1. Give a partial order set $(P, \sqsubseteq)$ and a subset S of P such that
   1) S has at Least one minimal element but no least element.
   2) S has exactly one minimal element but no least element.
   3) S has no minimal element.
2. Explain why the following definitions are not free.
   1) The set E, a subset of the universe Int, defined by
      (a) $0 \in E$,
      (b) if $n \in E$, then $n+2 \in E$ and $n-2 \in E$.
   2) The set Zn for some $n \geq 1$, a subset of the universe Int, defined by
      (a) $0 \in Zn$
      (b) if $m \in Zn$, then $(m+1) \bmod n \in Zn$.
3. Which of the following partial orders are well-founded sets?
   1) $(P(Nat), \subseteq)$.
   2) The set of all finite subsets of Nat with "$\subseteq$".
   3) The set of all non-negative rational numbers with "$\leq$".
   4) The set of all non-aagative numbers with finite decimal expansions with "$\leq$".
4. Give an example of a partial order $(D, \leq)$ and a subset S of D such that:
   (1) S has no upper bounds.
   (2) S has upper bounds but no lub.
   Is (2) possible if S is finite?
5. Which of the following partial orders $(D, \sqsubseteq)$ are complete?
   (i) $D = \{a, b\}^*$ where $v \sqsubseteq w$ iff $v$ is a prefix of w.
   *(ii)* $D = \{a, b\}^* \setminus \{\varepsilon\}$ with $v \sqsubseteq u$ iff v is a prefix of $u$ or there exist strings $x, y, z \in \{a,b\}^*$ such that $v = xay$ and $u = xbz$.
   (iii) $D = P\{H\}$ for an arbitrary set $H$ with the subset relation.
   (iv) $D = Nat \cup \{\propto\}$, where $d \sqsubseteq d'$ iff $d' = \propto$ or $(d, d' \neq \propto$ and $d \leq d')$.
   (v) $D = Q_+ \cup \{\propto\}$ (where $Q_+$ is the set of non-negative rational numbers) with $\sqsubseteq$ as defined in (iv).
   (vi) $D = R_+ \cup \{\propto\}$ (where $R_+$ is the set of non-negative real numbers) with $\sqsubseteq$ as defined in (iv).
6. Let $(D, \sqsubseteq)$ be as defined in Exercise 5-iv. Determine the lub of:
   (i) $S = \{(0,n) | n \in Nat\} \cup \{(n,0) | n \in Nat)$ in $(D^2, \sqsubseteq)$.
   (ii) $S = \{f_i \mid i \in D\}$ in $((D \to D), \sqsubseteq)$, where $f_i: D \to D$ is defined for each $i \in D$ by $f_i(i) = i$, and $f_i(d) = 0$ for $d \neq i$.
7. Let $(D, \sqsubseteq)$ be a partial order. $D$ is called a lattice if every two elements $a, b \in D$—and hence every finite number of elements—has a lub (denoted $a \sqcup b$) and a gib (denoted $a \sqcap b$). A lattice is *complete* if every subset of $D$ has a lub and gib. Prove:
   (i) Every complete lattice has a least and a greatest element (hence it is a cpo).
   (ii) A (complete) partial order for which every set has a lub is a complete lattice.
8. For all the *complete* partial orders $(D, \sqsubseteq)$ in Exercise 5 determine the set of continuous functions $f$: $D \to D$. In each case is there a monotonic function which is not continuous?
9. Consider P(Nat) ordered by the subset relation. Prove that each of the following functionals $\Phi$: $P(Nat) \to P(Nat)$ is continuous and determine the least fixpoint.
   (i) $\Phi(S) = S \cup T$ for a fixed set $T \subseteq Nat$.
   (ii) $\Phi(S) = S \cup \{0\} \cup \{n+2 \mid n \in S\}$.
10. Prove that the function $\Phi: (Nat_\omega \to Nat_\omega) \to Nat_\omega$ be defined by
    $\Phi(f) = 0$ if $f(n) \neq \omega$, for all $n \in Nat$,
    $\Phi(f) = \omega$ otherwise.
    is monotonic but not continuous.