Day 4
Advanced sets

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Warm up

- Check if the below proposition is true:

$$\forall x, (P(x) \vee Q(x)) \wedge (\neg P(x) \vee R(x)) \vee (\neg Q(x) \wedge \neg R(x)) \equiv 1$$

Outline

I. Special elements in Hasse diagram
II. Infimum, Supremum, and Lattice
III. Equivalence class
IV. Mappings
V. Laws

Revision

Set/Tuple/POSET

Relation

Reflexive

Symmetric

Antisymmetric

Transitive

Partial ordering

Equivalent

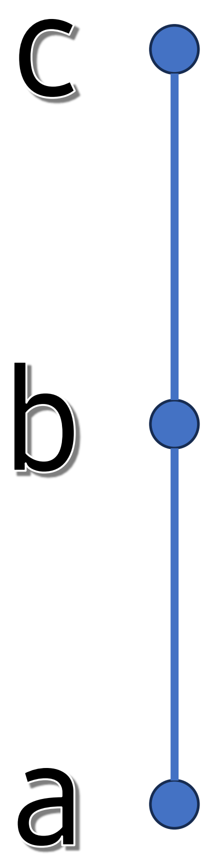
Hasse diagram

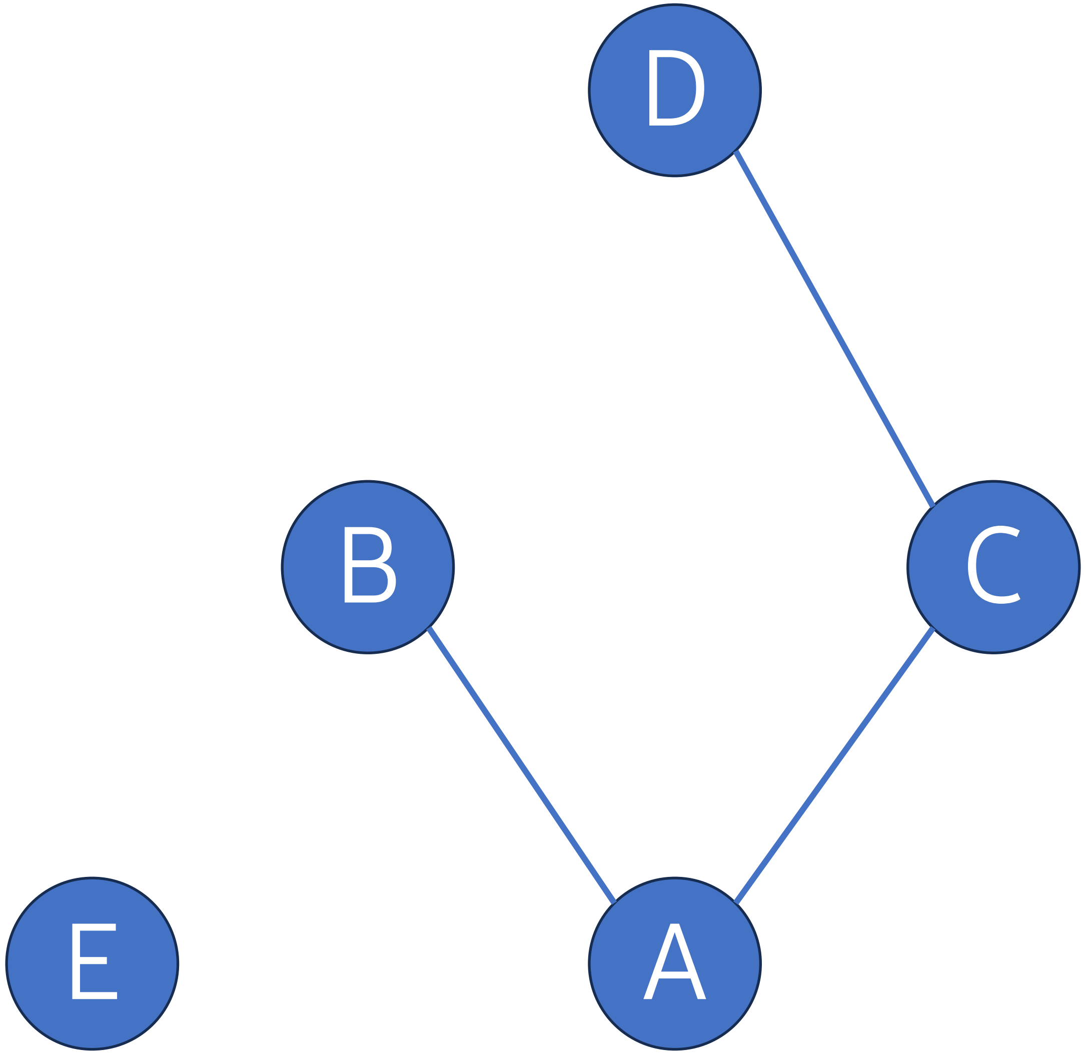
More to remember

- If R is partial ordering on A:
  - A is a partial ordering set (POSET)
  - $(A, \prec)$
  - $\prec$ is the relation expression

More to remember

- If R is partial ordering on A:
  - A is a partial ordering set (POSET)
  - $(a, b) \in R \Leftrightarrow a \prec b$ or $a \preccurlyeq b$
  - $(A, \prec)$ or $(A, \preccurlyeq)$: A is POSET
  - $\forall B \subset A, (B, \prec)$ or $(B, \preccurlyeq)$

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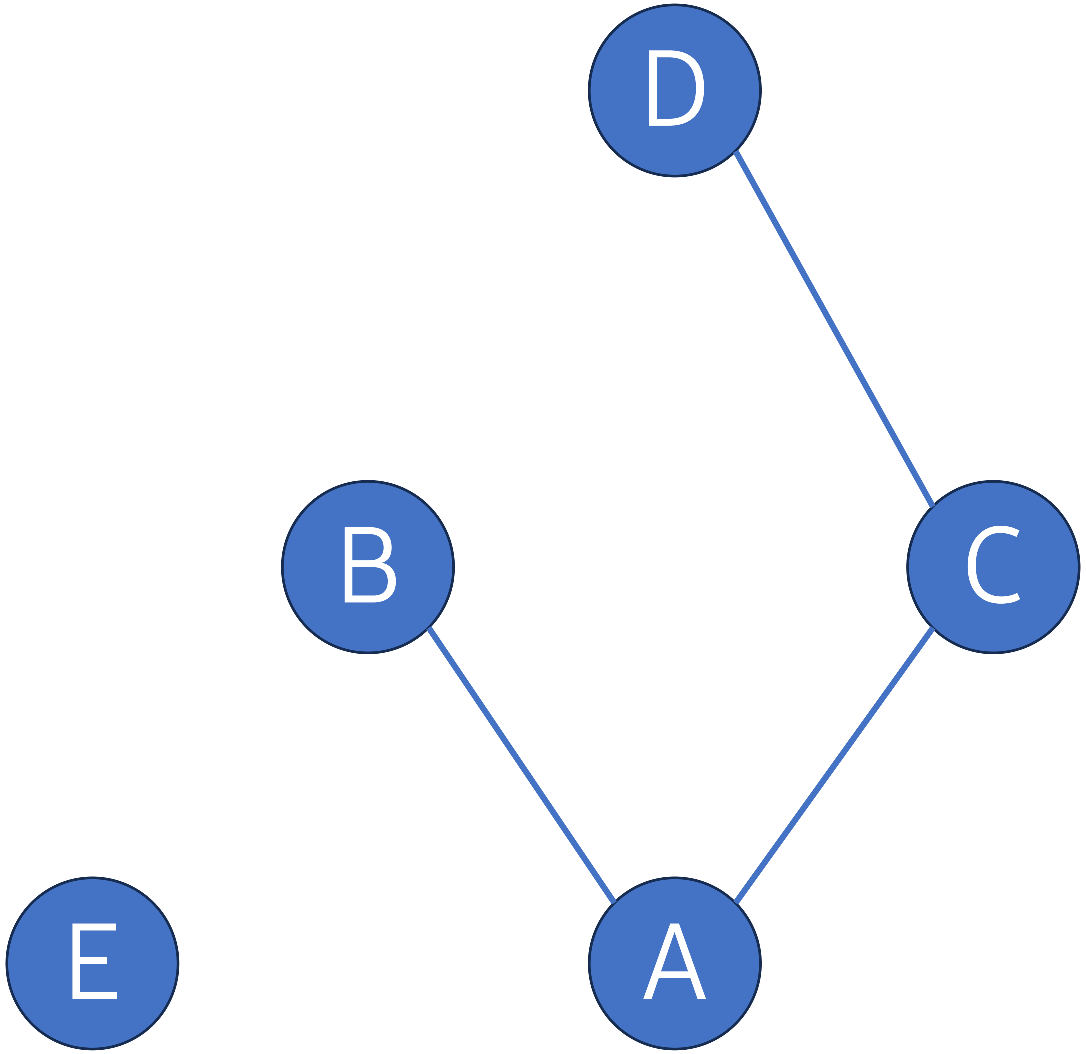
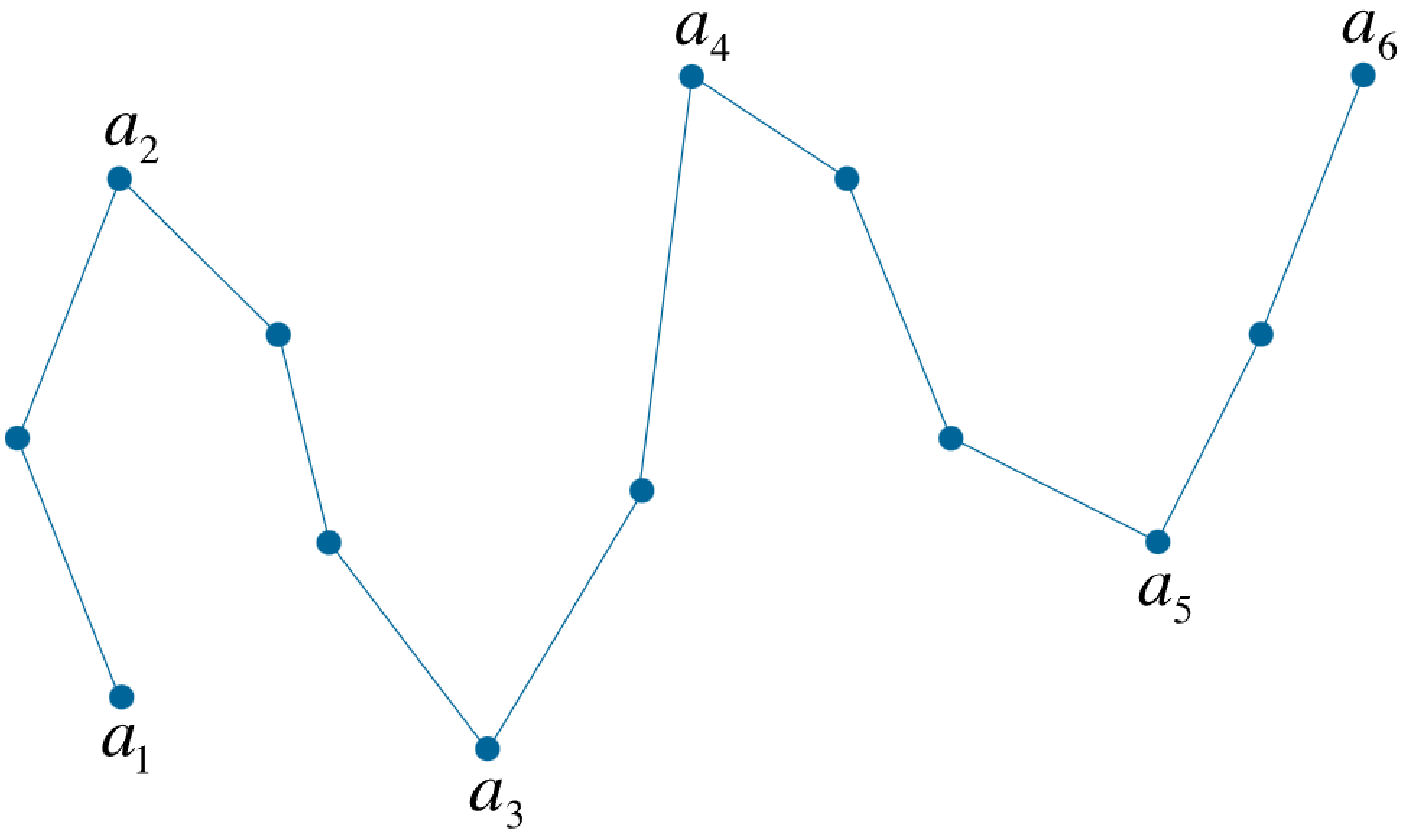
A



- a is **maximal element** if $\nexists b \in A, a \prec b$
  - Opposite of **minimal element**
- a is **greatest element** if $\forall b \in A, b \prec a$
  - Opposite of **least element**

- A is the **minimal element.**
- D is the **maximal element.**
- There is no **greatest element.**
- There is no **least element.**



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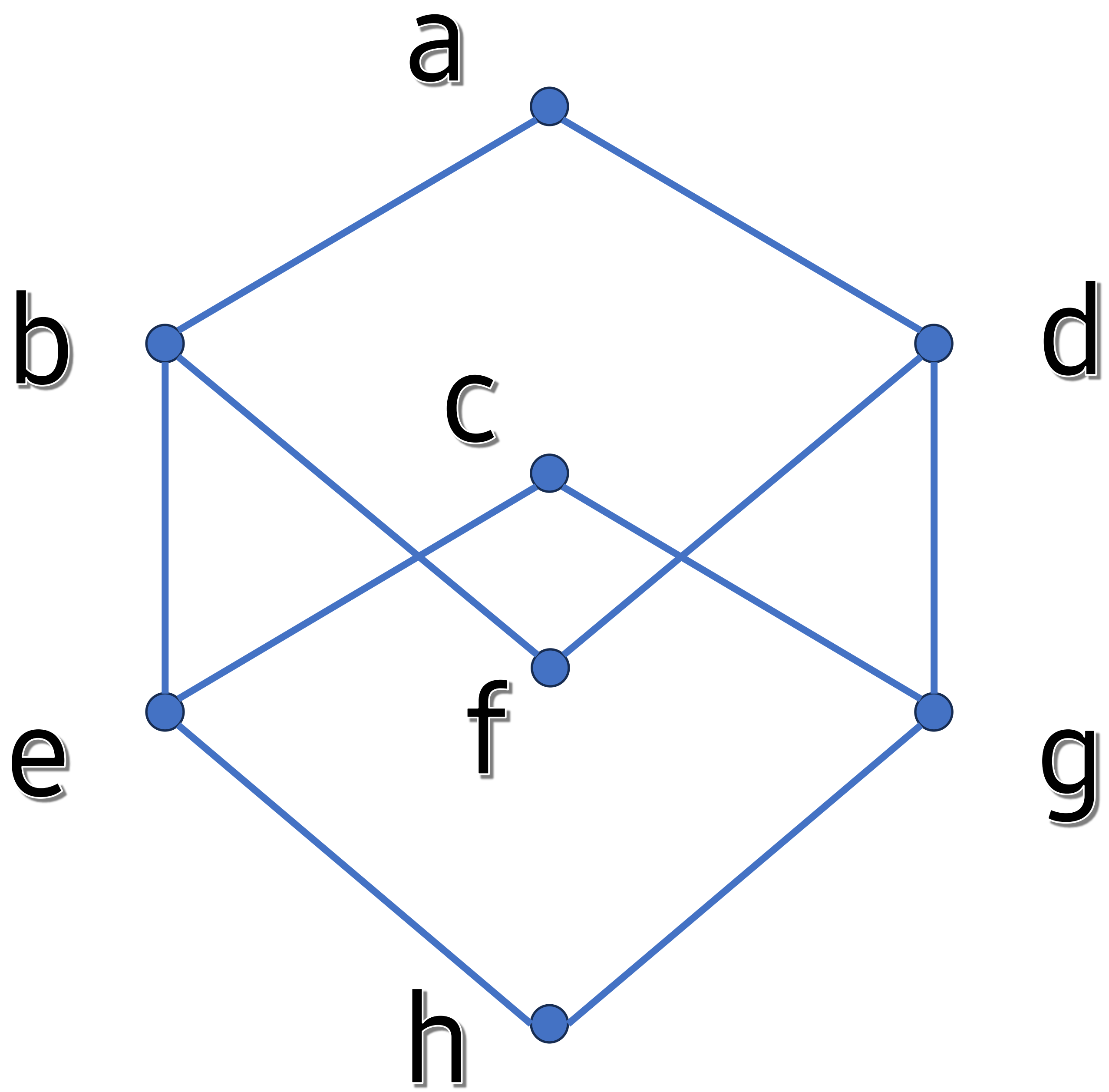
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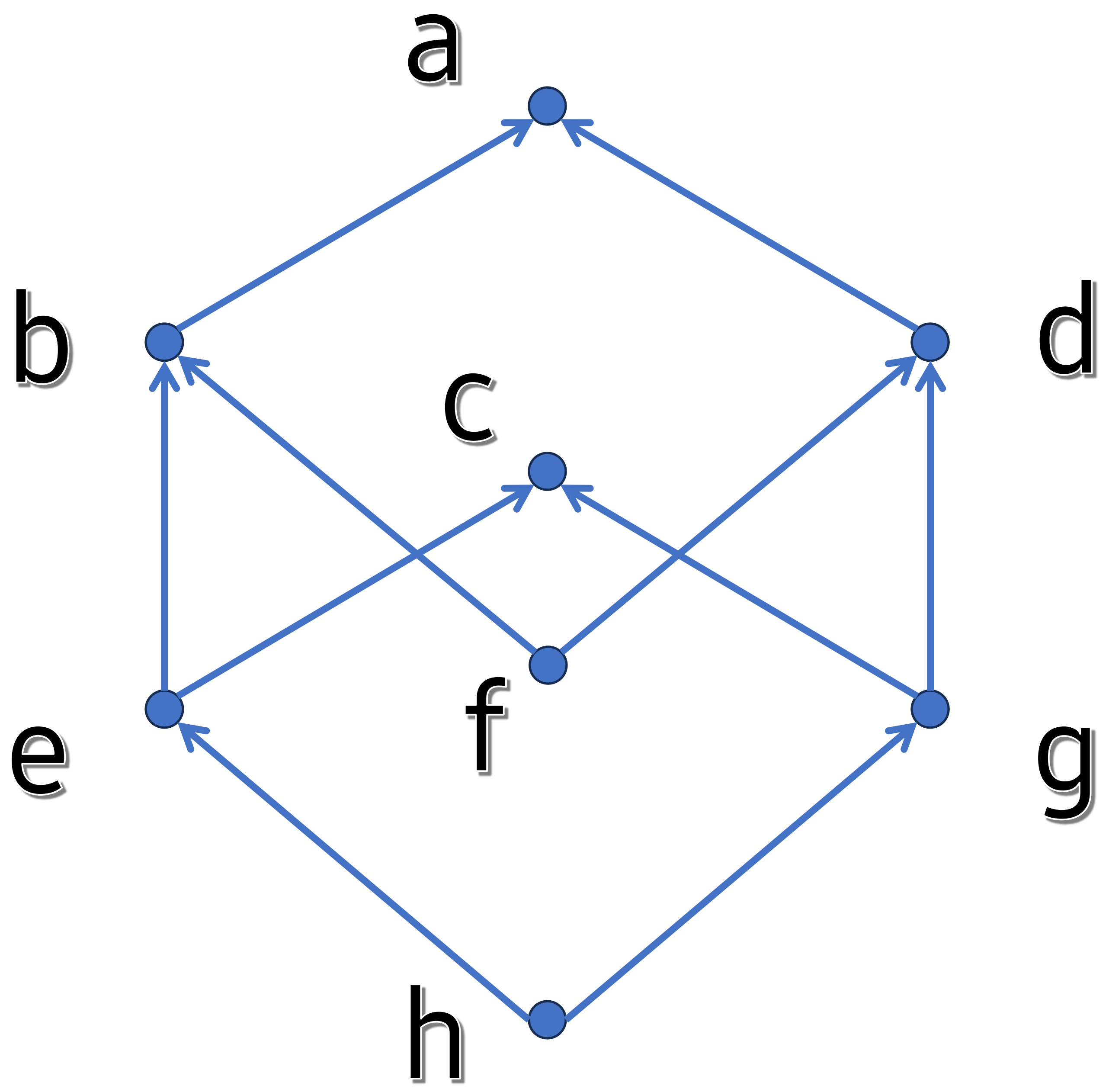
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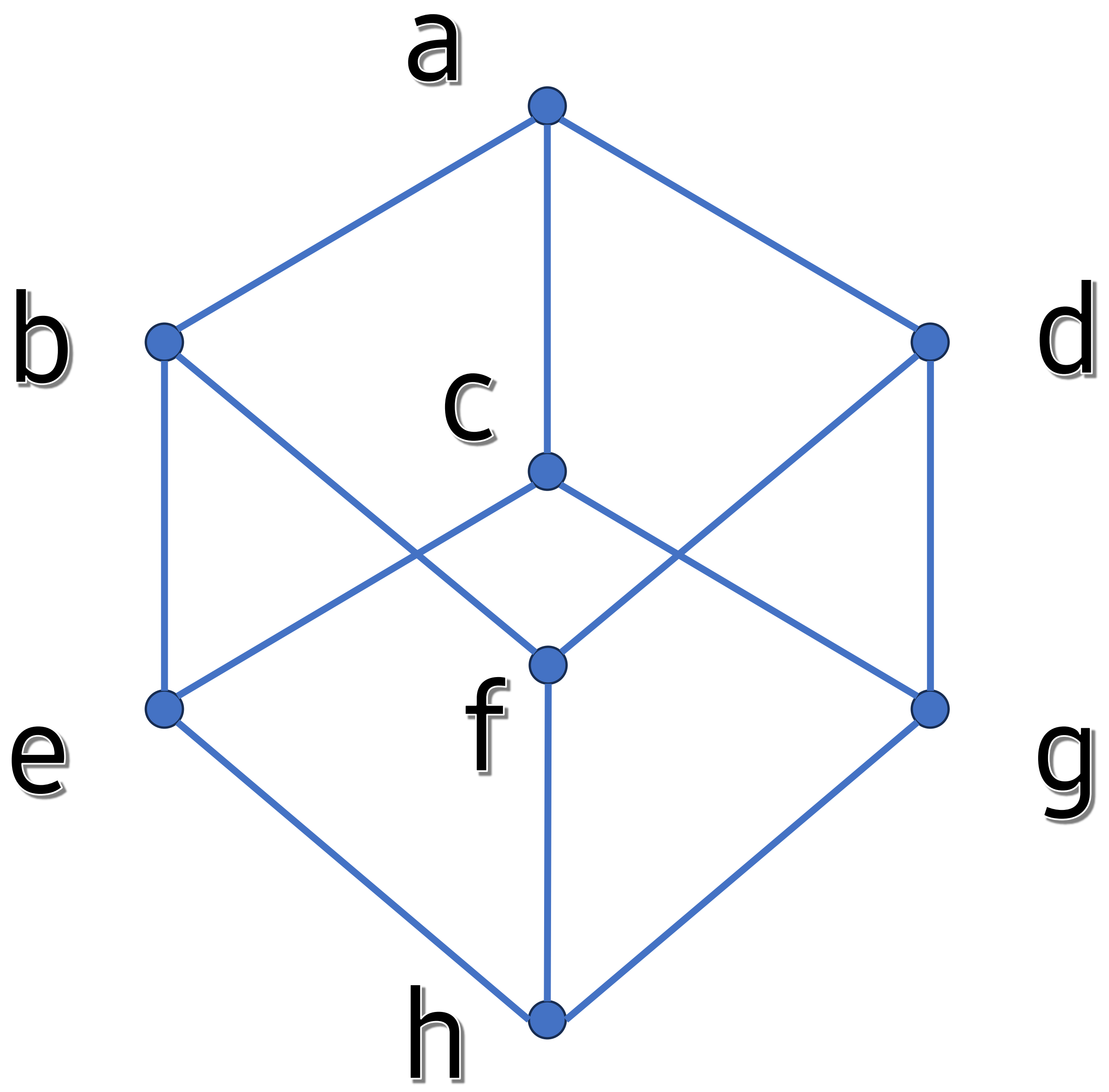
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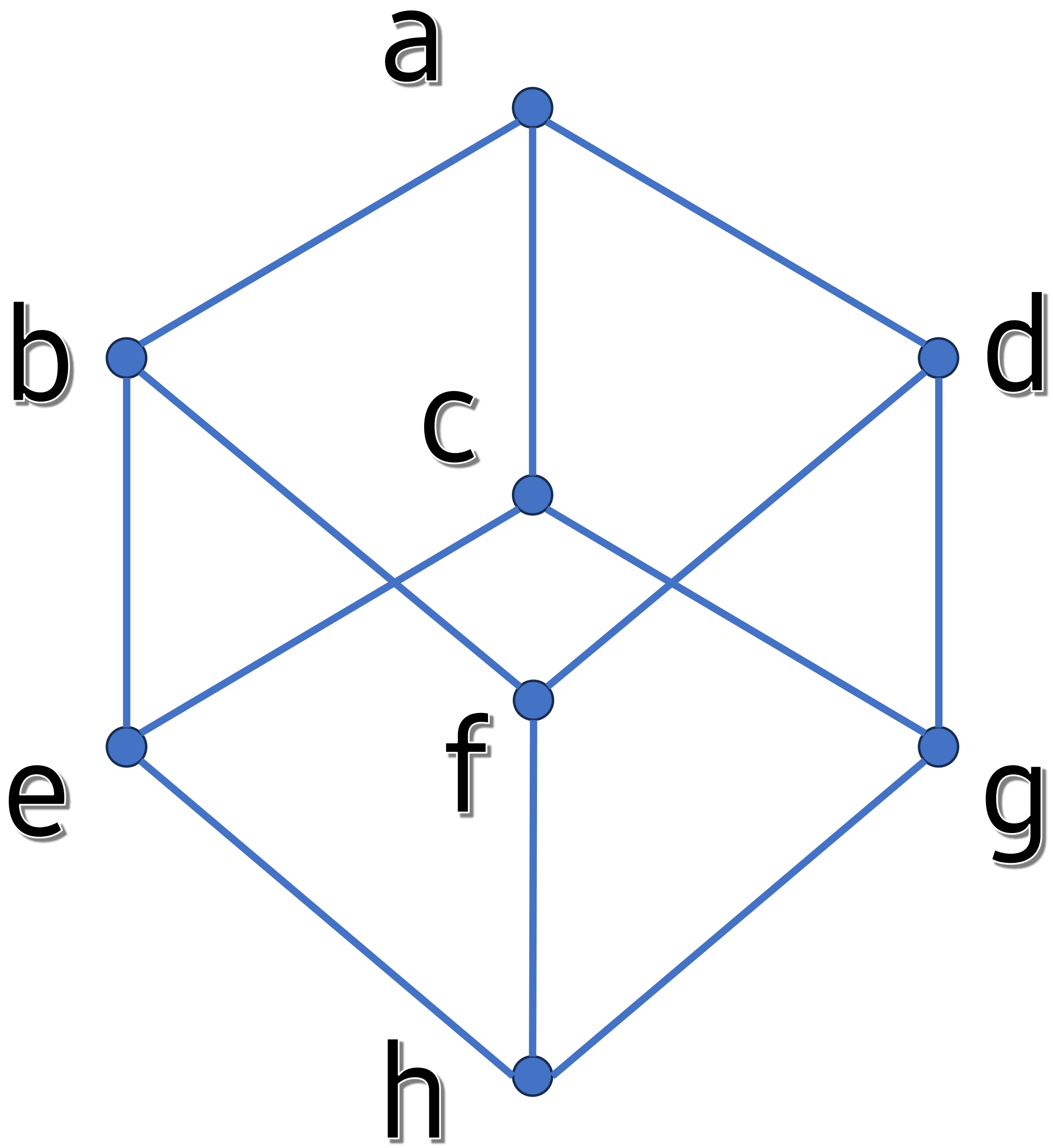
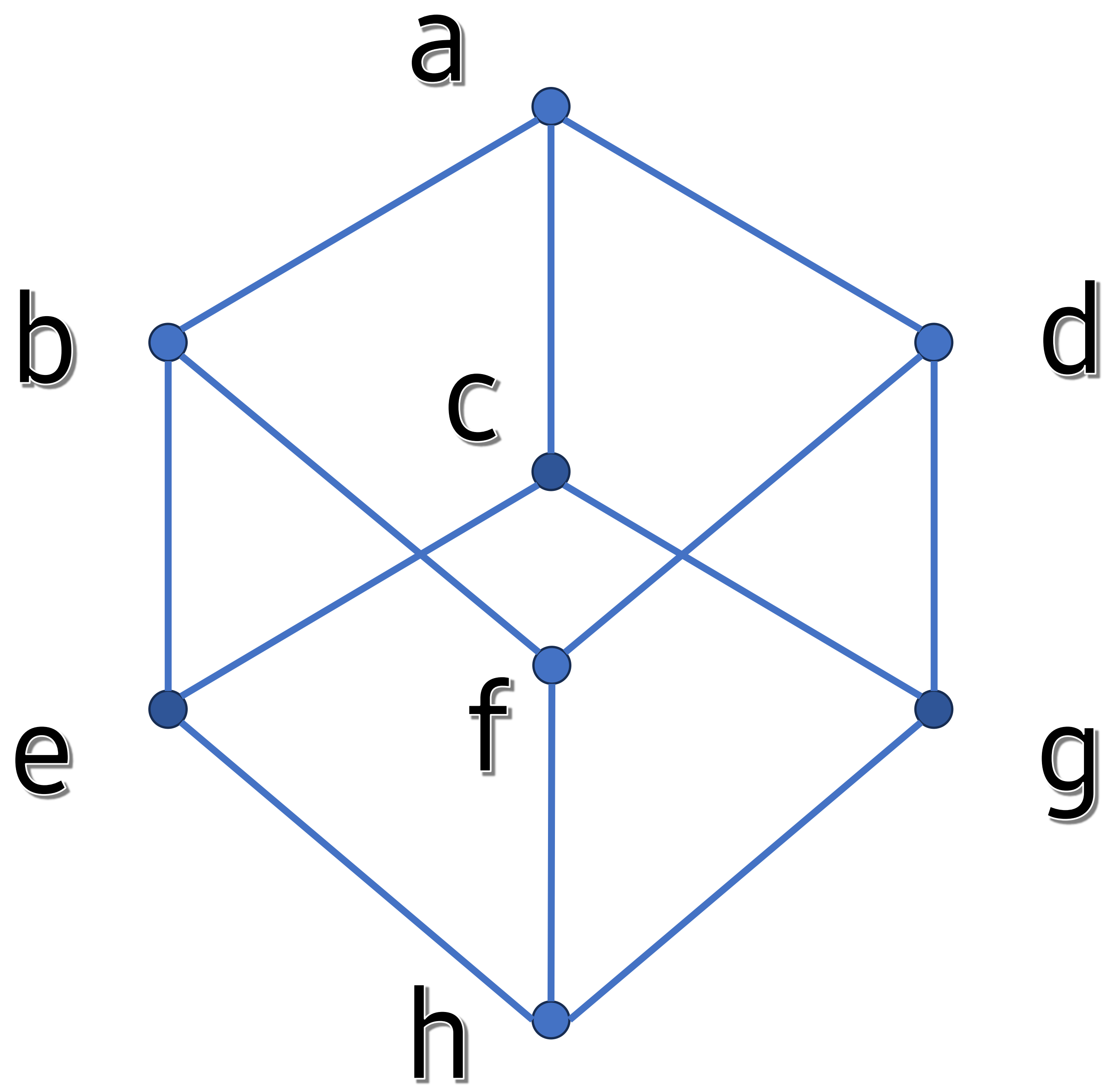


1. There is at most one greatest/least element.
2. There can be many maximal/minimal elements.
3. The greatest element is also and the only maximal element.
4. The least element is also and the only minimal element.

Let $B \subset A$,

1. $\forall b \in B$, if $\exists a, b \preccurlyeq a$, then a is the upper bound of B.
2. $\forall b \in B$, if $\exists a, a \preccurlyeq b$, then a is the lower bound of B.
3. The smallest upper bound is the least upper bound (sup B).
4. The largest lower bound is the greatest lower bound (inf B).

$$A = \{1, 3, 5, 7, 9\}$$
$$B = \{5, 7\}$$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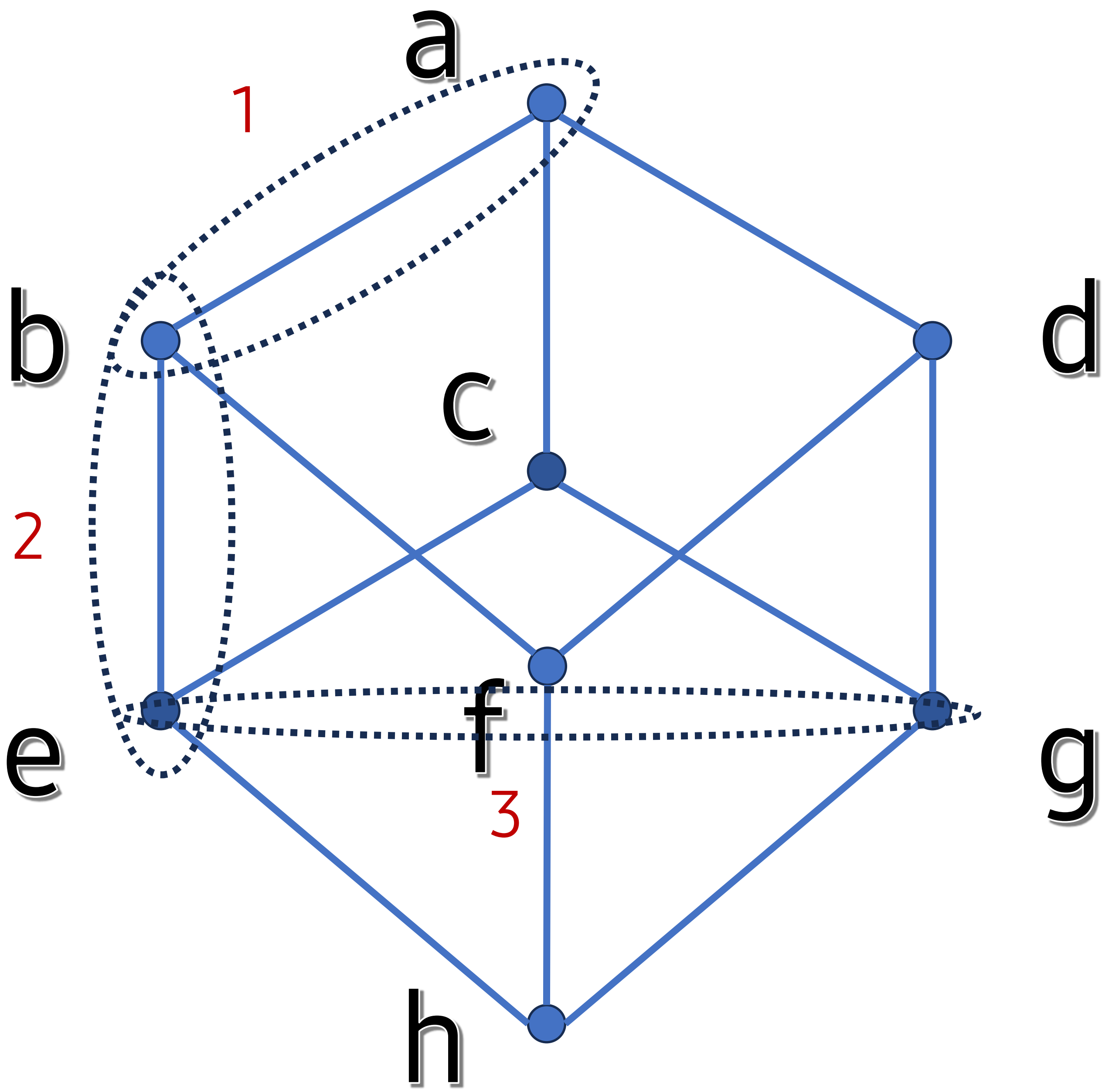
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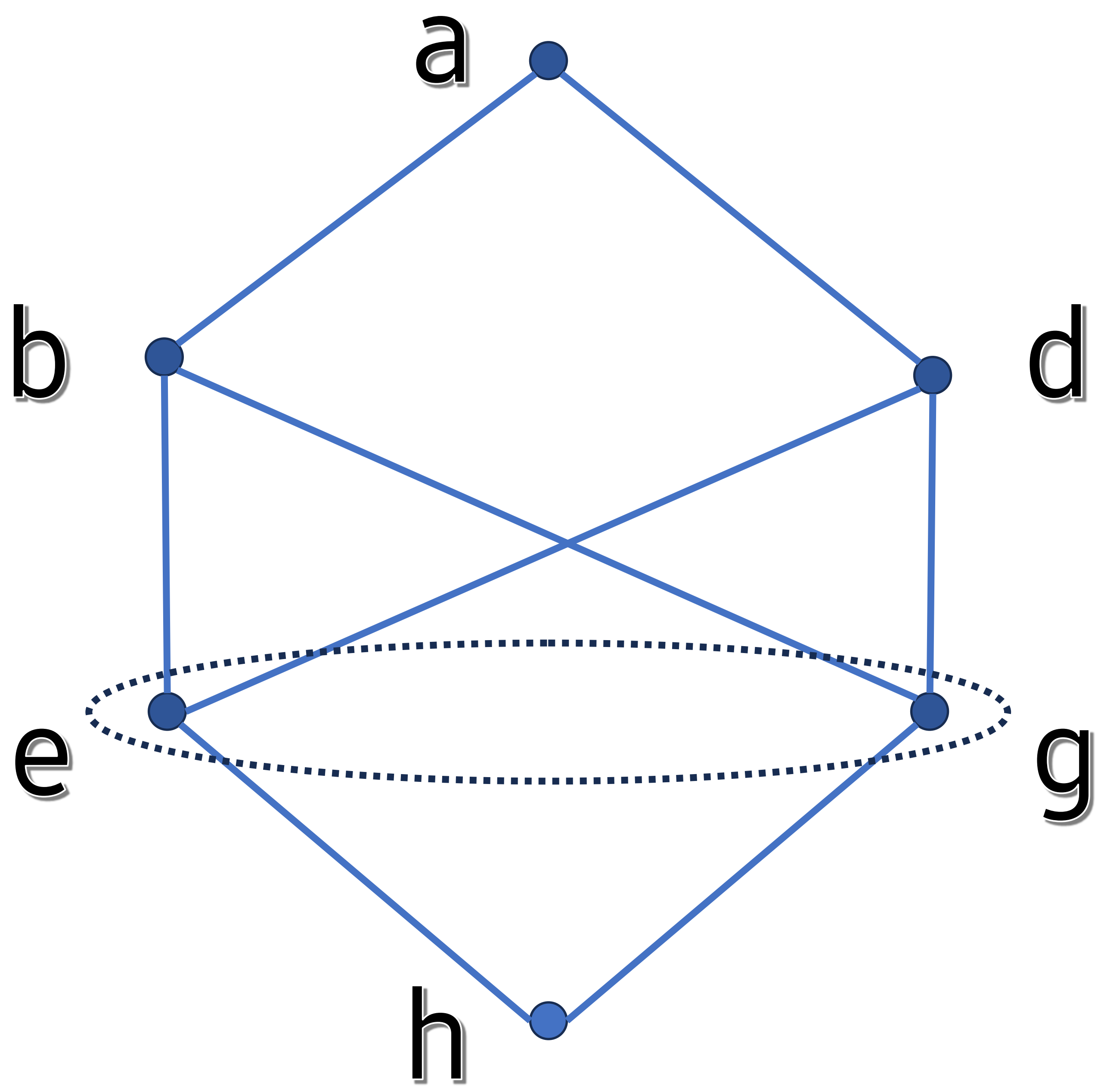
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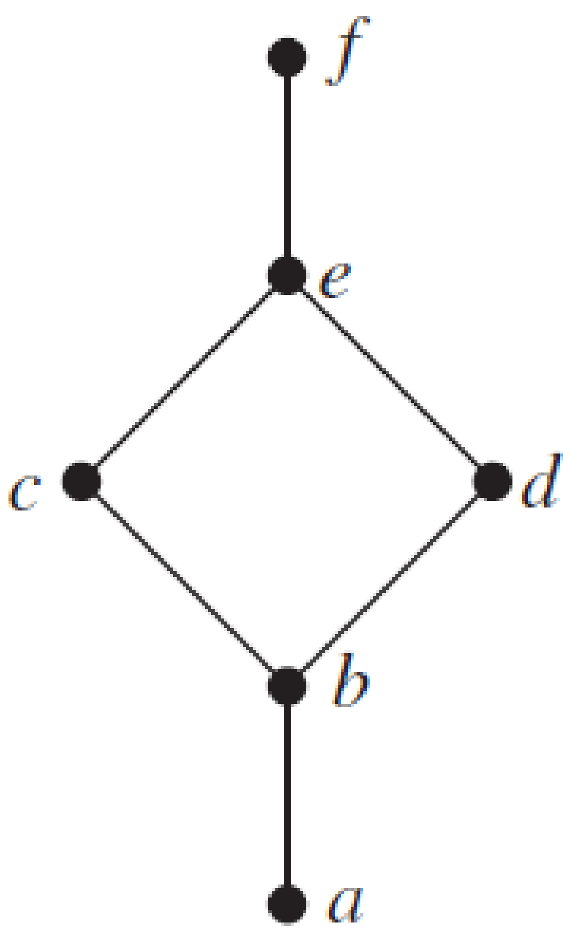
Definition

A POSET that every pair of its elements has both sup and inf is called a lattice.

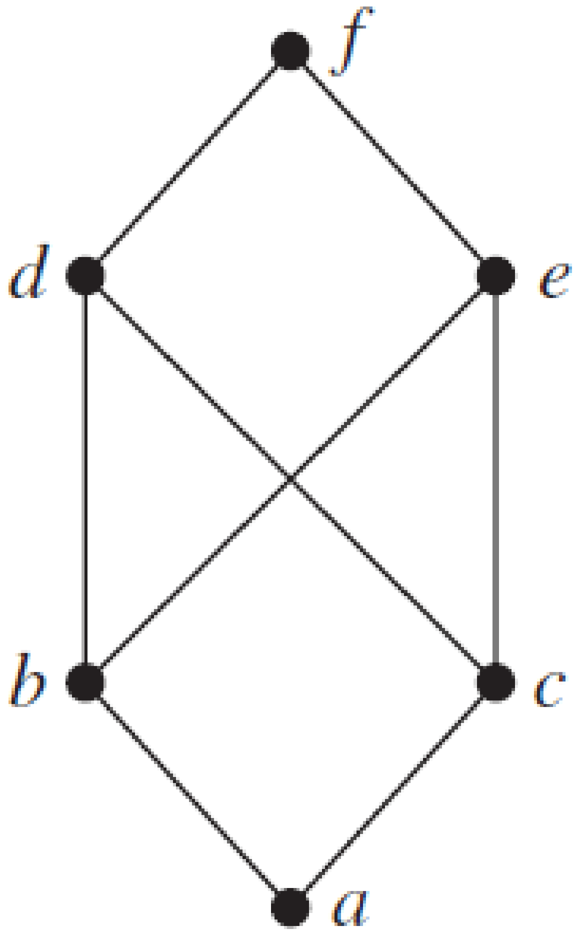
Tips to find sup (inf is similar):

1. Find all pair P of equal elements.
2. For $(a, b) \in P$, list all $c \in C$ so that $c \succcurlyeq a, c \succcurlyeq b$
3. If $\nexists \mathrm{sup} = \min(C) = \mathrm{join}$ $(\mathrm{max/meet}$ for inf)

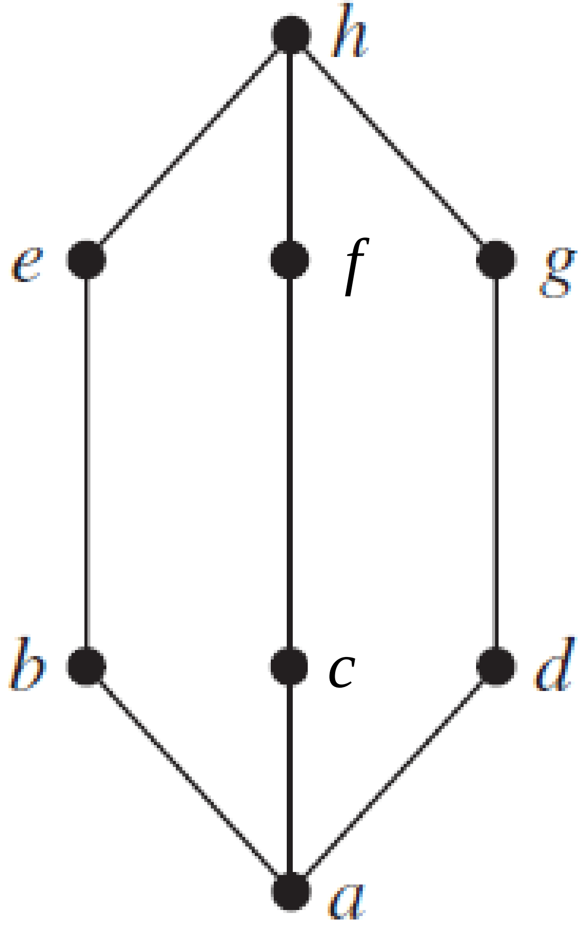
   => Not lattice.



(1)



(2)



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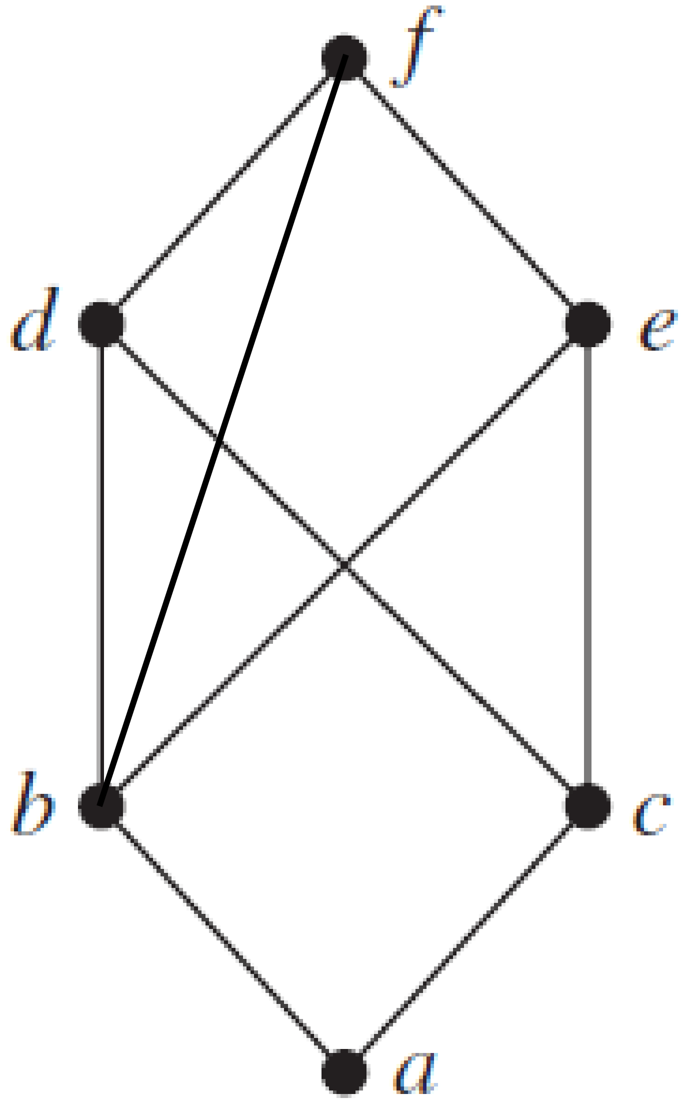
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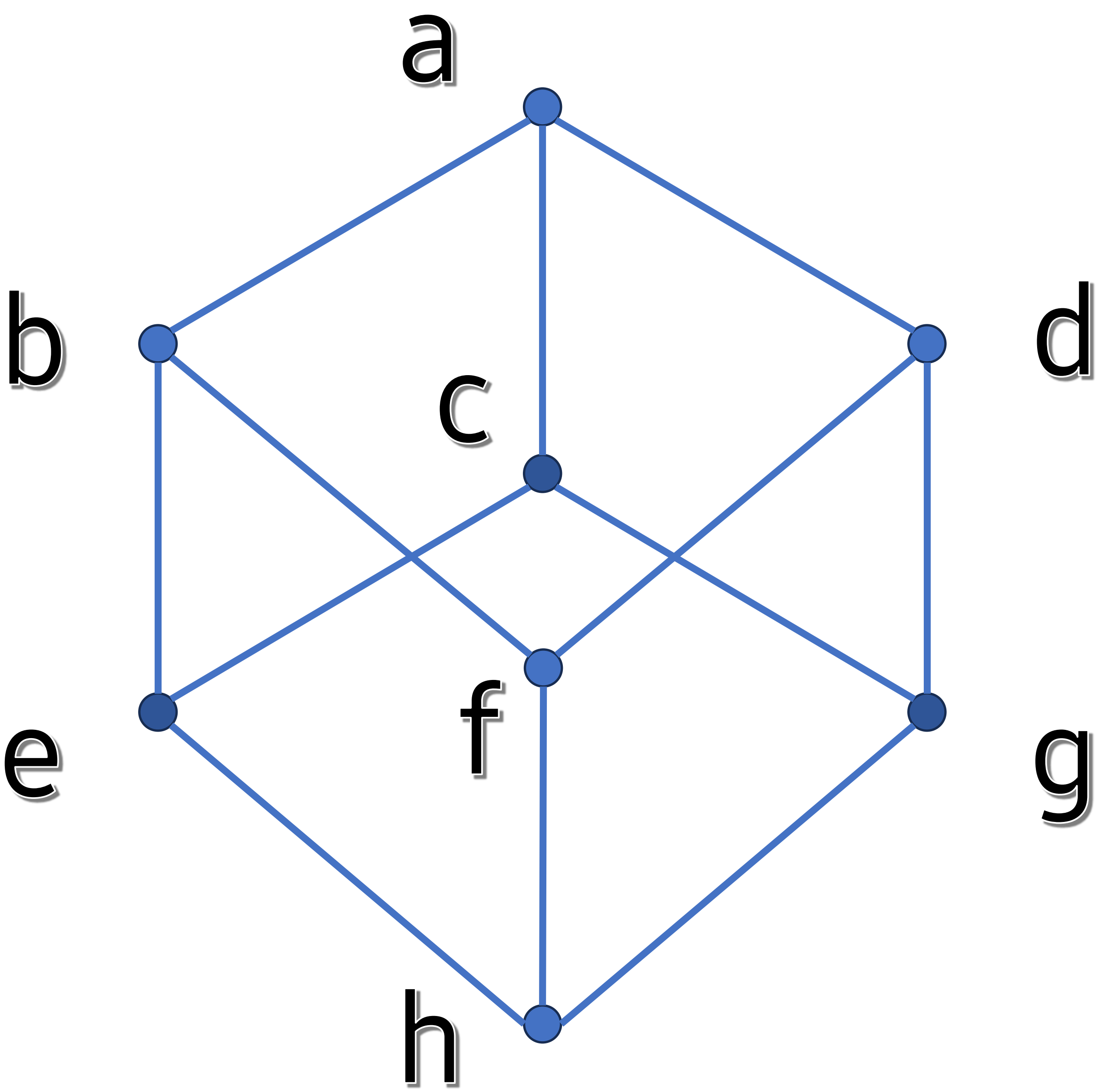
Moreover,...

- $\sup\{a, b\}$ can be written as $a \vee b$
- $\inf\{a, b\}$ can be written as $a \wedge b$
- Lattice is written as $(A, \vee, \wedge)$
- Laws: Commutative, associative, idempotent, absorption
  - Note: If distributive is valid, then lattice is distributive.



EXERCISE

Calculate:

$$c \wedge (a \wedge d) \wedge (g \vee h)$$

III. Equivalence class

- If R is equivalent on A and $a \in A$, $B = \{b \in A \mid (a, b) \in R\}$ is the equivalence class containing a, or $\overline{a}$, $[a]$.
- *Note: If* $(a, b) \in R$ *and* R *is equivalent on* A, *then we can write* $a\mathcal{R}b$.

$$A = \{0, 1, 2, 3, 4\}$$

$$R = \{(a, b) \mid a \bmod 3 = b \bmod 3\}$$

$$A = \{0, 1, 2, 3, 4\}$$

$$R = \{(0,0), (0,3), (3,0),$$

$$(1,1), (1,4), (4,1),$$

$$(2,2), (3,3), (4,4)\}$$

$$A = \{0, 1, 2, 3, 4\}$$

$$R = \{(0,0), (0,3), (3,0),$$

$$(1,1), (1,4), (4,1),$$

$$(2,2), (3,3), (4,4)\}$$

Equivalence class:

$$\overline{0} = \{0,3\} \qquad \overline{3} = \{0,3\}$$

$$\overline{1} = \{1,4\} \qquad \overline{4} = \{1,4\}$$

$$\overline{2} = \{2\}$$

$$A = \{0, 1, 2, 3, 4\}$$

$$R = \{(0,0), (0,3), (3,0),$$

$$(1,1), (1,4), (4,1),$$

$$(2,2), (3,3), (4,4)\}$$

Equivalence class:

$$\overline{0} = \overline{3} = \{0,3\}$$
$$\overline{1} = \overline{4} = \{1,4\}$$
$$\overline{2} = \{2\}$$

$$A = \{0, 1, 2, 3, 4\}$$

$$R = \{(0,0), (0,3), (3,0),$$

$$(1,1), (1,4), (4,1),$$

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Equivalence class:

$$\overline{0} = \overline{3} = \{0,3\}$$
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$P = \{\overline{0}, \overline{1}, \overline{2}\}$ is the partition of A

$$A = \{0, 1, 2, 3, 4\}$$

$$R = \{(0,0), (0,3), (3,0),$$
$$(1,1), (1,4), (4,1),$$
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Equivalence class:

$$\overline{0} = \{0,3\} \qquad \overline{3} = \{0,3\}$$
$$\overline{1} = \{1,4\} \qquad \overline{4} = \{1,4\}$$
$$\overline{2} = \{2\}$$

$P = \{\overline{0}, \overline{1}, \overline{2}\}$ is the partition of A

If $a \in \overline{x}$, then $a \notin \overline{y}, \overline{y} \neq \overline{x}$

No.	Name	Expression
1	Commutative (Giao hoán)	$\overline{x} + \overline{y} = \overline{y} + \overline{x}$
2	Associative (Kết hợp)	$(\overline{x} + \overline{y}) + \overline{z} = \overline{x} + (\overline{y} + \overline{z})$ $(\overline{x}\overline{\mathrm{y}})\overline{\mathrm{z}} = \overline{x}(\overline{\mathrm{yz}})$
3	Distributive (Phân bố)	$\overline{x}(\overline{\mathrm{y}} + \overline{\mathrm{z}}) = \overline{x}\overline{\mathrm{y}} + \overline{x}\overline{\mathrm{z}}$
4	Identity (Trung hòa)	$\overline{x} + \overline{0} = \overline{x}$ $\overline{x}\overline{1} = \overline{x}$

III. Mapping (function)

- A mapping f from A to B assigns $a \in A$ as input to <u>one and only</u> $b \in B$ as output.

$$f: A \to B$$

- Function is a **special kind of relation**.

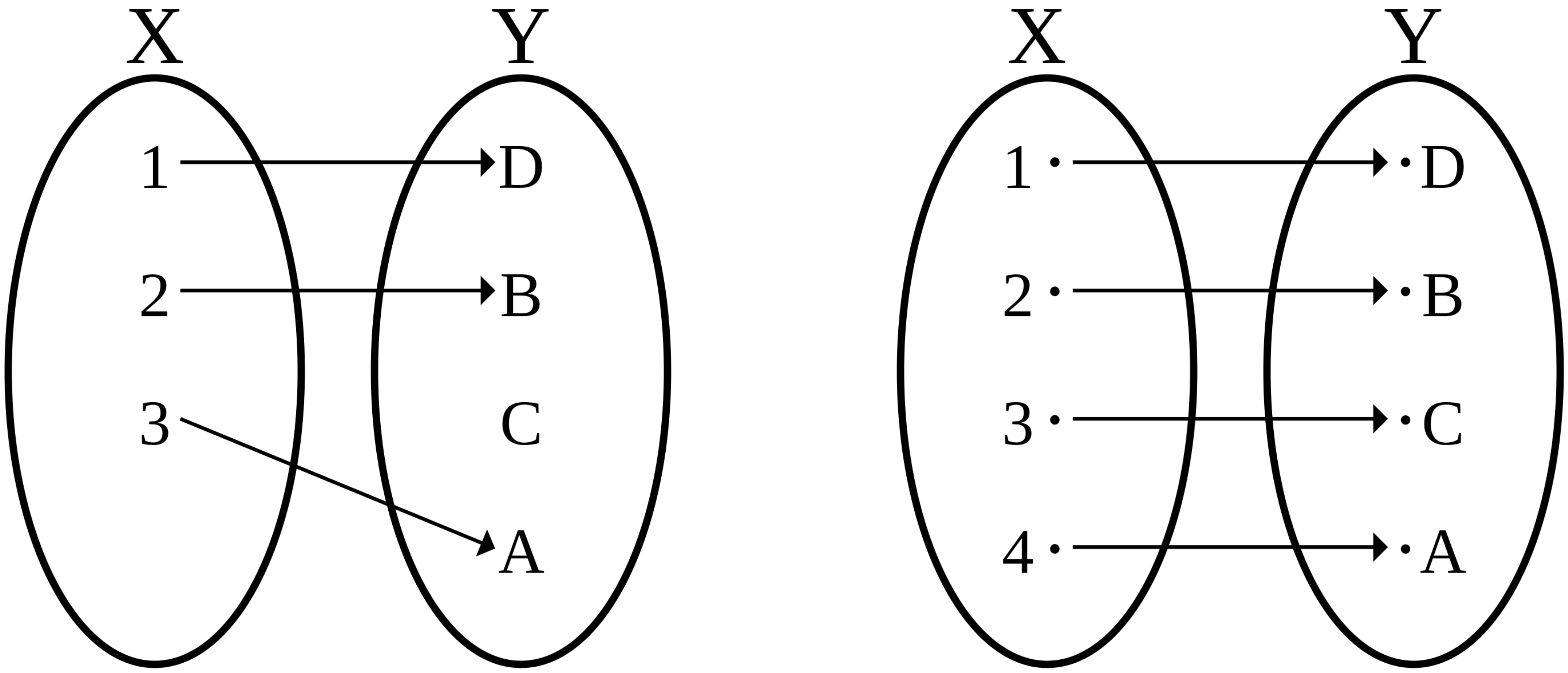
$$A = \{2, 4, 6, 8\}$$

$$R1 = \{(a, b), a <= b\}$$

$$R2=\{(a, b), b = a + 2\}$$

Types of mapping

- Injective: $\forall a_1, a_2 \in A, a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2)$
- Surjective: $\forall b \in B \rightarrow \exists a \in A, f(a) = b$
- Bijective: Injective + Surjective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(Không toàn ánh)</th></tr>
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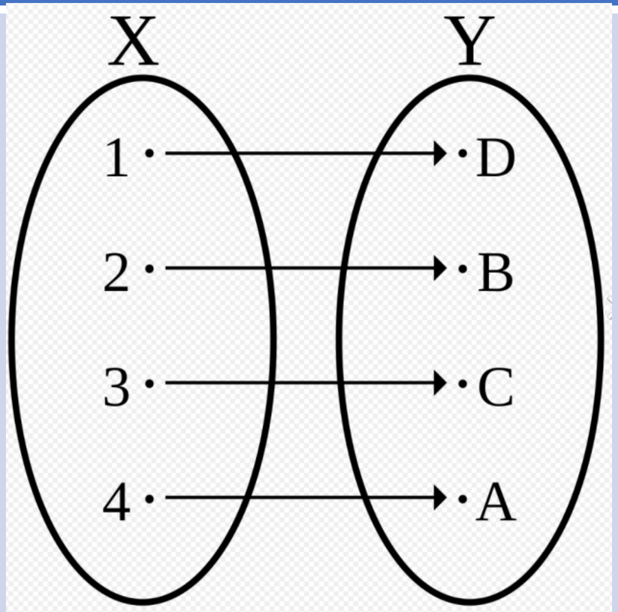
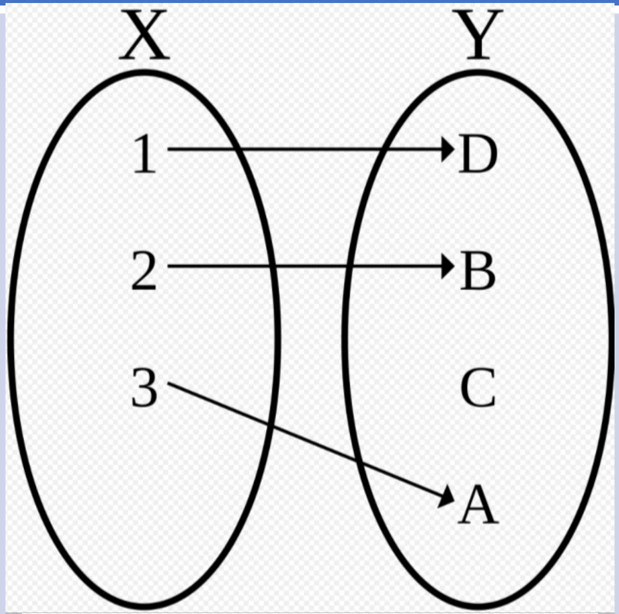
(Đơn ánh)</td><td>
</td><td>
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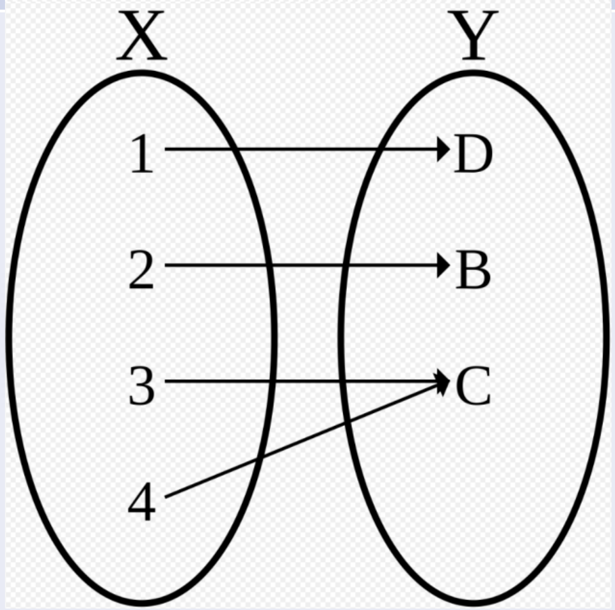
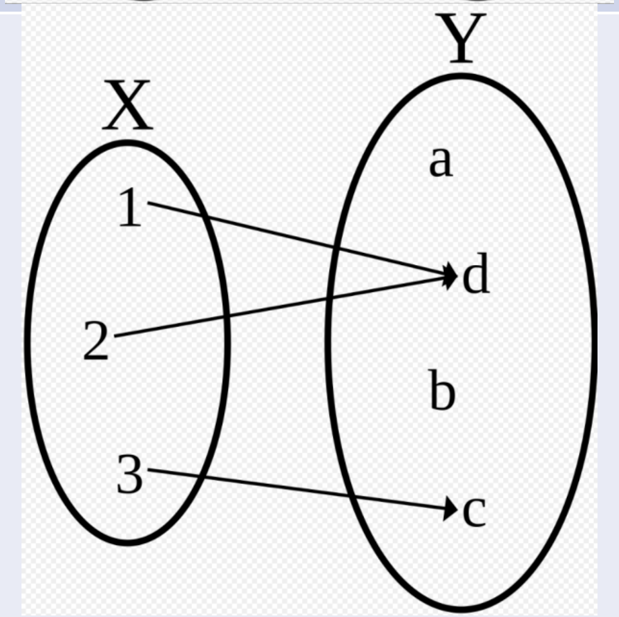
(Không đơn ánh)</td><td>
</td><td>
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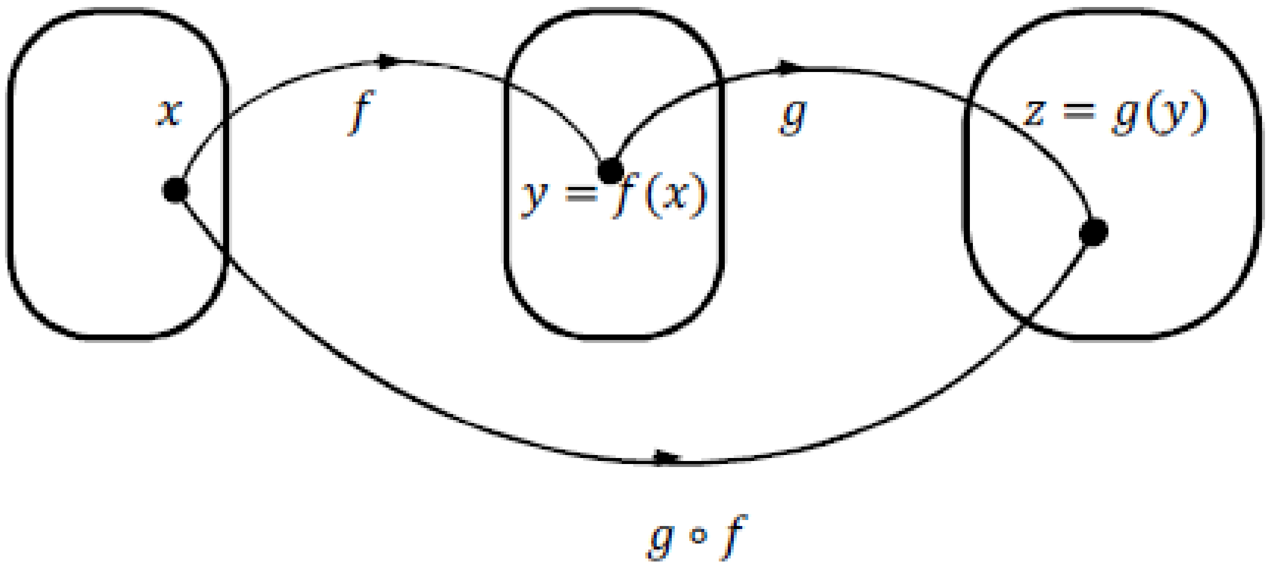
Beware of the sign!

$$f\colon A \to B$$
$$g\colon B \to C$$
$$\Rightarrow h\colon A \to C$$
$$\Rightarrow x \longmapsto h(x) = g(f(x))$$



IV. Laws

No.	Name	Expression
1	Union	$A \cup B$
2	Intersection	$A \cap B$
3	Minus, difference	$A - B, A\backslash \mathrm{B}$
4	Complement (of B in A)	$\overline{B} = C_A B = (A - B \text{ or } A\backslash \mathrm{B})$
5	Size	$\lvert\mathrm{A}\rvert$
6	Power set	$\mathcal{P}(S)$

No.	Name	Expression
1	Commutative (Giao hoán)	$A \cup B = B \cup A$ $A \cap B = B \cap A$
2	Associative (Kết hợp)	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
3	De Morgan	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$
4	Distributive (Phân bố)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5	Identity (Trung hòa)	$A \cup \emptyset = A$ $A \cap \mathcal{U} = A$
6	Complement (Phần bù)	$A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \emptyset$
7	Domination (Thống trị)	$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$

Summary

- Hasse diagram elements
  - Maximals, minimals, greatest, least
  - Upper bounds, lower bounds
  - least upper bound (sup), greatest lower bound (inf)
  - Lattices
    - Laws: Commutative, associative, idempotent, absorption, (distributive)
- Equivalence class
  - Partition
  - Laws
- Mapping: Injective, surjective, bijective, symbols
- Set laws

Exercises (ind.)

Let $A = \{1, 2, 3, 4, 5\}, B = \{3, 4, 5, 6, 7\}, C = \{2, 3, 5\}$

Find:

$a)$ $A \cap B$

$b)$ $A \cup B$

$c)$ $A \backslash \mathrm{B}$

$d)$ $A \cap \overline{B \cup C}$

Homework (group)

1. Let A $= \{1,2,3,4,5,6\}, B = \{2,4,6\}, C = \{1,2,3\}, D = \{7,8,9\}$,

$\mathcal{U} = \{1,2,\dots,10\}$, find:

$a)$ $A \cup B$

$b)$ $B \cap C$

$c)$ $\overline{B \cup C}$

$d)$ $|\mathcal{P}(C)|$

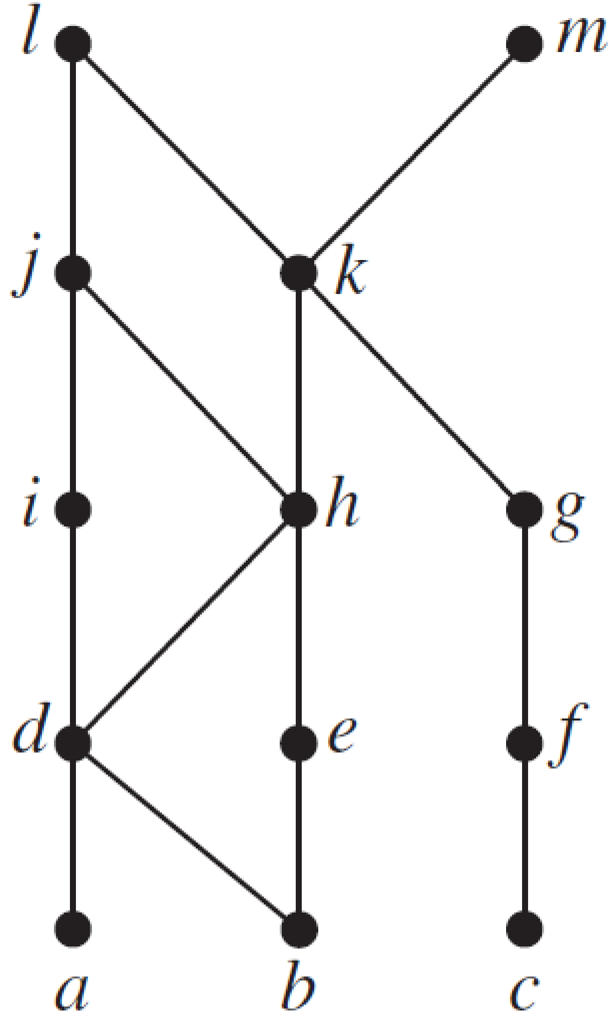
$e)$ $A \cap B$

$f)$ $\emptyset \cup C$

$g)$ $\emptyset \cap C$

$h)$ $(D \cap \bar{C}) \cup \overline{A \cap B}$

2.



Find:

a. The maximal elements

b. The minimal elements

c. The greatest element

d. The Least element

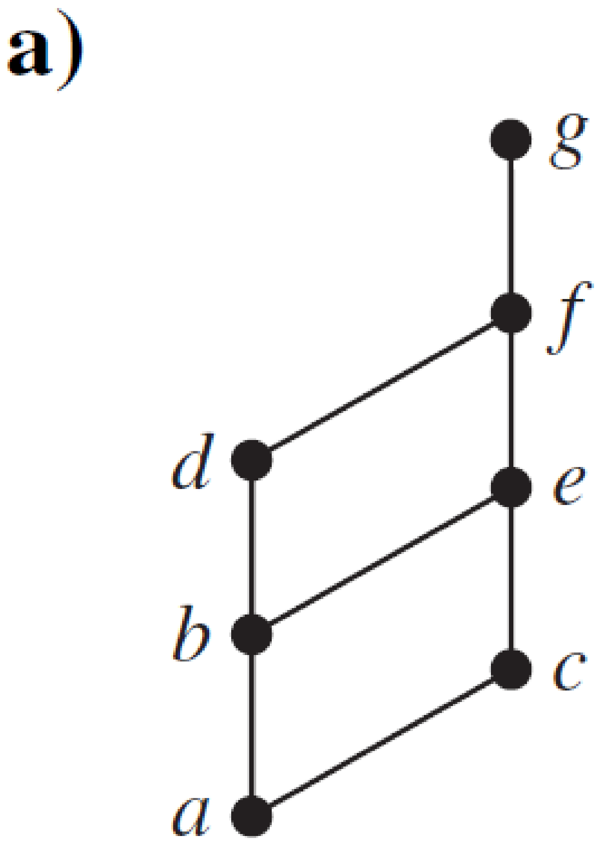
e. The upper bounds of $\{a, b, c\}$.

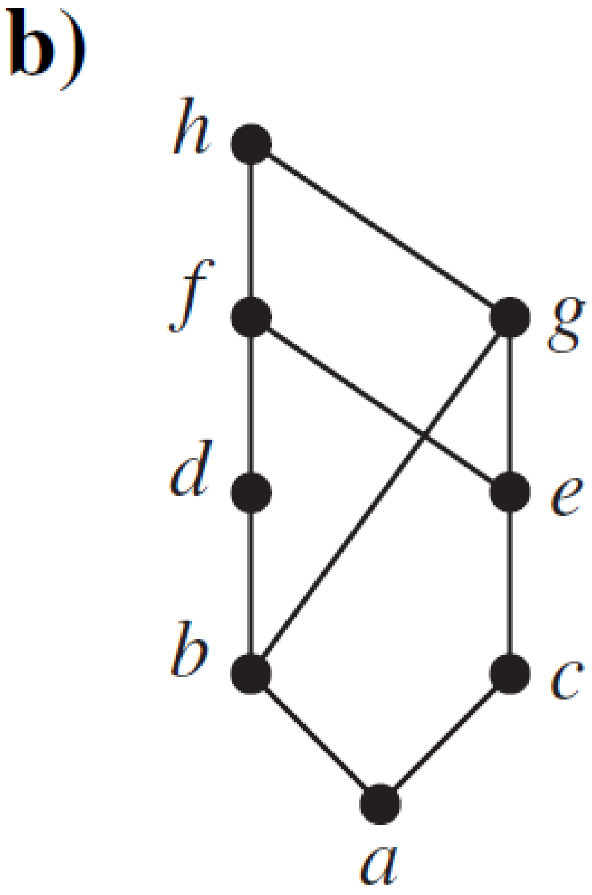
f. The least upper bound of $\{a, b, c\}$

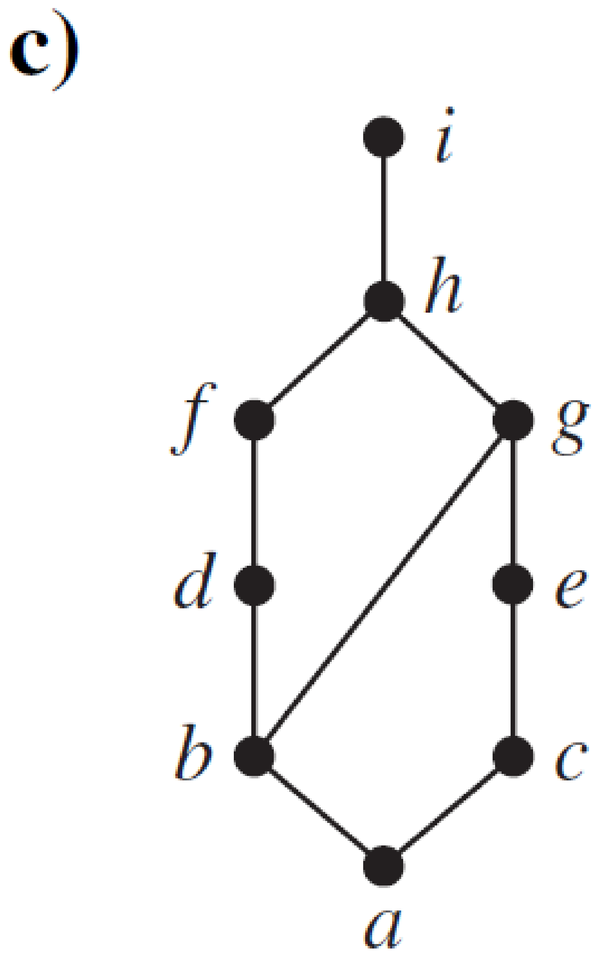
g. The lower bounds of $\{f, g, h\}$

h. The greatest lower bound of $\{f, g, h\}$

3. Which of these Hasse diagrams are lattices? If not, explain. *Example: (a) is not lattice as {a, b} doesn't have sup/inf.*







4. Which of the below are equvalence relations on $\{0, 1, 2, 3\}$?

a. $\{(0,0), (1,1), (2,2), (3,3)\}$

b. $\{(0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)\}$

c. $\{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$

5. Find the equivalence classes of the equivalence relations in exercise 4.