Solutions - Homework 6

1. (Griffiths 6.27)

$$\int (\vec{a}\cdot\hat{r})(\vec{b}\cdot\hat{r}) \sin\theta d\theta d\phi = \int \left(\sin\theta\cos\phi \, a_x + \sin\theta\sin\phi \, a_y + \cos\theta \, a_z\right)\cdot\left(\sin\theta\cos\phi \, b_x + \sin\theta\sin\phi \, b_y + \cos\theta \, b_z\right) \cdot \sin\theta d\theta d\phi$$

$$= \cancel{\int \sin^3\theta \cos\theta \, d\theta}$$

$$= \int \sin^3\theta d\theta \int \cos^2\phi \, d\phi \, a_x b_x + \int \sin^3\theta d\theta \int \cos\phi\sin\phi \, d\phi \left(a_x b_y + a_y b_x\right)$$

$$+ \int \sin^2\theta\cos\theta \, d\theta \int \cos\phi \, d\phi \left(a_x b_z + a_z b_x\right) + \int \sin^2\theta \cos\theta \, d\theta \int \sin\phi \, d\phi \left(a_y b_z + a_z b_y\right)$$

$$+ \int \sin^3\theta d\theta \int \sin^2\phi \, d\phi \, a_y b_y + \int \cos^2\theta\sin\theta \, d\theta \int d\phi \, a_z b_z$$

Now note that $\displaystyle\int_0^{2\pi} \cos\phi \, d\phi = + \int_0^0 d(\sin\phi) = 0$

Also, $\displaystyle\int_0^{2\pi} \cos\phi\sin\phi \, d\phi = \cancel{\int_1^{-1} \cos\phi \, d(\cos\phi)} = 0$

$$\therefore \; = \underbrace{\int \sin^3\theta \, d\theta \int \cos^2\phi \, d\phi \, (a_x b_x)}_{= \frac{4}{3}\cdot\pi} + \underbrace{\int \sin^3\theta \int \sin^2\phi \, d\phi \, (a_y b_y)}_{= \frac{4}{3}\cdot\pi} + \underbrace{\int \cos^2\theta\sin\theta \, d\theta \int d\phi \, a_z b_z}_{= \frac{2}{3}\cdot 2\pi}$$

$$\therefore \; = \frac{4\pi}{3}\left(a_x b_x + a_y b_y + a_z b_z\right) = \frac{4\pi}{3}\vec{a}\cdot\vec{b}$$

~~6.36~~

2. (Griffiths 6.36)

a)

$$\langle 100|H_s'|100\rangle = \int_0^\infty dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}} \left(e E_{ext}\, r\cos\theta\right) \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}$$

$$= \frac{eE_{ext}}{\pi a^3} \int_0^\infty dr\, r^2 r\, e^{-\frac{2r}{a}} \int_0^\pi d\theta \sin\theta\cos\theta \cdot 2\pi$$

$$= \frac{1}{2}\int_0^\pi d\theta \sin(2\theta)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cos(2\theta)\Big|_0^\pi = 0$$

$$\therefore \underline{\langle 000|H_s'|000\rangle = 0}$$

b) We'll use the convention,

$$\left\{\begin{array}{l} \psi_1 = \psi_{200} \\ \psi_2 = \psi_{211} \\ \psi_3 = \psi_{210} \\ \psi_4 = \psi_{21-1} \end{array}\right\}$$

$$\langle 1|H_s'|1\rangle = (\ldots)\cdot \underbrace{\int_0^\pi d\theta \sin\theta\cos\theta}_{=0} = 0$$

$$\langle 2|H_s'|2\rangle = (\ldots)\cdot \underbrace{\int_0^\pi d\theta \sin^2\theta\cos\theta}_{\frac{1}{3}\sin^3\theta\big|_0^\pi = 0}$$

$$\langle 4|H_s'|4\rangle = (\ldots)\cdot \int_0^\pi d\theta \sin^2\theta\cos\theta = 0$$

$$\langle 3|H_s'|3\rangle = (\ldots)\cdot \underbrace{\int_0^\pi d\theta \cos^3\theta}_{=0} = 0$$

Now also note that

$$\langle 2|H_s'|1\rangle = (\ldots)\cdot \underbrace{\int_0^{2\pi} e^{-i\phi} d\phi}_{=0} = 0$$

$$\langle 4|H_s'|1\rangle = (\ldots)\cdot \underbrace{\int_0^{2\pi} e^{+i\phi} d\phi}_{=0} = 0$$

$$\langle 2|H_s'|3\rangle = (\ldots)\cdot \underbrace{\int_0^{2\pi} e^{-i\phi} d\phi}_{=0}$$

$$\langle 4|H_s'|3\rangle = (\ldots)\cdot \underbrace{\int_0^{2\pi} e^{+i\phi} d\phi}_{=0}$$

The only terms left are W_{24} and W_{13}

$$W_{24} = (\cdots)\cdot \underbrace{\int_0^\pi d\theta \, \sin^2\theta \cos\theta}_{=0} = 0$$

~~Using ... I'll do this one carefully~~

$$W_{13} = \langle \psi_{200} | H'_S | \psi_{210} \rangle$$

$$= \int_0^\infty dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \cdot \sqrt{\frac{1}{a^3}\cdot\frac{1}{4\cdot 2^3}}\; e^{-\frac{r}{2a}} \left(-2\cdot\left(\frac{r}{a}\right)+4\right)\left(\frac{1}{4\pi}\right)^{1/2}$$

$$\cdot\, eE_{ext}\, r\cos\theta \cdot \sqrt{\frac{1}{a^3}\,\frac{1}{4\cdot 6^3}}\; e^{-\frac{r}{2a}}\cdot\left(\frac{r}{a}\right)(6)\cdot\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$= \frac{eE_{ext}}{a^3}\,\frac{1}{4}\cdot\frac{\cancel{6}}{\sqrt{8\cdot 6\cdot \cancel{36}}}\int_0^\infty dr\, e^{-\frac{r}{a}}\, r^3\cdot\frac{r}{a}\left(-\frac{2r}{a}+4\right)\underbrace{\int_0^\pi d\theta \sin\theta\cos^2\theta}_{=\frac{2}{3}}\;\underbrace{\int_0^{2\pi} d\phi}_{=2\pi}\cdot\sqrt{3}\cdot\frac{1}{4\pi}$$

$$= \frac{eE_{ext}}{\cancel{a^3}}\,\frac{1}{\cancel{4}16\sqrt{3}}\cdot\frac{\cancel{\sqrt{3}}}{\cancel{4\pi}}\cdot\cancel{2\pi}\cdot\frac{\cancel{2}}{3}\,\frac{1}{a}\underbrace{\int_0^\infty \frac{dr}{a}\cdot\left(\frac{r}{a}\right)^3 e^{-\frac{r}{a}}\cdot\frac{r}{a}\left(-2\frac{r}{a}+4\right)}_{=-144}$$

$$= \cancel{2}\, a e E_{ext}\cdot\frac{1}{16\cdot 3}\cdot(-144) = \underline{-3eaE_{ext}}$$

Note that $W_{31} = W_{13}^*$

$$\therefore \quad W = \begin{pmatrix} 0 & 0 & -3eaE_{ext} & 0 \\ 0 & 0 & 0 & 0 \\ -3eaE_{ext} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Swapping the ordering of the states, we could write W as,

$$W = \left(\begin{array}{cc|cc} 0 & -3eaE_{ext} & 0 & 0 \\ -3eaE_{ext} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Then we can clearly see that W has two eigenvalues $\lambda_1 = 0, \lambda_2 = 0$ and two nonzero eigenvalues (which are distinct, see below)

$\therefore$ The ~~n~~ states split into three energies.

Diagonalizing the 2×2 block gives,

$$0 = \det \begin{pmatrix} -\lambda & -3eaE_{ext} \\ -3eaE_{ext} & -\lambda \end{pmatrix} = \lambda^2 - 9(eaE_{ext})^2$$

$$\Rightarrow \quad \lambda = \pm 3eaE_{ext}$$

From above,

c) The good states are $\underset{a}{\psi_{211}}, \underset{b}{\psi_{21-1}}, \underset{c}{\frac{1}{\sqrt{2}}(\psi_{200} + \psi_{210})}, \underset{d}{\frac{1}{\sqrt{2}}(\psi_{200} - \psi_{210})}$

$$\langle \vec{p}_e \rangle = \langle p_x \rangle \hat{x} + \langle p_y \rangle \hat{y} + \langle p_z \rangle \hat{z}$$

Now note that $\frac{1}{\sqrt{2}}(\psi_{200} + \psi_{210})$ and $\frac{1}{\sqrt{2}}(\psi_{200} - \psi_{210})$ are symmetric w/ respect to ϕ rotations.

This implies $\langle p_x \rangle_c = 0$, $\langle p_x \rangle_d = 0$, $\langle p_y \rangle_c = 0$, $\langle p_y \rangle_d = 0$

Further, we know

$$H' = -p_z E_{ext}$$

$$\Rightarrow \langle p_z \rangle_c = -3ea$$

$$\langle p_z \rangle_d = 3ea$$

Now we need to do the same computations for ψ_{211} and ψ_{21-1}

$$\langle \vec{p} \rangle_{211} = -\frac{e}{\pi a}\frac{1}{64a^4}\int r^2 e^{-\frac{r}{a}} \sin^2\theta \left[r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k} \right] r^2 \sin\theta \, dr\, d\theta\, d\phi$$

Now we use the fact that

$$\int_0^{2\pi} \cos\phi = 0,$$

$$\int_0^{2\pi} \sin\phi = 0,$$

$$\int_0^{\pi} \sin^3\theta\cos\theta\, d\theta = \left.\frac{\sin^4\theta}{4}\right|_0^{\pi} = 0 \Rightarrow \langle \vec{p} \rangle_{211} = 0$$

Exactly the same reasoning goes through for $\langle \vec{p} \rangle_{21-1} \Rightarrow \langle \vec{p} \rangle_{21-1} = 0$

~~6.38~~

3. (Griffiths 6.38)

$$\vec{\mu}_d = \frac{g_d e}{2m_d}\vec{S}_d$$

$$\Rightarrow \quad E_{hf} = \frac{\mu_0 g_d e^2}{3\pi m_d m_e a^3}\langle \vec{S}_d \cdot \vec{S}_e \rangle$$

$$= \frac{\mu_0 g_d e^2}{3\pi m_d m_e a^3}\left\langle \frac{1}{2}\left(S_{TOT}^2 - S_d^2 - S_e^2\right)\right\rangle$$

Note that now the nucleus has spin 1
and " electron " " ½

$$\Rightarrow S_{TOT} = 3/2,\ 1/2$$

$$\Rightarrow E_{hf} = \frac{\mu_0 g_d e^2}{3\pi m_d m_e a^3}\cdot\frac{\hbar^2}{2}\cdot\begin{cases} \frac{3}{2}\cdot\frac{5}{2} - 1(2) - \frac{3}{4} = \frac{15-8-3}{4} = 1 \\ \frac{1}{2}\cdot\frac{3}{2} - 1(2) - \frac{3}{4} = \frac{3-8-3}{4} = -2 \end{cases}$$

Their difference is given by,

$$\Delta E_{hf} = \frac{\mu_0 g_d e^2}{3\pi m_d m_e a^3}\cdot\frac{3\hbar^2}{2}$$

$$\Delta E_{hf} = h\nu = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{\Delta E_{hf}} \approx 92\text{ cm}$$

4. (Griffiths 7.4)

a) $\langle \psi | \psi_{gs} \rangle = 0$

$$\Rightarrow |\psi\rangle = \sum_{n=2}^{\infty} c_n |n\rangle$$ From the orthogonality, we know that $c_1 = 0$

$$\langle \psi | \hat{H} | \psi \rangle = \sum_{n=2}^{\infty} |c_n|^2 E_n \geq \sum_{n=2}^{\infty} |c_n|^2 E_2 = E_2 \sum_{n=2}^{\infty} |c_n|^2 = E_2 = E_{fs}$$

first excited

$$\therefore \boxed{\langle \psi | \hat{H} | \psi \rangle \geq E_{fs}}$$

b)

$$\langle \psi | \hat{H} | \psi \rangle = \int_{-\infty}^{\infty} dx \, A x e^{-bx^2} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \right) A x e^{-bx^2}$$

$$= A^2 \int_{-\infty}^{\infty} dx \, x e^{-bx^2} \left[-\frac{\hbar^2}{2m} \frac{\partial}{\partial x} \left(e^{-bx^2} - 2bx^2 e^{-bx^2} \right) + \frac{1}{2} m\omega^2 x^3 e^{-bx^2} \right]$$

$$= A^2 \int_{-\infty}^{\infty} dx \, x e^{-bx^2} \left[-\frac{\hbar^2}{2m} \left(-2bx e^{-bx^2} - 4bx e^{-bx^2} + 4b^2 x^3 e^{-bx^2} \right) + \frac{1}{2} m\omega^2 x^3 e^{-bx^2} \right]$$

$$= A^2 \int_{-\infty}^{\infty} dx \, e^{-2bx^2} \left[-\frac{\hbar^2}{2m} \left(-6bx^2 + 4b^2 x^4 \right) + \frac{1}{2} m\omega^2 x^4 \right]$$

$$= A^2 \left[\frac{3\hbar^2 b}{m} \cdot \sqrt{\frac{\pi}{2}} \cdot \frac{1}{4b^{3/2}} + \left(\frac{1}{2} m\omega^2 - \frac{2\hbar^2 b^2}{m} \right) \cdot \frac{3\sqrt{\frac{\pi}{2}}}{16 b^{5/2}} \right]$$

We need to also determine the normalization,

$$1 = \int_{-\infty}^{\infty} |\psi|^2 dx = \int_{-\infty}^{\infty} dx\, A^2 x^2 e^{-2bx^2} = A^2 \cdot \frac{\sqrt{\frac{\pi}{2}}}{4b^{3/2}}$$

$$\Rightarrow \quad \boxed{A = \left(\frac{2}{\pi}\right)^{1/4} \cdot 2 \cdot b^{3/4}}$$

$$\Rightarrow E(b) = \frac{4b^{3/2}}{\sqrt{\frac{\pi}{2}}}\left[\frac{3\hbar^2 b}{m}\cdot\sqrt{\frac{\pi}{2}}\,\frac{1}{4b^{3/2}} + \left(\frac{1}{2}m\omega^2 - \frac{2\hbar^2 b^2}{m}\right)\cdot\frac{3\sqrt{\frac{\pi}{2}}}{4b\cdot 4b^{3/2}}\right]$$

$$= \frac{3\hbar^2 b}{m} + \frac{3}{4b}\left(\frac{1}{2}m\omega^2 - \frac{2\hbar^2 b^2}{m}\right)$$

$$= \frac{3\hbar^2 b}{m} + \frac{3m\omega^2}{8b} - \frac{3}{2}\frac{\hbar^2 b}{m}$$

$$\boxed{E(b) = \frac{3}{2}\frac{\hbar^2 b}{m} + \frac{3m\omega^2}{8b}}$$

$$0 = \frac{\partial E}{\partial b} = \frac{3\hbar^2}{2m} - \frac{3m\omega^2}{8b_*^2} \quad \Rightarrow \quad b_*^2 = \frac{3m\omega^2}{8\cdot 4}\cdot\frac{2m}{3\hbar^2} = \left(\frac{m\omega}{2\hbar}\right)^2$$

$$\boxed{b_* = \frac{m\omega}{2\hbar}}$$

$$\Rightarrow E(b_*) = \frac{3\hbar^2}{2m}\cdot\frac{m\omega}{2\hbar} + \frac{3m\omega^2}{8}\cdot\frac{2\hbar}{4m\omega}$$

$$= \frac{3}{4}\hbar\omega + \frac{3}{4}\hbar\omega = \boxed{\frac{3}{2}\hbar\omega}$$ ← Get exactly the correct energy

5. (Griffiths 7.13)

First, normalize the trial wavefunction

$$1 = \int_0^\infty dr\cdot(4\pi r^2)\, A^2 e^{-2br^2}$$

$$1 = 4\pi A^2 \int_0^\infty dr\, r^2 e^{-2br^2} = 4\pi A^2 \cdot \frac{\sqrt{\frac{\pi}{2}}}{8b^{3/2}}$$

$$\Rightarrow \quad A^2 = \frac{\cancel{8}^{\,2} b^{3/2}}{\cancel{4}\pi}\cdot\sqrt{\frac{2}{\pi}} = \left(\frac{2b}{\pi}\right)^{3/2} \quad \Rightarrow \quad A = \left(\frac{2b}{\pi}\right)^{3/4}$$

Now calculate $E(b)$,

$$E(b) = \langle\psi|\hat{H}|\psi\rangle = \int_0^\infty dr\, r^2\cdot 4\pi \cdot A e^{-br^2}\left(-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(A r^2 \frac{\partial}{\partial r} e^{-br^2}\right) + \frac{e^2}{4\pi\varepsilon_0 r} e^{-br^2}\right)$$

$$= 4\pi A^2 \int_0^\infty dr\, r^2 e^{-br^2}\left[-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(-2br^3 e^{-br^2}\right) + \frac{e^2}{4\pi\varepsilon_0 r} e^{-br^2}\right]$$

$$= 4\pi\left(\frac{2b}{\pi}\right)^{3/2}\int_0^\infty dr\, r^2 e^{-br^2}\left[-\frac{\hbar^2}{2mr^2}\left(-6br^2 e^{-br^2} + 4b^2 r^4 e^{-br^2}\right) + \frac{e^2}{4\pi\varepsilon_0 r} e^{-br^2}\right]$$

$$= 4\pi\left(\frac{2b}{\pi}\right)^{3/2}\int_0^\infty dr\, e^{-2br^2}\left[-\frac{\hbar^2}{2m}\left(-6br^2 + 4b^2 r^4\right) + \frac{e^2}{4\pi\varepsilon_0} r\right]$$

$$= 4\pi\left(\frac{2b}{\pi}\right)^{3/2}\left[\frac{3\hbar^2 b}{m}\,\frac{\sqrt{\frac{\pi}{2}}}{8b^{3/2}} - \frac{2\hbar^2 b^2}{m}\cdot\frac{3\sqrt{\frac{\pi}{2}}}{32\, b^{5/2}} + \frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{4b}\right]$$

$$= \cancel{4\pi\left(\frac{2b}{\pi}\right)^{3/2}}$$

$$= 4\cancel{\pi}\cdot \frac{2}{\cancel{\pi}}\cdot b^{3/2}\cdot\left(\frac{3\hbar^2}{8m\sqrt{b}} - \frac{3\hbar^2}{16m\sqrt{b}}\right) - \frac{e^2}{\varepsilon_0}\sqrt{\frac{2b}{\pi}}\cdot\frac{\cancel{2b}}{\pi}\cdot\frac{1}{4\cancel{b}\,2}$$

$$= + 8b^{3/2}\cdot\frac{3\hbar^2}{\underset{2}{\cancel{16}}m\sqrt{b}} - \frac{e^2}{2\pi\varepsilon_0}\sqrt{\frac{2b}{\pi}}$$

$$E(b) = + \frac{3\hbar^2 b}{2m} - \frac{e^2}{2\pi\varepsilon_0}\sqrt{\frac{2b}{\pi}}$$

$$0 = \frac{\partial E}{\partial b} = + \frac{3\hbar^2}{2m} - \frac{e^2}{4\pi\varepsilon_0}\sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sqrt{b}}$$

$$\sqrt{b} = \frac{e^2}{4\pi\varepsilon_0}\sqrt{\frac{2}{\pi}}\cdot\frac{2m}{3\hbar^2} = \frac{me^2}{6\pi\varepsilon_0\hbar^2}\sqrt{\frac{2}{\pi}}$$

$$\Rightarrow E(b_{min}) =$$

$$= \sqrt{b}\cdot\left(+\frac{\cancel{3\hbar^2}}{2\cancel{m}}\cdot\frac{\cancel{m}e^2}{\underset{2}{\cancel{6}}\pi\varepsilon_0\cancel{\hbar^2}}\sqrt{\frac{2}{\pi}} - \frac{e^2}{2\pi\varepsilon_0}\sqrt{\frac{2}{\pi}}\right)$$

$$= \sqrt{b}\left(+\frac{e^2}{4\pi\varepsilon_0}\sqrt{\frac{2}{\pi}} - \frac{e^2}{2\pi\varepsilon_0}\sqrt{\frac{2}{\pi}}\right)$$

$$= \sqrt{b}\cdot\left(-\frac{e^2}{4\pi\varepsilon_0}\sqrt{\frac{2}{\pi}}\right) = \frac{me^2}{6\pi\varepsilon_0\hbar^2}\cdot\sqrt{\frac{2}{\pi}}\cdot\left(-\frac{e^2}{4\pi\varepsilon_0}\sqrt{\frac{2}{\pi}}\right)$$

$$= -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\cdot\frac{8}{3\pi}$$

$$\approx -11.5\,\mathrm{eV}$$

6. (Saxon 4)

Note that it only really makes sense to choose a trial ~~poly~~ wavefunction so that $\psi(0)=0$, $\psi(L)=0$ since we know the ground state has this property.

$\therefore$ Most general wavefunc that is a 2nd order polynomial:

$$\psi(x) = a + bx + cx^2$$

$$\psi(0) = 0 \Rightarrow \underline{a=0}$$

$$\psi(L) = 0 \Rightarrow bL + cL^2 = 0 \Rightarrow bL\left(1+\frac{cL}{b}\right) = 0$$

$$\Rightarrow b = -cL$$

$$\therefore \psi(x) = c\left(xL - x^2\right)$$
$$= c\,x(L-x)$$

Finally we can fix c by requiring that the solution is properly normalized.

$$1 = c^2\int_0^L x^2(L-x)^2dx = c^2\int_0^L dx\left[x^4 - 2Lx^3 + L^2x^2\right] = c^2\cdot\left(\frac{x^5}{5} - \frac{Lx^4}{2} + \frac{L^2x^3}{3}\right)\Big|_0^L = c^2\left(\frac{L^5}{5} - \frac{L^5}{2} + \frac{L^5}{3}\right)$$

$$= c^2\cdot\frac{6-15+10}{30}L^5$$

$$1 = c^2\cdot\frac{L^5}{30}$$

$$\Rightarrow c = \sqrt{\frac{30}{L^5}}$$

Now we calculate

$$\langle\psi|\hat{H}|\psi\rangle = \frac{30}{L^5}\int_0^L dx\, x(L-x)\left(-\frac{\hbar^2}{2m}\right)\frac{\partial^2}{\partial x^2}\left(x(L-x)\right)$$

$$= \frac{30}{L^5}\int_0^L dx\, x(L-x)\cdot\frac{\hbar^2}{2m}\cdot 2$$

$$= \frac{30\hbar^2}{L^5 m}\cdot\left(\frac{x^2L}{2} - \frac{x^3}{3}\right)\Big|_0^L = \frac{30\hbar^2}{L^5 m}\cdot\left(\frac{3L^3}{6} - \frac{2L^3}{6}\right) = \frac{5\hbar^2}{mL^2} = \frac{10\hbar^2}{2mL^2}$$

Compare w/ actual value, $\frac{\pi^2\hbar^2}{2mL^2}$ $\quad \pi^2 < 10$ but they're close, so the estimate is pretty good.

7. (Saxon 20)

Take $\psi = Ae^{-\alpha x^2}$

Need to carefully solve for A.

$$1 = A^2 \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} = A^2 \cdot \sqrt{\frac{\pi}{2\alpha}} \Rightarrow A = \left(\frac{2\alpha}{\pi}\right)^{1/4}$$

$$\therefore \quad \psi = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2}$$

Now we need to compute $\langle \psi | \hat{H} | \psi \rangle$

$$\langle \psi | \hat{H} | \psi \rangle = \sqrt{\frac{2\alpha}{\pi}} \int_{-\infty}^{\infty} dx \left(e^{-\alpha x^2} \cdot \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\left(e^{-\alpha x^2}\right) + e^{-2\alpha x^2} V \right)$$

$$= \sqrt{\frac{2\alpha}{\pi}} \int_{-\infty}^{\infty} dx \left[\left(e^{-\alpha x^2} \cdot \left(-\frac{\hbar^2}{2m}\right)\left(-2\alpha + 4\alpha^2 x^2\right) e^{-\alpha x^2} \right) + e^{-2\alpha x^2} V \right]$$

$$\underset{\langle \psi | \hat{H} | \psi \rangle}{\overset{E(\alpha)}{\shortparallel}} = \underbrace{\sqrt{\frac{2\alpha}{\pi}} \frac{\hbar^2}{m} \cdot \frac{1}{2} \cdot \sqrt{\frac{\alpha\pi}{2}}}_{=\alpha} + \sqrt{\frac{2\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-2\alpha x^2} V = \frac{\hbar^2 \alpha}{2m} + \sqrt{\frac{2\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-2\alpha x^2} V$$

$$\cancel{= \sqrt{\frac{2\alpha}{\pi}}}$$

Now we need to take the variation, and set it to zero

$$0 = \frac{\partial E}{\partial \alpha} = \frac{\hbar^2}{2m} + \frac{1}{2}\sqrt{\frac{2}{\pi \alpha_*}} \int_{-\infty}^{\infty} dx\, e^{-2\alpha_* x^2} V - 2\sqrt{\frac{2\alpha}{\pi}} \int_{-\infty}^{\infty} dx\, e^{-2\alpha_* x^2} x^2 V(x)$$

$$0 = \frac{\hbar^2}{2m} + \sqrt{\frac{2\alpha_*}{\pi}} \int_{-\infty}^{\infty} e^{-2\alpha_* x^2} V \cdot \left(\frac{1}{2\alpha_*} - 2x^2\right)$$

$$0 = \frac{\hbar^2 \alpha_*}{2m} + \frac{1}{2}\sqrt{\frac{2\alpha_*}{\pi}} \int_{-\infty}^{\infty} dx\, e^{-2\alpha_* x^2} V - 2\sqrt{\frac{2}{\pi}}\, \alpha_*^{3/2} \int_{-\infty}^{\infty} dx\, e^{-2\alpha_* x^2} x^2 V$$

Now we can plug in...

$$\Rightarrow \quad E(\alpha_*) = \underbrace{\frac{1}{2}\sqrt{\frac{2\alpha}{\pi}} \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} V + 2\sqrt{\frac{2}{\pi}}\, \alpha^{3/2} \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} x^2 V}$$

Note, both term are negative since $V(x) < 0 \quad \forall x$ (using the fact that the potential is attractive)

$\Rightarrow \quad E(\alpha_*) < 0 \quad \Rightarrow$ A bound state exists!

(since $E_{ground} < E(\alpha_*)$)