Model Free Reinforcement Learning
ε-Greedy Exploration
Q-Learning
Function Approximation

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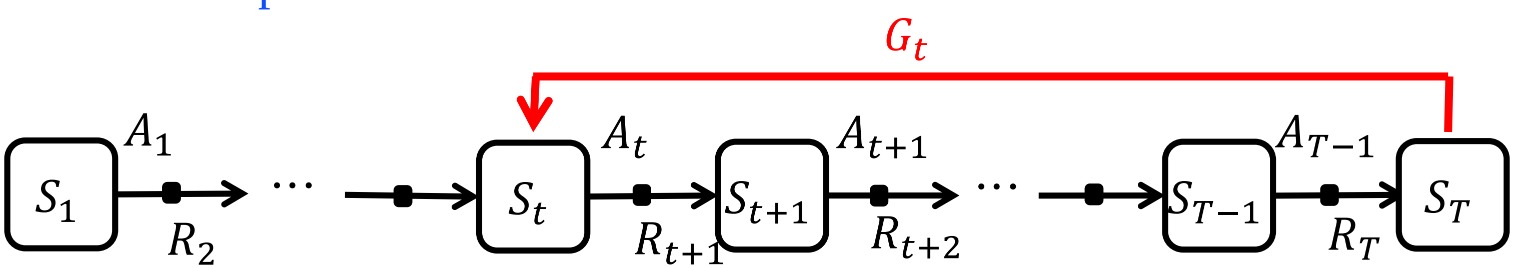
Model Free Reinforcement Learning

Model Free Reinforcement Learning

- No model
- Learn value function (and/or policy) from experience
- Common Model Free RL
  - Monte-Carlo (MC) Reinforcement Learning
  - Temporal Difference (TD) Reinforcement Learning
  - TD(λ)

Monte-Carlo Reinforcement Learning

- MC methods learn directly from episodes of experience
- MC is model-free:
  - no knowledge of MDP transitions / rewards
- MC learns from complete episodes:
  - no bootstrapping
- MC uses the simplest possible idea:
  - value = mean return
- Caveat: can only apply MC to episodic MDPs
  - All episodes must terminate



Monte-Carlo Policy Evaluation

- Goal: learn v_π from episodes of experience under policy π

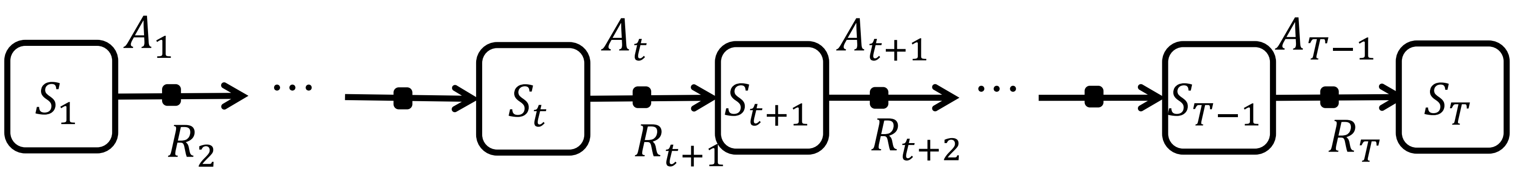
  $S_1, A_1, R_2, \ldots, S_T \sim \pi$
- Recall that the return is the total discounted reward:

  $$G_t = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-1} R_T$$
- Recall that the value function is the expected return:

  $$v_\pi(s) = \mathbb{E}_\pi[G_t | S_t = s]$$
- Monte-Carlo policy evaluation uses empirical mean return instead of expected return

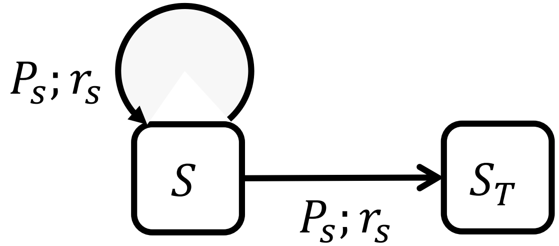
Monte-Carlo Policy Evaluation (cont.)

- To evaluate $v_\pi(s)$ at state s
  - Increment counter $N(s) \leftarrow N(s) + 1$
  - Increment total return $S(s) \leftarrow S(s) + G_t$
  - Value is estimated by mean return $V(s) \leftarrow S(s)/N(s)$
- By law of large numbers, $V(s) \to v_\pi(s)\ as\ N(s) \to \infty$



First Visit vs. Every Visit

- To evaluate $v_\pi(s)$ at state s
  - Increment counter $N(s) \leftarrow N(s) + 1$
  - Increment total return $S(s) \leftarrow S(s) + G_t$
  - Value is estimated by mean return $V(s) \leftarrow S(s)/N(s)$
- By law of large numbers, $V(s) \rightarrow v_\pi(s)\ as\ N(s) \rightarrow \infty$
- What if the same state s is visited in an episode?
  - Do the above for every visit or first visit?
    - What happen for the case in the figure?
    - Both converge quadratically, so this issue is ignored in this course.



Incremental Mean

The mean $\mu_1, \mu_2, \ldots$ of a sequence $x_1, x_2, \ldots$ can be computed incrementally,

$$\mu_k = \frac{1}{k} \sum_{j=1}^{k} x_j$$

$$= \frac{1}{k}\left(x_k + \sum_{j=1}^{k} x_j\right)$$

$$= \frac{1}{k}(x_k + (k-1)\mu_{k-1})$$

$$= \mu_{k-1} + \frac{1}{k}(x_k - \mu_{k-1})$$

Incremental Monte-Carlo Updates

- Update $V(s)$ incrementally after episode $S_1, A_1, R_2, \ldots, S_T$
- For each state S_t with return G_t

$$N(S_t) \leftarrow N(S_t) + 1$$

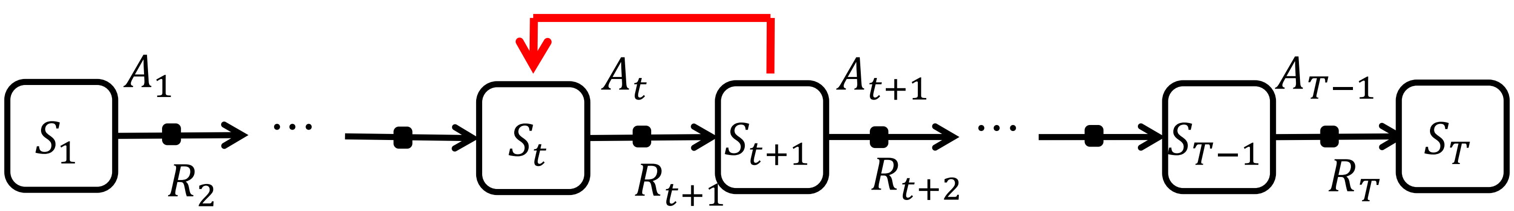
$$V(S_t) \leftarrow V(S_t) + \frac{1}{N(S_t)}\left(G_t - V(S_t)\right)$$

- In non-stationary problems, it can be useful to track a running mean, i.e. forget old episodes.

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t - V(S_t)\right)$$

Temporal-Difference Learning

- TD methods learn directly from episodes of experience
- TD is model-free:
  - no knowledge of MDP transitions / rewards
- TD learns from incomplete episodes,
  - by bootstrapping
- TD updates a guess towards a guess



MC vs. TD

- Goal: learn v_π online from experience under policy π
- Incremental every-visit Monte-Carlo
  - Update value $V(S_t)$ toward actual return G_t

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t - V(S_t)\right)$$

- Simplest temporal-difference learning algorithm: TD(0)
  - Update value $V(S_t)$ toward estimated return $R_{t+1} + \gamma V(S_{t+1})$

$$V(S_t) \leftarrow V(S_t) + \alpha\left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right)$$

  - $R_{t+1} + \gamma V(S_{t+1})$ is called the TD target
  - $\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$ is called the TD error

TD vs. MC (I)

- TD can learn before knowing the final outcome
  - TD can learn online after every step
  - MC must wait until end of episode before return is known
- TD can learn without the final outcome
  - TD can learn from incomplete sequences
  - MC can only learn from complete sequences
  - TD works in continuing (non-terminating) environments
  - MC only works for episodic (terminating) environments

Bias/Variance Trade-Off

- TD target $R_{t+1} + \gamma\, V(S_{t+1})$ is biased estimate of $v_\pi(S_t)$
  - Return $G_t = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-1} R_T$ is unbiased estimate of $v_\pi(S_t)$
  - True TD target $R_{t+1} + \gamma\, v_\pi(S_{t+1})$ is unbiased estimate of $v_\pi(S_t)$
- TD target is much lower variance than the return:
  - Return depends on many random actions, transitions, rewards
  - TD target depends on only one random action, transition, reward

MC vs. TD (II)

- MC has high variance, zero bias
  - Good convergence properties (even with function approximation)
  - Not very sensitive to initial value
  - Very simple to understand and use
- TD has low variance, some bias
  - Usually more efficient than MC
  - TD(0) converges to $v_\pi(s)$ (but not always with function approximation)
  - More sensitive to initial value

Batch MC and TD

- MC and TD converge: $V(s) \rightarrow v_\pi(s)$ as experience $\rightarrow \infty$
- But what about batch solution for finite experience?

$$s_1^1, a_1^1, r_2^1, \ldots, s_{T_1}^1$$
$$\vdots$$
$$s_1^k, a_1^k, r_2^k, \ldots, s_{T_k}^k$$

  - e.g. Repeatedly sample episode $k \in [1, K]$
  - Apply MC or TD(0) to episode k

AB Example

Two states A, B; no discounting; 8 episodes of experience

A, 0, B, 0

B, 1

B, 1

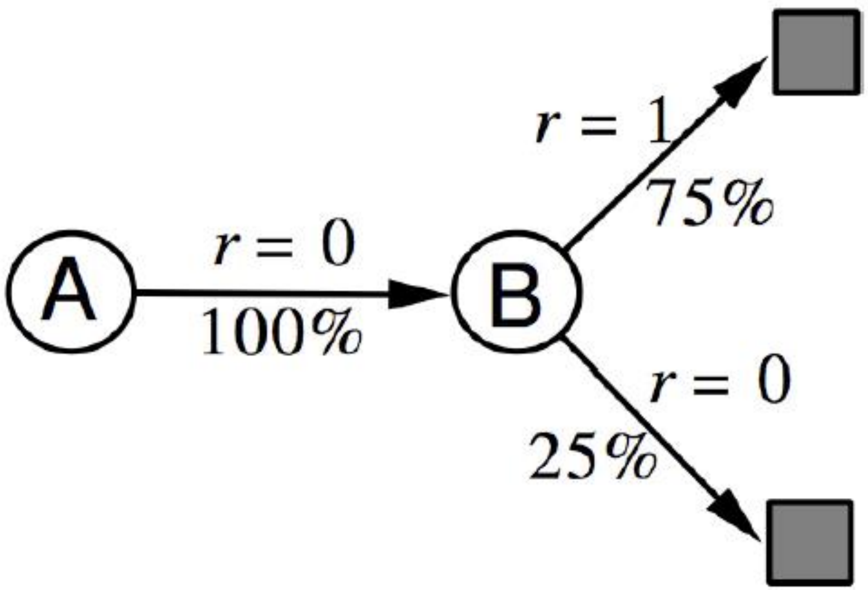
B, 1

B, 1

B, 1

B, 1

B, 0



What is $V(A), V(B)$?

Both MC and TD will obtain different values!!

Certainty Equivalence

- MC converges to solution with minimum mean-squared error
  - Best fit to the observed returns

$$\sum_{k=1}^{K}\sum_{t=1}^{T_k}\left(G_t^k - V(s_t^k)\right)^2$$

  - In the AB example, $V(A) = 0$, $V(B) = 0.75$
- TD(0) converges to solution of max likelihood Markov model
  - Solution to the MDP $<\mathcal{S}, \mathcal{A}, \hat{\mathcal{P}}, \hat{\mathcal{R}}, \gamma>$ that best fits the data

$$\hat{\mathcal{P}}_{s}^{a} = \frac{1}{N(s,a)}\sum_{k=1}^{K}\sum_{t=1}^{T_k} 1\left(s_t^k, a_t^k, s_{t+1}^k = s, a, s'\right)$$

$$\hat{\mathcal{R}}_{s}^{a} = \frac{1}{N(s,a)}\sum_{k=1}^{K}\sum_{t=1}^{T_k} 1\left(s_t^k, a_t^k = s, a\right) r_t^k$$

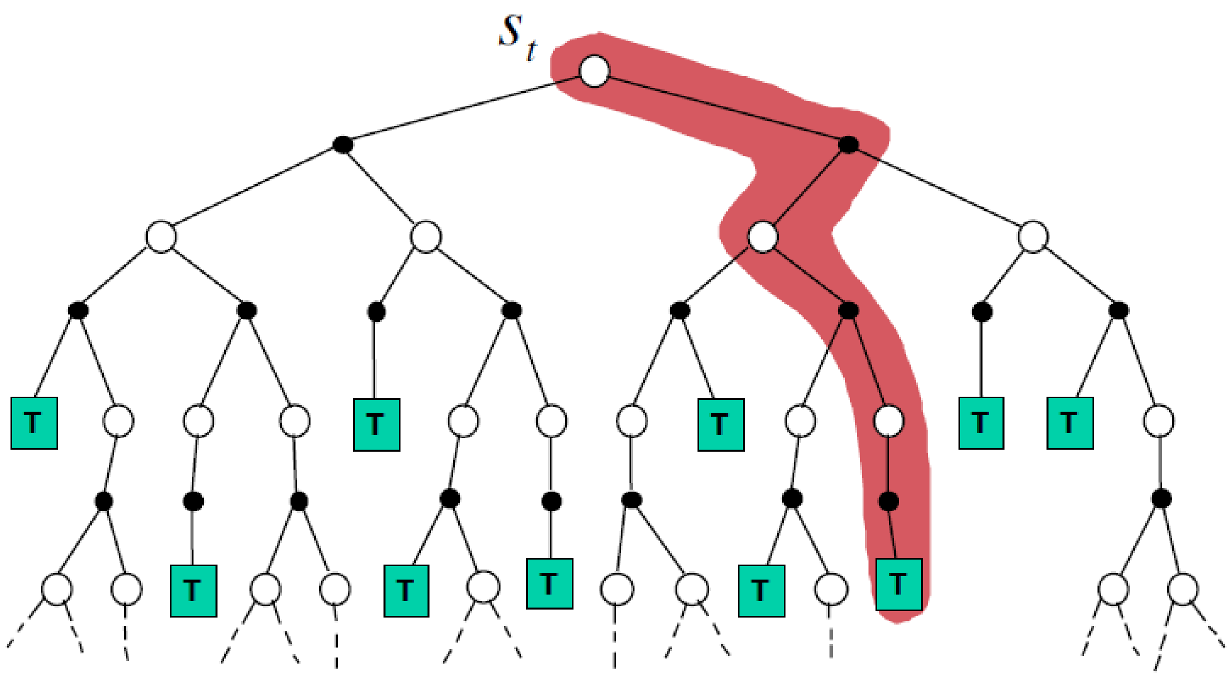
  - In the AB example, $V(A) = 0.75$, $V(B) = 0.75$

MC vs. TD (III)

- TD exploits Markov property
  - Usually more efficient in Markov environments
    - So, TD works well for MDP problems like 2048.
- MC does not exploit Markov property
  - Usually more effective in non-Markov environments
    - MC works fine for non-MDP too.

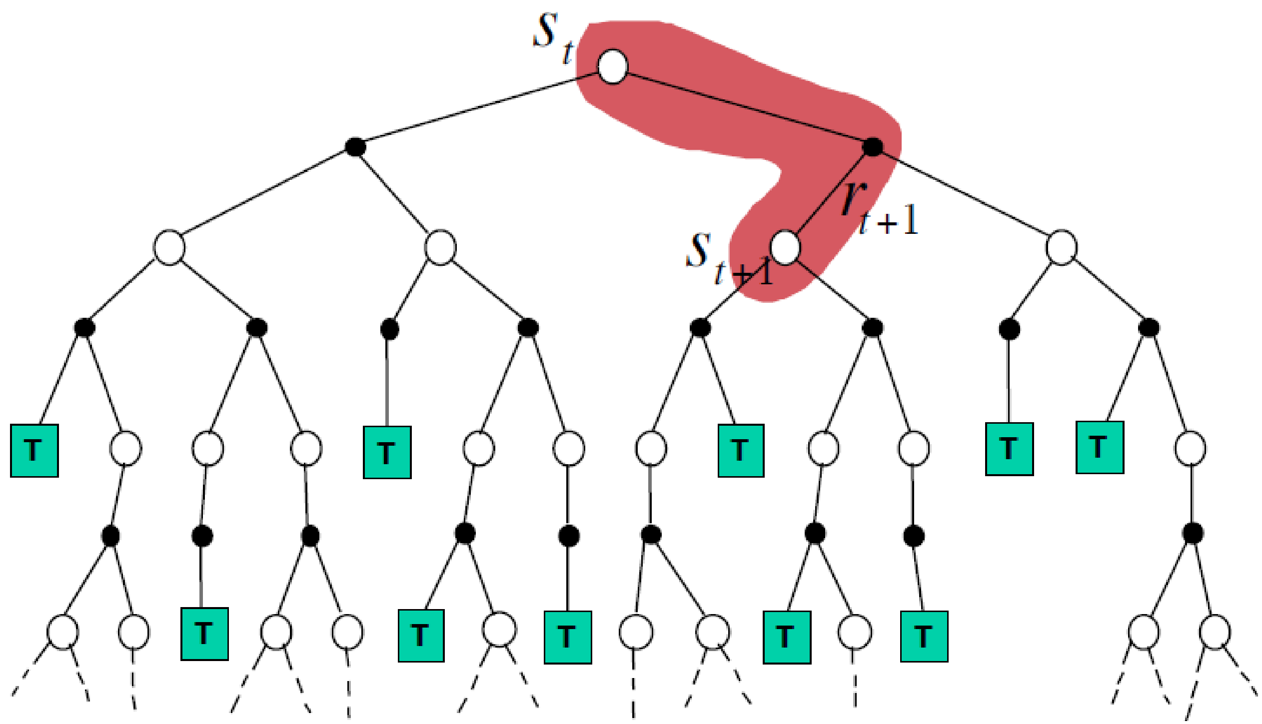
Monte-Carlo Backup

$$V(S_t) \leftarrow V(S_t) + \alpha\big(G_t - V(S_t)\big)$$



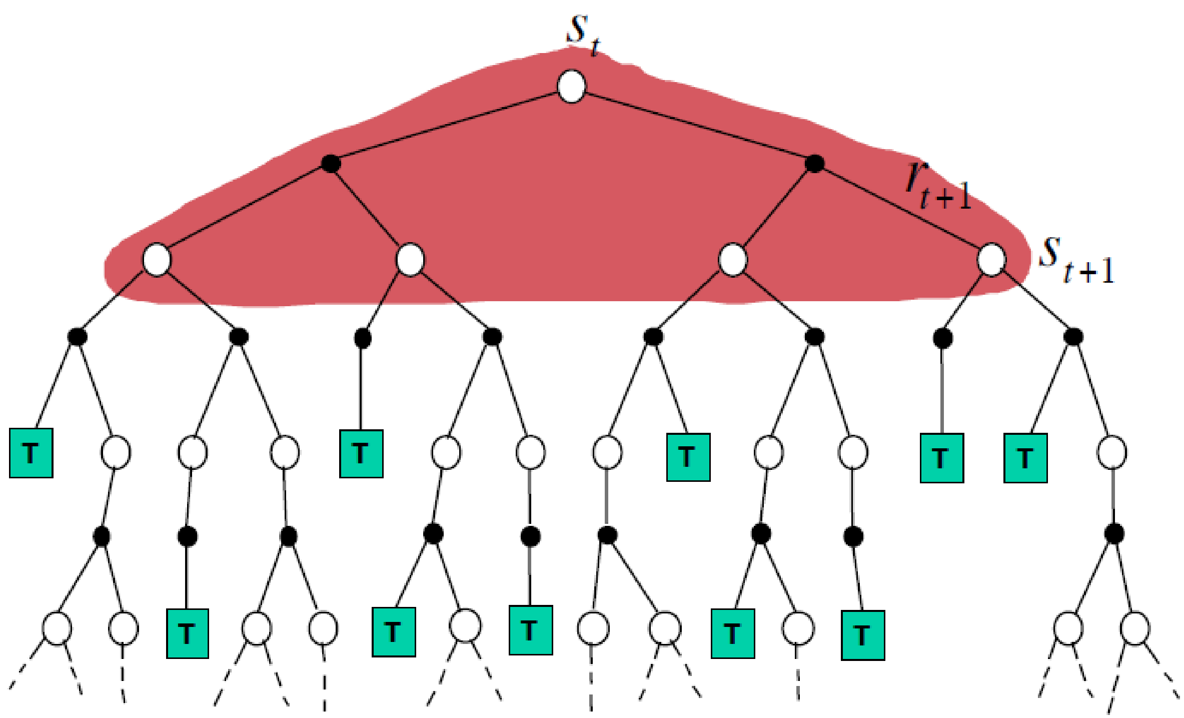
Temporal-Difference Backup

$$V(S_t) \leftarrow V(S_t) + \alpha\left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right)$$



Dynamic Programming Backup

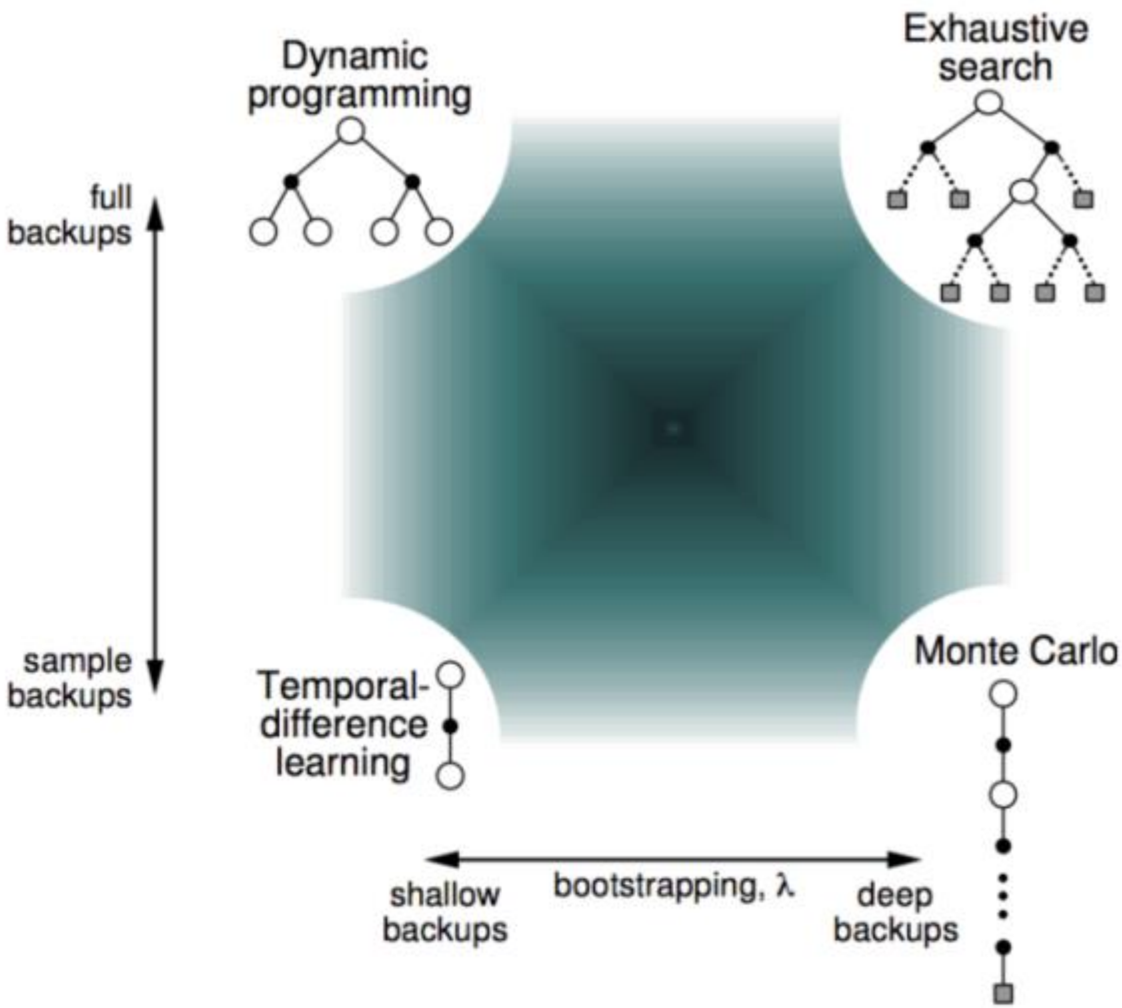
$$V(S_t) \leftarrow \mathbb{E}_\pi[R_{t+1} + \gamma V(S_{t+1})]$$



Bootstrapping and Sampling

- Bootstrapping: update involves an estimate
  - MC does not bootstrap
  - DP bootstraps
  - TD bootstraps
- Sampling: update samples an expectation
  - MC samples
  - DP does not sample
  - TD samples

Unified View of Reinforcement Learning

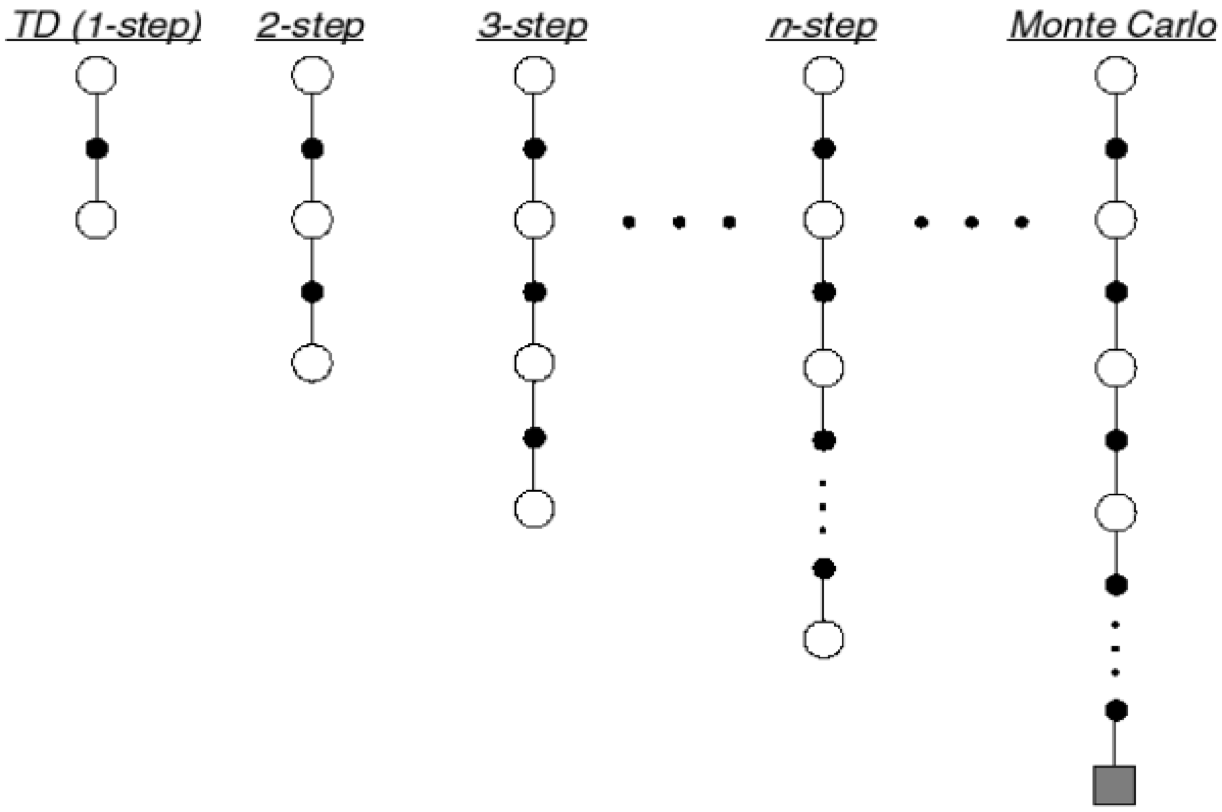


General TD Learning

- Review TD
  - Update value $V(S_t)$ toward estimated return $R_{t+1} + \gamma V(S_{t+1})$
    $$V(S_t) \leftarrow V(S_t) + \alpha\big(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\big)$$
  - $R_{t+1} + \gamma V(S_{t+1})$ is called the TD target
  - For MC learning, the TD target is replaced by G_t
    $$V(S_t) \leftarrow V(S_t) + \alpha\big(G_t - V(S_t)\big)$$
- Question: a more general TD target?
- Investigate TD in a more general manner.
- A typical one: TD(λ)

n-Step Prediction

- Let TD target look n steps into the future



n-Step Return

- Define the n-step return

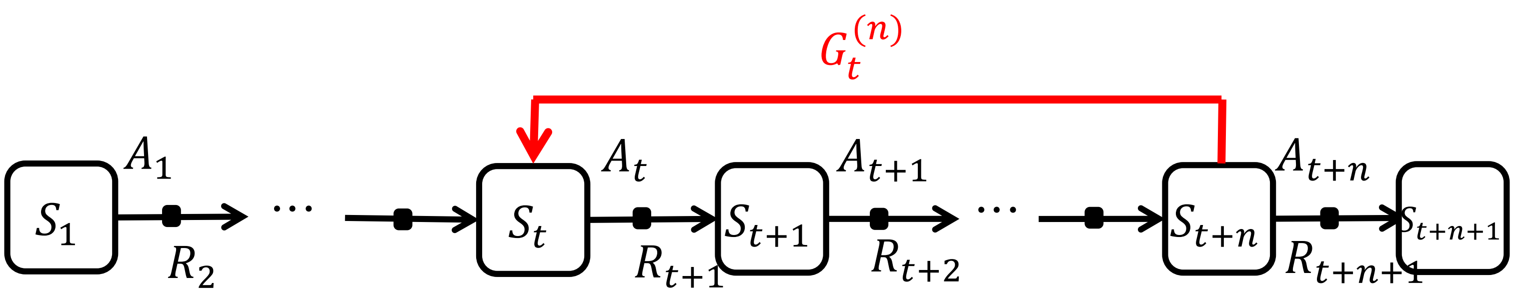
$$G_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

- Consider the following n-step returns for $n = 1,2,\infty$

$$
\begin{array}{ll}
n = 1 & G_t^{(1)} = R_{t+1} + \gamma V(S_{t+1}) \\
n = 2 & G_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2}) \\
\vdots & \vdots \\
n = \infty & G_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{T-1} R_T, \text{ if ends at } T.
\end{array}
$$

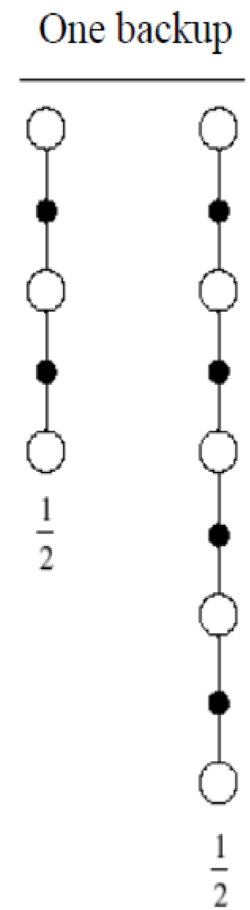
- n-step temporal-difference learning

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t^{(n)} - V(S_t)\right)$$



Example of Averaging n-Step Returns

- We can average n-step returns over different n
- Example:
  - average the 2-step and 4-step returns
$$\frac{1}{2}G^{(2)} + \frac{1}{2}G^{(4)}$$
  - Combines information from two different time-steps
- Next:
  - combine information from all time-steps?



λ-return

- λ-return G_t^{λ}:
  - combines all n-step returns $G_t^{(n)}$
- Using weight$(1-\lambda)\,\lambda^{n-1}$

$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}\,G_t^{(\mathrm{n})}$$

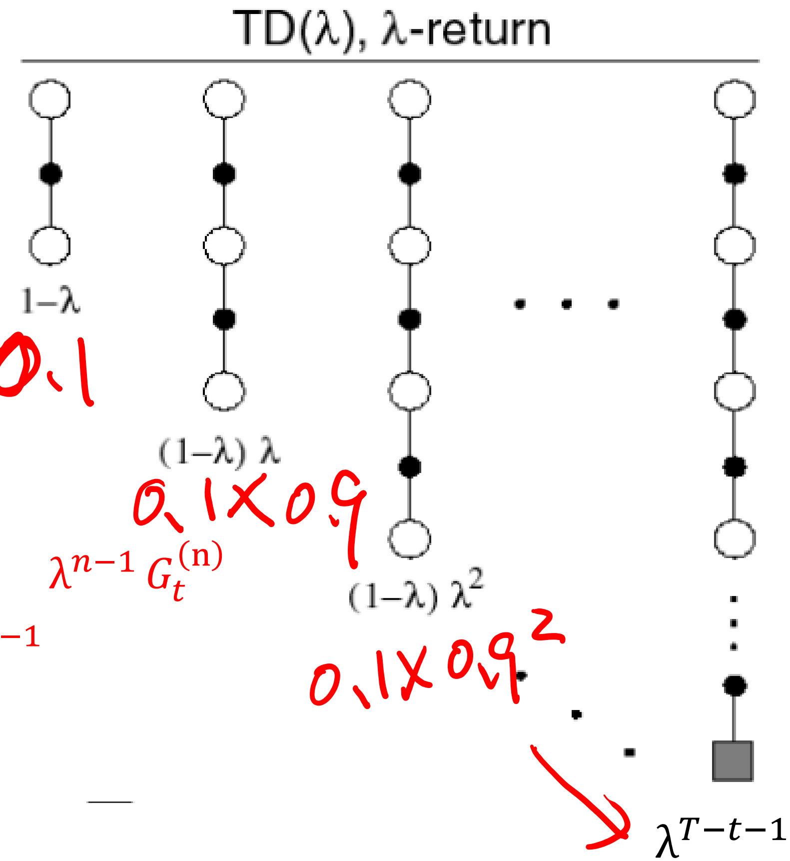
or (in the case of termination)

$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{T-t-1}\lambda^{n-1}\,G_t^{(\mathrm{n})} + (1-\lambda)\sum_{n=T-t-1}^{\infty}\lambda^{n-1}\,G_t^{(\mathrm{n})}$$

$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{T-t-1}\lambda^{n-1}\,G_t^{(\mathrm{n})} + \lambda^{T-t-1}G_t$$

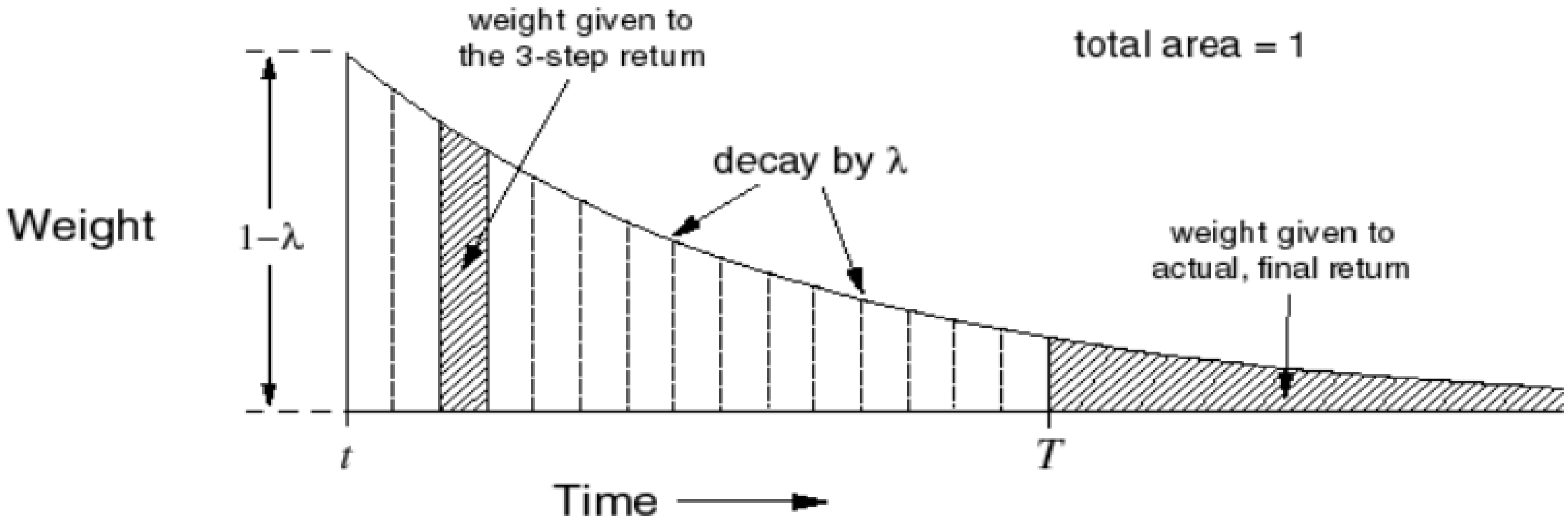
- Forward-view TD(λ)

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t^{\lambda} - V(S_t)\right)$$



TD(λ) Weighting Function

$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1}\, G_t^{(n)}$$



TD(λ) and TD(0)

- When $\lambda = 0$, only current state is updated

  $$V(s) \leftarrow V(s) + \alpha\delta_t$$

  – This is exactly equivalent to TD(0) update

TD(λ) and MC

- When $\lambda = 0$, only current state is updated, → TD(0)=TD

$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{T-t-1} \lambda^{n-1} G_t^{(n)} + \lambda^{T-t-1} G_t = G_t^{(1)}$$

  – This is exactly equivalent to TD target.

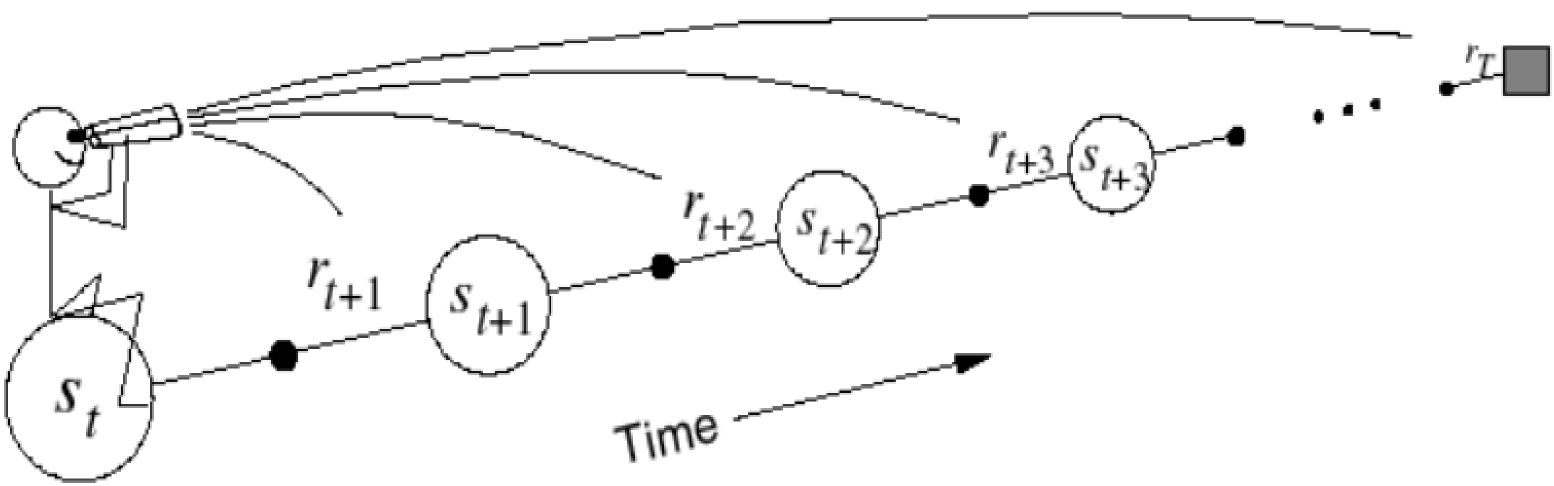
- When $\lambda = 1$, TD(1) = MC

$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{T-t-1} \lambda^{n-1} G_t^{(n)} + \lambda^{T-t-1} G_t = G_t$$

  – This is exactly equivalent to MC target.

Forward-view TD(λ)

- Update value function towards the λ-return
- Forward-view looks into the future to compute G_t^λ
- Like MC, can only be computed from complete episodes



Backward View TD(λ)

- Forward view provides theory
- Backward view provides mechanism
  - Update online, every step, from incomplete sequences

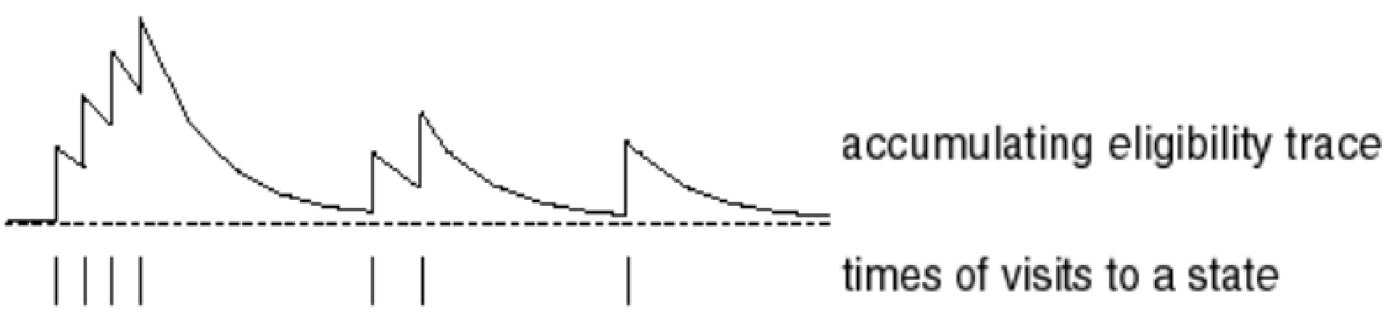
- Notes:
  - You may ignore it now.
  - Consider backward (eligible traces) only when you try to implement it.

Eligibility Traces

- Credit assignment problem: did bell or light cause shock?
- Frequency heuristic: assign credit to most frequent states
- Recency heuristic: assign credit to most recent states
- Eligibility traces combine both heuristics

$$E_0(s) = 0$$

$$E_t(s) = \gamma\, \lambda E_{t-1}(s) + 1(S_t = s)$$

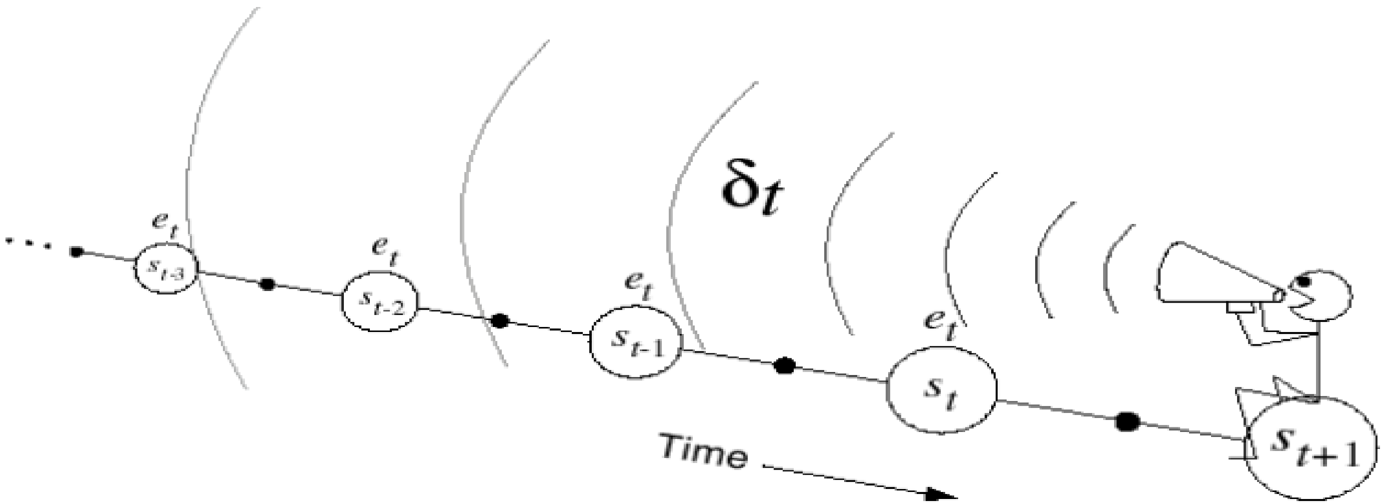


Backward View TD(λ)

- Keep an eligibility trace for every state *s*
- Update value $V(s)$ for every state *s*
- In proportion to TD-error δ_t and eligibility trace $E_t(s)$

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$

$$V(s) \leftarrow V(s) + \alpha\delta_t E_t(s)$$



Eligibility Trace

- Explain more in policy-based reinforcement learning for GAE (Generalized Advantage Estimator).
  – (See Page 37, "Advantages and TD(λ)" in the chapter of policy-based RL.)

ε-Greedy Exploration

Example of Greedy Action Selection

- There are two doors in front of you,
  Always apply the greedy action selection:
  - You open the left door and get reward 0
    $V(left) = 0$
  - You open the right door and get reward +1
    $V(right) = +1$
  - You open the right door and get reward +3
    $V(right) = +2$
  - You open the right door and get reward +2
    $V(right) = +2$
  - $\vdots$
- Are you sure you've chosen the best door?

ε-Greedy Exploration

- ε-greedy policy:

$$\pi(a|s) = \begin{cases} \varepsilon/m + 1 - \varepsilon & if\ a^* = \underset{a \in \mathcal{A}}{\text{argmax}}\ Q(s,a) \\ \epsilon/m & \text{otherwise} \end{cases}$$

- Exploration
  - If you always try the best, you don't explore a real better one.
  - With probability ε choose an action at random
    - Simplest idea for ensuring continual exploration
  - All m actions are tried with non-zero probability
- Exploitation
  - If you always choose at random, you don't exploit the best
  - With probability $1 - \varepsilon$ choose the greedy action

ε-Greedy Policy Improvement

(for reference only; can be skipped)

- Theorem
  - For any policy π, the ε-greedy policy π' with respect to q_π is an improvement,$v_{\pi'}(s) \geq v_\pi(s)$
- Proof:

$$
\begin{aligned}
v_{\pi'}(s) &= q_\pi\big(s, \pi'(s)\big) \qquad \text{(follow new policy } \pi' \text{ using old } q_\pi\text{.)} \\
&= \textstyle\sum_{a\epsilon\mathcal{A}} \pi'(a|s)\, q_\pi(s,a) \\
&= \frac{\varepsilon}{m} \textstyle\sum_{a\epsilon\mathcal{A}} q_\pi(s,a) + (1-\varepsilon) \max\limits_{a\epsilon A}\ q_\pi(s,a) \\
&\geq \frac{\varepsilon}{m} \textstyle\sum_{a\epsilon\mathcal{A}} q_\pi(s,a) + (1-\varepsilon) \textstyle\sum_{a\epsilon\mathcal{A}} \frac{\pi(a|s)-\frac{\varepsilon}{m}}{1-\varepsilon} q_\pi(s,a) \text{ (Lemma)} \\
&= \frac{\varepsilon}{m} \textstyle\sum_{a\epsilon\mathcal{A}} q_\pi(s,a) + \textstyle\sum_{a\epsilon\mathcal{A}} \left(\pi(a|s) - \frac{\varepsilon}{m}\right) q_\pi(s,a) \\
&= \textstyle\sum_{a\epsilon\mathcal{A}} \pi(a|s) q_\pi(s,a) = v_\pi(s)
\end{aligned}
$$

- Therefore from policy improvement theorem, $v_{\pi'}(s) \geq v_\pi(s)$

A Lemma in the Previous Proof

(for reference only; can be skipped)

(the sum is a weighted average with nonnegative weights summing to 1, and as such it must be less than or equal to the largest number averaged)

- Lemma: For the previous proof, assume $\pi(a|s) - \frac{\varepsilon}{m} \geq 0$.

  $$\max_{a \epsilon A} q_\pi(s,a) \geq \sum_{a \epsilon \mathcal{A}} \frac{\pi(a|s) - \frac{\varepsilon}{m}}{1-\varepsilon} q_\pi(s,a)$$

- Proof: Assume all weights $w_a \geq 0$, and $\sum_{a \epsilon \mathcal{A}} w_a = 1$.

  Then, $\max_{a \epsilon A} q_\pi(s,a) \geq \sum_{a \epsilon \mathcal{A}} w_a \, q_\pi(s,a)$

  Since weights $\frac{\pi(a|s) - \frac{\varepsilon}{m}}{1-\varepsilon} \geq 0$ and their summation = 1,

  we have $\max_{a \epsilon A} q_\pi(s,a) \geq \sum_{a \epsilon \mathcal{A}} \frac{\pi(a|s) - \frac{\varepsilon}{m}}{1-\varepsilon} q_\pi(s,a)$

Key Idea

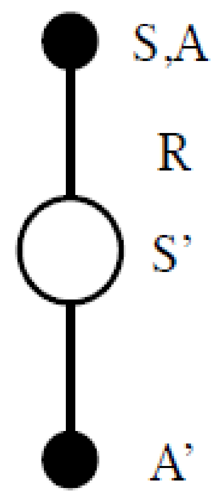
- Key idea:
  - $\pi(a|s) - \frac{\varepsilon}{m}$ is non-negative, as long as ε is monotonically decreasing.
- Example:
  Assume ε = 0.4 and m=4 (4 actions, a_1, a_2, a_3, a_4).
  - $\pi(a_1|s) = 0.4$, and $q_\pi(a_1|s) = 20$
  - $\pi(a_2|s) = 0.3$, and $q_\pi(a_2|s) = 30$ (max in the new policy π')
  - $\pi(a_3|s) = 0.2$, and $q_\pi(a_3|s) = 15$
  - $\pi(a_4|s) = 0.1$, and $q_\pi(a_4|s) = 15$
  - Works when ε remains the same or drops to a smaller number, say 0.3.

Q-Learning

Updating Action-Value Functions with Sarsa

$$Q(S, A) \leftarrow Q(S, A) + \alpha(R + \gamma Q(S', A') - Q(S, A))$$

- Notice: Interesting naming



Sarsa Algorithm for On-Policy Control

Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Choose A from S using policy derived from Q (e.g., ε-greedy)
  Repeat (for each step of episode):
    Take action A, observe R, S'
    Choose A' from S' using policy derived from Q (e.g., ε-greedy)
    $Q(S,A) \leftarrow Q(S,A) + \alpha\big[R + \gamma Q(S',A') - Q(S,A)\big]$
    $S \leftarrow S'$; $A \leftarrow A'$;
  until S is terminal

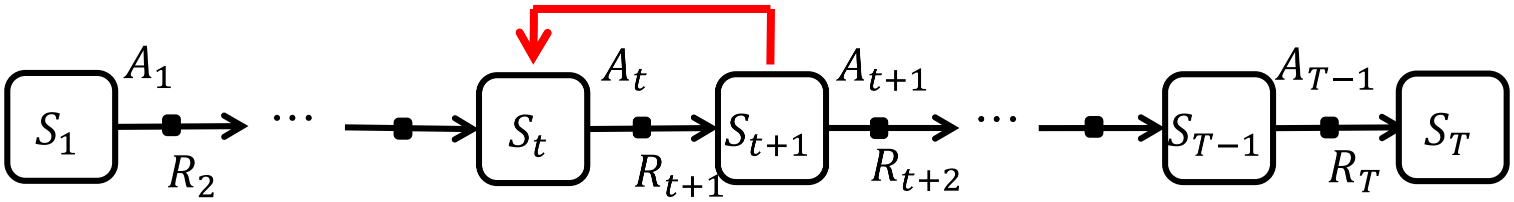
- Sarsa converges to the optimal action-value function
- n-step Sarsa – like n-step return
- Sarsa(λ) – like TD(λ)

Off-Policy Learning

- Evaluate current policy $\pi(a|s)$ to compute $V_\pi(s)$ or $q_\pi(s,a)$, while following an old policy $\mu(a|s)$

$$\{S_1, A_1, R_2, \ldots, S_T\} \sim \mu$$

- Why is this important?
  - Learn from observing humans or other agents
  - Re-use experience generated from old policies $\pi_1, \pi_2, \ldots, \pi_{t-1}$
  - Learn about optimal policy while following exploratory policy
  - Learn about multiple policies while following one policy



Current Policy π

Importance Sampling

- Estimate the expectation of a different distribution

$$
\begin{aligned}
&\mathbb{E}_{X\sim P}[f(X)] \\
&= \sum P(X) f(X) \\
&= \sum Q(X) \frac{P(X)}{Q(X)} f(X) \\
&= \mathbb{E}_{X\sim Q}\left[\frac{P(X)}{Q(X)} f(X)\right]
\end{aligned}
$$

Importance Sampling for Off-Policy Monte-Carlo

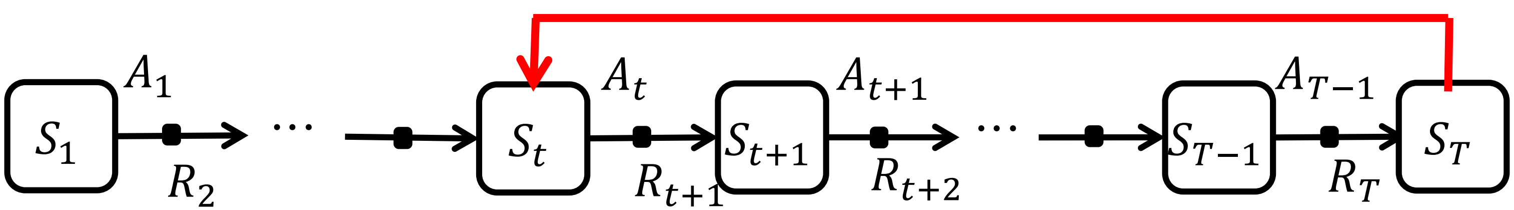
- Use returns generated from μ to evaluate π
- Weight return G_t according to similarity between policies
- Multiply importance sampling corrections along whole episode

$$G_t^{\pi/\mu} = \frac{\pi(A_t|S_t)\pi(A_{t+1}|S_{t+1})}{\mu(A_t|S_t)\mu(A_{t+1}|S_{t+1})} \cdots \frac{\pi(A_T|S_T)}{\mu(A_T|S_T)} G_t$$

- Update value towards corrected return

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t^{\pi/\mu} - V(S_t)\right)$$

- Cannot use if μ is zero when π is non-zero
- Importance sampling can dramatically increase variance



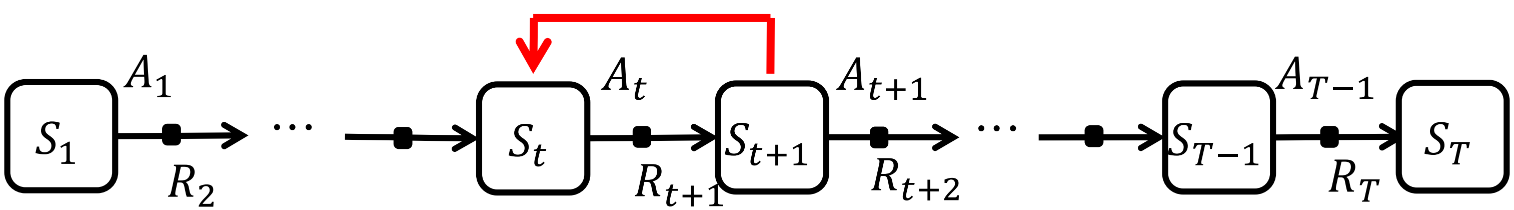
Current Policy π

Importance Sampling for Off-Policy TD

- Use TD targets generated from μ to evaluate π
- Weight TD target $R + \gamma V(S')$ by importance sampling
- Only need a single importance sampling correction

$$V(S_t) \leftarrow V(S_t) + \alpha\left(\frac{\pi(A_t|S_t)}{\mu(A_t|S_t)}\left(R_{t+1} + \gamma V(S_{t+1})\right) - V(S_t)\right)$$

- Much lower variance than Monte-Carlo importance sampling (since just one step)
- Policies only need to be similar over a single step



Current Policy π

Q-Learning

- We now consider off-policy learning of action-values $Q(s, a)$
- No importance sampling is required
- Next action is chosen using the old policy

  $$A_{t+1} \sim \mu(\cdot \mid S_{t+1})$$
- But we consider alternative successor action

  $$A' \sim \pi(\cdot \mid S_{t+1})$$
- And update $Q(S_t, A_t)$ towards value of alternative action

  $$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma Q(S_{t+1}, A') - Q(S_t, A_t))$$

Off-Policy Control with Q-Learning

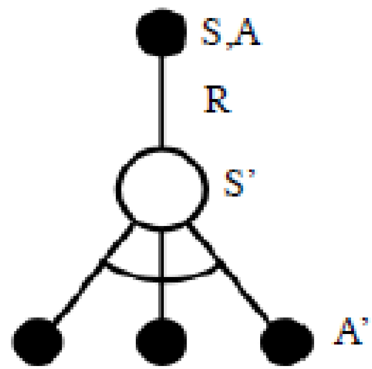
- We now allow both old and current policies to improve
- The current policy π is greedy w.r.t. $Q(s, a)$

$$\pi(S_{t+1}) = \underset{a'}{\operatorname{argmax}}\ Q(S_{t+1}, a')$$

- The old policy μ is e.g. ε-greedy w.r.t. $Q(s, a)$
- The Q-learning target then simplifies:

$$\begin{aligned} & R_{t+1} + \gamma Q(S_{t+1}, A') \\ &= R_{t+1} + \gamma Q\left(S_{t+1}, \underset{a'}{\operatorname{argmax}}\ Q(S_{t+1}, a')\right) \\ &= R_{t+1} + \max_{a'} \gamma Q(S_{t+1}, a') \end{aligned}$$

Q-Learning Control Algorithm



- $Q(S,A) \leftarrow Q(S,A) + \alpha\left(R + \gamma \max_{a'} Q(S',a') - Q(S,A)\right)$
- Theorem
  - Q-learning control converges to the optimal action-value function, $Q(s,a) \rightarrow q_*(s,a)$

Q-Learning Algorithm for Off-Policy Control

Initialize $Q(s,a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\textit{terminal-state}, \cdot) = 0$
Repeat (for each episode):
  Initialize S
  Repeat (for each step of episode):
    Choose A from S using policy derived from Q (e.g., ε-greedy)
    Take action A, observe R, S'
    $Q(S,A) \leftarrow Q(S,A) + \alpha\big[R + \gamma \max_a Q(S',a) - Q(S,A)\big]$
    $S \leftarrow S'$;
  until S is terminal

Function Approximation

Large-Scale Reinforcement Learning

- Reinforcement learning can be used to solve large problems, e.g.
  - Backgammon: 10^{20} states
  - Computer Go: 10^{170} states
  - Helicopter: continuous state space

- How can we scale up the model-free methods for prediction and control from the last two lectures?

Value Function Approximation

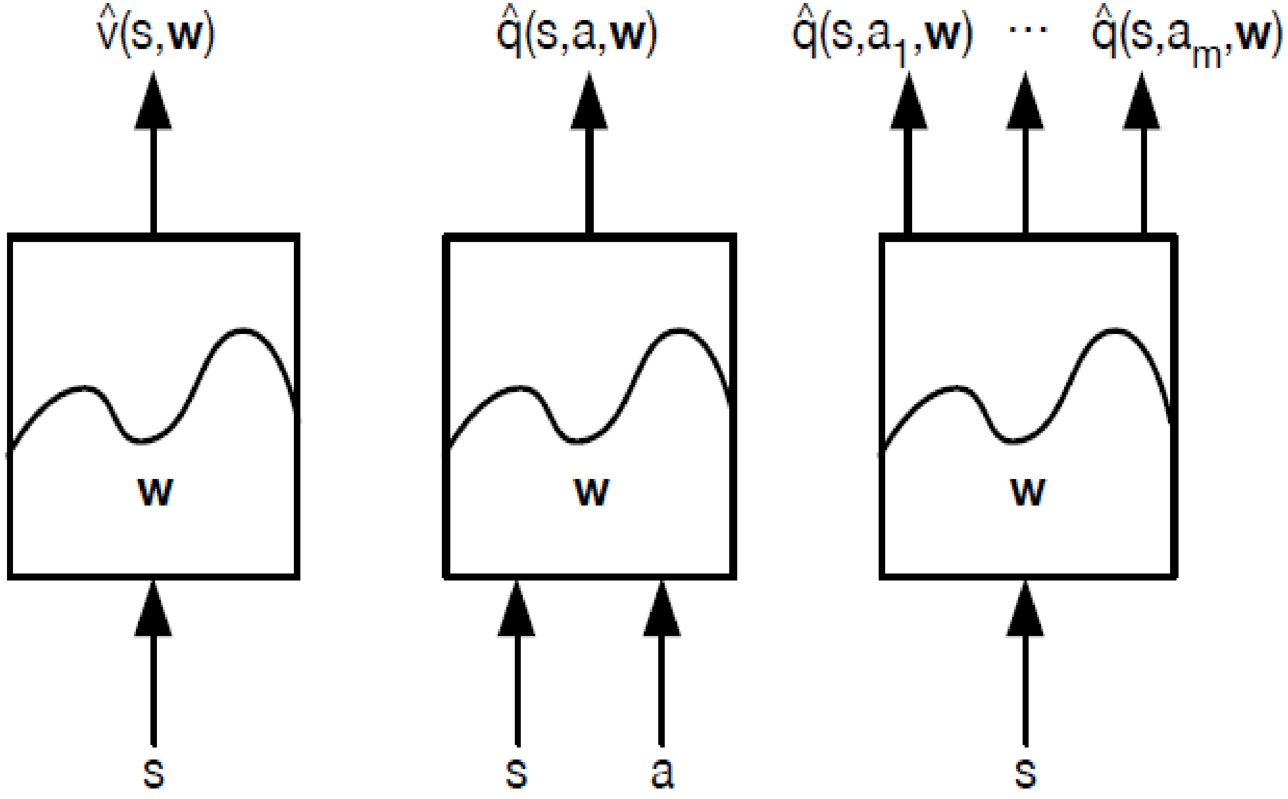
- So far we have represented value function by a lookup table
  - Every state s has an entry $V(s)$
  - Or every state-action pair s; a has an entry $Q(s,a)$
- Problem with large MDPs:
  - There are too many states and/or actions to store in memory
  - It is too slow to learn the value of each state individually
- Solution for large MDPs:
  - Estimate value function with function approximation

$$\hat{v}(s,w) \approx v_\pi(s)$$

$$\text{or } \hat{q}(s,a,w) \approx q_\pi(s,a)$$

  - Generalize from seen states to unseen states
  - Update parameter w using MC or TD learning

Types of Value Function Approximation



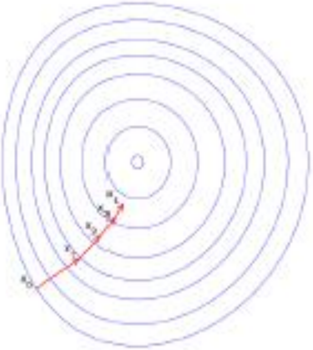
Which Function Approximator?

- There are many function approximators, e.g.
  - Linear combinations of features
  - Neural network
  - Decision tree
  - Nearest neighbour
  - Fourier / wavelet bases
  - ...
- Better to consider differentiable function approximators (in red above)
- Furthermore, we require a training method that is suitable for non-stationary, non-iid data

Gradient Descent

- Let $J(w)$ be a differentiable function of parameter vector w
- Define the gradient of $J(w)$ to be

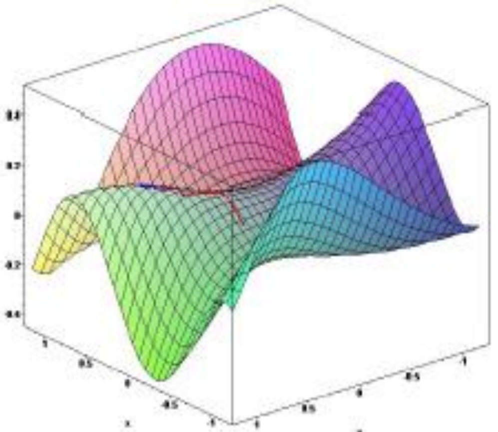
$$\nabla_w J(w) = \begin{pmatrix} \frac{\partial J(w)}{\partial w_1} \\ \vdots \\ \frac{\partial J(w)}{\partial w_n} \end{pmatrix}$$



- To find a local minimum of $J(w)$
- Adjust w in direction of -ve gradient

$$\Delta w = -\frac{1}{2}\alpha \nabla_w J(w)$$

  – where α is a step-size parameter



Value Function Approx. By Stochastic Gradient Descent

- Goal: find parameter vector w
  - minimizing mean-squared error between approximate value function $\hat{v}(s,w)$ and true value function $v_\pi(s)$

$$J(w) = \mathbb{E}_\pi[(v_\pi(S) - \hat{v}(S,w))^2]$$

- Gradient descent finds a local minimum

$$\Delta\mathrm{w} = -\frac{1}{2}\alpha\nabla_w J(w)$$
$$= \alpha\mathbb{E}_\pi[\big(v_\pi(S) - \hat{v}(S,w)\big)\nabla_w\hat{v}(S,w)]$$

- Stochastic gradient descent samples the gradient

$$\Delta\mathrm{w} = \alpha\big(v_\pi(S) - \hat{v}(S,w)\big)\nabla_w\hat{v}(S,w)$$

- Expected update is equal to full gradient update

Linear Value Function Approximation

- Represent value function by a linear combination of features

$$\hat{v}(S, w) = x(S)^T w = \sum_{j=1}^{n} x_j(S) w_j$$

- Objective function is quadratic in parameters *w*

$$J(w) = \mathbb{E}_\pi[(v_\pi(S) - x(S)^T w)^2]$$

- Stochastic gradient descent converges on global optimum
- Update rule is particularly simple

$$\nabla_w \hat{v}(S, w) = x(S)$$

$$\Delta \mathrm{w} = \alpha\left(v_\pi(S) - \hat{v}(S, w)\right)x(S)$$

- Update = step-size × prediction error × feature value

Incremental Prediction Algorithms

- Have assumed true value function $v_\pi(s)$ given by supervisor
- But in RL there is no supervisor, only rewards
- In practice, we substitute a target for $v_\pi(s)$
  - For MC, the target is the return G_t

$$\Delta \mathrm{w} = \alpha\big(G_t - \hat{v}(S_t, w)\big)\nabla_w \hat{v}(S_t w)$$

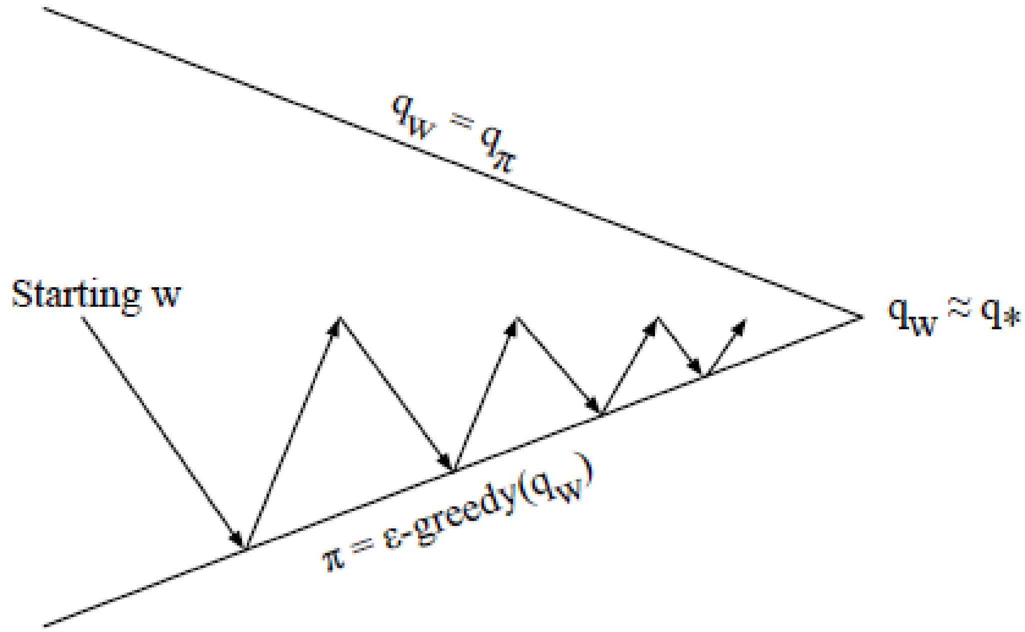
  - For TD(0), the target is the TD target $R_{t+1} + \gamma \hat{v}(S_{t+1}, w)$

$$\Delta \mathrm{w} = \alpha(R_{t+1} + \gamma \hat{v}(S_{t+1}, w) - \hat{v}(S_t, w))\nabla_w \hat{v}(S_t w)$$

  - For TD(λ), the target is the λ-return G_t^λ

$$\Delta \mathrm{w} = \alpha(G_t^\lambda - \hat{v}(S_t, w))\nabla_w \hat{v}(S_t w)$$

Control with Value Function Approximation



- Policy evaluation
  - Approximate policy evaluation, $\hat{q}(\cdot,\cdot,w) \approx q_\pi$
- Policy improvement
  - ε-greedy policy improvement

Action-Value Function Approximation

- Approximate the action-value function

$$\hat{q}(S, A, w) \approx q_\pi(S, A)$$

- Minimize mean-squared error between approximate action-value function $\hat{q}(S, A, w)$ and true action-value function $q_\pi(S, A)$

$$J(w) = \mathbb{E}_\pi[(q_\pi(S, A) - \hat{q}(S, A, w))^2]$$

- Use stochastic gradient descent to find a local minimum

$$-\frac{1}{2}\nabla_w J(w) = \big(q_\pi(S, A) - \hat{q}(S, A, w)\big)\nabla_w \hat{q}(S, A, w)$$

$$\Delta \mathrm{w} = \alpha\big(q_\pi(S, A) - \hat{q}(S, A, w)\big)\nabla_w \hat{q}(S, A, w)$$

Incremental Control Algorithms

- Like prediction, we must substitute a target for $q_\pi(S, A)$
  - For MC, the target is the return G_t

$$\Delta w = \alpha\big(G_t + \hat{q}(S_t, A_t, w)\big)\nabla_w \hat{q}(S_t, A_t, w)$$

  - For TD(0), the target is the TD target $R_{t+1} + \gamma Q(S_{t+1}, A_{t+1})$

$$\Delta w = \alpha\big(R_{t+1} + \gamma\hat{q}(S_{t+1}, A_{t+1}, w) - \hat{q}(S_t, A_t, w)\big)$$

$$\nabla_w \hat{q}(S_t, A_t, w)$$

  - For forward-view TD(λ), target is the action-value λ-return

$$\Delta w = \alpha\big(q_t^\lambda - \hat{q}(S_t, A_t, w)\big)\nabla_w \hat{q}(S_t, A_t, w)$$

  - For backward-view TD(λ), equivalent update is

$$\delta_t = R_{t+1} + \gamma\hat{q}(S_{t+1}, A_{t+1}, w) - \hat{q}(S_t, A_t, w)$$

$$E_t = \gamma\lambda E_{t-1} + \nabla_w \hat{q}(S_t, A_t, w)$$

$$\Delta w = \alpha\delta_t E_t$$

Batch Reinforcement Learning

- Gradient descent is simple and appealing
- But it is not sample efficient
- Batch methods seek to find the best fitting value function
- Given the agent's experience ("training data")

Least Squares Prediction

- Given value function approximation $\hat{v}(s, w) \approx v_\pi(s)$
- And experience D consisting of $\langle \text{state}, \text{value} \rangle$ pairs

$$D = \{\langle s_1, v_1^\pi \rangle, \langle s_2, v_2^\pi \rangle, \ldots, \langle s_T, v_T^\pi \rangle\}$$

- Which parameters w give the best fitting value fn $\hat{v}(s, w)$?
- Least squares algorithms find parameter vector w minimizing sum-squared error between $\hat{v}(s_t, w)$ and target values v_t^π,

$$LS(w) = \sum_{t=1}^{T} (v_t^\pi - \hat{v}(s_t, w))^2$$

$$= \mathbb{E}_D[(v^\pi - \hat{v}(s, w))^2]$$

Stochastic Gradient Descent with Experience Replay

- Given experience consisting of $\langle \text{state}, \text{value} \rangle$ pairs

$$D = \{\langle s_1, v_1^\pi \rangle, \langle s_2, v_2^\pi \rangle, \ldots \langle s_T, v_T^\pi \rangle\}$$

- Repeat:
  - Sample state, value from experience

$$\langle s, v^\pi \rangle \sim D$$

  - Apply stochastic gradient descent update

$$\Delta \mathrm{w} = \alpha(v^\pi - \hat{v}(s, w))\, \nabla_w \hat{v}(s, w)$$

- Converges to least squares solution

$$\mathrm{w}^\pi = \underset{w}{\operatorname{argmin}}\, LS(\mathrm{w})$$

  - Similar for action value function q^π