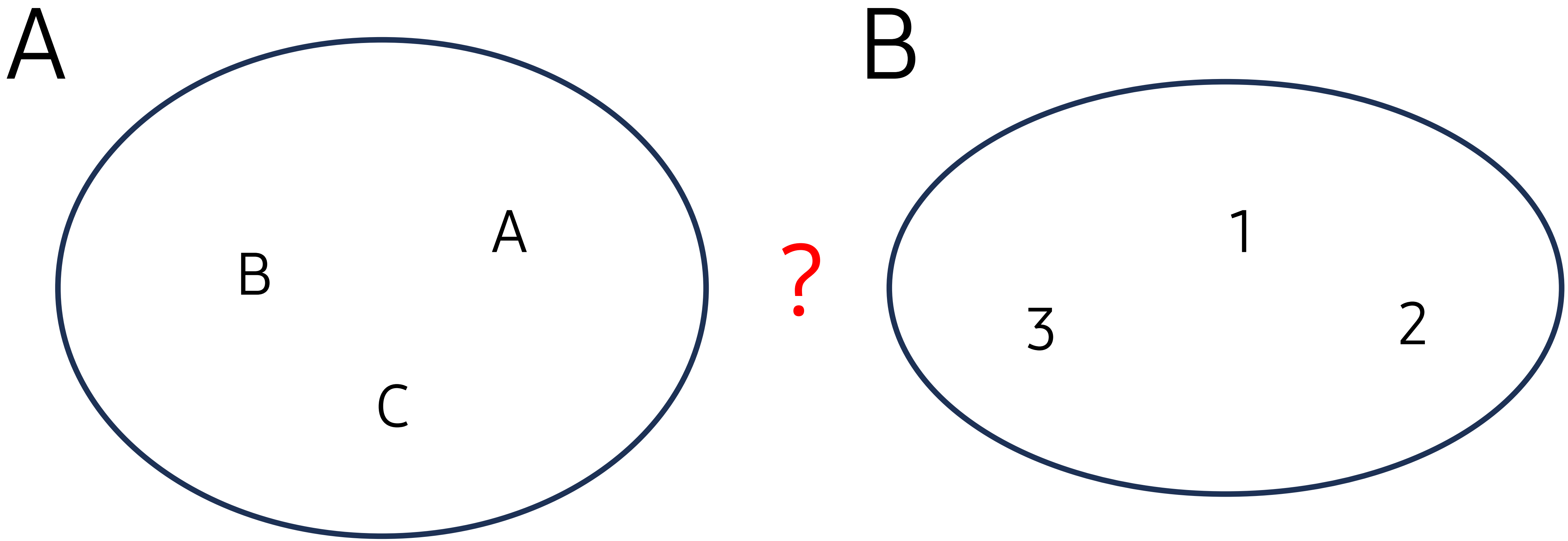
- The Cartesian product of sets A and B is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \land b \in B\}$$

Example

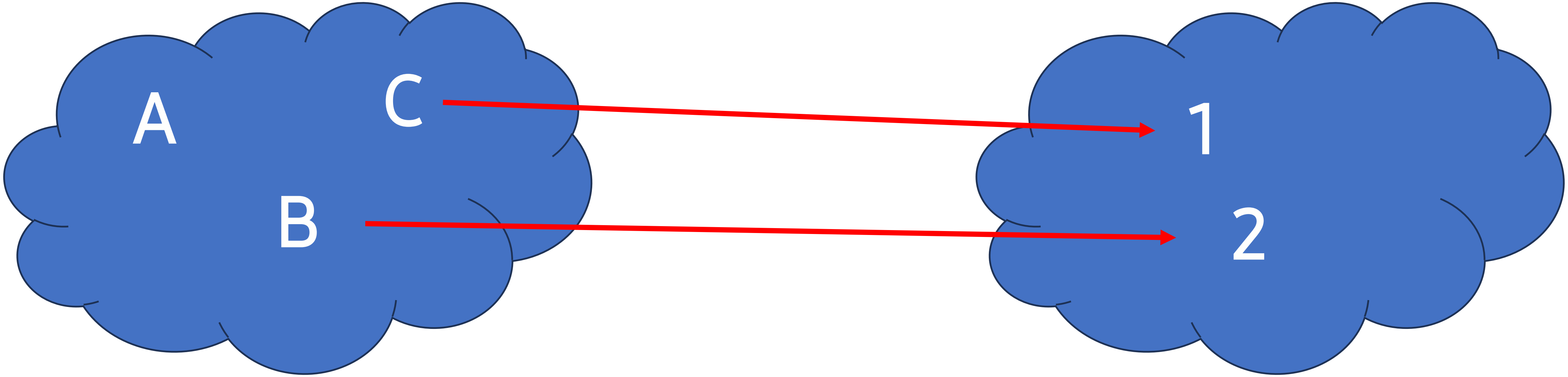
What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

- It’s not really useful having multiple separated sets.
- Relation is the structure that represent relationships.

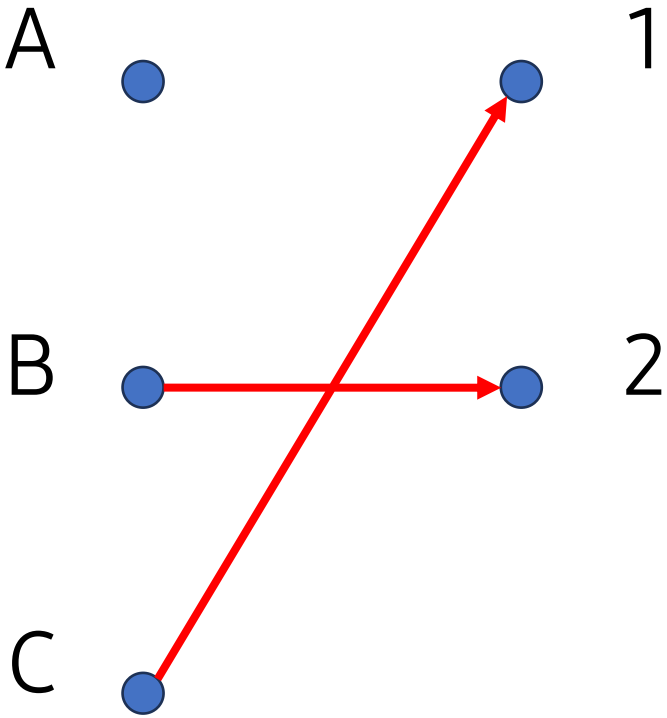


Relations and their properties

- A binary relation from A to B, which is a subset of $A \times B$, consists of some ordered pairs of objects: First element from A and second from B.



$$R = \{(C, 1), (B, 2)\} \subseteq A \times B$$



R	1	2
A		
B		✖
C	✖	

Special case of relation

- A **relation on a set** A is a relation from A to A.

$$R \subseteq A \times A$$

- Example: Given $A = \{1, 3\}$, a relation on A can be:

$$R = \{(1,1), (1,3), (3,1)\}$$

As R is the subset of $A \times A$:

$$A \times A = \{(1,1), (1,3), (3,1), (3,3)\}$$

Can we have relations for multiple sets?

- Yes! It's called **n-ary relation**.

$$R = A_1 \times A_2 \times \cdots \times A_n$$

- $A_1, A_2, \ldots, A_n$ are the **domains** of the relation, n is the **degree**.

Example

Let $A = \{2, 4, 5, 6, 8\}$

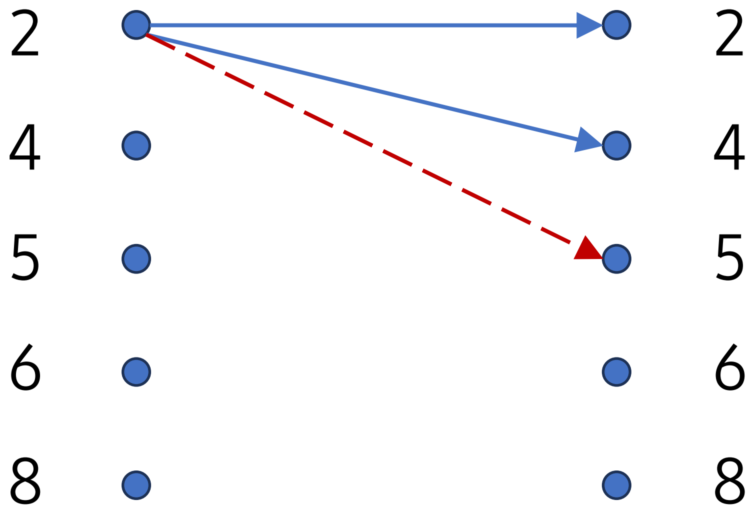
Find all the elements of relation $R = \{(a, b) | b \text{ completely divided by } a\}$

Example

Let $A = \{2, 4, 5, 6, 8\}$

Find all the elements of relation $R = \{(a, b)|b \text{ completely divided by } a\}$

Answer: $R = \{(2,2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8), (5, 5), (6, 6), (8, 8)\}$



Properties

- With $a, b, c \in A$, a relation R on set A is called:
  - **Reflexive**: $\forall a \rightarrow (a, a) \in R$.
  - **Symmetric**: $\forall a \forall b((a, b) \in R, a \neq b) \rightarrow (b, a) \in R$
  - **Antisymmetric**: $\forall a \forall b((a, b) \in R, a \neq b) \rightarrow (b, a) \notin R$
  - **Transitive**: $\forall a \forall b \forall c\big((a, b) \in R \wedge (b, c) \in R, a \neq b, b \neq c\big) \rightarrow (a, c) \in R$

Example

$$A = \{2, 4, 5, 6, 8\}$$

$$R = \{(2,2), (2,4), (2,6), (2,8), (4,4), (4,8), (5,5), (6,6), (8,8)\}$$

Reflexive?

$$\forall a((a, a) \in R)$$

Example

$$A = \{2, 4, 5, 6, 8\}$$

$$R = \{(2,2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8), (5, 5), (6, 6), (8, 8)\}$$

Symmetric?

$$\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$$

Example

$$A = \{2, 4, 5, 6, 8\}$$

$$R = \{(2,2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8), (5, 5), (6, 6), (8, 8)\}$$

Antisymmetric?

$$\forall a \forall b((a, b) \in R \rightarrow (b, a) \notin R)$$

Example

$$A = \{2, 4, 5, 6, 8\}$$

$$R = \{(2,2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8), (5, 5), (6, 6), (8, 8)\}$$

Transitive?

$$\forall a \forall b \forall c \big((a, b) \in R \wedge (b, c) \in R\big) \rightarrow (a, c) \in R$$

$A = \{1, 2, 3, 4\}$

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},

Determine and prove the properties of the above relations.

What if we can’t find any false statement?
Does that mean it’s true?

IV. Matrix of relation

- In order to demonstrate relation, we build a matrix.
  - Each row & column represents .
  - Each scalar contains binary value.

	a_1	…	a_n
a_1			
…			
a_n			

$$A = \{2, 4, 5, 6, 8\}$$
$$R = \{(2,2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8), (5, 5), (6, 6), (8, 8)\}$$

	2	4	5	6	8
2	1	1	0	1	1
4	0	1	0	0	1
5	0	0	1	0	0
6	0	0	0	1	0
8	0	0	0	0	1

	2	4	5	6	8
2	1	1	0	1	1
4	0	1	0	0	1
5	0	0	1	0	0
6	0	0	0	1	0
8	0	0	0	0	1

Reflexive

Diagonal

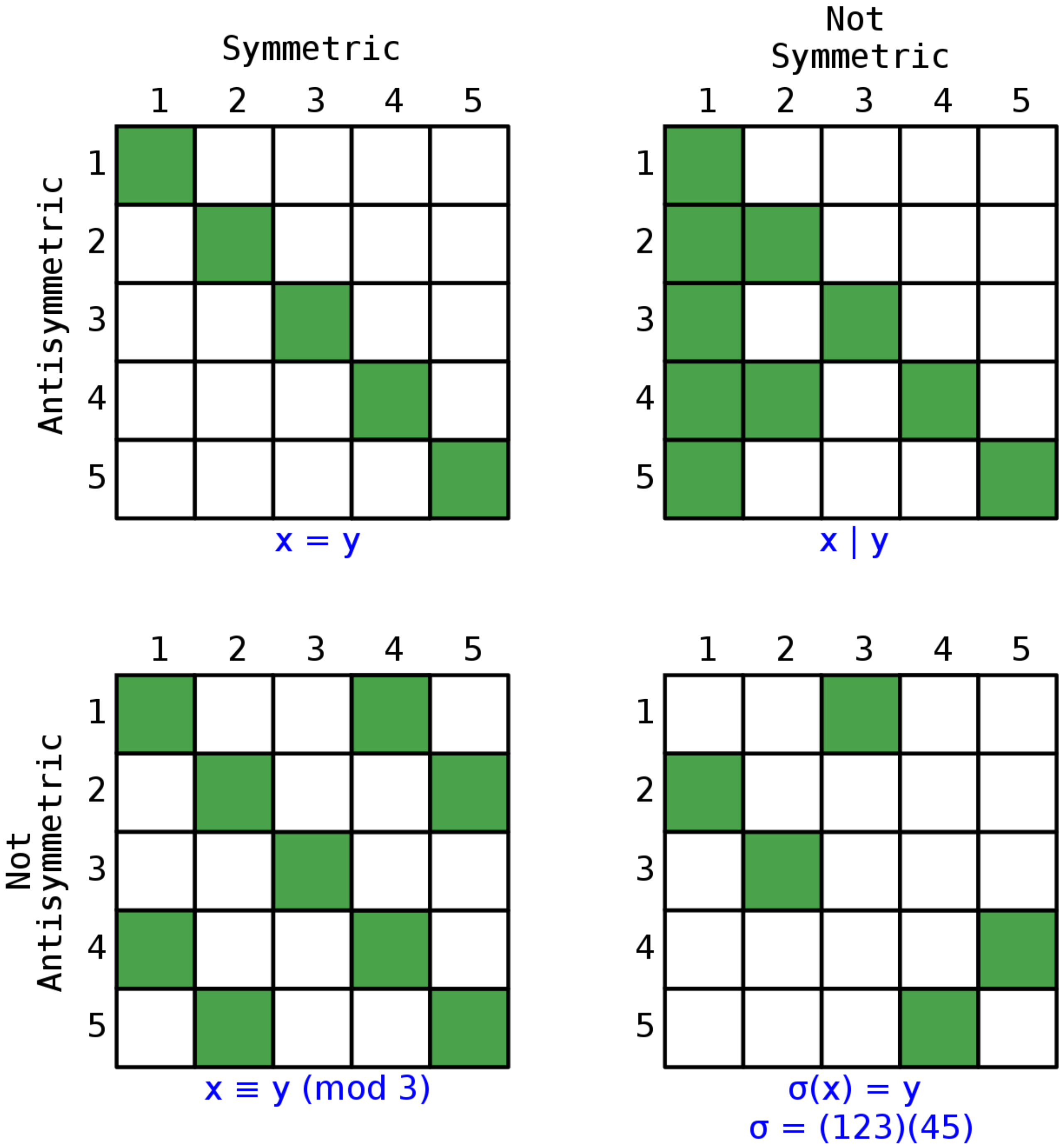
	2	4	5	6	8
2	1	1	0	1	1
4	0	1	0	0	1
5	0	0	1	0	0
6	0	0	0	1	0
8	0	0	0	0	1

Symmetric

Diagonal

	2	4	5	6	8
2	1	1	0	1	1
4	0	1	0	0	1
5	0	0	1	0	0
6	0	0	0	1	0
8	0	0	0	0	1

Antisymmetric


Symmetric
Not Symmetric
Antisymmetric
Not Antisymmetric
1 2 3 4 5
x = y
x | y
x ≡ y (mod 3)
σ(x) = y
σ = (123)(45)

$$A = \{2, 4, 5, 6, 8\}$$

$$R = \{(2,2), \boxed{(2,4)}, (2,6), \boxed{(2,8)}, (4,4), \boxed{(4,8)}, (5,5), (6,6), (8,8)\}$$

A =

	2	4	5	6	8
2	1	1	0	1	1
4	0	1	0	0	1
5	0	0	1	0	0
6	0	0	0	1	0
8	0	0	0	0	1

Transitive

$$A = \{2, 4, 5, 6, 8\}$$

$$R = \{(2,2), (2,4), \boxed{(2,6)}, \boxed{(2,8)}, (4,4), \boxed{(4,8)}, (5,5), (6,6), (8,8)\}$$

A =

	2	4	5	6	8
2	1	1	0	1	1
4	0	1	0	0	1
5	0	0	1	0	0
6	0	0	0	1	0
8	0	0	0	0	1

Transitive

$$A = \{2, 4, 5, 6, 8\}$$

$$R = \{(2,2), (2,4), \boxed{(2,6)}, \boxed{(2,8)}, (4,4), \boxed{(4,8)}, (5,5), (6,6), (8,8)\}$$

A =

	2	4	5	6	8
2	1	1	0	1	1
4	0	1	0	0	1
5	0	0	1	0	0
6	0	0	0	1	0
8	0	0	0	0	1

Will be transitive if this is 1, or *(6, 8)* exists

Transitive

$$AA = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$AA = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
\begin{array}{cccccc}
 & & 4 & & & 8 \\
2 & 1 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 0 & 0 & 1 \\
AA=B= & 0 & 0 & 1 & 0 & 0 \\
 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{ccccc}
1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}
$$

$B_{red} > 0$ means (a, b) and (b, c) exists!

$$AA=B=\begin{array}{cccccc}
 & & 4 & & & 8 \\
2 & 1 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 0 & 0 & 1 \\
 & 0 & 0 & 1 & 0 & 0 \\
 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{ccccc}
1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}$$

$B_{red} > 0$ means (a, b) and (b, c) exists!

Moreover, if $A_{red} \neq 0$, R is transitive.

Step-by-step

1. **Square matrix $B = AA$.**
2. **Check if there is any entry equal non-zero in B**

At the equivalent position in A, does the entry has zero?

If yes => Non-transitive.

3. **If no entry meets (2) => Transitive.**

There's more...

- A relation is **equivalent** if it's **reflexive, symmetric and transitive**.
- A relation is **partial ordering** if it's **reflexive, antisymmetric and transitive.**
- You can use directed graph instead of matrix, but it can be tough in some situations.

Operators for relations

- Union:
  - $a \in (R_1 \cup R_2)$ if $\forall a(a \in R_1 \vee a \in R_2)$
- Intersection:
  - $a \in (R_1 \cap R_2)$ if $\forall a(a \in R_1 \wedge a \in R_2)$
- Minus:
  - $a \in (R_1 - R_2)$ if $\forall a(a \in R_1 \wedge a \notin R_2)$

Summary

- Propositional function = Predicate(s) + variable(s).
- the function becomes a proposition if:
  - variable(s) are set.
  - Quantifier(s) are used: All, some, at least one, none of,…
- Sets and tuples are collections:
  - Sets are unordered.
  - Tuples are in order.
- Relation is a subset (or equal) of Cartesian product.
- 4 properties of relation, which can be proved by:
  - A false statement from the function ($f(\ldots) \rightarrow 0$).
  - Matrix of relation.

Homework

1. Write a C/C++ console app that:
    - Read set A (2pt).
    - Read relation R on set A from input in matrix form (2pt).
    - Verify these properties: Reflexive & symmetric (1pt).

#2

$A = \{1, 2, 3, 4\}$

R_2 = {(1, 1), (1, 2), (2, 1)},

R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},

R_4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},

R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},

R_6 = {(3, 4)}

Determine and prove the properties of the above relations by giving a false statement.