AI(Fall 2020) - Assignment 2

CSP and KRR

1. Provide formulations for each of the following problems as CSPs: specify the variables, domains and constraints.

   (a) Magic square: An order 3 magic square is a 3×3 square grid filled with distinct positive integers in the range $1, 2, ..., 9$ such that the sums of the integers in each row, column and diagonal are equal.

   (b) Independent set: Given a graph and a number k, find an independent set of size k, that is, a set of k vertices, no two of which are adjacent.

   (c) Crypto-arithmetic puzzle: $\texttt{INT} \times \texttt{L} = \texttt{AAAI}$. We want to replace each letter by a different digit so that the equation is correct.

Solution:

(a)记V_k表示第k个方格，其中$k=3(i-1)+j, k\in\{1,2,......9\}$,即代表第$i$行第$j$列的方格$(i,j=1,2,3)$。

论域：$Dom[V_k]=\{1,2,......，9\}$

约束条件：$V_m \neq V_n$，对于任意的$m \neq n(m,n=1,2,......，9)$

$V_1+V_5+V_9=V_3+V_5+V_7=V_1+V_2+V_3=V_4+V_5+V_6=V_7+V_8+V_9=V_1+V_4+V_7=V_2+V_5+V_8=V_3+V_6+V_9$

(b)记$V_1,V_2,......，V_k$为独立集里的k个顶点。

论域：图G的所有顶点ν。

约束条件：V_i与V_j不相邻且不重合，对于任意的$i \neq j(i,j=1,......,k)$

(c)V_I,V_N,V_T,V_L,V_A分别代表I,N,T,L,A所对应的数字。

论域：$Dom[V_k]=\{0,1,......,9\}(k=I,N,T,L,A)$

约束条件：$(100V_I+10V_N+V_T)V_L=1110V_A+V_I$，且$V_i \neq V_j(i,j=I,N,T,L,A)$

2. Consider the following CSP with binary constraints. There are 4 variables: A, B, C, D with their respective domains:

   $D_A = \{1,2,3,4\}$, $D_B = \{3,4,5,8,9\}$, $D_C = \{2,3,5,6,7,9\}$, $D_D = \{3,5,7,8,9\}$.

   The constraints are:

   - $C_1 : A \geq B$
   - $C_2 : B > C$ or $C - B = 2$
   - $C_3 : C \neq D$

   (a) Find the first solution by using the Forward Checking algorithm with the MRV heuristics, *i.e.*, always choose the variable with smallest remaining number of elements in the domain to instantiate, breaking ties in the alphabetic order. Assign values in the current domain of each variable in increasing order. At each node indicate:

      i. The variable being instantiated and the value being assigned to it.
      ii. The CurDom for each variable.
      iii. Mark any node with an empty CurDom with DWO.

   (b) Enforce GAC on the constraints and give the resultant variable domains. You should show which values of a domain are removed at each step, and which arc is responsible for removing the value. After this first step, use the GAC algorithm to find the first solution.

Solution:

(a)以表格的形式给出过程：

`Constraints`	`CurDomA`	`CurDomB`	`CurDomC`	`CurDomD`
`START`	{1,2,3,4}	{3,4,5,8,9}	{2,3,5,6,7,9}	{3,5,7,8,9}
$A \geq B$	1	`DWO`		
$A \geq B$	2	`DWO`		
$A \geq B$	3	3	{2,3,5,6,7,9}	{3,5,7,8,9}
$B > C \vee C - B = 2$	3	3	$min(2,5) = 2$	{3,5,7,8,9}
$C \neq D$	3	3	2	$min(3,5,7,8,9) = 3$

故找到的第一组解为$A = 3, B = 3, C = 2, D = 3$。

(b)首先执行`Enforce GAC`算法，划线表示删除：

`Constraints`	`CurDomA`	`CurDomB`	`CurDomC`	`CurDomD`
`START`	{1,2,3,4}	{3,4,5,8,9}	{2,3,5,6,7,9}	{3,5,7,8,9}
$A \geq B$	{~~1,2~~,3,4}	{3,4,5,8,9}	{2,3,5,6,7,9}	{3,5,7,8,9}
$A \geq B$	{3,4}	{3,4,~~5,8,9~~}	{2,3,5,6,7,9}	{3,5,7,8,9}
$B > C \vee C - B = 2$	{3,4}	{3,4,~~5,8,9~~}	{2,3,5,6,~~7,9~~}	{3,5,7,8,9}
$C \neq D$	{3,4}	{3,4,~~5,8,9~~}	{2,3,5,6,~~7,9~~}	{3,5,7,8,9}

故`Enforce GAC`后每个变量的论域如下：

$Dom[A] =$$\{3, 4\}$;$Dom[B] =$$\{3, 4\}$;$Dom[C] =$$\{2, 3, 5, 6\}$;$Dom[D] =$$\{3, 5, 7, 8, 9\}$

接下来找第一组解，括号里的值代表与该变量相关的另一变量的取值。

首先扩展$A = 3$，将三个约束条件加入队列：

`Constraints`	`CurDomA`	`CurDomB`	`CurDomC`	`CurDomD`
`START`	{3}	{3,4}	{2,3,5,6}	{3,5,7,8,9}
$A \geq B$	{3(3)}	{3(3)}	{2,3,5,6}	{3,5,7,8,9}
$B > C \vee C - B = 2$	{3}	{3(2)}	{2(3),5(3)}	{3,5,7,8,9}
$C \neq D$	{3}	{3}	{2(3),5(3)}	{3(2),5(2),7(2),8(2),9(2)}

然后扩展$B = 3$，将前两个约束条件加入队列：

`Constraints`	`CurDomA`	`CurDomB`	`CurDomC`	`CurDomD`
`START`	{3}	{3}	{2,5}	{3,5,7,8,9}
$A \geq B$	{3(3)}	{3(3)}	{2,5}	{3,5,7,8,9}
$B > C \vee C - B = 2$	{3}	{3(2)}	{2(3),5(3)}	{3,5,7,8,9}

然后扩展$C = 2$，将后两个约束条件加入队列：

`Constraints`	`CurDomA`	`CurDomB`	`CurDomC`	`CurDomD`
`START`	{3}	{3}	{2}	{3,5,7,8,9}
$B > C \vee C - B = 2$	{3}	{3(2)}	{2(3)}	{3,5,7,8,9}
$C \neq D$	{3}	{3}	{2(3)}	{3(2),5(2),7(2),8(2),9(2)}

最后扩展$D=3$，此时只需将最后一个约束条件加入队列即可：

`Constraints`	`CurDomA`	`CurDomB`	`CurDomC`	`CurDomD`
`START`	{3}	{3}	{2}	{3}
$C \neq D$	{3}	{3}	{2(3)}	{3(2)}

故找到的第一组解为$A=3, B=3, C=2, D=3$。

3. Consider the following facts about the Elm Street Bridge Club:

   > Joe, Sally, Bill, and Ellen are the only members of the club. Joe is married to Sally. Bill is Ellens brother. The spouse of every married person in the club is also in the club.

   From these facts, most people would be able to determine that Ellen is not married.

   (a) Represent these facts as sentences in FOL, and show semantically that by themselves they do not entail that Ellen is not married.

(b) Write in FOL some additional facts that most people would be expected to know, and show that the augmented set of sentences now entails that Ellen is not married.

Solution:

(a)

记$club(x)$表示x在该俱乐部内，$spouse(x,y)$表示x,y是夫妻，$brother(x,y)$表示x是y的兄弟，则：

$$
\begin{aligned}
&KB:\\
&club(Joe)①\\
&club(Sally)②\\
&club(Ellen)③\\
&club(Bill)④\\
&spouse(Joe, Sally)⑤\\
&brother(Bill, Ellen)⑥\\
&\forall x \forall y(club(x) \wedge spouse(x,y) \rightarrow club(y))⑦\\
&\forall x \forall y(club(y) \wedge spouse(x,y) \rightarrow club(x))⑧\\
&\text{证明：}\forall w(\neg spouse(Ellen, w))⑨
\end{aligned}
$$

归结方法无法证明结论的不可满足性，因此需要构造一个解释，来证明⑨式是不可满足的。

构造解释s，该解释将常量映射到自己，将谓词符号按照如下进行映射：

s[club] $=$\{$Joe, Sally, Bill, Ellen$\}

s[brother] $=$\{$(Bill, Ellen)$\}

s[spouse] $=$\{$(Joe, Sally), (Bill, Ellen)$\}

显然该解释满足知识库，但是我们可以发现⑨式显然不可满足，因为有$spouse(Bill, Ellen)$。

(b)

首先将⑨式取反得到⑩式：

$$\exists w(spouse(Ellen, w))⑩$$

由于后面会用到该式，因此放在这里。

在上述知识库的基础上添加事实如下：

$$
\begin{gathered}
\forall x \forall y(spouse(x,y) \rightarrow \neg self(x,y))⑪\\
\forall x \forall y(brother(x,y) \rightarrow \neg spouse(x,y))⑫\\
\forall x \forall y \forall z(spouse(x,y) \rightarrow \neg spouse(y,w))⑬\\
\forall x \forall y \forall z(spouse(x,y) \wedge spouse(x,z) \rightarrow self(y,z))⑭
\end{gathered}
$$

其中$self(x,y)$表示x和y是同一人，上一问没有用到而这一问有用到，因此添加在这里。

这些条件主要是说自己和自己不能成为配偶、兄弟不能成为配偶、不能重婚，显然成立。

用归结的作法会很复杂，要讨论4种情况各自推出矛盾，因此这里简单演绎推理一下：

要证明$KB \to ⑨$成立，只需证明$\neg KB \wedge ⑩$不可满足。

根据③、⑦、⑩我们可以知道，一定有$club(w)$成立。故根据知识库，w的论域只能为$\{Joe, Sally, Bill, Ellen\}$。

接下来用反证法，若$w = Ellen$,则⑩式与⑪式推出矛盾；若$w = Bill$，则⑩式与⑫式推出矛盾；若$w = Joe$，根据②、⑩、⑬式可推出矛盾；若$w = sally$，同$w = Joe$一样也可推出矛盾。因此$\neg KB \wedge ⑩$不可满足，故$KB \to ⑨$为真，即$Ellen$没结婚。

4. Consider the following formulae asserting that a binary relation is symmetric, transitive, and serial:

$$
\begin{aligned}
S_1 &: \forall x \forall y (P(x,y) \to P(y,x)) \\
S_2 &: \forall x \forall y \forall z (P(x,y) \wedge P(y,z) \to P(x,z)) \\
S_3 &: \forall x \exists y P(x,y)
\end{aligned}
$$

Prove by resolution that

$$S_1 \wedge S_2 \wedge S_3 \models \forall x P(x,x)$$

In other words, if a binary relation is symmetric, transitive and serial, then it is reflexive.

Solution：

消去蕴含并`skolemize`：

$$
\begin{aligned}
S1 &: \forall x \forall y (\neg P(x,y) \vee P(y,x)) \\
S2 &: \forall x \forall y \forall z (\neg P(x,y) \vee \neg P(y,z) \vee P(x,z)) \\
S3 &: \forall x P(x, g(x))
\end{aligned}
$$

故子句为：

$$
\begin{aligned}
&\neg P(x,y) \vee P(y,x) ① \\
&\neg P(x,y) \vee \neg P(y,z) \vee P(x,z) ② \\
&P(x, g(x)) ③
\end{aligned}
$$

结论的否定为：

$$\neg P(a,a) ④$$

首先将①与②归结$R[1b, 2b]\{z = x\}$：

$$\neg P(x,y) \vee P(x,x) ⑤$$

将③与⑤归结$R[3b, 5a]\{y = g(x)\}$：

$$P(x,x) ⑥$$

将④与⑥归结$R[4, 6]\{x = a\}$：

$$() ⑦$$

可以看到得到了空子句。故结论成立。