Question 1

For a fair 6-face dice, the probability of getting any integer of range [1, 6] is equal. The sampe space is of size 6, therefore:

$$P(X=1) = P(X=2) = P(X=3) = P(X=4) = P(X=5) = P(X=6) = 1/6$$

1) Expectation

$E(X) = \Sigma f(x)P(X=x)$

$= 1 \times P(X=1) + 2 \times P(X=2) + 3 \times P(X=3) + 4 \times P(X=4) + 5 \times P(X=5) + 6 \times P(X=6)$

$= 1 \times 1/6 + 2 \times 1/6 + 3 \times 1/6 + 4 \times 1/6 + 5 \times 1/6 + 6 \times 1/6$

$= (1 + 2 + 3 + 4 + 5 + 6)/6$

$= 7/2$

2) Variance

$Var(X) = E[(X-m)^2] = \Sigma f[(x-m)^2] P[X]$, where $m = (1+6)/2 = 7/2$

$= [(1 - 7/2)^2 + (2 - 7/2)^2 + (3 - 7/2)^2 + (4 - 7/2)^2 + (5 - 7/2)^2 + (6 - 7/2)^2] \times (1/6)$

$= [25/4 + 9/4 + 1/4 + 1/4 + 9/4 + 25/4] \times (1/6)$

$= (70/4) \times (1/6)$

$=35/12$

Question 2

By definition, we have:

- vector u: $u = \langle u_1, u_2, \dots, u_d \rangle$
- vector v: $v = \langle v_1, v_2, \dots, v_d \rangle$
- matrix A: $A =$

$$\begin{matrix} a_{11}, a_{12}, \dots, a_{1d} \\ a_{21}, a_{22}, \dots, a_{2d} \\ \dots\dots\dots\dots\dots \\ a_{n1}, a_{n2}, \dots, a_{nd} \end{matrix}$$

<u>Euclidean norm of *u*</u>

$$\|u\| = \sqrt{(\Sigma u_i^2)} = \sqrt{(u_1^2 + u_2^2 + u_3^2 + \dots + u_d^2)}$$

<u>Dot product of *u* and *v*</u>

$$u \bullet v = \Sigma\, u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 + \dots + u_d v_d$$

<u>Matrix-vector product *Au*</u>

$$Au = [a_{11}u_1 + a_{12}u_2 + a_{13}u_3 + \dots + a_{1d}u_d,$$

$$a_{21}u_1 + a_{22}u_2 + a_{23}u_3 + \dots + a_{2d}u_d,$$

$$\dots\dots$$

$$a_{n1}u_1 + a_{n2}u_2 + a_{n3}u_3 + \dots + a_{nd}u_d]$$

Where Au is a column vector of dimension $n \times 1$

Question 3

NOTE: In this question, we only compare time complexity of algorithms irrespective of space complexity during implementation. Time complexity is denoted in big-O notations.

Algorithm 1

It iterates through the array of length *n* and update the result every single iteration. This means that it is guaranteed to take *n* operations to obtain the final result. The algorithm therefore is of order *O(n)* in terms of running time.

Algorithm 2

It does exactly three arithmetic operations, based on precedence rules of operators, following the sequence:

1. Calculate *n + 1* - addition
2. Calculate *(n + 1) * n* – multiplication
3. Calculate *[(n + 1) * n]/2* - division

Each step takes exactly *1* operation, or *O(1)*. Three such operations sequentially means that the entire algorithm takes *3* operations, which is still of *O(1)* in terms of running time.

Comparing the two algorithms, it is clear that ***Algorithm 2*** has shorter running time and therefore executes faster time-wise.

Question 4

i) $f(x, \beta) = x^2 \cdot e^{-\beta x}$

$$\frac{\partial f}{\partial x} = x^2 \cdot \partial/_{\partial x}[e^{-\beta x}] + \partial/_{\partial x}[x^2] \cdot e^{-\beta x}$$

$$= x^2 \cdot [(-\beta) \cdot e^{-\beta x}] + 2x \cdot e^{-\beta x}$$

$$= -\beta \cdot x^2 \cdot e^{-\beta x} + 2x \cdot e^{-\beta x}$$

or, group everything:

$$\frac{\partial f}{\partial x} = e^{-\beta x}[2x - \beta x^2]$$

$$= e^{-\beta x} \cdot x \cdot (2 - \beta x)$$

ii) $f(x, \beta) = x \cdot e^{-\beta x}$

$$\frac{\partial f}{\partial \beta} = \partial/_{\partial \beta}[x] \cdot e^{-\beta x} + x \cdot \partial/_{\partial \beta}[e^{-\beta x}]$$

$$= 0 + x \cdot (\overset{-x}{\cancel{-\beta}}) \cdot e^{-\beta x}$$

$$= -x^2 \cdot e^{-\beta x}$$

iii) $f(x) = \sin(e^{x^2})$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}, \quad \text{where } u = e^{x^2}$$

$$\frac{df}{du} = d/_{du}[\sin u] = \cos u = \cos(e^{x^2}) \quad \ldots \quad ①$$

$$\frac{du}{dx} = d/_{dx}[e^{x^2}]$$

$$= \frac{du}{dv} \cdot \frac{dv}{dx} \quad \text{where } v = x^2$$

$$\frac{du}{dx} = d/_{dv}[e^v] \cdot d/_{dx}[x^2]$$

$$= e^v \cdot 2x$$

$$= 2 \cdot x \cdot e^v$$

$$= 2 \cdot x \cdot e^{x^2} \quad \ldots \quad ②$$

① & ②:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = 2x e^{x^2} \cdot \cos(e^{x^2})$$

Question 5

$$E|X^2| = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

where $\mu = \mu$, $\sigma = 1$

$$\therefore E|X^2| = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\mu)^2}{2}} dx$$

solve $I = \int x^2 e^{-\frac{(x-\mu)^2}{2}} dx$, by Integration by Parts

$u(x) = x$ $\quad v(x) = -e^{-\frac{(x-\mu)^2}{2}}$

$du/dx = 1 \cdot dx$ $\quad dv/dx = x \cdot e^{-\frac{(x-\mu)^2}{2}} dx$

$$I = -x \cdot e^{-\frac{(x-\mu)^2}{2}} + \int e^{-\frac{(x-\mu)^2}{2}} dx$$

$$\cancel{= -x \cdot e^{-\frac{(x-\mu)^2}{2}}}$$

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\mu)^2}{2}} dx = -x \cdot e^{-\frac{(x-\mu)^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2}} dx$$

$$= \lim_{x\to\infty} -\frac{x}{e^{\frac{(x-\mu)^2}{2}}} - \lim_{x\to-\infty} -\frac{x}{e^{\frac{(x-\mu)^2}{2}}}$$

$$+ \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2}} dx$$

$$= \lim_{x\to\infty} -\frac{x}{e^{\frac{(x-\mu)^2}{2}}} + \lim_{x\to-\infty} \frac{x}{e^{\frac{(x-\mu)^2}{2}}} + \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2}} dx$$

L'hôpital

$$\lim_{x\to\infty} \frac{x}{e^{\frac{(x-\mu)^2}{2}}} = \cancel{\lim} \lim_{x\to\infty} -\frac{1}{2\cdot(\frac{1}{2}) e^{\frac{(x-\mu)^2}{2}} \cdot x} = -\frac{1}{\infty} = 0$$

$$\lim_{x\to-\infty} \frac{x}{e^{\frac{(x-\mu)^2}{2}}} = \lim_{x\to-\infty} \frac{1}{2\cdot(\frac{1}{2}) e^{\frac{(x-\mu)^2}{2}} \cdot x} = \frac{1}{-\infty} = 0$$

$$\therefore E|X^2| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} (\sqrt{2\pi})$$

$$= 1$$