Reinforcement Learning for Lightweight Model

- Fundamentals of Reinforcement Learning
  - Markov Decision Process (MDP)
  - Dynamic Programming (Tabular RL)

Outline

- Introduction
- Markov Property
- Markov Process
- Markov Reward Process (MRP)
- Markov Decision Process (MDP)

The purpose of this chapter:

- Introduce different Markov processes

Introduction

- Markov decision processes formally describe an environment for reinforcement learning
  - where the environment is fully observable.
  - i.e. The current state completely characterizes the process
  - E.g., 2048.
- Almost all RL problems can be formalized as MDPs, e.g.
  - Optimal control primarily deals with continuous MDPs
  - Partially observable problems can be converted into MDPs
  - Bandits are MDPs with one state

Markov Property

- Markov Property:
  - "The future is independent of the past given the present"
  - Definition: A state S_t is Markov if and only if
    $$\mathbb{P}[S_{t+1} \mid S_t] = \mathbb{P}[S_{t+1} \mid S_1, \ldots, S_t]$$
- Comments:
  - The state captures all relevant information from the history
  - Once the state is known, the history may be thrown away
  - i.e. The state is a sufficient statistic of the future
- But, what if the history does matter?
  - Simply let S_t carry all information of history, $H_t = (S_1, \ldots, S_{t-1})$.
    - E.g., the castling rule for chess.
  - Then, it satisfies Markov Property.

Markov Process

- A Markov process is a memoryless random process,
  - i.e. a sequence of random states S_1, S_2, ... with the Markov property.

Definition:

- A Markov Process (or Markov Chain) is a tuple $<\mathcal{S}, \mathcal{P}>$
  - $\mathcal{S}$ is a (finite) set of states
  - $\mathcal{P}$ is a state transition probability matrix (part of the environment), $\mathcal{P}_{ss'} = \mathbb{P}[S_{t+1} = s' | S_t = s]$

State Transition Matrix

- For a Markov state s and successor state s', the state transition probability is defined by

$$\mathcal{P}_{ss'} = \mathbb{P}[S_{t+1} = s' \mid S_t = s]$$

- State transition matrix $\mathcal{P}$: (assume n states)

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \dots & & \dots \\ \mathcal{P}_{n1} & \dots & \mathcal{P}_{nn} \end{bmatrix}$$

  - Each row of matrix sums to 1.
- Stationary distribution:
  - Let π be the stationary distribution of states.
  - Then, $\pi\mathcal{P} = \pi$. → $\pi\mathcal{P} - \pi = 0$ → $\pi\mathcal{P} - \pi I = \pi(\mathcal{P} - I) = 0$ →
    - Use eigenvectors to derive it. (But not the scope of this course)

Markov Reward Process (MRP)

- A Markov reward process is a Markov chain with values.

Definition:

- A Markov Reward Process is a tuple <$\mathcal{S}$, $\mathcal{P}$, $\mathcal{R}$, γ>
  - $\mathcal{S}$ is a (finite) set of states
  - $\mathcal{P}$ is a state transition probability matrix (part of the environment),
    $\mathcal{P}_{ss'} = \mathbb{P}[S_{t+1} = s' \mid S_t = s]$
  - $\mathcal{R}$ is a reward function,
    $\mathcal{R}_s = \mathbb{E}[R_{t+1} \mid S_t = s]$
  - γ is a discount factor $\gamma \in [0, 1]$.
-

Return

Definition

- The return G_t is the total discounted reward from time-step t.

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

Notes:

- The discount $\gamma \in [0, 1]$ is the present value of future rewards
- The value of receiving reward R is diminishing
  - $\gamma^k R$, after $k+1$ time-steps.
- This values immediate reward above delayed reward.
- Discount:
  - γ close to 0 leads to ”myopic” evaluation
  - γ close to 1 leads to ”far-sighted” evaluation

Value Function

- The value function $v(s)$ gives the long-term value of s
- Definition
  - The state value function $v(s)$ of an MRP is the expected return starting from state s
  - $v(s) = \mathbb{E}[G_t \mid S_t = s]$

Bellman Equation for MRPs

- The value function can be decomposed into two parts:
  - immediate reward R_{t+1}
  - discounted value of successor state $\gamma\, v(S_{t+1})$
- $$\begin{aligned} v(s) &= \mathbb{E}[G_t \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma (R_{t+2} + \gamma R_{t+3} + \cdots) \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma\, G_{t+1} \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma\, v(S_{t+1}) \mid S_t = s] \end{aligned}$$
- For a transition (s, r, s'), we have

$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}\, v(s')$$

Bellman Equation in Matrix Form

- The Bellman equation can be expressed concisely using matrices, (closed form)

$$v = \mathcal{R} + \gamma \mathcal{P} v$$

  - where v is a column vector with one entry per state.

$$\begin{bmatrix} v(1) \\ \dots \\ v(n) \end{bmatrix} = \begin{bmatrix} R_1 \\ \dots \\ R_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \dots & & \dots \\ \mathcal{P}_{n1} & \dots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \dots \\ v(n) \end{bmatrix}$$

Solving the Bellman Equation

- The Bellman equation is a linear equation
- It can be solved directly:

$$v = \mathcal{R} + \gamma \mathcal{P} v$$
$$v = (1 - \gamma \mathcal{P})^{-1} \mathcal{R}$$

- Computational complexity is $O(n^3)$ for n states
- Direct solution only possible for small MRPs
- There are many iterative methods for large MRPs, e.g.
  - Dynamic programming
  - Monte-Carlo evaluation
  - Temporal-Difference learning

Markov Decision Processes (MDP)

- A (Finite) Markov Decision Process is a tuple $<\mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma>$
  - $\mathcal{S}$ is a (finite) set of states
  - $\mathcal{A}$ is a (finite) set of actions
  - $\mathcal{P}$ is a state transition probability matrix (part of the environment), $\mathcal{P}_{ss'}^{a} = \mathbb{P}[S_{t+1} = s' \mid S_t = s, A_t = a]$
    - Let $\mathcal{P}^{a}$ denote the matrix $\mathcal{P}_{::}^{a}$.
  - $\mathcal{R}$ is a reward function, $\mathcal{R}_{s}^{a} = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$
  - γ is a discount factor $\gamma \in [0, 1]$.

Example: Recycling Robot

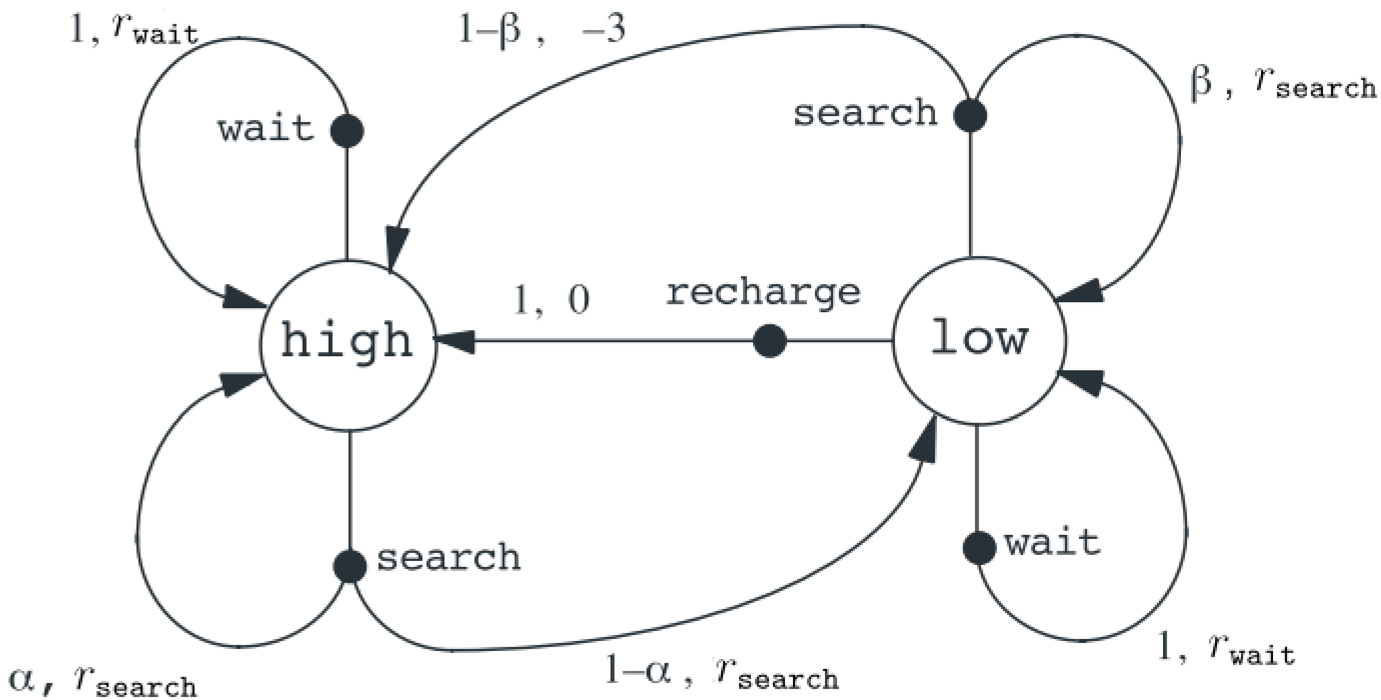


Figure 3.3: Transition graph for the recycling robot example.

Example: Recycling Robot

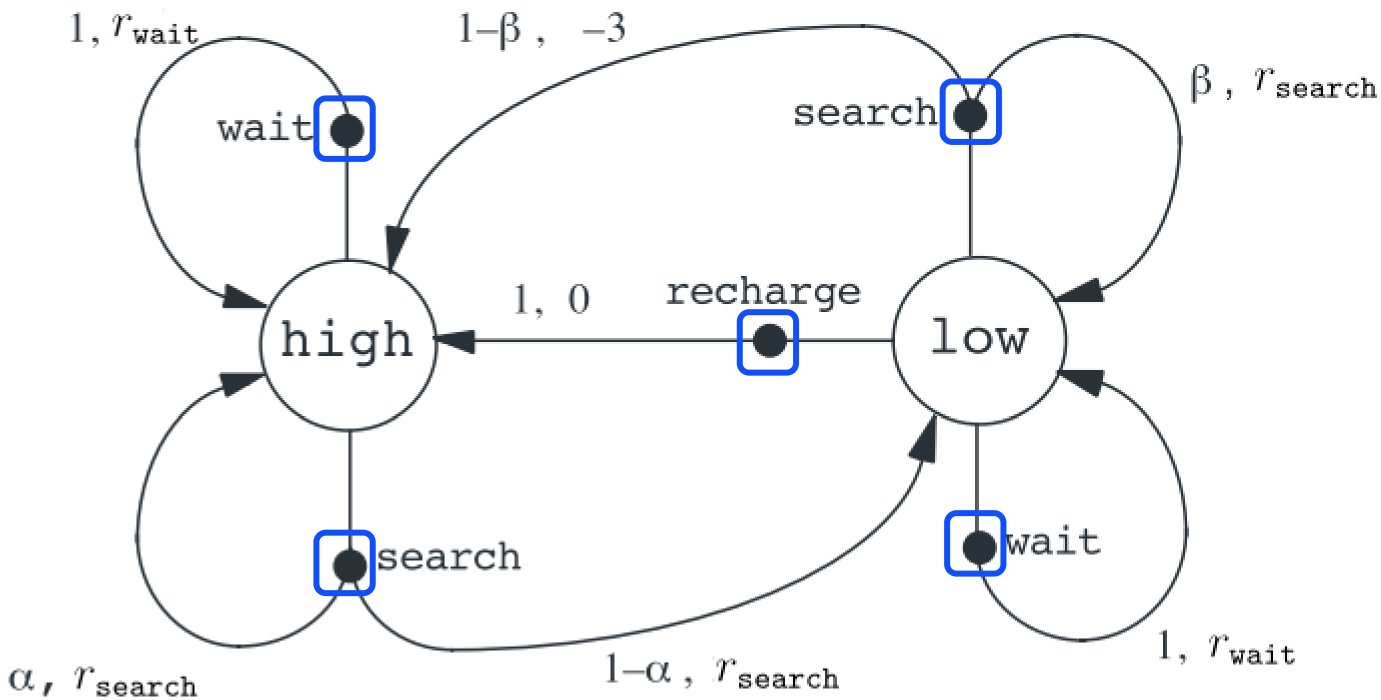


Figure 3.3: Transition graph for the recycling robot example.

Example: Recycling Robot

- Transition and Rewards:

s	s'	a	$p(s' \mid s, a)$	$r(s, a, s')$
high	high	search	α	$r_{\texttt{search}}$
high	low	search	$1-\alpha$	$r_{\texttt{search}}$
low	high	search	$1-\beta$	-3
low	low	search	β	$r_{\texttt{search}}$
high	high	wait	1	$r_{\texttt{wait}}$
high	low	wait	0	$r_{\texttt{wait}}$
low	high	wait	0	$r_{\texttt{wait}}$
low	low	wait	1	$r_{\texttt{wait}}$
low	high	recharge	1	0
low	low	recharge	0	$0.$

Policies

- A policy is the agent's behavior
  - It is a map from state to action
  - A policy fully defines the behavior of an agent
  - MDP policies depend on the current state (not the history)
    - i.e. Policies are stationary (time-independent),
      $A_t \sim \pi(\cdot \mid S_t), \forall t > 0$
- Policy types:
  - Deterministic policy: $a = \pi(s_i)$
  - Stochastic policy: $\pi(a|s) = \mathbb{P}[A_t = a \mid S_t = s]$
    - Sometimes, written in $\pi(s, a)$.
    - Note: for deterministic policy,
      - if $a = \pi(s_i)$, $\pi(a|s) = 1$. otherwise, $\pi(a|s) = 0$.
- Examples:
  - In 2048: Up/down/left/right
  - In robotics: angle/force/…

Policy and MRP

- Given an MDP $<\mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma>$ and a policy π
- The state sequence $S_1, S_2, \ldots$ is a Markov process $<\mathcal{S}, \mathcal{P}^{\pi}>$
- The state and reward sequence $S_1, R_2, S_2, R_3, \ldots$ becomes a Markov reward process (MRP) $<\mathcal{S}, \mathcal{P}^{\pi}, \mathcal{R}^{\pi}, \gamma>$
  - $\mathcal{P}^{\pi}$ is a state transition probability matrix (part of the environment),

$$\mathcal{P}^{\pi}_{ss'} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{P}^{a}_{ss'}$$

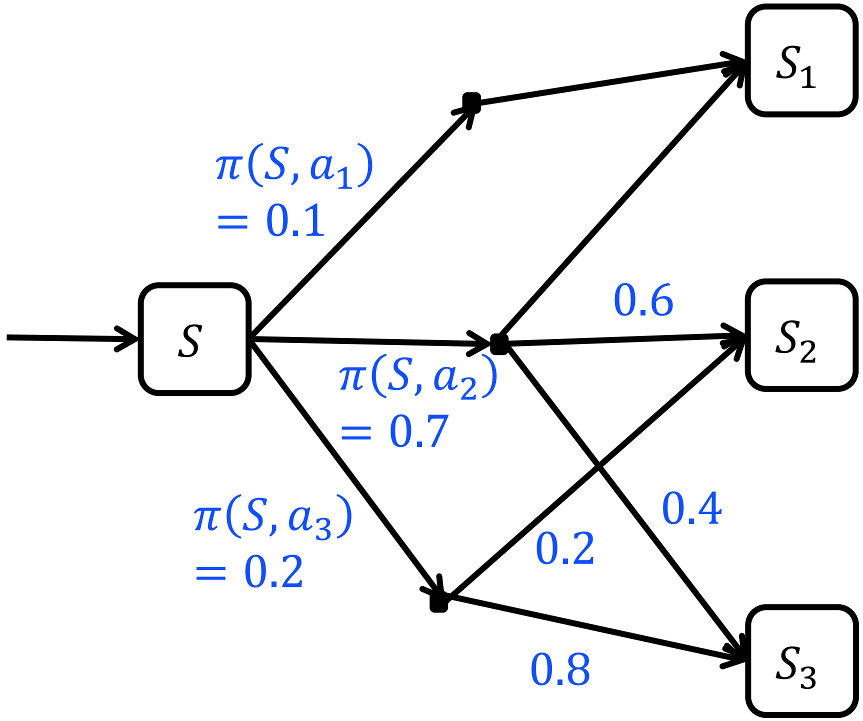
  - $\mathcal{R}^{\pi}$ is a reward function,

$$\mathcal{R}^{\pi}_{s} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{R}^{a}_{s}$$

- So, the property of MRP can be applied.

Example

- We have $\mathcal{P}_{ss_3}^{\pi} = 0.7 * 0.4 + 0.2 * 0.8 = 0.44$



Value Function

- A value function is a prediction of future reward
  - Used to evaluate the goodness/badness of states
    - therefore to select between actions.
  - Return $G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots$
- Types of value functions under policy π:
  - State value function: the expected return from s.

$$\begin{aligned} v_\pi(s) &= \mathbb{E}_\pi[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots \mid S_t = s] \\ &= \mathbb{E}_\pi[G_t \mid S_t = s] \end{aligned}$$

  - Q-Value function: the expected return from s taking action a.

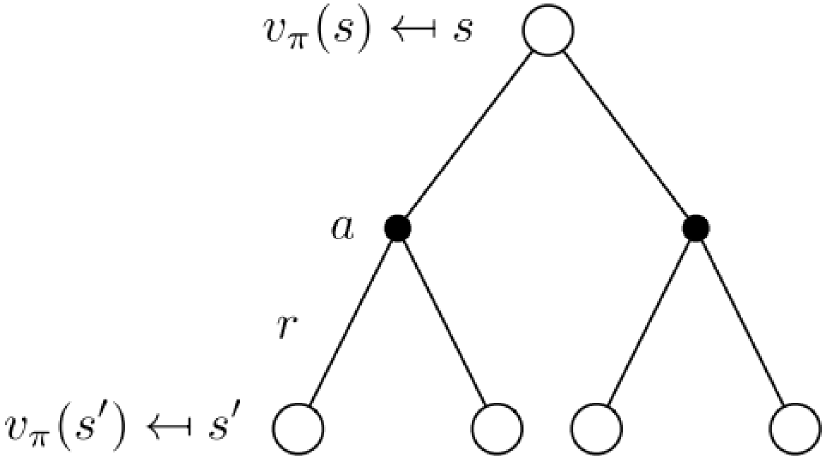
$$q_\pi(s, a) = \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a]$$

- Examples:
  - In 2048, the expected score from a board S_t.

Bellman Expectation Equation for π

- State value function:

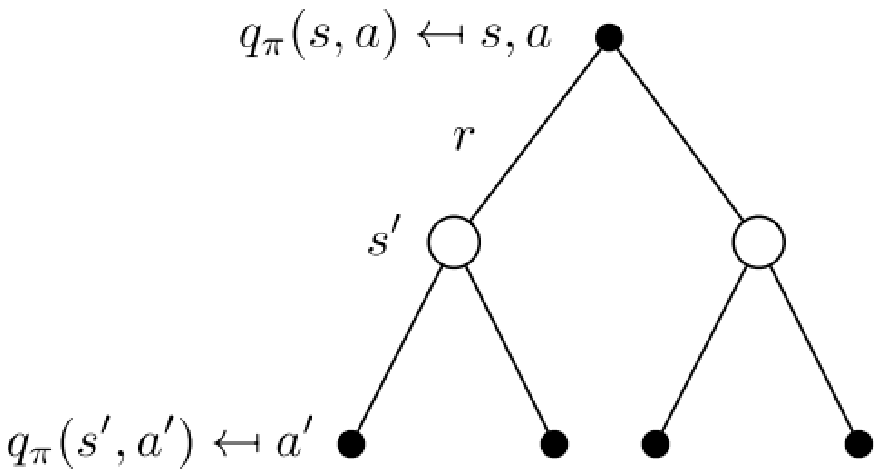
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$



Bellman Expectation Equation for π

- Q value

$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a \sum_{a' \in \mathcal{A}} \pi(a'|s') q_\pi(s',a')$$



Bellman Expectation Equation in Matrix

- The Bellman expectation equation can be expressed concisely using the induced MRP.
- So, it can be solved directly:

$$v_\pi = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v_\pi$$
$$v_\pi = (1 - \gamma \mathcal{P}^\pi)^{-1} \mathcal{R}^\pi$$

Optimal Value Function

- The optimal state-value function $v_*(s)$ is the maximum value function over all policies

$$v_*(s) = \max_{\pi} v_\pi(s)$$

- The optimal action-value function $q_*(s,a)$ is the maximum action-value function over all policies

$$q_*(s,a) = \max_{\pi} q_\pi(s,a)$$

- Notes:
  - The optimal value function specifies the best possible performance in the MDP.
  - An MDP is “solved” when we know the optimal value function.

Optimal Policy

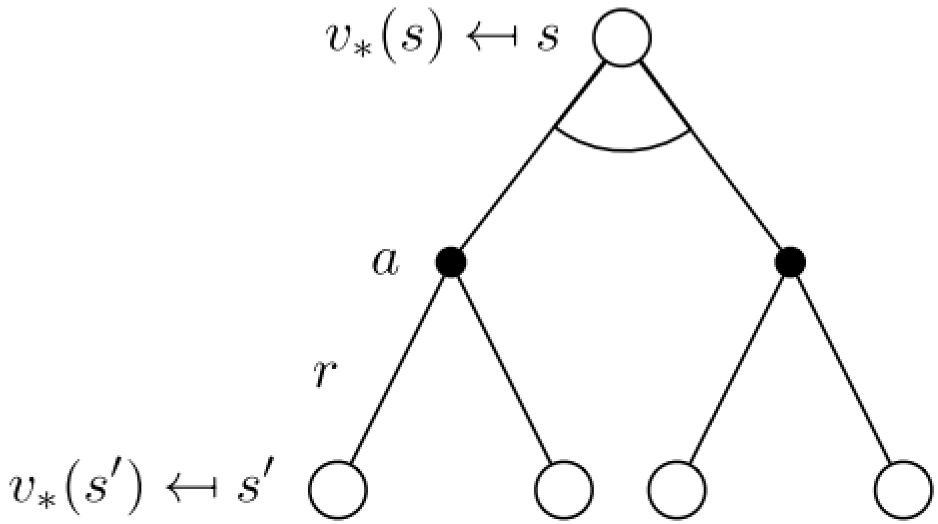
- Define a partial ordering over policies
  $\pi \geq \pi'$ if $v_\pi(s) \geq v_{\pi'}(s), \forall s$

- Theorem: For any Markov Decision Process,
  - There exists an optimal policy π_* that is better than or equal to all other policies, $\pi_* \geq \pi, \forall \pi$.
  - All optimal policies achieve the optimal value function,
    $$v_{\pi_*}(s) = v_*(s)$$
  - All optimal policies achieve the optimal action-value function,
    $$q_{\pi_*}(s,a) = q_*(s,a)$$

Finding an Optimal Policy

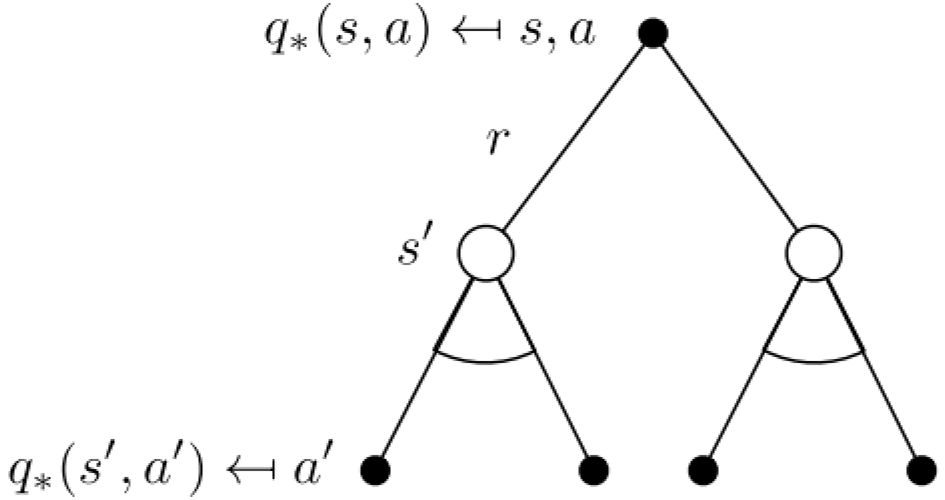
- An optimal policy can be found by maximizing over $q_*(s,a)$,
  - $\pi(a|s) = 1$, if $a = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\, q_*(s,a)$
  - $\pi(a|s) = 0$, otherwise.
- There is always a deterministic optimal policy for any MDP
- If we know $q_*(s,a)$, we immediately have the optimal policy
- What about state value function $v_*(s)$?
  - Similar, but we need to know model, $\mathcal{P}_{ss'}^{a}$. → not model free.

Bellman Optimality Equation for V*



$$v_*(s) = \max_a \left(\mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a v_*(s') \right)$$

Bellman Optimality Equation for Q*



$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a \max_{a' \in \mathcal{A}} q_\pi(s,a')$$

Solving the Bellman Optimality Equation

- Bellman Optimality Equation is non-linear
- No closed form solution (in general)
- Many iterative solution methods
  - Value Iteration
  - Policy Iteration
  - Q-learning
  - Sarsa

Extensions to MDPs

- Infinite and continuous MDPs
  - Countably infinite state and/or action spaces
    - Straightforward
  - Continuous state and/or action spaces
    - Closed form for linear quadratic model (LQR)
  - Continuous time
    - Requires partial differential equations
    - Hamilton-Jacobi-Bellman (HJB) equation
    - Limiting case of Bellman equation as time-step
- Partially observable MDPs
  - E.g., Mahjong (as we mentioned)
- Undiscounted, average reward MDPs (ignored)

Prediction vs. Control

- For prediction: evaluate values
  - Input: MDP <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$, $\mathcal{R}$, γ> and policy π
    or: MRP <$\mathcal{S}$, $\mathcal{P}^{\pi}$, $\mathcal{R}^{\pi}$, γ>
  - Output: value function v_{π} or q_{π}
- For control: find the optimal policy.
  - Input: MDP <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$, $\mathcal{R}$, γ>
  - Output: optimal value function v_{*} or q_{*}
    and: optimal policy, π_{*}

	state values	action values
prediction	v_π	q_π
control	v_*	q_*

Reinforcement Learning for Lightweight Model

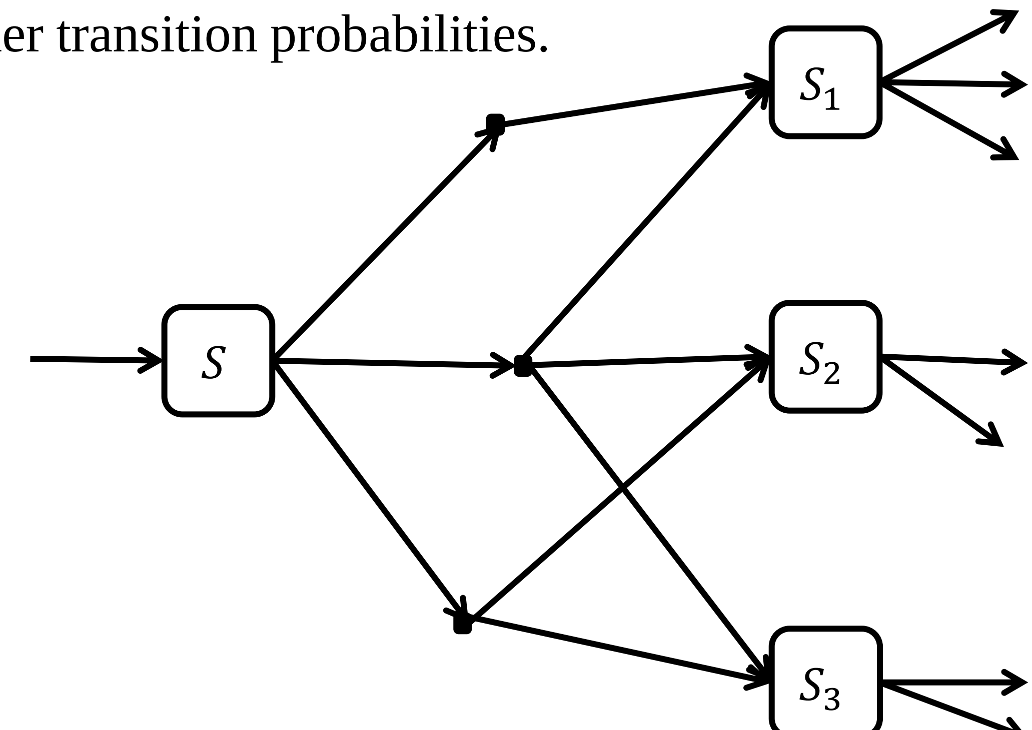
- Fundamentals of Reinforcement Learning
  - Markov Decision Process (MDP)
  - Dynamic Programming (Tabular RL)

Dynamic Programming

- (Sutton) The term dynamic programming (DP) refers to a collection of algorithms that
  - compute optimal policies given a perfect model of the environment as a Markov decision process (MDP).
- (Silver) A method for solving complex problems by breaking them down into subproblems
  - Solve the subproblems,
  - Combine solutions to subproblems
- (Algorithm textbook by Cormen et al.) says
  - DP, like the divide-and-conquer method, solves problems by combining the solutions to subproblems.
  - DP is typically applied to optimization problems.
  - Applications:
    - String algorithms (e.g. sequence alignment)
    - Graph algorithms (e.g. shortest path algorithms)
    - Bioinformatics (e.g. lattice models)

Example

- By dynamic programming, we don't have to repeat calculate the state values, such as S_1, S_2, S_3.
- In most algorithms given in Algorithms course
  - Rarely consider transition probabilities.



Why is DP related?

- Sequential or temporal component to the problem optimizing
  - a "program", i.e. a policy,
  - values, i.e., state values and state action values
- Like solving LCS (longest common sequence) problem.
  - The optimal actions.
  - The optimal values.
  - $\mathcal{P}$ and π are deterministic.
  - Exercise: shortest path problem.

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑0	↑0	↑0	↖1	←1	↖1
2	B	0	↖1	←1	←1	↑1	↖2	←2
3	C	0	↑1	↑1	↖2	←2	↑2	↑2
4	B	0	↖1	↑1	↑2	↑2	↖3	←3
5	D	0	↑1	↖2	↑2	↑2	↑3	↑3
6	A	0	↑1	↑2	↑2	↖3	↑3	↖4
7	B	0	↖1	↑2	↑2	↑3	↖4	↑4

Requirements for Dynamic Programming

- Dynamic Programming is a very general solution method for problems which have two properties:
  - Optimal substructure
    - Principle of optimality applies
    - Optimal solution can be decomposed into subproblems
  - Overlapping subproblems
    - Subproblems recur many times
    - Solutions can be cached and reused
- Markov decision processes satisfy both properties
  - Bellman equation gives recursive decomposition
  - Value function stores and reuses solutions

Planning by Dynamic Programming

- Dynamic programming assumes full knowledge of the MDP
  - It is used for planning in an MDP
- For prediction: evaluate values
  - Input: MDP <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$, $\mathcal{R}$, γ> and policy π
    or: MRP <$\mathcal{S}$, $\mathcal{P}^{\pi}$, $\mathcal{R}^{\pi}$, γ>
  - Output: value function v_{π}
- For control: find the optimal policy.
  - Input: MDP <$\mathcal{S}$, $\mathcal{A}$, $\mathcal{P}$, $\mathcal{R}$, γ>
  - Output: optimal value function v_{*}
    and: optimal policy, π_{*}

Three Approaches

- Policy Evaluation
  - Directly solve Bellman Equation in matrix form (see above)
    - Given an MDP $<\mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma>$ and a policy π,
      it becomes a MRP problem $<\mathcal{S}, \mathcal{P}^{\pi}, \mathcal{R}^{\pi}, \gamma>$.
  - Use Iterative Policy Evaluation
- Policy Iteration
- Value Iteration

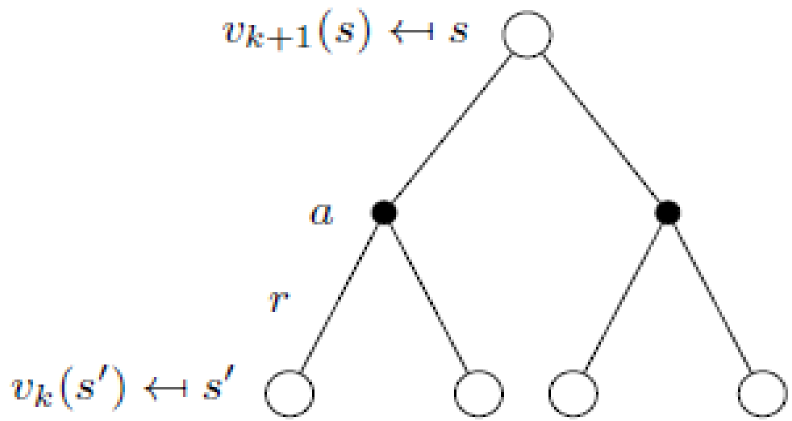
Iterative Policy Evaluation

- Problem: evaluate a given policy π
- Solution: iterative application of Bellman expectation backup

$$v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_*$$

- Using synchronous backups,
  - At each iteration $k+1$,
    - for all states $s \in S$, update $v_{k+1}(s)$ from $v_k(s')$ where s' is a successor state of s
- Notes:
  - We will discuss asynchronous backups later
  - Convergence to v_π will be proven at the end of the lecture
  - Review the Bellman-Ford algorithm for the shortest path problem.

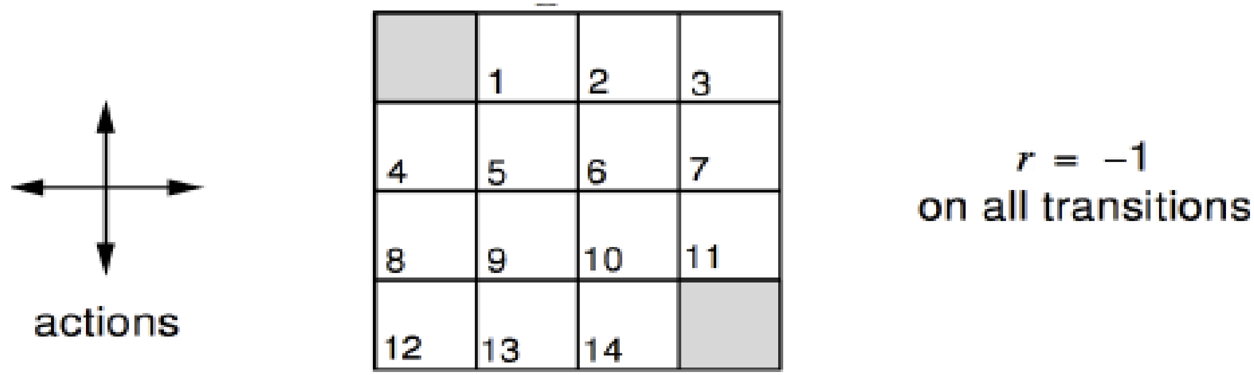
Iterative Policy Evaluation



$$v_{k+1}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a v_k(s') \right)$$

$$\boldsymbol{v}^{k+1} = \boldsymbol{\mathcal{R}}^{\pi} + \gamma \boldsymbol{\mathcal{P}}^{\pi} \boldsymbol{v}^{k}$$

Example: Evaluating a Random Policy in the Small Gridworld



- States:
  - Nonterminal states 1, …, 14
  - One terminal state (shown twice as shaded squares)
- Actions
  - Four directional moves
  - leading out of the grid leave state unchanged
- Reward
  - -1 until the terminal state is reached
- Undiscounted: episodic MDP ($\gamma = 1$)
- Agent follows uniform random policy

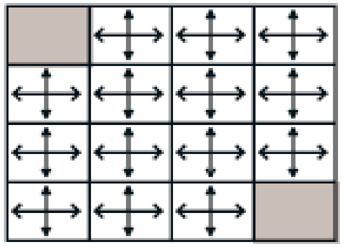
$$\pi(n|\cdot) = \pi(e|\cdot) = \pi(s|\cdot) = \pi(w|\cdot) = 0.25$$

Iterative Policy Evaluation in Small Gridworld (I)

v_k for the Random Policy | Greedy Policy w.r.t. v_k

$k = 0$

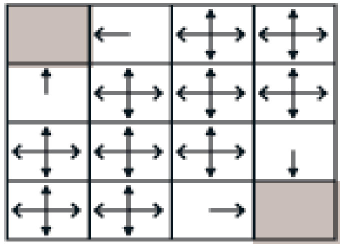
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



random policy

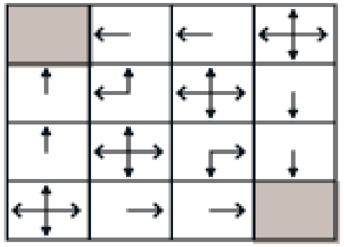
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0



$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0



Iterative Policy Evaluation in Small Gridworld (2)

$k = 3$

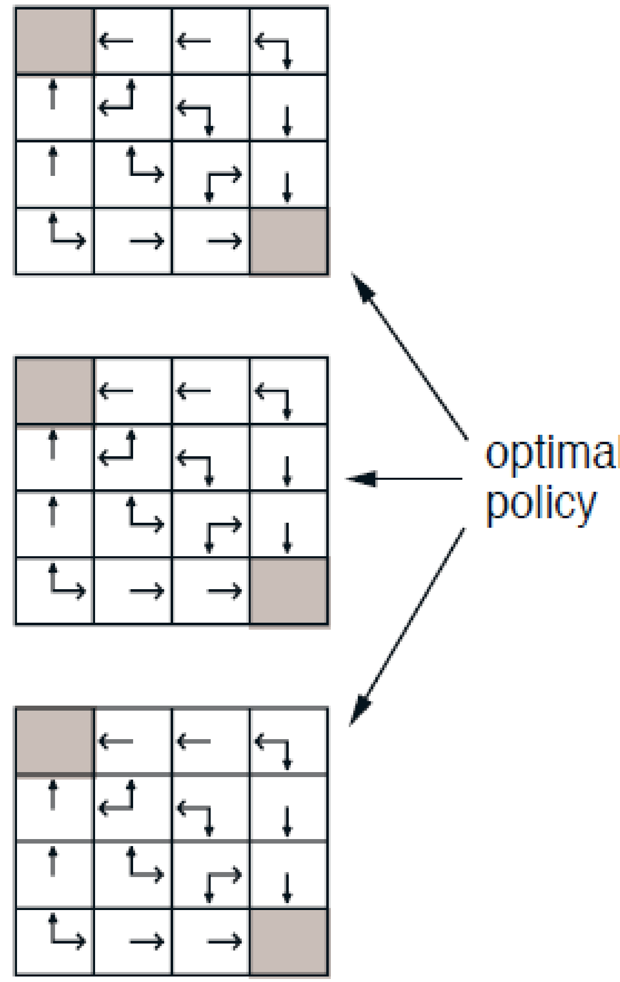
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



How to Improve a Policy

- Definition of policy improvement
  - Let π and π' be any pair of deterministic policies
    - For all $s \in S$, "$\pi(s)$ performs better than $\pi'(s)$". (We will see example)
- Given a policy π
  - Evaluate the policy π

$$v_\pi(s) = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \cdots | S_t = s]$$

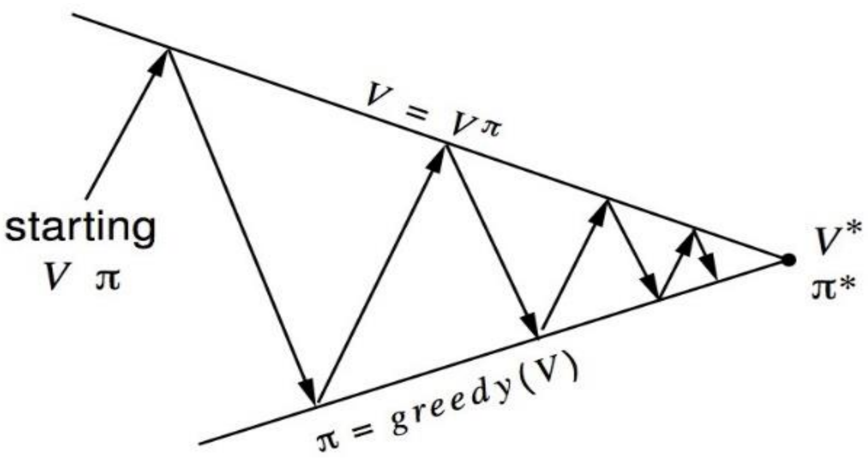
  - Improve the policy by acting greedily with respect to v_π

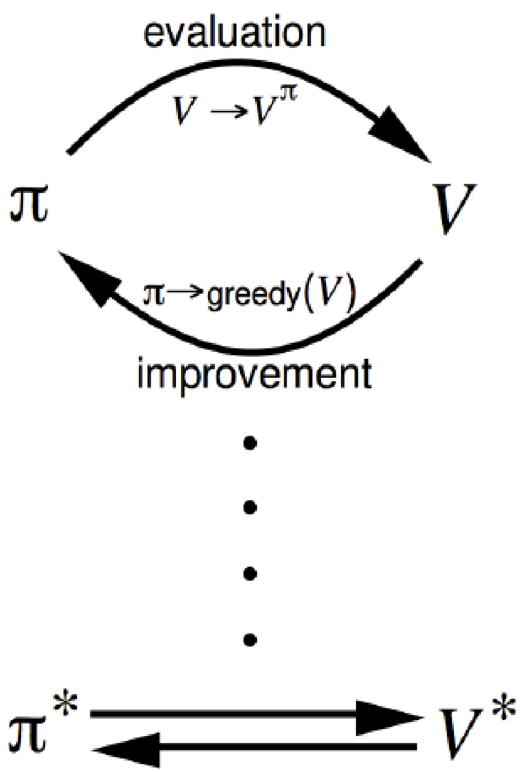
$$\pi' = \text{greedy}(v_\pi)$$

- Notes:
  - In Small Gridworld improved policy was optimal, $\pi' = \pi^*$
  - In general, need more iterations of improvement / evaluation
  - But this process of policy iteration always converges to π^*

Policy Iteration

- Policy evaluation → Estimate v_π
  - Iterative policy evaluation
- Policy improvement → Generate $\pi' \geq \pi$
  - Greedy policy improvement





Proof of Policy Improvement

- Consider a deterministic policy, $a = \pi(s)$
- We can improve the policy by acting greedily

$$\pi'(s) = \underset{a \in A}{\operatorname{argmax}}\, q_\pi(s, a)$$

- This improves the value from any state s over one step,

$$q_\pi\big(s, \pi'(s)\big) = \max_{a \in A} q_\pi(s, a) \geq q_\pi\big(s, \pi(s)\big) = v_\pi(s)$$

- It therefore improves the value function, $v_{\pi'}(s) \geq v_\pi(s)$.

$$\begin{aligned} v_\pi(s) &\leq q_\pi\big(s, \pi'(s)\big) = \mathbb{E}_{\pi'}[R_{t+1} + \gamma\, v_\pi(S_{t+1}) \mid S_t = s] \\ &\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma q_\pi\big(S_{t+1}, \pi'(S_{t+1})\big) \mid S_t = s\big] \\ &\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi\big(S_{t+2}, \pi'(S_{t+2})\big) \mid S_t = s\big] \\ &\leq \mathbb{E}_{\pi'}[R_{t+1} + \gamma R_{t+2} + \cdots \mid S_t = s] = v_{\pi'}(s) \end{aligned}$$

Converge of Policy Improvement

- If improvements stop,
  - That is, for $q_\pi\big(s,\pi'(s)\big) = \max_{a\in A} q_\pi(s,a) \geq q_\pi\big(s,\pi(s)\big) = v_\pi(s)$
    - "≥" becomes "=" when stopping.
- Then the Bellman optimality equation has been satisfied

$$v_\pi(s) = \max_{a\in A} q_\pi(s,a)$$

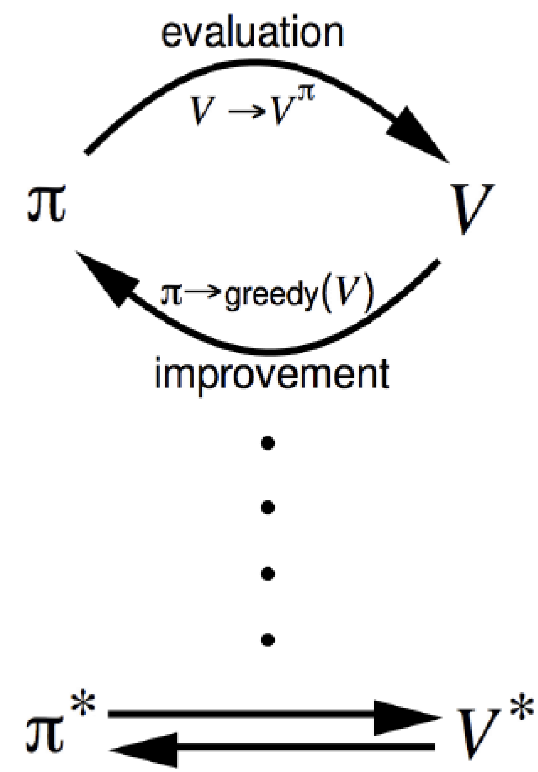
- This implies $v_\pi(s) = v_*(s)$ for all $s \in S$
- The above proves that π will converge to an optimal policy.

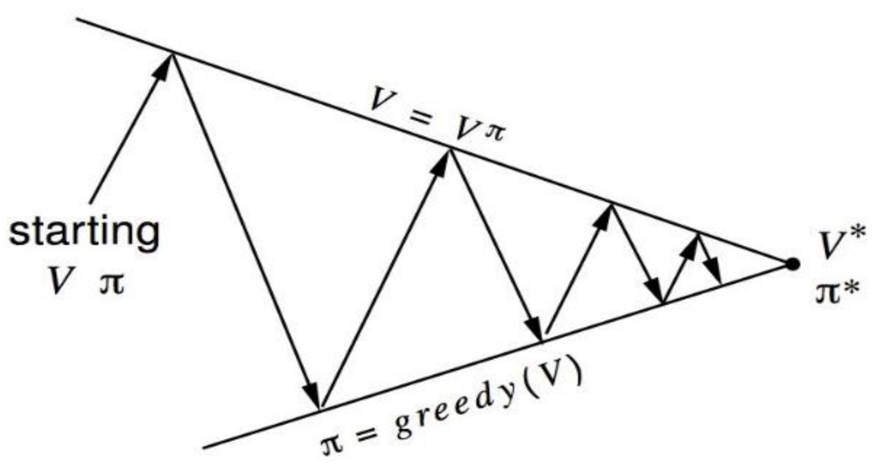
Variations of Policy Iteration

- Questions:
  - Does policy evaluation need to converge to v_π?
  - Should we introduce a stopping condition, e.g. ∈-convergence of value function?
  - Simply stop after k iterations of iterative policy evaluation?
    - For example, in the small gridworld $k = 3$ was sucient to achieve optimal policy
    - Why not update policy every iteration? i.e. stop after $k = 1$

Generalized Policy Iteration

- Policy evaluation → Estimate v_π
  - Any policy evaluation algorithm
- Policy improvement → Generate $\pi' \geq \pi$
  - Any policy improvement algorithm





Principle of Optimality

- Theorem (Principle of Optimality)
  - A policy $\pi(a|s)$ achieves the optimal value from state s, $v_\pi(s) = v_*(s)$, if and only if
  - For any state s' reachable from s, π achieves the optimal value from state s', $v_\pi(s') = v_*(s')$

Deterministic Value Iteration

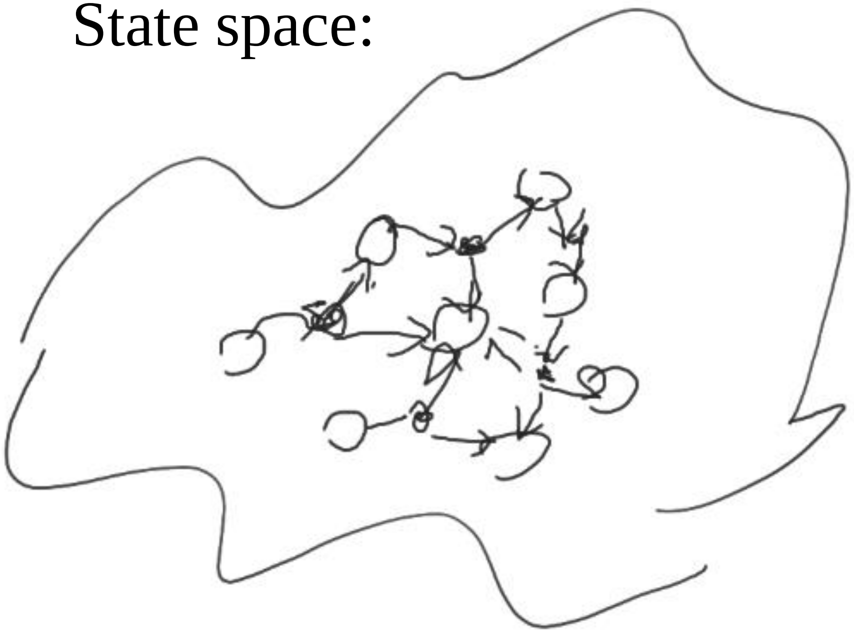
- If we know the (optimal) solution to subproblems $v_*(s')$
- Then solution $v_*(s)$ can be found by one-step lookahead

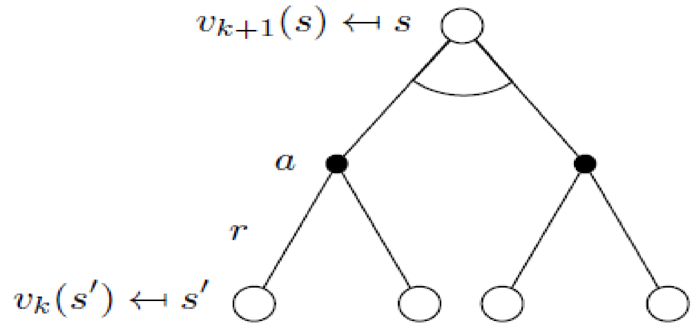
$$v_*(s) \leftarrow \max_{a \in A} \left(R_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a v_*(s') \right)$$

- Intuition:
  - Start with final rewards and work backwards
  - apply these updates iteratively
- Notes:
  - Still works with loopy, stochastic MDPs
  - Like most DP problems. (e.g., shortest path problem)

Bellman Optimality

State space:





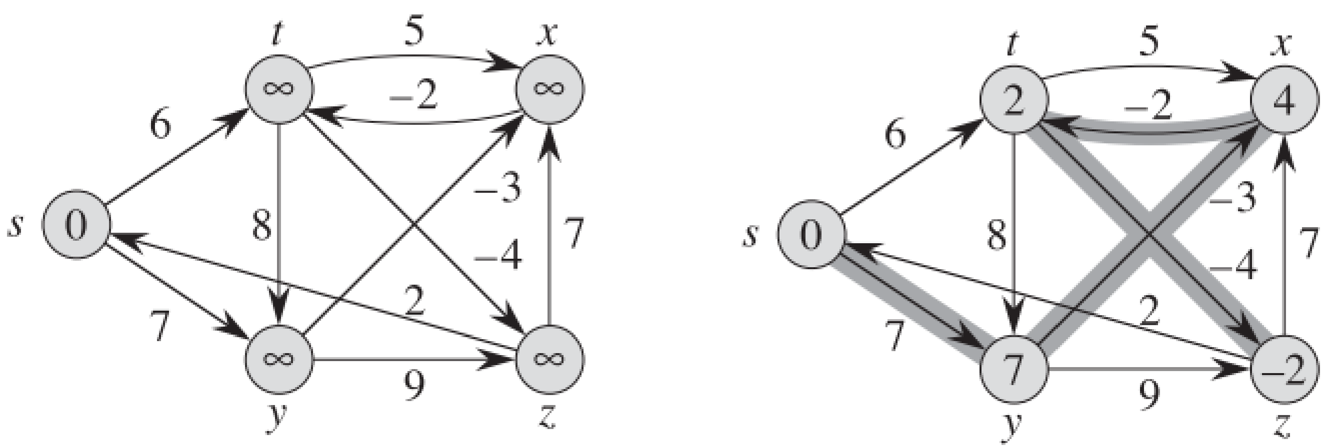
$$v_{n+1}(s) = \max_{a \in A}\left(\mathcal{R}_s^a + \gamma \sum_{s' \in S} \mathcal{P}_{ss'}^a\, v_n(s')\right)$$

or:

$$V^{(n+1)}(s) = \max_{a \in \mathcal{A}}\left(\mathbb{E}_{s'|s,a}\left[r + \gamma V^{(n)}(s')\right]\right)$$

The Shortest Path Problem

- A very simple MDP problem with
  - deterministic state transition $\mathcal{P}$.
    (Just consider the case without state-action or black dots)
- A good example to get a quick idea about why it works.
  (see Cormen’s Algorithm textbook)



Algorithms for the Shortest Path Problem

```
RELAX(u, v, w)
if v.d > u.d + w(u, v)
    v.d = u.d + w(u, v)
    v.π = u
```

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| - 1
    for each edge (u, v) ∈ G.E
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

- Bellman-Ford Algorithm:
  - Simple, but it works.
    - All are based on Relexation
    - Complexity for all pairs: $O(n^2 e)$, n: vertex count, e: edge count.
- Dijkstra Algorithm:
  - Faster, but complex and no negative values
    - Complexity for all pairs: $O(ne + n^2 \log n)$
- Note:
  - The concept of Value Iterative is based on Bellman-Ford.

Value Iteration

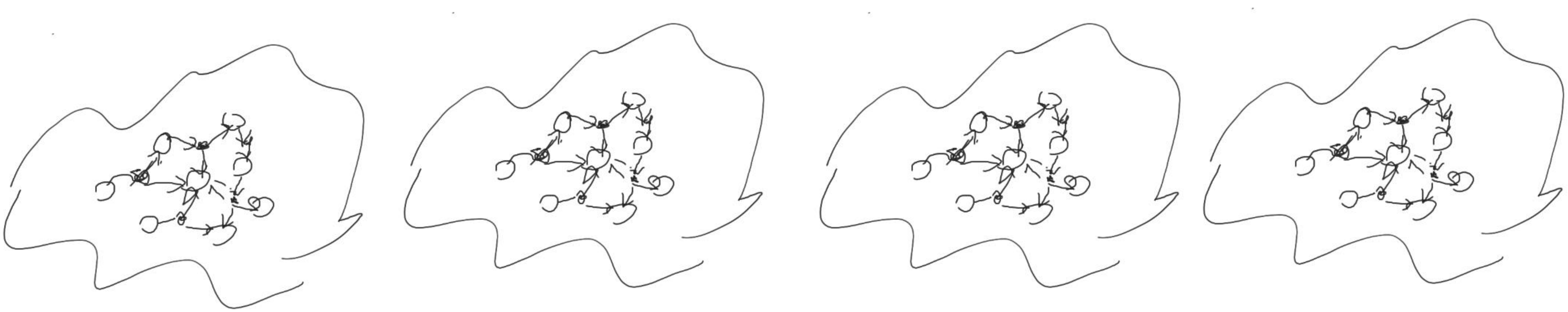
- Problem:
  - find optimal policy π
- Solution: directly find the optimal v_* without π.
  - iterative application of Bellman optimality backup

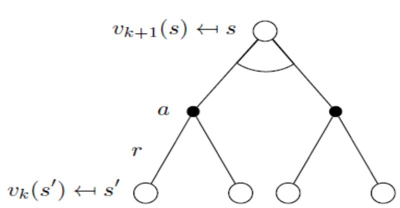
$$v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_*$$

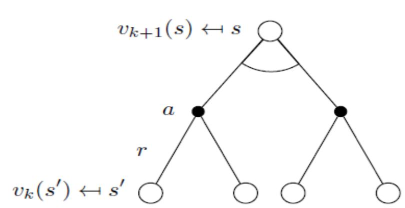
- Using synchronous backups (like Bellman-Ford)
  - At each iteration $k+1$
    - For all states $s \in S$
      - Update $v_{k+1}(s)$ from $v_k(s')$
- Convergence to v_* will be proven later
- Unlike policy iteration, there is no explicit policy

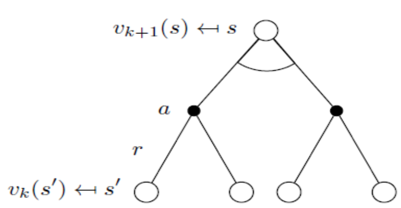
Value Iteration

$v_1 \rightarrow$ $v_2 \rightarrow$ … $\rightarrow v_*$





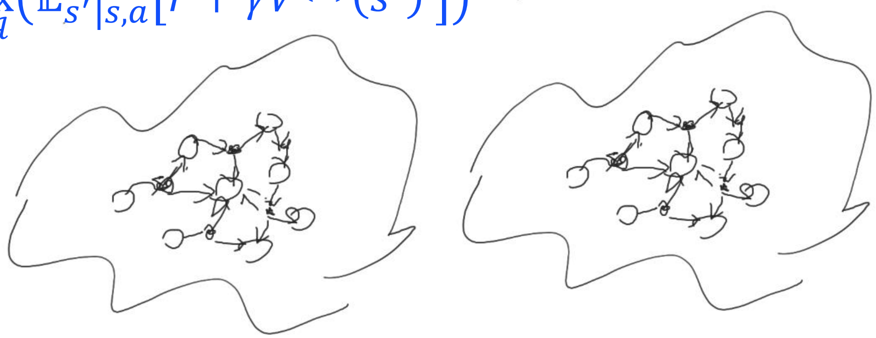




Operator View

- Value iteration update

$$V^{(n+1)}(s) = \max_{a\in\mathcal{A}}\left(\mathbb{E}_{s'|s,a}\left[r + \gamma V^{(n)}(s')\right]\right)$$



- It can be viewed as:
  - A function $\mathcal{T}: \mathcal{S} \to \mathcal{S}$.
  - Called backup operator.

$$[\mathcal{T}V](s) = \max_{a\in\mathcal{A}}\left(\mathbb{E}_{s'|s,a}\left[r + \gamma V(s')\right]\right)$$

$$V^{(n+1)} = \mathcal{T}V^{(n)}$$

(Let V be an array of $v(s)$)

Algorithm Value Iteration
Initialize $V^{(0)}$ arbitrarily.
for n = 0, 1, 2, . . . until termination condition do
$V^{(n+1)} = \mathcal{T}V^{(n)}$
end

Value Function Space

- Consider the vector space V over value functions
  - There are $|S|$ dimensions
  - Each point in this space fully species a value function $v(s)$
- What does a Bellman backup do to points in this space?
  - It brings value functions closer
  - Therefore the backups must converge on a unique solution

Value Function ∞-Norm

- We will measure distance between state-value functions u and v by the ∞-norm
  - i.e. the largest difference between state values,
  $$\|U - V\|_\infty = \max_s |u(s) - v(s)|$$
- Let $\delta = ||(U - V)||_\infty$
  - $u(s) - v(s) \leq \delta$ for all s

Contraction for Bellman Optimality Backup

- Bellman optimality backup operator $\mathcal{T}$ is a γ-contraction.
- Proof: Since

$$\max_{a\in\mathcal{A}}\big(x(a)\big) - \max_{a\in\mathcal{A}}\big(y(a)\big) \le \max_{a\in\mathcal{A}}(x(a) - y(a))$$

- we have $\left|\left|\mathcal{T}U - \mathcal{T}V\right|\right|_\infty$

$$= ||\max_{a\in\mathcal{A}}(\mathcal{R}^a + \gamma\,\mathcal{P}^a U) - \max_{a\in\mathcal{A}}(\mathcal{R}^a + \gamma\,\mathcal{P}^a V)||_\infty$$

$$\le ||\max_{a\in\mathcal{A}}[(\mathcal{R}^a + \gamma\,\mathcal{P}^a U) - (\mathcal{R}^a + \gamma\,\mathcal{P}^a V)]||_\infty$$

$$= ||\max_{a\in\mathcal{A}}[\gamma\,\mathcal{P}^a(U - V)]||_\infty = \gamma||\max_{a\in\mathcal{A}}[\,\mathcal{P}^a(U - V)]||_\infty$$

$$\le \gamma\delta = \gamma||(U - V)||_\infty$$

  - Note: $(\mathcal{P}^a_{s:,}(U - V)) \le \delta$ for all s
    → $||\mathcal{P}^a(U - V)||_\infty \le \delta$
    - For $\mathcal{P}^a$, each row of matrix sums to 1.

Contraction Mapping Theorem

- Backup operator $\mathcal{T}$ is a γ-contraction with modulus $\gamma(< 1)$ under ∞-norm

$$||\mathcal{T}U - \mathcal{T}V||_\infty \le \gamma\, ||U - V||_\infty$$

- By contraction-mapping principle, it has a fixed point V^*
  - by iterating

$$V, \mathcal{T}V, \mathcal{T}^2V, \ldots \to V^*$$

- Proof:

$$||\mathcal{T}V - \mathcal{T}V^*||_\infty \le \gamma\, ||V - V^*||_\infty$$

  - Since $\mathcal{T}V^* = V^*$,

$$||\mathcal{T}V - V^*||_\infty \le \gamma\, ||V - V^*||_\infty$$

  - By recurrence,
    $||\mathcal{T}^nV - V^*||_\infty \le \gamma\, ||\mathcal{T}^{n-1}V - V^*||_\infty \le \cdots \le \gamma^n\, ||V - V^*||_\infty$
  - Since $\gamma^n \to 0$, $||\mathcal{T}^nV - V^*||_\infty \to 0$.
  - That is, $\mathcal{T}^nV \to V^*$

Policy Evaluation

- Problem: how to evaluate fixed policy π:

$$V^{\pi}(s) = \mathbb{E}_{\pi}[G_t \,|\, S_t = s\,] = \mathbb{E}_{\pi}[R_{t+1} + \gamma V^{\pi}(S_{t+1}) \mid S_t = s]$$

- Backwards recursion involves a backup operation

$$V^{(k+1)} = \mathcal{T}^{\pi} V^{(k)}$$

  – $\mathcal{T}^{\pi}$ is defined as:

$$[\mathcal{T}^{\pi} V](s) = \mathbb{E}_{s'|s,a=\pi(s)}[r + \gamma V(s')\,]$$

- $\mathcal{T}^{\pi}$ is also a contraction with modulus γ, sequence

$$V, \mathcal{T}^{\pi}V, (\mathcal{T}^{\pi})^2 V, (\mathcal{T}^{\pi})^3 V, \ldots \to V^{\pi}$$

- $V = \mathcal{T}^{\pi} V$ is a linear equation that we can solve directly.

Contraction for Bellman Expectation Backup

- Bellman Expectation Backup operator $\mathcal{T}^{\pi}$ is a γ-contraction,
- Proof:

$$\left|\left|\mathcal{T}^{\pi}U - \mathcal{T}^{\pi}V\right|\right|_{\infty} = \|(\mathcal{R}^{\pi} + \gamma\, \mathcal{P}^{\pi}U) - (\mathcal{R}^{\pi} + \gamma\, \mathcal{P}^{\pi}V)\|_{\infty}$$
$$= \|\gamma\, \mathcal{P}^{\pi}(U - V)\|_{\infty}$$
$$\leq \gamma\delta = \gamma\|(U - V)\|_{\infty}$$

  – Note:
    - $(\mathcal{P}^{\pi}_{s:,}(U - V)) \leq \delta$ for all s
      $\rightarrow \|\mathcal{P}^{\pi}(U - V)\|_{\infty} \leq \delta$
      – For $\mathcal{P}^{\pi}$, each row of matrix sums to 1.

Policy Iteration: Overview

- Alternate between
  - Evaluate policy $\pi \Rightarrow V^{\pi}$
  - Set new policy to be greedy policy for V^{π}

$$\pi(s) = \underset{a}{\operatorname{argmax}}\, \mathbb{E}_{s'|s,a}[R_{t+1} + \gamma V^{\pi}(s')\ ]$$

- Guaranteed to converge to optimal policy and value function in a finite number of iterations, when $\gamma < 1$
- Value function converges faster than in value iteration

Algorithm Policy Iteration

Initialize $\pi^{(0)}$ arbitrarily.

for n = 1, 2, . . . until termination condition do

$V^{(n+1)} = \text{Solve}\ [V = \mathcal{T}^{\pi^{(n-1)}}V]$

$\pi^{(n)} = \mathcal{G}V^{(n)}$ $\mathcal{G}$: a greedy mapping function.

end

Modified Policy Iteration

- Update π to be the greedy policy, then value function with k backups (k-step lookahead)

Algorithm Modified Policy Iteration
Initialize $V^{(0)}$ arbitrarily.
for n = 1, 2, . . . until termination condition do
$\pi^{(n+1)} = \mathcal{G}V^{(n)}$
$V^{(n+1)} = \left(\mathcal{T}^{\pi^{(n+1)}}\right)^k V^{(n)}$, for integer $k \geq 1$.
end

- $k = 1$: value iteration
- $k = \infty$: policy iteration

Exercise

- What if $\gamma = 1$?
  - Hint: Like The Shortest Path Problem
    - The shortest path to node 0.