Day 5
Boolean algebra

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Outline

I. Operators, expressions, functions
II. Logic gates & layout tips
III. Minimization
IV. SOP expansion

Warm up

- What is Boolean actually?
- In which scenarios Boolean is useful?

Some operators in Boolean algebra

Order	Propositional logic	Boolean algebra	Definition
1	$\neg$, $-$	$-$	NOT/Complementation
2	$\wedge$	$\cdot$ (Can be omitted)	AND/Boolean product
3	$\vee$	$+$	OR/Boolean sum

Note: Calculate by order

Expression vs. function

$$x \cdot 0 + \overline{0 + y}$$

An expression with variables

$$f(x, y) = x \cdot 0 + \overline{0 + y}$$

A function

Find all values of Boolean function

x	y	$0+y$	$\overline{0+y}$	$x \cdot 0$	$f(x,y)$
1	1	1	0	0	0
1	0	0	1	0	1
0	1	1	0	0	0
0	0	0	1	0	1

$$f(x,y) = x \cdot 0 + \overline{0+y}$$

Laws (identities)

No.	Law	Expression
1	$\overline{\overline{x}} = x$	Negative of negative (double complement)
2	$x + x = x$ $x \cdot x = x$	Idempotent
3	$x + 0 = x$ $x \cdot 1 = x$	Identity
4	$x + 1 = 1$ $x \cdot 0 = 0$	Domination
5	$x + y = y + x$ $xy = yx$	Commutative

No.	Law	Expression
6	$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative
7	$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive
8	$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\overline{y}$	De Morgan
9	$x + xy = x$ $x(x + y) = x$	Absorption
10	$x + \overline{x} = 1$	Unit property
11	$x\overline{x} = 0$	Zero property

A *Boolean algebra* is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation $\overline{}$ such that these properties hold for all x, y, and z in B:

$$\left.\begin{aligned} x \vee 0 &= x \\ x \wedge 1 &= x \end{aligned}\right\} \quad \text{Identity laws}$$

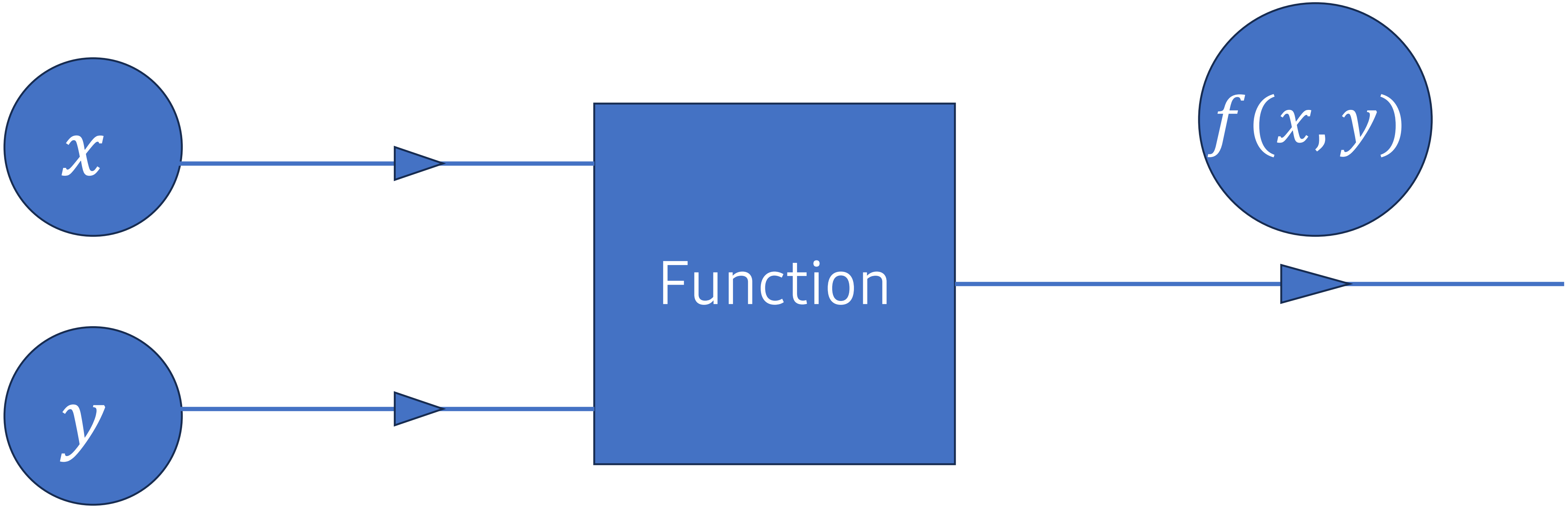
$$\left.\begin{aligned} x \vee \overline{x} &= 1 \\ x \wedge \overline{x} &= 0 \end{aligned}\right\} \quad \text{Complement laws}$$

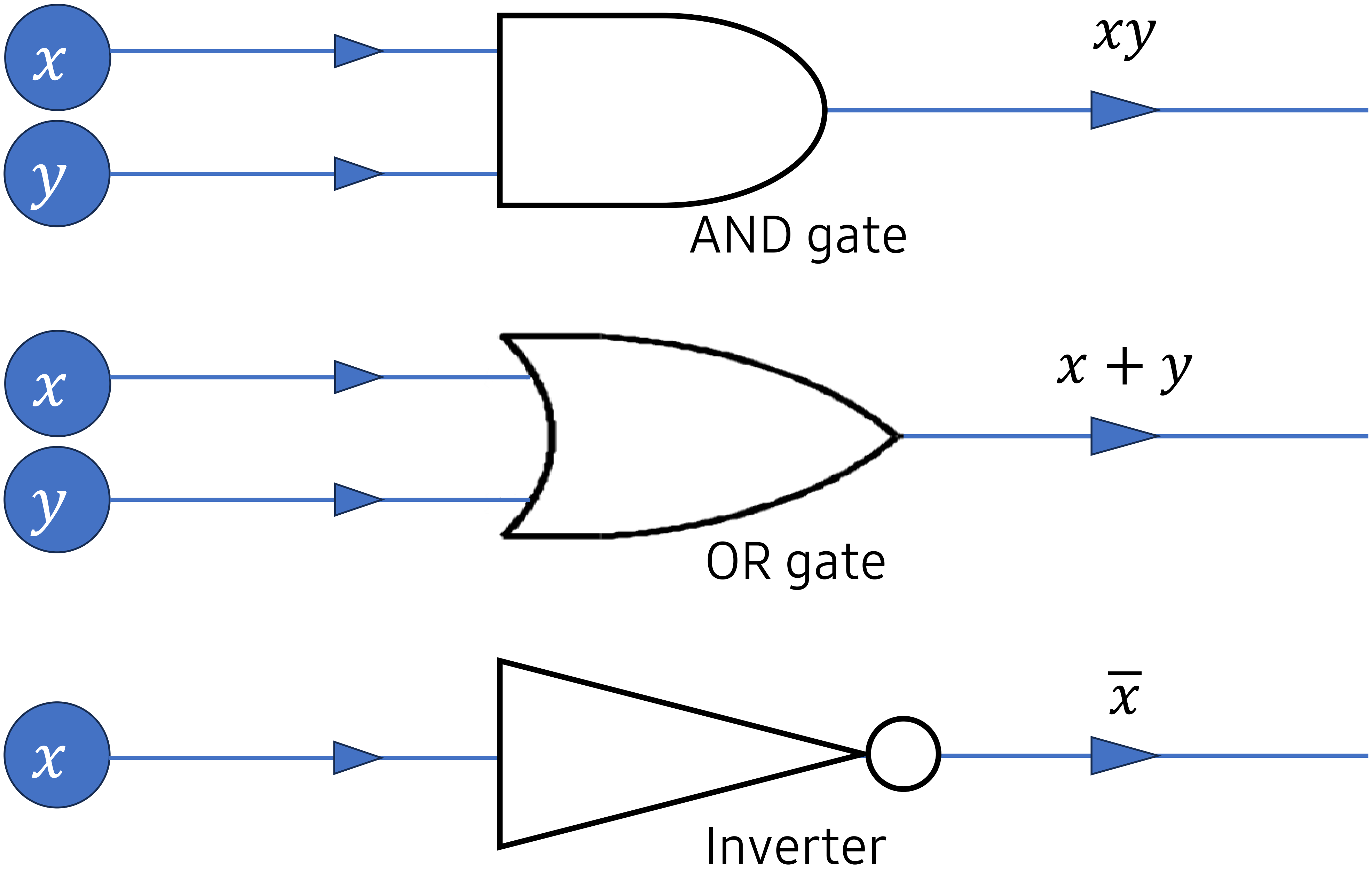
$$\left.\begin{aligned} (x \vee y) \vee z &= x \vee (y \vee z) \\ (x \wedge y) \wedge z &= x \wedge (y \wedge z) \end{aligned}\right\} \quad \text{Associative laws}$$

$$\left.\begin{aligned} x \vee y &= y \vee x \\ x \wedge y &= y \wedge x \end{aligned}\right\} \quad \text{Commutative laws}$$

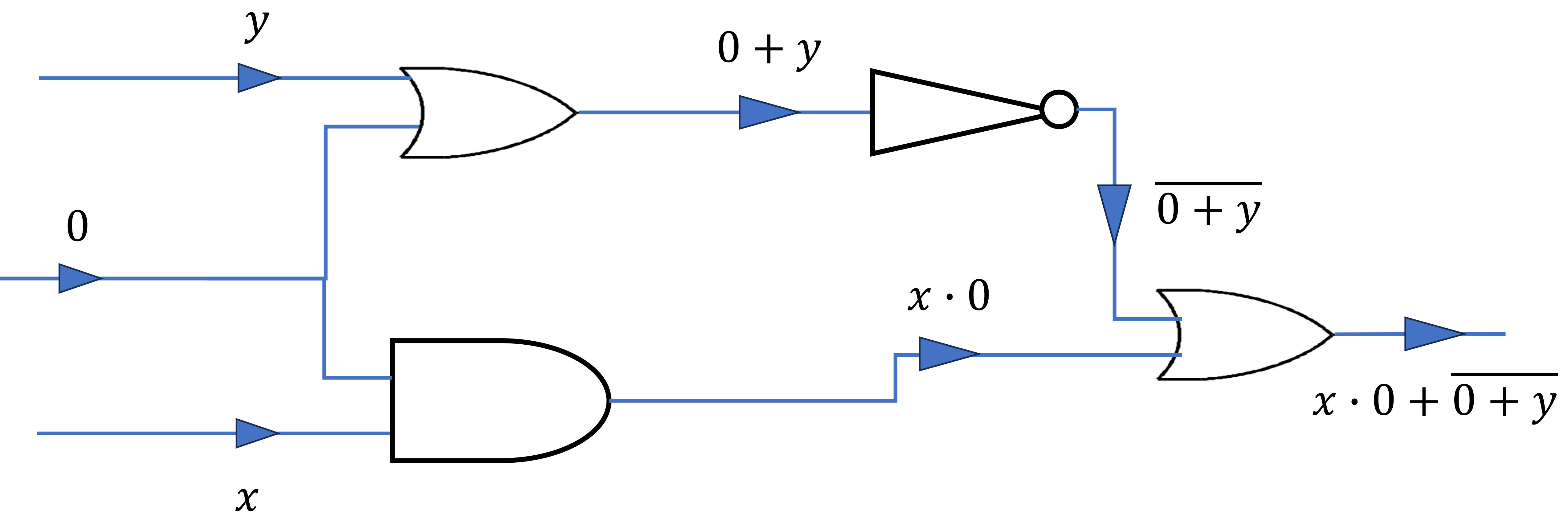
$$\left.\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \end{aligned}\right\} \quad \text{Distributive laws}$$

p. 817, Discrete Mathematics & It's Application – Kenneth H. Rosen


x
y
Function
$f(x, y)$



x
y
xy
AND gate
x
y
$x + y$
OR gate
x
$\overline{x}$
Inverter

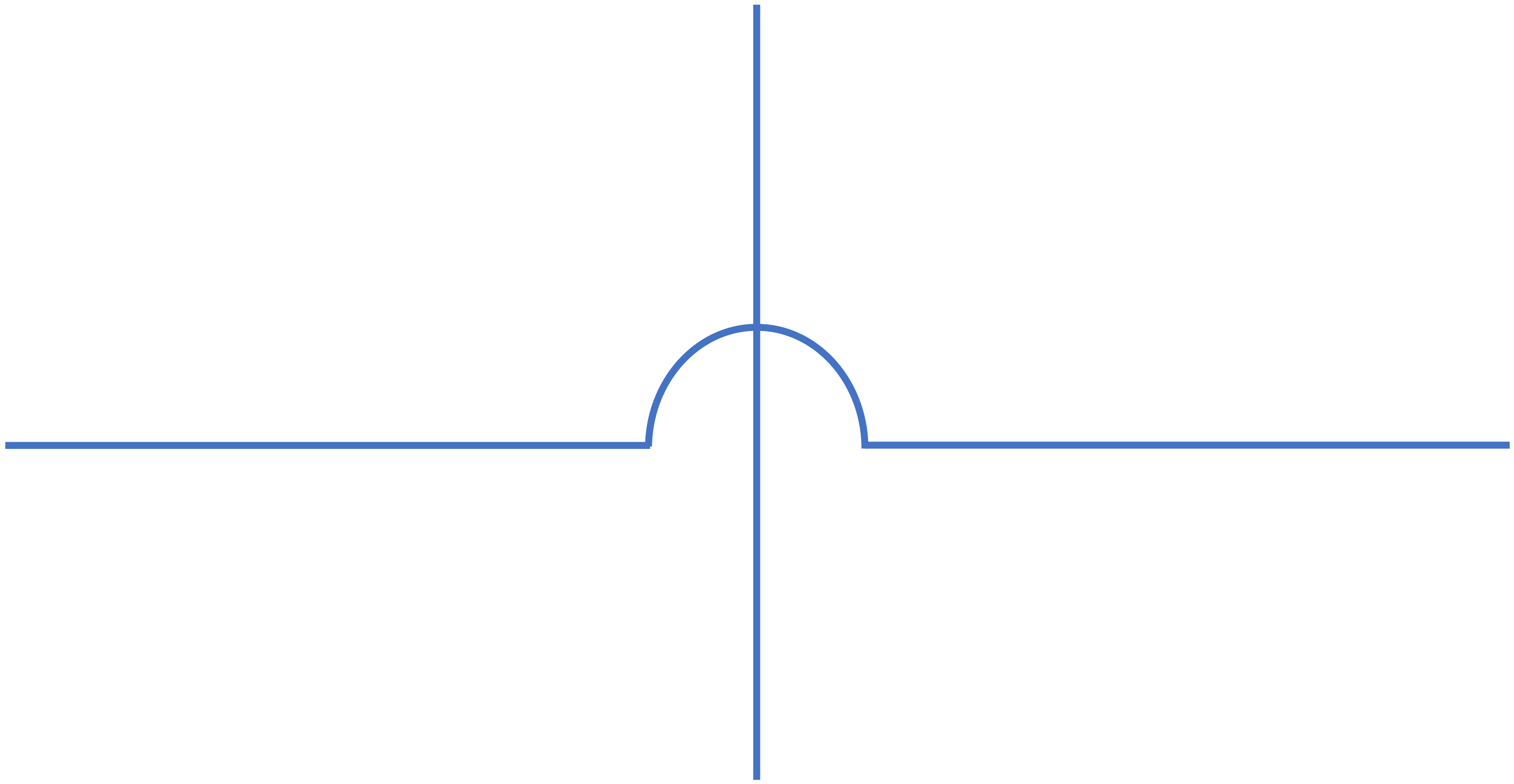


$$f(x, y) = x \cdot 0 + \overline{0 + y}$$

Layout problems

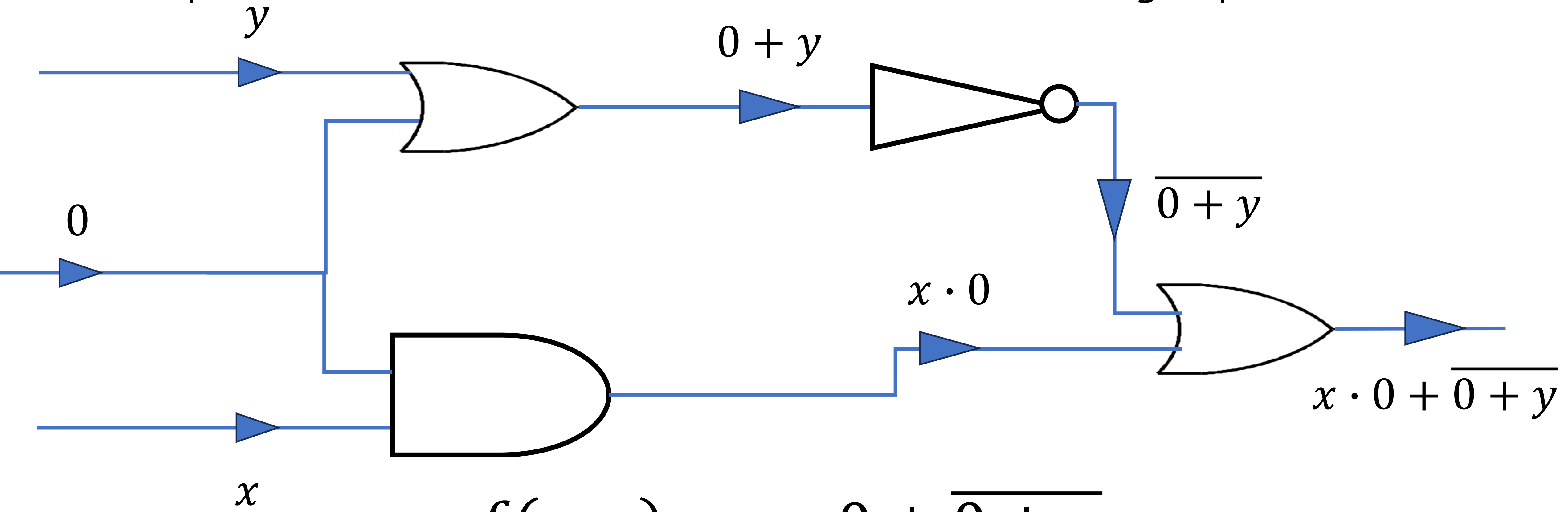
- How to improve readability?
    - Easier for student to double-check.
    - Easier for teacher to grade.
    - => Best of both worlds!

#1. Use bridge



#2. Estimate the complexity

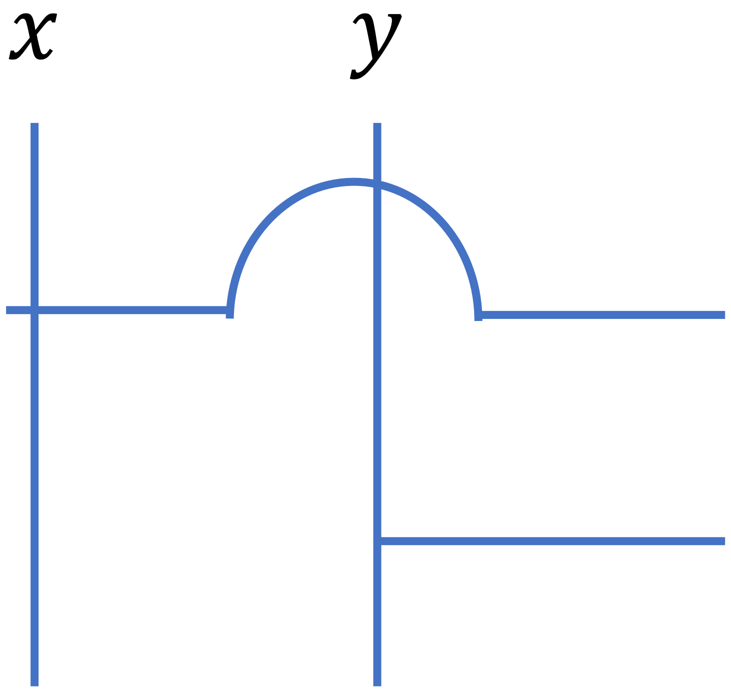
- Tips: First calculated variables should have enough spaces.



$$f(x, y) = x \cdot 0 + \overline{0 + y}$$

#3. Initial lines can be horizontal/vertical

- Vertical layout is for function with reused variable(s).
- This is to make sure the circuit is readable.



Exercise

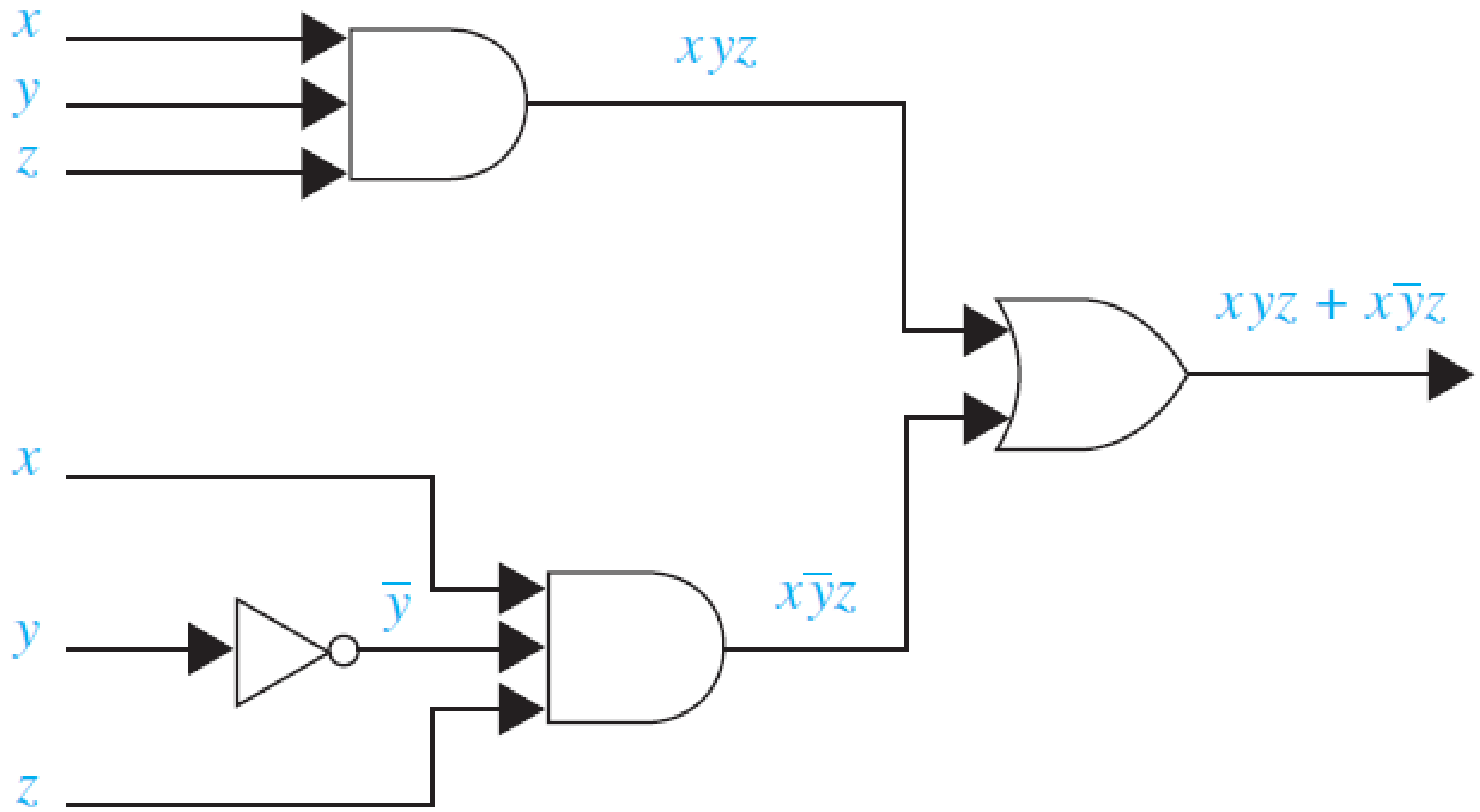
Draw the circuit of the below function:

$$f(x, y, z) = xy + \overline{x}z$$

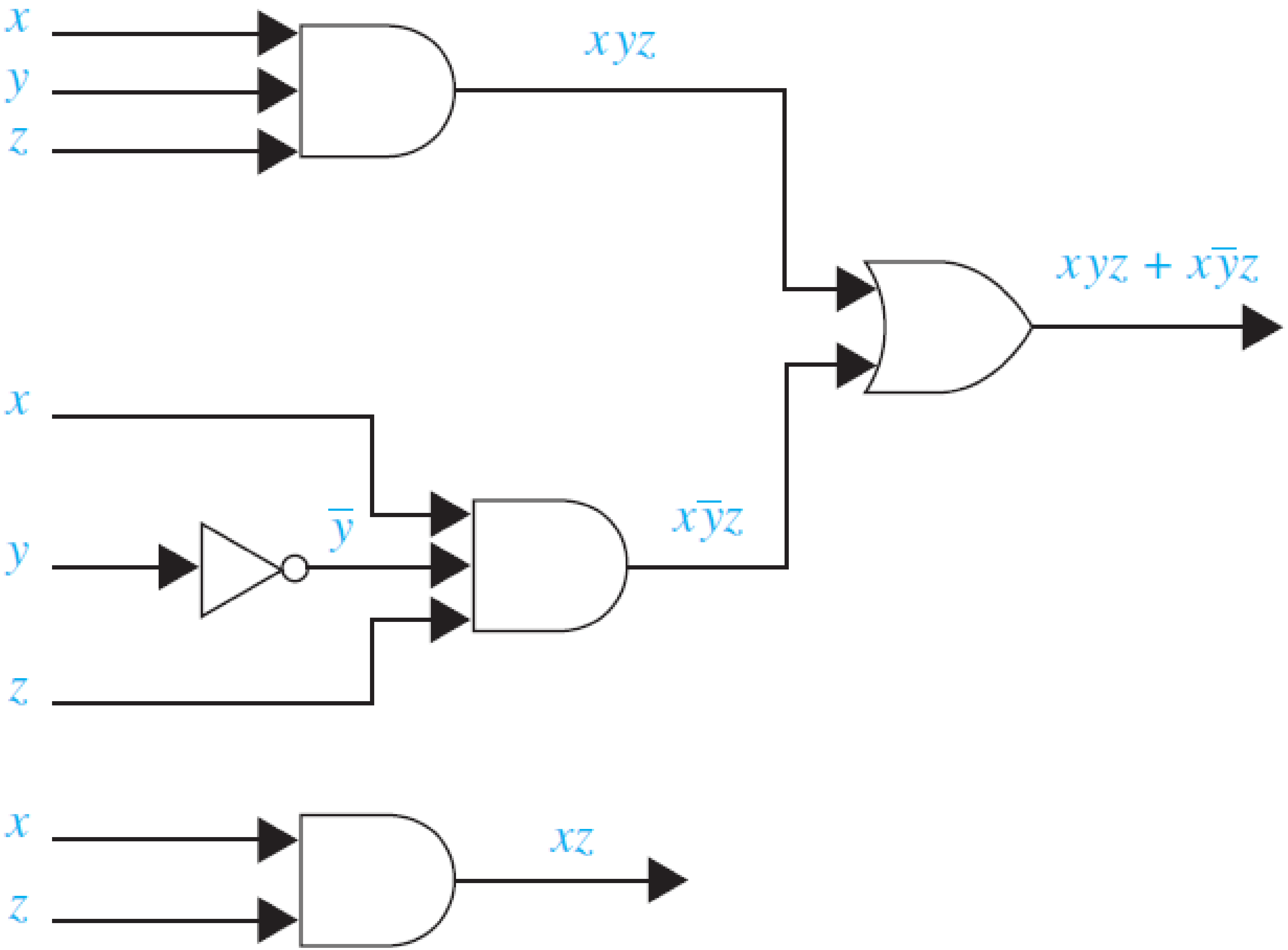
It's quite obvious...

- The more complicated the function is, the bigger the circuit is.
- Can we **minimize** the circuit by **simplifying** the function?
- Let's try this function:

$$f(x, y, z) = xyz + x\overline{y}z$$


x
y
z
xyz
xyz + x$\overline{y}$z
x
$\overline{y}$
y
x$\overline{y}$z
z

$$f(x, y, z) = xyz + x\overline{y}z$$



x
y
z
xyz
xyz + x$\bar{y}$z
x
y
$\bar{y}$
z
x$\bar{y}$z
x
z
xz

Function types

- A **sum of products** (SOP) are multiple product terms (*) which are added (+) later.
  - Also called **sum of minterm**.
- A **product of sum** (POS) are multiple sum terms (+) which are producted (*) later.
  - Also called **product of maxterm**.

- Our 1st goal is to find the simplest SOP expression.
- There’s a graphical method known as Karnaugh map or K-map.
- Drawback: Function with 4 variables or less is recommended.

$$f(x, y) = xy + \overline{x}y$$

Step 1: Truth table (optional)

$$f(x,y) = xy + \overline{x}y$$

$\boldsymbol{x}$	$\boldsymbol{y}$	$\overline{\boldsymbol{x}}$	$\boldsymbol{xy}$	$\overline{\boldsymbol{x}}\boldsymbol{y}$	$\boldsymbol{f(x,y)}$
1	1	0	1	0	1
1	0	0	0	0	0
0	1	1	0	1	1
0	0	1	0	0	0

Step 2: Create Karnaugh map

$$f(x, y) = xy + \overline{x}y$$

x	y	$\overline{x}$	xy	$\overline{x}y$	f
1	1	0	1	0	1
1	0	0	0	0	0
0	1	1	0	1	1
0	0	1	0	0	0

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{y} \\
 & & 1 & 0 \\
\hline
\multirow{2}{*}{x} & 1 & 1 & 0 \\
\cline{3-4}
 & 0 & 1 & 0 \\
\hline
\end{array}$$

Step 3: Find the (large) cell (1's area)

$$f(x, y) = xy + \overline{x}y$$

x	y	$\overline{x}$	xy	$\overline{x}y$	f
1	1	0	1	0	1
1	0	0	0	0	0
0	1	1	0	1	1
0	0	1	0	0	0

		y	
		1	0
x	1	1	0
	0	1	0

Step 4: Find the minimum sum-of-product that satisfies the map

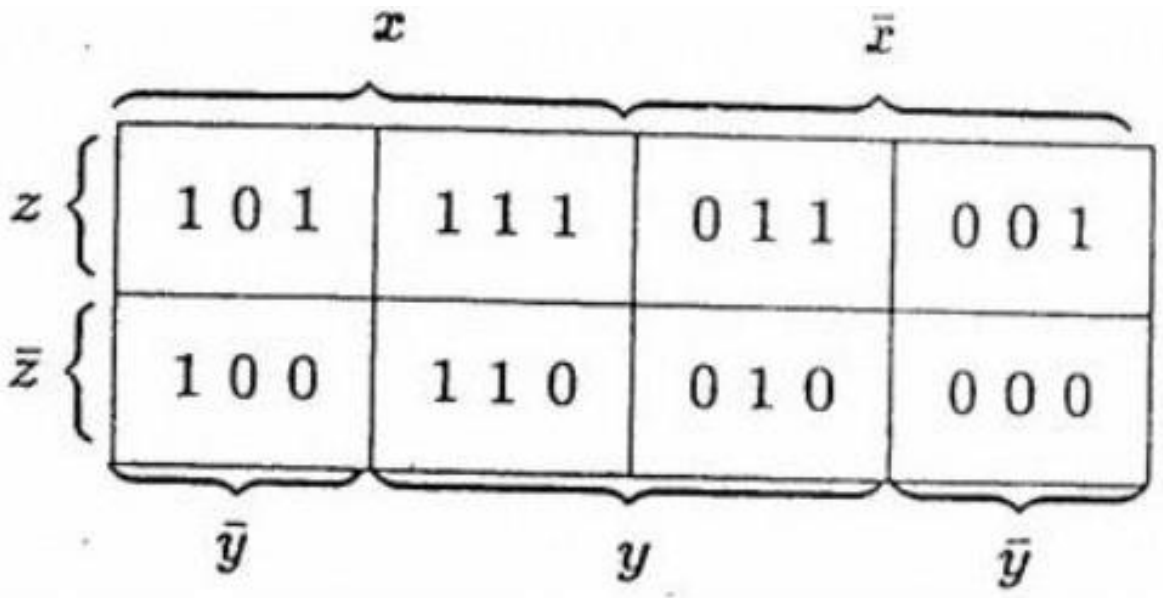
$$f(x, y) = xy + \bar{x}y$$

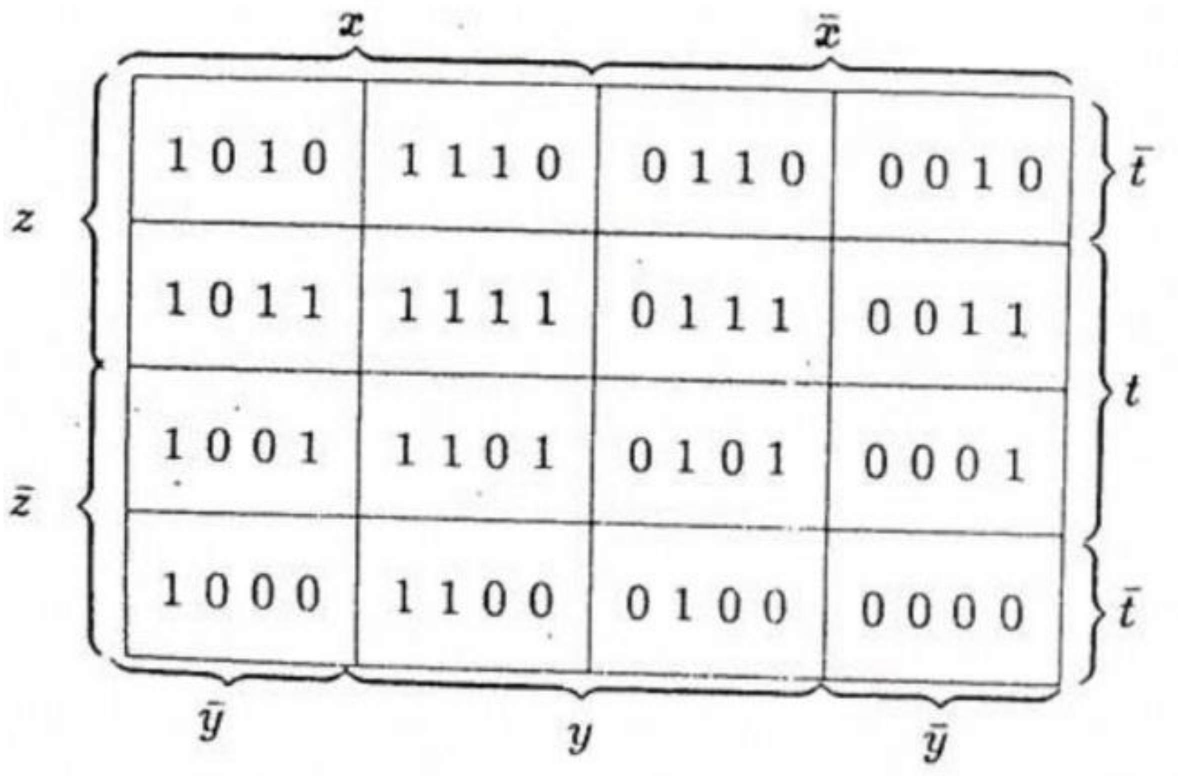
x	y	$\bar{x}$	xy	$\bar{x}y$	f
1	1	0	1	0	1
1	0	0	0	0	0
0	1	1	0	1	1
0	0	1	0	0	0

		y	
		1	0
x	1	1	0
	0	1	0

$$f(y) = y$$

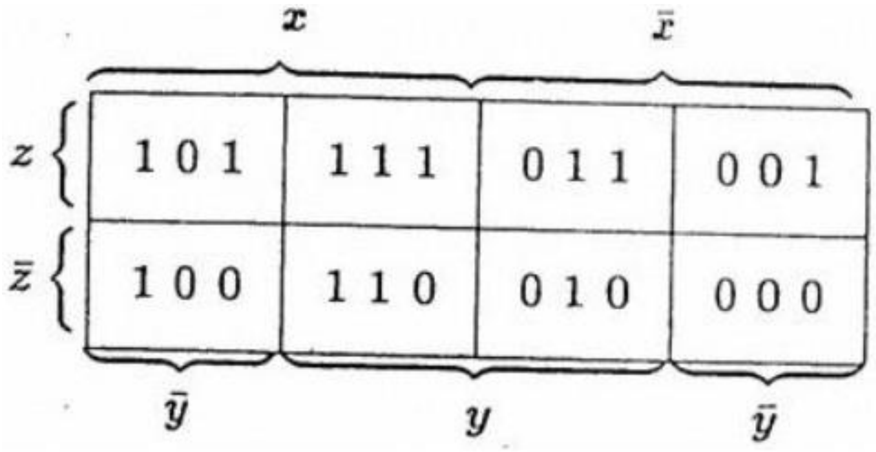
Map with n variables





$$f(x, y, z) = xyz + x\overline{y}z$$

$$f(x,y,z) = xyz + x\overline{y}z$$

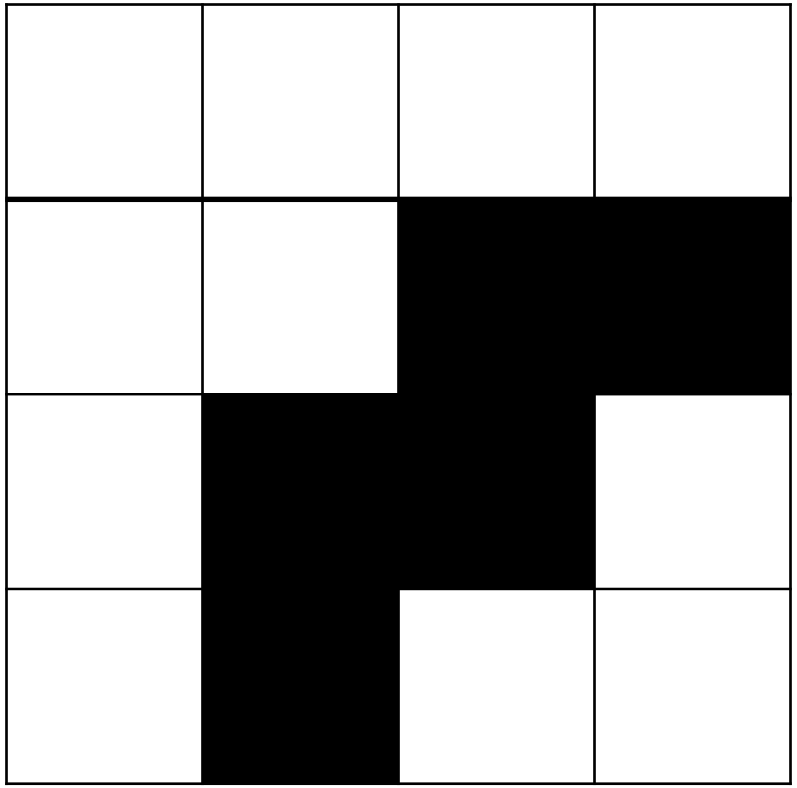


1	1	0	0
0	0	0	0

$$f(x, y, z) = xyz + x\overline{y}z$$

	x		$\overline{x}$	
z	1	1	0	0
$\overline{z}$	0	0	0	0
	$\overline{y}$	y	y	$\overline{y}$

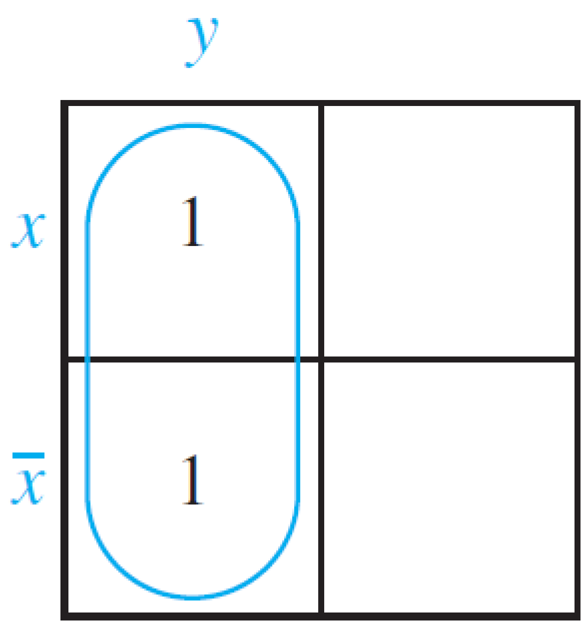
Exercise

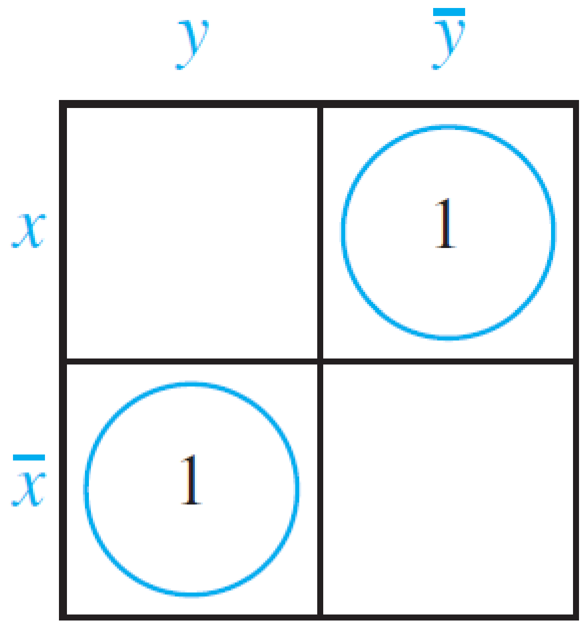


Notes: Large cells in same 1’s area are overlapped.

Notes

- Cells must be formed by squares in horizontal or vertical.
- Cells can be partially or fully overlapped.
- Combine multiple large cells using + (OR) => SOP.





Summary of grouping rules

1. No zeros allowed.
2. No diagonals.
3. Only power of 2 (2^n) number of cells in each group.
4. Group is expanded as large as possible.
5. Every 1 (black) must be in at least one group.
6. Overlapping allowed.
7. Wrapping around is allowed.
8. Fewest number of groups possible.

So... how to find POS?

- Our 2nd goal: Find POS.
- Every Karnaugh steps are kept, except:
  - For POS, we put 0 in grouped blocks.

$$f(x, y) = x\overline{y} + \overline{x}y + \overline{xy}$$

Simplify the above function using Karnaugh map

So far, we are solving problems with pre-defined SOP expression

But some may ask you to find one.

Example: Find the function $F(x, y, z)$

satisfying the truth table.

x	y	z	F
1	1	1	0
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

#1 assuming $F(x, y, z)$ only has AND and negation

- Something like:
  - xyz
  - $\overline{x}yz$
  - $\overline{xy}z$
  - …

#2: Assuming $F(x, y, z) = xyz$, which one is wrong?

Tips: Focus rows that $F = 1$

x	y	z	F
1	1	1	0
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

x	y	z		G
1	1	1		0
1	1	0		1
1	0	1		0
1	0	0		0
0	1	1		0
0	1	0		1
0	0	1		0
0	0	0		0

#2: Assuming $G(x, y, z) = xyz$, which one is wrong?

x	y	z		G
1	1	1		0
1	1	0		1
1	0	1		0
1	0	0		0
0	1	1		0
0	1	0		1
0	0	1		0
0	0	0		0

Homework

- 6, 7, 12, 13/841, 842 (group)
- Submission: Docx file.