FINAL EXAM, Physics 105, Spring 2021, Reinsch

Write this information in the upper left on the first page of your solutions:

NAME:

STUDENT ID NUMBER:

SIGNATURE:

- The exam is open-book, open-notes, and you may look at all files on our bCourses site. You are not allowed to use the internet or consult with anyone except the instructors.
- No Mathematica, Wolfram Alpha, or any other computer help. You may use a calculator (a small device that computes numerical results like 1 / 7 = 0.1428571428571429).
- We will not be doing Zoom monitoring or any sort of video proctoring.
- Explain how you got your answers. Put a box around your answers.
- At the end of the exam you must create a pdf file that contains all of your work. After you upload it you must download it to verify that it is legible and contains all of your work.

Problem 1

A wedge of mass M can slide freely on a horizontal frictionless table (the surface of the table is shown in green). A block of mass m can slide on the frictionless slope of the ramp. The natural (unstretched) length of the massless spring is D_0. The spring constant is k. The spring is compressed to a length D as shown in the diagram, and released at time $t = 0$. At time $t = 0$, both the block and wedge are motionless.

Throughout this problem we ignore gravity (set $g = 0$).

We use generalized coordinates s and x, where s is the length of the spring, and x is a coordinate that specifies the location of the wedge on the table (See orange arrow from a blue reference mark on the table to a blue reference mark on the wedge). Note: as stated above, the initial value of s is D.

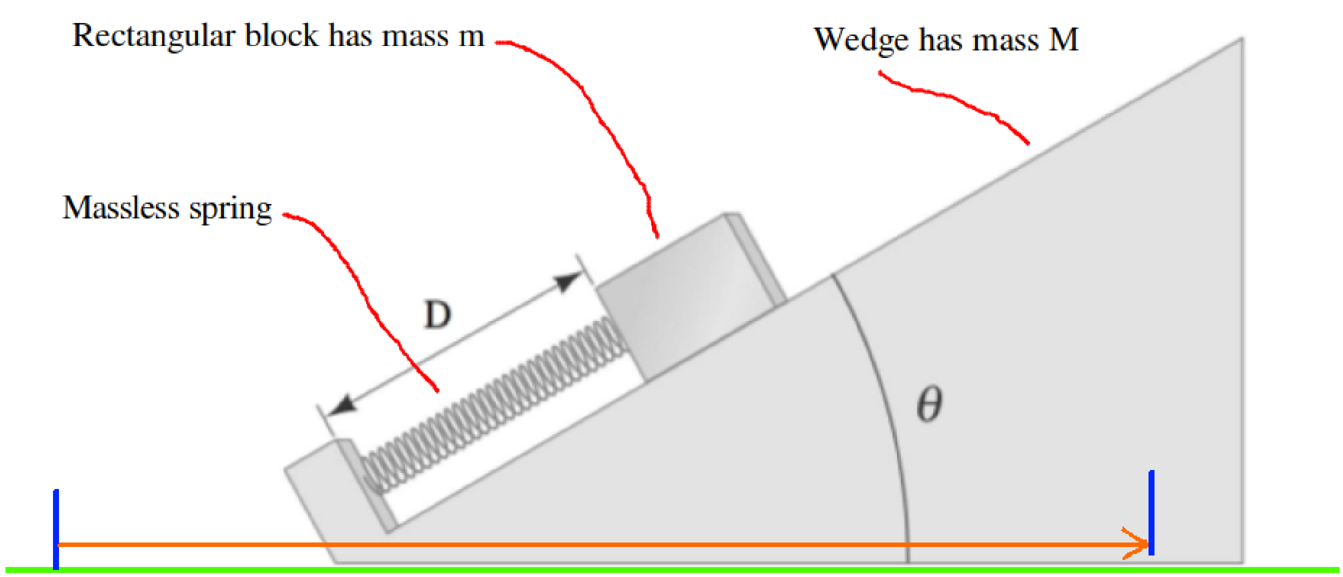
(a) Calculate the Lagrangian.

(b) From the Lagrangian, get the equations of motion. You do not have to solve the equations of motion.

(c) There are two conserved quantities (also called constants of the motion). What are they physically? Write expressions for them using our generalized coordinates.

(d) As discussed above, the spring is initially compressed. Later, when the spring's length is equal to D_0, what is the velocity vector of the block relative to the table.

(e) In the limit where the mass of the wedge goes to infinity, what happens to your answer to part (d)? Is the result what you expect?



Problem 2

A point mass m slides without friction on the surface of a horizontal table. There is an xy coordinate system on the table, and one end of an ideal massless spring is attached to the origin. The other end of the spring is attached to the point mass. The magnitude of the force is proportional to the length of the spring, and the spring constant is k. (The unstretched length of the spring is zero, an idealization.)

(a) Using x and y as coordinates, write out the Lagrangian.

(b) From the Lagrangian, deduce the equations of motion, and solve for the case where the initial conditions are $x = x_0$, $y = y_0$, $dx/dt = v_{x0}$, $dy/dt = v_{y0}$.

(c) Using plane polar coordinates (r and ϕ) as generalized coordinates write out the Lagrangian and the equations of motion. If one of the coordinates is ignorable (that means the Lagrangian does not depend on this coordinate), work out a formula for the corresponding conserved quantity.

(d) Now the table is tilted by putting wooden blocks beneath two of the legs. The result is that the potential you used above is modified by adding c times y, where c is a constant (the value of c could be computed in terms of g and the slope of the table, but there is no need to do that). Using x and y as coordinates, write out the Lagrangian. Because the Lagrangian is time-independent, there is a conserved quantity. Write a formula for this quantity.

Problem 3

Two particles with masses m_1 and m_2 move under the influence of Newtonian gravity in an otherwise empty space. A Cartesian coordinate system is used, and the initial conditions at $t = 0$ are $\boldsymbol{r}_1 = (0, 0, 0)$, $\boldsymbol{r}_2 = (b, 0, 0)$, $d\boldsymbol{r}_1/dt = (0, 0, v_0)$ and $d\boldsymbol{r}_2/dt = (0, v_0, 0)$, where b and v_0 are positive constants.

(a) Calculate the location of the center of mass as a function of time.

(b) Calculate the total angular momentum about the center of mass. This is a vector.

(c) Assuming v_0 is small enough (if it were too large the particles would go out to infinity) that the particles orbit their common center of mass, what is the orbital period?

(d) Calculate r_{min} and r_{max}, the minimal and maximal values of the distance between the particles. You do not have to simplify the expressions and show that one of these is equal to b.

Problem 4

We study a long river in the vicinity of a city in the northern hemisphere at co-latitude θ (the latitude is $\pi/2 - \theta$). In this region, the river flows due north (that means the direction is exactly north). All the water in the river has the same speed. What is the difference in height at the two banks? At which bank is the water higher? (A river bank is the terrain at the edge of the water.) The river has width w (the distance from one bank to the other), and the Earth rotates with angular velocity Ω. The water height is measured relative to the horizontal, which is defined to be orthogonal to the vector $\mathbf{g}$, the observed free-fall acceleration discussed in the textbook. In your answer, you can use the magnitude g. [Hint: study a small parcel of water at the surface of the river, similar to what we did several times in Chapter 9.]

The speed of the water is v.
Hint: In the rotating frame of the Earth, think about the forces on the small parcel of water (with mass m). The buoyant force is perpendicular to the surface of the water. It remains to think about $m\mathbf{g}$ and the Coriolis force. The vector sum of the forces is zero.

Problem 5

Study Problem 5 of Homework 7, and its solution, available on bCourses.

(A) If at $t = 0$ the $\boldsymbol{\omega}$ vector points in the y direction, what is the angular momentum?

(B) Now assume we have steady rotation with the constant angular velocity vector defined part (A). At time $t = \pi/(2\omega)$ calculate the moment of inertia tensor in the space frame. Hint: You should get a diagonal matrix.

(C) Repeat the previous calculation for $t = \pi/(4\omega)$. Hint: You should not get a diagonal matrix.

Problem 6

Half of a uniform disk of radius R is cut out of sheet metal. This half of the disk (shown in blue in the diagram) has mass M. It is suspended by a massless thread of length b (shown in red). The upper end of the thread is attached to a fixed point on the ceiling (green). The motion of the system is in a fixed vertical plane containing the green point. Thus we have a two degree-of-freedom system.

The diagram shows the system in equilibrium.

(a) Show how to calculate the distance from the center of mass of the blue half disk (that is the black dot) to the intersection of the two black straight lines. Call this h. For the rest of the problem use the symbol h, and do not plug in whatever result you got for h from the first calculation.

(b) Show how to calculate the moment of inertia of the blue half disk about its center of mass. (Hint: use the Parallel Axis Theorem to do this very quickly). Call this I_{CM}. For the rest of the problem use this symbol.

(c) Calculate the eigenfrequencies for small oscillations near equilibrium.