Homework 9 Problem 1

Part a

As calculated in Equation 11.5:

$$\mathbf{M} = \begin{bmatrix} 2m & 0 \\ 0 & 5m \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} \frac{3}{2}k & -\frac{1}{2}k \\ -\frac{1}{2}k & \frac{3}{2}k \end{bmatrix}$$

Thus:

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} \frac{3}{2}k - 2m\omega^2 & -\frac{1}{2}k \\ -\frac{1}{2}k & \frac{3}{2}k - 5m\omega^2 \end{bmatrix}$$

For there to be a mode, the determinant of this matrix must be zero:

$$\begin{aligned} 0 &= \det\big(\mathbf{K} - \omega^2\mathbf{M}\big) \\ &= \left(\frac{3}{2}k - 2m\omega^2\right)\left(\frac{3}{2}k - 5m\omega^2\right) - \left(-\frac{1}{2}k\right)^2 \\ &= \frac{9}{4}k^2 - \frac{21}{2}km\omega^2 + 10m^2\omega^4 - \frac{1}{4}k^2 \\ &= 2k^2 - \frac{21}{2}km\omega^2 + 10m^2\omega^4 \end{aligned}$$

The roots are:

$$\begin{aligned} \omega^2 &= \frac{\frac{21}{2} \pm \sqrt{\left(\frac{21}{2}\right)^2 - 4\,(10)\,(2)}}{2\,(10)}\frac{k}{m} \\ &= \frac{\frac{21}{2} \pm \frac{11}{2}}{20}\frac{k}{m} \end{aligned}$$

$$\omega_1 = \sqrt{\frac{k}{4m}} \qquad \omega_2 = \sqrt{\frac{4k}{5m}}$$

For the two normal modes:

$$\mathbf{K} - \omega_1^2\mathbf{M} = \begin{bmatrix} k & -\frac{1}{2}k \\ -\frac{1}{2}k & \frac{1}{4}k \end{bmatrix} \qquad \mathbf{K} - \omega_2^2\mathbf{M} = \begin{bmatrix} -\frac{1}{10}k & -\frac{1}{2}k \\ -\frac{1}{2}k & -\frac{5}{2}k \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} a_1^{(1)} \\ a_2^{(1)} \end{bmatrix} = 0 \qquad \begin{bmatrix} -\frac{1}{10} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} a_1^{(2)} \\ a_2^{(2)} \end{bmatrix} = 0$$

$$a_1^{(1)} - \frac{1}{2}a_2^{(1)} = 0 \qquad -\frac{1}{10}a_1^{(2)} - \frac{1}{2}a_2^{(2)} = 0$$

$$-\frac{1}{2}a_1^{(1)} + \frac{1}{4}a_2^{(1)} = 0 \qquad -\frac{1}{2}a_1^{(2)} - \frac{5}{2}a_2^{(2)} = 0$$

The solutions are $a_1^{(1)} = \frac{1}{2}a_2^{(1)}$ and $a_1^{(2)} = -5a_2^{(2)}$ for the modes, respectively. This yields modes:

$$\mathbf{z}^{(1)}(t) = \begin{bmatrix} A^{(1)} \\ 2A^{(1)} \end{bmatrix} e^{i(\omega_1 t - \delta_1)} \qquad \mathbf{z}^{(2)}(t) = \begin{bmatrix} 5A^{(2)} \\ -A^{(2)} \end{bmatrix} e^{i(\omega_2 t - \delta_2)}$$

$$x_1^{(1)}(t) = A^{(1)}\cos(\omega_1 t - \delta_1) \qquad x_1^{(2)}(t) = 5A^{(2)}\cos(\omega_2 t - \delta_2)$$

$$x_2^{(1)}(t) = 2A^{(1)}\cos(\omega_1 t - \delta_1) \qquad x_2^{(2)}(t) = -A^{(2)}\cos(\omega_2 t - \delta_2)$$

Hence, the general solution is:

$$\mathbf{x}(t) = A^{(1)} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A^{(2)} \begin{bmatrix} 5 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2)$$

Part b

Differentiating:

$$\dot{\mathbf{x}}(t) = -A^{(1)}\omega_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(\omega_1 t - \delta_1) - A^{(2)}\omega_2 \begin{bmatrix} 5 \\ -1 \end{bmatrix} \sin(\omega_2 t - \delta_2)$$

Plugging in initial conditions:

$$\begin{bmatrix} 0 \\ x_{20} \end{bmatrix} = A^{(1)} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos\delta_1 + A^{(2)} \begin{bmatrix} 5 \\ -1 \end{bmatrix} \cos\delta_2$$
$$\begin{bmatrix} v_{10} \\ 0 \end{bmatrix} = A^{(1)}\omega_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin\delta_1 + A^{(2)}\omega_2 \begin{bmatrix} 5 \\ -1 \end{bmatrix} \sin\delta_2$$

Note that:

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ -\frac{11}{4} \end{bmatrix}$$
$$\begin{bmatrix} -\frac{1}{10} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{11}{10} \\ -\frac{11}{2} \end{bmatrix} \qquad \begin{bmatrix} -\frac{1}{10} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Multiplying these equations by these matrices, respectively, gives:

$$\begin{bmatrix} -\frac{1}{2}x_{20} \\ -\frac{5}{2}x_{20} \end{bmatrix} = A^{(1)}\cos\delta_1 \begin{bmatrix} -\frac{11}{10} \\ -\frac{11}{2} \end{bmatrix} \qquad \begin{bmatrix} -\frac{1}{2}x_{20} \\ \frac{1}{4}x_{20} \end{bmatrix} = A^{(2)}\cos\delta_2 \begin{bmatrix} \frac{11}{2} \\ -\frac{11}{4} \end{bmatrix}$$
$$\begin{bmatrix} -\frac{1}{10}v_{10} \\ -\frac{1}{2}v_{10} \end{bmatrix} = A^{(1)}\omega_1\sin\delta_1 \begin{bmatrix} -\frac{11}{10} \\ -\frac{11}{2} \end{bmatrix} \qquad \begin{bmatrix} v_{10} \\ -\frac{1}{2}v_{10} \end{bmatrix} = A^{(2)}\omega_2\sin\delta_2 \begin{bmatrix} \frac{11}{2} \\ -\frac{11}{4} \end{bmatrix}$$

Canceling terms:

$$A^{(1)}\cos\delta_1 = \frac{5}{11}x_{20} \qquad A^{(2)}\cos\delta_2 = -\frac{1}{11}x_{20}$$
$$A^{(1)}\sin\delta_1 = \frac{1}{11}\frac{v_{10}}{\omega_1} \qquad A^{(2)}\sin\delta_2 = \frac{2}{11}\frac{v_{10}}{\omega_2}$$

The amplitudes can be found by summing squares of the equations; the phases can be found by dividing equations:

$$A^{(1)} = \frac{1}{11}\sqrt{25x_{20}^2 + \frac{v_{10}^2}{\omega_1^2}} \qquad A^{(2)} = \frac{1}{11}\sqrt{x_{20}^2 + 4\frac{v_{10}^2}{\omega_2^2}}$$
$$\delta_1 = \arctan\frac{v_{10}}{5\omega_1 x_{20}} \qquad \delta_2 = \pi - \arctan\frac{2v_{10}}{\omega_2 x_{20}}$$

(The choices of phase assume that $x_{20} \geq 0$ and $v_{10} \geq 0$, due to the non-uniqueness of the arctangent function. Other sign choices may lead to π's being placed elsewhere.)

Physics 105, Homework Assignment 9

Problem 2 solutions, summary

(a) Calculate the exact Lagrangian.

The original answer in the book is given in equations (11.37) and (11.38).
Now we add this to the kinetic energy:

$$\frac{1}{6}\,\mu\,\dot{\phi_1}^2\,L_1^3 + \frac{1}{2}\,\mu\,\dot{\phi_1}^2\,L_1^2\,L_2 + \frac{1}{2}\,\mu\,\mathrm{Cos}[\phi_1 - \phi_2]\,\dot{\phi_1}\,\dot{\phi_2}\,L_1\,L_2^2 + \frac{1}{6}\,\mu\,\dot{\phi_2}^2\,L_2^3$$

And we add this to the potential energy (an overall additive constant has no effect):

$$-\frac{1}{2}\,g\,\mu\,\mathrm{Cos}[\phi_1]\,L_1^2 - g\,\mu\,\mathrm{Cos}[\phi_1]\,L_1\,L_2 - \frac{1}{2}\,g\,\mu\,\mathrm{Cos}[\phi_2]\,L_2^2$$

(b) Calculate the new matrices in Eq. (11.44).

$$\begin{pmatrix} \frac{1}{3} L_1^2 (\mu L_1 + 3 (\mu L_2 + m_1 + m_2)) & \frac{1}{2} L_1 L_2 (\mu L_2 + 2 m_2) \\ \frac{1}{2} L_1 L_2 (\mu L_2 + 2 m_2) & \frac{1}{3} L_2^2 (\mu L_2 + 3 m_2) \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} g L_1 (\mu L_1 + 2 (\mu L_2 + m_1 + m_2)) & 0 \\ 0 & \frac{1}{2} g L_2 (\mu L_2 + 2 m_2) \end{pmatrix}$$

(c) For the case of equal lengths and masses (which refers to the point masses), repeat the calculations on pages 435 and 436. Use the value $\mu = m/(2L)$.

The values for omega squared are

$$\frac{3\left(157-12\sqrt{86}\right)g}{223\,L}$$

$$\frac{3\left(157+12\sqrt{86}\right)g}{223\,L}$$

Eigenvectors are

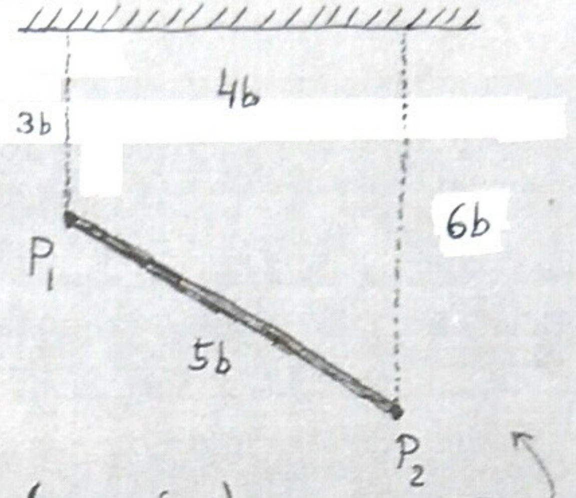
$$\left\{\frac{1}{55}\left(1-4\sqrt{86}\right),\ 1\right\}$$

$$\left\{\frac{1}{55}\left(1+4\sqrt{86}\right),\ 1\right\}$$

To compare the frequencies with those printed in the text, the following numbers are to be compared with the numbers after Eq.(11.47)

0.784233
1.89979

Homework 9 Problem 3 Phys 105



(x_1, y_1, z_1) = displacement of P_1 relative to equilibrium location

(x_2, y_2, z_2) = same for P_2 (rel. to equil. of P_2)

x to the right, y into page, z up. (space-fixed)

You can literally build this on a shoestring budget

$\boxed{\text{Generalized coordinates : } x_1,\ y_1,\ y_2}$

z_1, x_2, z_2 can be computed from these.

$$z_1 = 3b - \sqrt{9b^2 - x_1^2 - y_1^2} \approx 3b - 3b\left(1 - \frac{x_1^2+y_1^2}{18b^2}\right) = \frac{x_1^2+y_1^2}{6b}$$

Similarly, $z_2 = \dfrac{x_2^2+y_2^2}{12b}$ + higher order terms (3rd order + higher)

An important relation is $x_2 = x_1$ + higher order terms. (2nd order + higher)

This can be proven by direct calculation, or by observing that to first order $z_1 = z_2 = 0 = \text{Const}$ so the problem reduces to studying horizontal displacements; the y-displacements are orthogonal to the x-direction, so they contribute at 2nd order.

In the equilibrium position, $\vec{v}_1 = (\dot{x}_1, \dot{y}_1, 0)$ and $\vec{v}_2 = (\dot{x}_1, \dot{y}_2, 0)$.

Thus $\vec{V}_{CM} = \left(\dot{x}_1, \tfrac{1}{2}(\dot{y}_1+\dot{y}_2), 0\right)$ and $\vec{v}_1 - \vec{V}_{CM} = \left(0, \tfrac{1}{2}(\dot{y}_1-\dot{y}_2), 0\right)$.

Therefore $|\vec{\omega}| = \left|\tfrac{1}{2}(\dot{y}_1-\dot{y}_2)\right| / (5b/2)$ and direction of $\vec{\omega}$ = orthog. to rod. In summary,

$$\left.\begin{aligned} T_{CM} &= \tfrac{1}{2} m \left[\dot{x}_1^2 + (\dot{y}_1+\dot{y}_2)^2/4\right] \\ T_{\text{about CM}} &= \frac{1}{2}\left(\frac{m(5b)^2}{12}\right)\left(\frac{(\dot{y}_1-\dot{y}_2)/2}{5b/2}\right)^2 \end{aligned}\right\} \Longrightarrow \boxed{\overleftrightarrow{M} = \frac{m}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}}$$

For convenience, we choose the direction of $\vec{\omega}$ to be an eigendirection of the mass distribution. For a thin rod, $\vec{\omega}$ is not uniquely determined by the state of motion

Homework 9 Problem 3 continued

potential energy to second order: $U = mg\,\dfrac{z_1 + z_2}{2}$

$$= \frac{mg}{2}\left(\frac{x_1^2 + y_1^2}{6b} + \frac{x_1^2 + y_2^2}{12b} \right)$$

$$\Rightarrow \boxed{\overleftrightarrow{K} = \frac{mg}{24b}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

With $\vec{q} = \begin{pmatrix} x_1 \\ y_1 \\ y_2 \end{pmatrix}$, we have $\boxed{\mathcal{L} = \frac{1}{2}\dot{\vec{q}}\,\overleftrightarrow{M}\,\dot{\vec{q}} - \frac{1}{2}\vec{q}\,\overleftrightarrow{K}\,\vec{q}}$

We need to solve $\det\left(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M}\right) = 0$.

Define λ such that $\omega^2 = \lambda g/b$

The solutions are $\boxed{\lambda_1 = \frac{1}{4}, \quad \lambda_{2,3} = \frac{3 \pm \sqrt{3}}{6}}$ all positive

The frequencies are $\boxed{\omega_i = \sqrt{\lambda_i g/b}, \quad i = 1, 2, 3}$

Eigenvectors $\boxed{\vec{a}_1 = (1, 0, 0), \quad \vec{a}_{2,3} = \left(0, \frac{-1 \mp \sqrt{3}}{2}, 1\right)}$

could also use $(0, 1 \pm \sqrt{3}, -2)$ — not normalized

Mode 1: Swings left/right, stays in plane of the page.

Mode 2: No left/right motion. P_1 and P_2 move perpendicular to the page, 180° out of phase

Mode 3: Like mode 2, but in-phase (different amplitudes)

The CM is not motionless in any of these modes.

Homework 9 Problem 4 Phys 105

(a) $F_1 + F_2 = mg$. $F_1 = F_2$ because net torque about CM is zero.

Thus $F_1 = F_2 = mg/2 \Rightarrow$ $\boxed{k_1 = \frac{mg}{6b}, \quad k_2 = \frac{mg}{12b}}$

(b) Cartesian coordinates: x to the right, y up.

Generalized coordinates: $\boxed{x_1, y_1, y_2}$ See diagram for Prob. 3

Length of rod = const $\Rightarrow (5b)^2 = (4b + x_2 - x_1)^2 + (3b + y_1 - y_2)^2$

Apply $\frac{d}{dt}$ and then set coordinates to zero: $0 = 4(\dot{x}_2 - \dot{x}_1) + 3(\dot{y}_1 - \dot{y}_2)$

Solve to get $\dot{x}_2 = \dot{x}_1 + \frac{3}{4}(\dot{y}_2 - \dot{y}_1)$.

Thus $x_2 = x_1 + \frac{3}{4}(y_2 - y_1)$ + second order and higher

In the equilibrium position, $\vec{v}_1 = (\dot{x}_1, \dot{y}_1)$, $\vec{v}_2 = (\dot{x}_1 + \frac{3}{4}(\dot{y}_2 - \dot{y}_1), \dot{y}_2)$

Thus $\vec{V}_{cm} = (\dot{x}_1 + \frac{3}{8}(\dot{y}_2 - \dot{y}_1), \frac{1}{2}(\dot{y}_1 + \dot{y}_2))$

and $\vec{v}_1 - \vec{V}_{cm} = (\dot{y}_1 - \dot{y}_2)(\frac{3}{8}, \frac{1}{2})$

From this we compute T and $\overleftrightarrow{M}$:

$$\boxed{\overleftrightarrow{M} = \frac{m}{48}\begin{pmatrix} 48 & -18 & 18 \\ -18 & 25 & -1 \\ 18 & -1 & 25 \end{pmatrix}}$$

Next, $U = mg(y_1 + y_2)/2 + \frac{1}{2}k_1((3b - y_1)^2 + x_1^2) + \frac{1}{2}k_2((6b - y_2)^2 + x_2^2)$

From this we get $\boxed{\overleftrightarrow{K} = \frac{mg}{192b}\begin{pmatrix} 48 & -12 & 12 \\ -12 & 41 & -9 \\ 12 & -9 & 25 \end{pmatrix}}$

Now proceed as in Problem 3. Remarkably the eigenvalues are the same! The eigenvectors are (not normalized)

$$\boxed{\vec{a}_1 = (-4, 3, 3), \quad \vec{a}_{2,3} = \left(\frac{3}{4}, 1, 1 \mp \sqrt{3}\right)}$$

Mode 1: Motion in the direction of the rod, no rotation.

Modes 2, 3: More general, including rotation

Physics 105, Homework Assignment 9
Problem 5 sample numbers

For a pencil, b = 35 mm (remember b is one-fifth the length)

The frequencies in cycles per second are:
1.33, 1.22, 2.37

The periods in seconds are:
0.751, 0.817, 0.423

For a broomstick, b = 30 cm (remember b is one-fifth the length)

The frequencies in cycles per second are:
0.455, 0.418, 0.808

The periods in seconds are:
2.20, 2.39, 1.24

Due to the covid pandemic, the code for this problem was given to the students as part of the problem statement. So in that sense, the solutions were part of the problem statement.