Question 1

a) Conditional probability of discrete random variable X given a discrete random variable Y is the probability distribution of X for any particular value of that of , i.e. Y = y, for any given value y. From here, we can write the formula for the conditional probability mass function of X given Y=y accordingly, as:

$$p_{X|Y}(x|y)=P(X=x|Y=y)=\frac{P(Y=y\cap X=x)}{P(Y=y)}$$

Note that for this expression to be valid, P(Y=y) must not be 0. The numerator is the intersection of two discrete random variables (the collection of elements that both X and Y have).

b) P(X: x = h) = 2/3, P(X: x = t) = 1/3
For three tosses, there are 8 combinations of results (h as ‘head’, t as ‘tail’):

h-h-h
h-h-t
h-t-h
h-t-t
t-h-h
t-h-t
t-t-h
t-t-t

We know that the first one is a ‘head’ and there are two ‘head’s, so the two possible combinations are: h-h-t and h-t-h. Therefore:

$$P(\sum X=2:x_i=h|X:x_1=h)$$
$$=\ (P(X:x_1=h)\times P(X:x_2=h)\times P(X:x_3=t))+(P(X:x_1=h)\times P(X:x_2=t)\times P(X:x_3=h))$$
$$=\ (\frac{2}{3})\times(\frac{2}{3})\times(\frac{1}{3})+(\frac{2}{3})\times(\frac{1}{3})\times(\frac{2}{3})$$
$$=\ \frac{8}{27}$$

c)
(i) P(X, Y) in terms of P(X) and P(Y|X) (joint probability in terms of conditional probability)

P(X ∩ Y) = P(X) P(Y|X)

(ii) P(X, Y) = P(Y), P(X|Y)

P(X ∩ Y) = P(Y) P(X|Y)

d) Bayes' Theorem Proof

The probability of two events X and Y happening, $P(X \cap Y)$, is the probability of X, $P(X)$, times that of Y given that X has occurred, $P(Y|X)$

$$P(X \cap Y) = P(X)\, P(Y|X) \;\ldots\ldots\; (1)$$

Additionally, the probability of Y and X happening could also be expressed as the probability of Y, $P(Y)$, times that of X given Y has occurred, $P(X|Y)$

$$P(Y \cap X) = P(Y)\, P(X|Y) \;\ldots\ldots\; (2)$$

(1) and (2) refer to the same quantity (probability of two same events), just with different wording,

$$P(X \cap Y) = P(Y \cap X) = P(X)\, P(Y|X) = P(Y)\, P(X|Y) \;\ldots\ldots\; (3)$$

Rearranging (3),

$$P(X|Y) = \frac{P(Y|X)\, P(X)}{P(Y)}$$

Which is the Bayes' Theorem

e)

i). We define the two discrete random variables as: student – S {Udem, McGill}, and billingualism – B {true, false}. This problem is solved under the assumption that students belong to either McGill or UdeM. From problem statement, we have: McGill 55% (P(S: s = McGill) = 0.55), then for UdeM:

$$P(S: s = UdeM) = 1 - 0.55 = 0.45$$

So the probability for a student to be from UdeM is 45%

ii). Billingualism: UdeM students 80%, McGill students 50%. We can re-write that as:

$$P(B: b = true \mid S: s = UdeM) = 0.8$$

$$P(B: b = true \mid S: s = McGill) = 0.5$$

Then the probability of this student being McGill student is:

$$\begin{aligned} & P(B: b = true, S: s = McGill) \\ &= P(B: b = true \mid S: s = McGill) \times P(S: s = McGill) \\ &= 0.5 \times 0.55 \\ &= 0.275 \end{aligned}$$

Which is 27.5%