1. Let $B = (F,P)$ be a basis for predicate logic with $F = \{0,1,\ldots,+,*,\ldots\}$ and $P = \{<,\leq,\ldots\}$. Furthermore, let I be the usual interpretation of the function and predicate symbols over the set Nat of natural numbers. Determine for each of the following formulas $w \in WFF_B$ the set of all assignments $\sigma: V \rightarrow Nat$ with $I(w)(\sigma) = true$:
   (a) $\neg\exists y.(x<y \wedge y < z)$
   (b) $\exists x.\forall x.y \leq x$
   (c) $x = 0 \rightarrow x = 1 \rightarrow x = 2$
   (d) $\exists y.x*x = x \rightarrow x \leq x$
   Which of these formulas are valid in I?
2. Let B = (F, P) and I be as in Exercise 1. Give a formula $w \in WFF_B$ such that, for each assignment $\sigma$, $I(w)(\sigma) = true$ exactly when
   (a) $\sigma(z)$ divides $\sigma(x)$
   (b) $\sigma(x)$ and $\sigma(y)$ are relatively prime
   (c) $\sigma(z)$ is the gcd of $\sigma(x)$ and $\sigma(y)$.
3. For each of the following formulas give the set of all interpretations in which this formula is valid:
   (a) $x = y$ (b) $\forall x,y.x = y$
   (c) $\forall x.\exists y.x = y$ (d) $\exists y.\forall x.x = y$
   (e) $\exists x,y.\forall z(z = x \vee z = y)$ (f) $x = y \wedge y = z \rightarrow x = z$
   (g) $f(x) = f(y) \rightarrow x = y$ (h) $x = y \rightarrow f(x) = f(y)$
   Which of these formulas are logically valid ?
4. Show that for all formulas w,w1,w2 each of the following pairs of formulas are logically equivalent:
   (a) $w1 \rightarrow w2$ and $\neg w1 \vee w2$
   (b) $\forall x.w$ and $\neg\exists x.\neg w$
   (c) true and $w \vee \neg w$
5. Substitute the variable y by the term $y + z + x$ in the formula $\exists z.y * z = x$.
6. Let V be the set of variables in predicate logic, and let P(V) denote the power set of V, i.e. the set of all subsets of variables. Give an inductive definition of function
   *free*: $WFF_B \rightarrow P(V)$,
   that maps every predicate logic formula $w \in WFF_B$ to the set *free*(w) of all variables occurring free in w.
7. Let B = (F, P) be a basis for predicate logic with $F = \{0,1,+\}$ and $P = \emptyset$. Suppose $W \subseteq WFF_B$ consists of the following formulas:
   $\forall x,y,z.\ (x + (y + z) = (x + y) + z)$
   $\forall x.\ (x + 0 = x \wedge 0 + x = x)$
   $\forall x.\ \exists y.\ (x + y = 0 \wedge y + x = 0)$
   Give a model of W such that the following formula is also valid over this model:
   $\forall x, y.\ \exists z, z'.\ (x + z = y \wedge z' + x = y)$
8. Let $Z = \{ x+x = x, \forall y.y+0=y\}$. Which of the following formulas are the logical consequences of Z in predicate logic?
   (a) x=0 (b) 1+0=1 (c) 0+1=1 (d) 1+1=1
9. Give a deduction of the formula $\forall x.w \rightarrow w_x^t$ from the empty set in the predicate calculus.
10. Let $W \subseteq WFF_B$ be a set of formulas with at least one model. Show that Cn(W) is a theory.