and,

$$\int dx_1 dx_2 dx_3 x_1 x_2 = \frac{1}{4}(a_1 a_2 a_3) a_1 a_2, \tag{23}$$

by cyclic symmetry, we can write Eq. 20 directly as

$$\mathbf{I} = M \begin{pmatrix} \frac{1}{3}\left(a_2^2 + a_3^2\right) & -\frac{1}{4}a_1 a_2 & -\frac{1}{4}a_1 a_3 \\ -\frac{1}{4}a_2 a_1 & \frac{1}{3}\left(a_3^2 + a_1^2\right) & -\frac{1}{4}a_2 a_3 \\ -\frac{1}{4}a_3 a_1 & -\frac{1}{4}a_3 a_2 & \frac{1}{3}\left(a_1^2 + a_2^2\right) \end{pmatrix} = \frac{M}{12} \begin{pmatrix} 4\left(a_2^2 + a_3^2\right) & -3a_1 a_2 & -3a_1 a_3 \\ -3a_2 a_1 & 4\left(a_1^2 + a_3^2\right) & -3a_2 a_3 \\ -3a_3 a_1 & -\frac{1}{4}a_3 a_2 & 4\left(a_1^2 + a_2^2\right) \end{pmatrix}. \tag{24}$$

If $a_1 = a_2 = a_3 = a$, we shall recover the result in [Taylor] Example 10.4,

$$\mathbf{I} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}, \tag{25}$$

which is the same as [Taylor] (10.49). Now substitute $a_1 = a_2 = \frac{1}{2}a_3 = a$, we obtain

$$\mathbf{I} = \frac{Ma^2}{12} \begin{pmatrix} 20 & -3 & -6 \\ -3 & 20 & -6 \\ -6 & -6 & 8 \end{pmatrix}. \tag{26}$$

We shall diagonal the matrix to find the principal axes. From the eigenfunction,

$$\mathbf{I\Omega} = \lambda \mathbf{\Omega}, \tag{27}$$

we can solve for the eigenvalues and eigenvectors,

$$\begin{aligned} \lambda_1 &= \tfrac{1}{24}(25 + 3\sqrt{41})Ma^2, \quad \tilde{\mathbf{\Omega}}_1 = \tfrac{\Omega}{2\sqrt{41+3\sqrt{41}}}\left(-3 - \sqrt{41}, -3 - \sqrt{41}, 8\right); \\ \lambda_2 &= \tfrac{1}{24}(25 - 3\sqrt{41})Ma^2, \quad \tilde{\mathbf{\Omega}}_2 = \tfrac{\Omega}{2\sqrt{41-3\sqrt{41}}}\left(-3 + \sqrt{41}, -3 + \sqrt{41}, 8\right); \\ \lambda_3 &= \tfrac{23}{12}Ma^2, \qquad\qquad\qquad \tilde{\mathbf{\Omega}}_3 = \tfrac{\Omega}{\sqrt{2}}\left(-1, 1, 0\right), \end{aligned} \tag{28}$$

with Ω_i representing the principal axes and λ_i the corresponding moments. Here, $i = 1, 2, 3$.

III. TAYLOR 10.42: A SPINNING BOOK

Eq. 24 gives the inertia tensor of an arbitrary cuboid rotating about a corner. In this problem, the book spins freely in the air and thus is a rigid body without any fixed point. In the center of mass reference frame, the rotational motion is about the CM. We shall first derive the general formula for transforming from the inertia tenor about an arbitrary point on the rigid body to the one about the CM, i. e., the so called "Parallel Axis Theorem".

Denote I_{ij} the inertia tensor with respective to the CM of the rigid body. We choose the coordinate system such that the CM is the origin. Due to the same derivations in Section I, we have

$$I_{ij} = \int dm \left(r^2 \delta_{ij} - r_i r_j\right), \tag{29}$$

as the continuous version of Eq. 13. (Refer to Eq. 20 as well.) Here $\vec{r}$ denotes the integrate variable.

Considering an axis through an arbitrary point $\vec{R}$ on the rigid body. The inertial tensor is then

$$\begin{aligned} J_{ij} &= \int dm \left[(\vec{r} - \vec{R})^2 \delta_{ij} - (r_i - R_i)(r_j - R_j)\right] \\ &= \int dm \left[\left(r^2 + R^2 - 2\vec{r}\cdot\vec{R}\right)\delta_{ij} - r_i r_j - R_i R_j + (R_i r_i + R_j r_i)\right] \\ &= \int dm \left(r^2 \delta_{ij} - r_i r_j\right) + \left(R^2 \delta_{ij} - R_i R_j\right)\int dm - 2\delta_{ij}\vec{R}\cdot\int dm\vec{r} + R_i \int dm r_j + R_j \int dm r_i. \end{aligned} \tag{30}$$

Note that, with the origin of the coordinate system the CM,

$$\int dm = M, \quad \int dm r_i = 0. \tag{31}$$

Therefore,

$$J_{ij} = I_{ij} + M\left(R^2\delta_{ij} - R_iR_j\right). \tag{32}$$

In the present problem, we have

$$\mathbf{J} = M\begin{pmatrix} \frac{1}{3}\left(a_2^2 + a_3^2\right) & -\frac{1}{4}a_1a_2 & -\frac{1}{4}a_1a_3 \\ -\frac{1}{4}a_2a_1 & \frac{1}{3}\left(a_3^2 + a_1^2\right) & -\frac{1}{4}a_2a_3 \\ -\frac{1}{4}a_3a_1 & -\frac{1}{4}a_3a_2 & \frac{1}{3}\left(a_1^2 + a_2^2\right) \end{pmatrix}, \tag{33}$$

and

$$\vec{R} = -\frac{1}{2}\left(a_1, a_2, a_3\right). \tag{34}$$

Substitute Eq. 33 & 34 into Eq. 32, we can solve for $\mathbf{I}$,

$$\mathbf{I} = \frac{M}{12}\begin{pmatrix} a_2^2 + a_3^2 & 0 & 0 \\ 0 & a_3^2 + a_1^2 & 0 \\ 0 & 0 & a_1^2 + a_2^2 \end{pmatrix}. \tag{35}$$

The above result can be obtained much more straightforwardly if we calculate the integral over the cuboid directly: the moment of inertia around a principal axis, say, along a_1 direction, through the CM of the cuboid is equivalent to the moment of inertia of a uniform rectangular plate of height a_2 and width a_3:

$$\frac{1}{12}M(a_2^2 + a_3^2), \tag{36}$$

which can be computed directly from the moments of inertia of uniform rods of length a_2 and a_3 by the Perpendicular Axis Theorem. A schematic description of such Theorem is given as following: Treating the cuboid equivalently as a flat plane, while computing its moment of inertia I around the axis through its center and normal to one of its face (then the slices of planes perpendicular to the axis all give the same moment of inertia and thus can be treated equivalently as compressed into a single plane), we have

$$I = \int dm(x^2 + y^2) = \int dm x^2 + \int dm y^2 = I_x + I_y. \tag{37}$$

with the present context we have $I_x = \frac{1}{12}a_2^2$ and $I_y = \frac{1}{2}a_3^2$, which result in an I the same as given in Eq. 36.

We take $a_1 = 30$ cm, $a_2 = 20$ cm, and $a_3 = 3$ cm. If the spinning axis is close to the book's shortest symmetry axis, i. e., the direction along a_3. Meanwhile, the Euler's equations with zero torque reads ([Taylor] (10.89)),

$$\begin{aligned} \lambda_1\dot{\omega}_1 &= (\lambda_2 - \lambda_3)\omega_2\omega_3, \\ \lambda_2\dot{\omega}_2 &= (\lambda_3 - \lambda_1)\omega_3\omega_1, \\ \lambda_3\dot{\omega}_3 &= (\lambda_1 - \lambda_2)\omega_1\omega_2. \end{aligned} \tag{38}$$

Since the spinning axis is close to the third direction, we have ω_1, $\omega_2 \ll \omega_3$. By [Taylor] (10.91), we have,

$$\Omega_3^2 = \frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1\lambda_2}\omega_3^2 = \frac{(a_2^2 - a_3^2)(a_1^2 - a_3^2)}{(a_2^2 + a_3^2)(a_3^2 + a_1^2)}\omega_3^2 = 0.937\omega_3^2. \tag{39}$$

Therefore,

$$\Omega_3 = 0.968\omega = 174 \text{ rpm}. \tag{40}$$

If the book spins about the longest axis, i. e., along a_1, we have

$$\Omega_1^2 = \frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{\lambda_3\lambda_2}\omega_1^2 = \frac{(a_2^2-a_1^2)(a_3^2-a_1^2)}{(a_2^2+a_1^2)(a_1^2+a_3^2)}\omega_1^2 = 0.377\omega_1^2, \tag{41}$$

i. e.,

$$\Omega_1 = 0.614\omega_1 = 111 \text{ rpm}. \tag{42}$$

IV. TAYLOR 10.44:

The Euler's equations given in [Taylor] (10.88) now reads

$$\begin{aligned}
&\lambda\dot{\omega}_1 - (\lambda-\lambda_3)\omega_2\omega_3 = 0,\\
&\lambda\dot{\omega}_2 - (\lambda_3-\lambda)\omega_3\omega_1 = 0,\\
&\lambda_3\dot{\omega}_3 = \Gamma.
\end{aligned} \tag{43}$$

We have taken $\lambda = \lambda_1 = \lambda_2$, and $\vec{\Gamma} = (0,0,\Gamma)$ constant. The third equation can be then readily solved,

$$\omega_3 = \omega_{30} + \frac{\Gamma}{\lambda_3}t, \tag{44}$$

with the initial condition $\omega(t=0) = \omega_{30}$ taken into consideration.

Denote $w = \omega_1 + i\omega_2$, from 43 we have

$$\dot{w} = -i\frac{\lambda-\lambda_3}{\lambda}\omega_3 w. \tag{45}$$

Separating the variables, we obtain

$$\frac{dw}{w} = -i\frac{\lambda-\lambda_3}{\lambda}\left(\omega_{30} + \frac{\Gamma}{\lambda_3}t\right)dt. \tag{46}$$

Integrate both sides of the differential equation, it yields

$$w = w_0 \exp\left[-i\frac{\lambda-\lambda_3}{\lambda}\left(\omega_{30}t + \frac{\Gamma}{2\lambda_3}t^2\right)\right], \tag{47}$$

with w_0 a complex constant. At $t=0$, the initial condition $\vec{\omega} = (\omega_{10}, 0, \omega_{30})$ requires, $w(t=0) = w_{10}$, i. e., $w_0 = \omega_{10}$.

Collecting the results we obtained above, the exact solutions for $\vec{\omega}$ are

$$\vec{\omega} = (\omega_{10}\cos(\Omega_b t), \omega_{10}\sin(\Omega_b t), \omega_3)\,. \tag{48}$$

with

$$\Omega_b = \frac{\lambda_3-\lambda}{\lambda}\left(\omega_{30} + \frac{\Gamma}{2\lambda_3}t\right), \tag{49}$$

and ω_3 given by Eq. 44. It is evident that the rotational axis $\vec{e}_3$ is preserved while the rotation along $\vec{e}_3$ accelerates via time. The total moment in this motion rotates around $\vec{e}_3$ and gets closer and closer to $\vec{e}_3$ while elongating itself at the meantime. By comparing with [Taylor] (10.94), we know that the motion is actually a non-uniform precession.

V. TAYLOR: 10.47

We set up the coordinate system such that the origin coincides with the center of the earth, and z-axis passing through the mountain. Assume that the mass of the mountain is m, the mass of the earth is M, and the radius of the earth is R. The inertia tensor is then

$$\mathbf{I} = m\begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda\mathcal{I}, \tag{50}$$

Solution using Euler's equations (10.88)

Define $\mathcal{O}$ as shown in diagram.

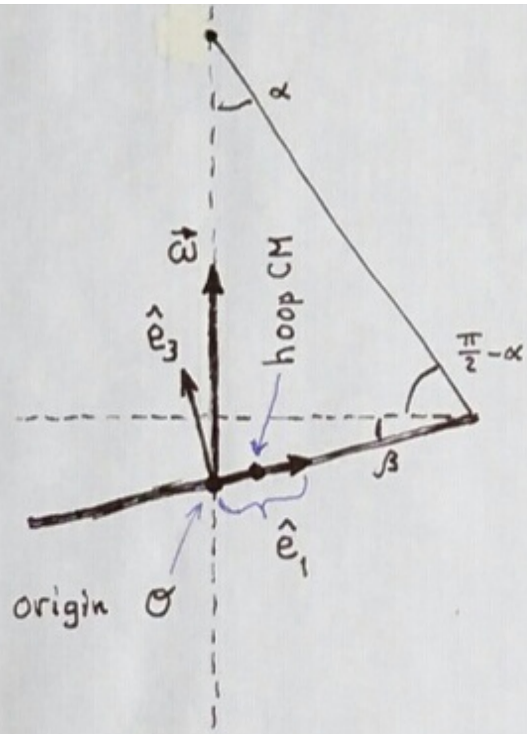
Define body frame as shown.

$\hat{e}_2$ points into page.

$\vec{\omega}$ = const in space frame.

$\vec{\omega} \cdot \hat{e}_i$ = const in space frame.

Thus $\vec{\omega}$ = const in body frame.



<u>In body frame</u>

$$\vec{\omega} = \omega(\sin\beta, 0, \cos\beta)$$

$$\lambda_1 = \frac{1}{2} M R^2$$

R_c is calculated in the box at left

R_c = radius of circle traced out by CM

T = tension in string

Look at vertical forces: $T\cos\alpha = Mg$

Look at horizontal force: $T \sin\alpha = M\omega^2 R_c$

Thus $R_c = g\tan(\alpha)/\omega^2$

distance from $\mathcal{O}$ to CM $= R_c / \cos\beta$

$$\lambda_3 = MR^2 + M(R_c/\cos\beta)^2$$

$$\Gamma_2 = R_c M g - (R + R_c/\cos\beta)\, T\, \overbrace{\sin\left(\tfrac{\pi}{2} - \alpha + \beta\right)}^{\cos(\alpha-\beta)}$$

Now plug into 2^{nd} equation in (10.88)

$$-\left(\tfrac{1}{2}MR^2 + M(R_c/\cos\beta)^2\right)\omega^2 \sin\beta\cos\beta = \Gamma_2$$

Everything has been exact so far. Now begin approximations.

Using $R_c \ll R$ we get

$$-\frac{1}{4} MR^2\omega^2 \sin(2\beta) = -R\frac{Mg}{\cos\alpha}\cos(\alpha-\beta)$$

$$R\omega^2\sin(2\beta) = 4g\frac{\cos(\alpha-\beta)}{\cos\alpha}$$

Using $\beta \ll 1$ we get

$$\beta \approx \frac{2g}{\omega^2 R}$$

Note very tiny (yet very strong) people standing on the inside edge of the hoop would be leaning with angle $\beta/2$ to the horizontal.

(a) If $m_\alpha = m_\beta$ and $\vec{r}_{(\alpha)} = -\vec{r}_{(\beta)}$ then the contributions to $\overleftrightarrow{I}$ are the same.

$$\overleftrightarrow{I} = 2m\begin{pmatrix} a^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a^2 \end{pmatrix} + 4m\begin{pmatrix} 4a^2 & 0 & 0 \\ 0 & 4a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 6m\begin{pmatrix} a^2 & 0 & -a^2 \\ 0 & 2a^2 & 0 \\ -a^2 & 0 & a^2 \end{pmatrix}$$

$$= 2ma^2\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 & -3 \\ 0 & 6 & 0 \\ -3 & 0 & 3 \end{pmatrix}\right]$$

$$= 2ma^2\begin{pmatrix} 12 & 0 & -3 \\ 0 & 14 & 0 \\ -3 & 0 & 4 \end{pmatrix}$$

(b) Define $\beta = 2ma^2$

$$\det\left(\overleftrightarrow{I} - \lambda\overleftrightarrow{1}\right) = \det\begin{pmatrix} 12\beta-\lambda & 0 & -3\beta \\ 0 & 14\beta-\lambda & 0 \\ -3\beta & 0 & 4\beta-\lambda \end{pmatrix}$$

$$= (12\beta-\lambda)(14\beta-\lambda)(4\beta-\lambda) + 0 + 0 - 9\beta^2(14\beta-\lambda) - 0 - 0$$

$$= (14\beta-\lambda)\left[(12\beta-\lambda)(4\beta-\lambda) - 9\beta^2\right]$$

$$= (14\beta-\lambda)\left[39\beta^2 - 16\beta\lambda + \lambda^2\right]$$

$$= (14\beta-\lambda)(13\beta-\lambda)(3\beta-\lambda)$$

Eigenvalues	$14\beta = 28ma^2$	$3\beta = 6ma^2$	$13\beta = 26ma^2$
A choice of eigenvectors	$(0,1,0)$	$\frac{1}{\sqrt{10}}(1,0,3)$	$\frac{1}{\sqrt{10}}(-3,0,1)$

each of these could be multiplied by -1

Students do not have to normalize the eigenvectors

We borrow some notation from the finance world

"H1" = first half = time interval $0 \le t \le \tau_0$

"H2" = second half = time interval $\tau_0 \le t \le 2\tau_0$

(a) <u>During H1</u> we have rotation about the z-axis.

The angle of rotation is

$$\alpha_{H1}(t) = \int_0^t \frac{3\omega_0}{2\tau_0^2}\, t'(\tau_0 - t')\, dt'$$

$$= \frac{3\omega_0}{2\tau_0^2}\left(\frac{1}{2}\tau_0 t^2 - \frac{1}{3}t^3\right) \qquad \left[\text{Note } \alpha_{H1}(\tau_0) = \frac{\pi}{2}\right]$$

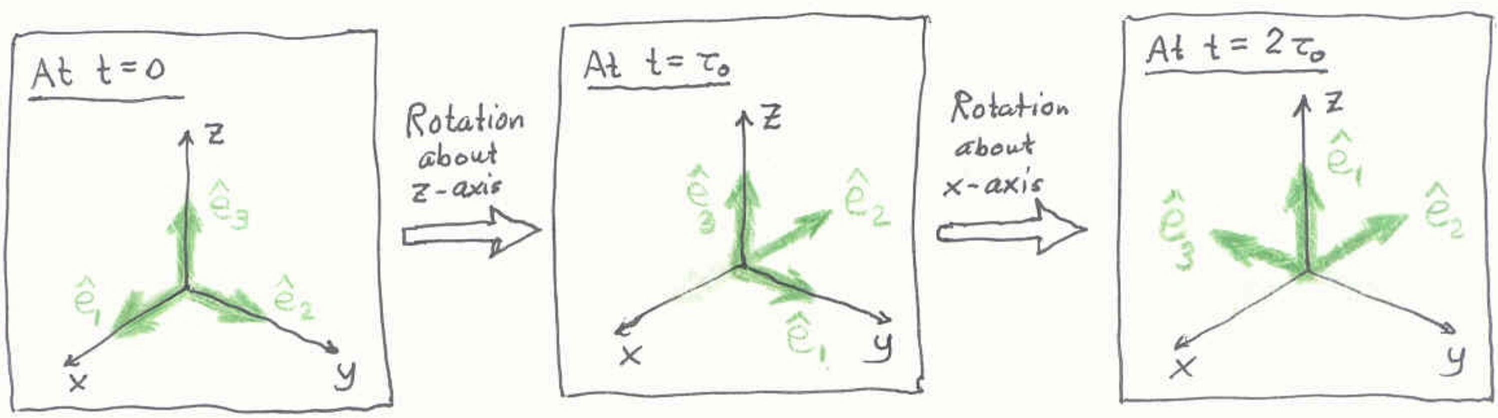
<u>During H2</u> we have rotation about the x-axis.

The angle of rotation is

$$\alpha_{H2}(t) = \int_{\tau_0}^t \frac{3\omega_0}{2\tau_0^2}(t'-\tau_0)(2\tau_0 - t')\, dt'$$

$$= \int_0^{t-\tau_0} \frac{3\omega_0}{2\tau_0^2}\, t''(\tau_0 - t'')\, dt'', \quad \text{where } t'' = t' - \tau_0$$

$$= \alpha_{H1}(t - \tau_0) \qquad \left[\text{Note } \alpha_{H2}(2\tau_0) = \frac{\pi}{2}\right]$$



$$\hat{e}_1(t) = \begin{cases} \cos(\alpha_{H1}(t))\,\hat{x} + \sin(\alpha_{H1}(t))\,\hat{y}, & 0 \le t \le \tau_0 \\ \cos(\alpha_{H2}(t))\,\hat{y} + \sin(\alpha_{H2}(t))\,\hat{z}, & \tau_0 \le t \le 2\tau_0 \end{cases}$$

(b)

$$\hat{e}_2(t) = \begin{cases} \cos(\alpha_{H1}(t))\,\hat{y} - \sin(\alpha_{H1}(t))\,\hat{x}, & 0 \le t \le \tau_0 \\ -\hat{x}, & \tau_0 \le t \le 2\tau_0 \end{cases}$$

$$\hat{e}_3(t) = \begin{cases} \hat{z}, & 0 \le t \le \tau_0 \\ \cos(\alpha_{H2}(t))\,\hat{z} - \sin(\alpha_{H2}(t))\,\hat{y}, & \tau_0 \le t \le 2\tau_0 \end{cases}$$

(c) Define scalar functions

$$\omega_{H1}(t) = \frac{3\omega_0}{2\tau_0^2}\, t(\tau_0 - t), \qquad 0 \le t \le \tau_0$$

$$\omega_{H2}(t) = \frac{3\omega_0}{2\tau_0^2}(t-\tau_0)(2\tau_0 - t), \qquad \tau_0 \le t \le 2\tau_0$$

$$= \omega_{H1}(t-\tau_0)$$

During H1, $\vec{\omega}(t) = \omega_{H1}(t)\,\hat{z} = \omega_{H1}(t)\,\hat{e}_3(t)$

During H2, $\vec{\omega}(t) = \omega_{H2}(t)\,\hat{x} = -\omega_{H2}(t)\,\hat{e}_2(t)$

$$\boxed{\omega_1(t) = 0} \qquad \boxed{\omega_2(t) = \begin{cases} 0 & \text{in } H1 \\ -\omega_{H2}(t) & \text{in } H2 \end{cases}} \qquad \boxed{\omega_3(t) = \begin{cases} \omega_{H1}(t) & \text{in } H1 \\ 0 & \text{in } H2 \end{cases}}$$

(d) $\boxed{\Gamma_1 = 0}$, $\Gamma_2 = \lambda_2 \dot{\omega}_2$, $\Gamma_3 = \lambda_3 \dot{\omega}_3$. $\dot{\omega}_{H1} = \frac{3\omega_0}{2\tau_0^2}(\tau_0 - 2t)$, $\dot{\omega}_{H2}(t) = \dot{\omega}_{H1}(t-\tau_0)$

$$\boxed{\Gamma_2(t) = \begin{cases} 0 & \text{in } H1 \\ -\lambda_2 \dot{\omega}_{H2}(t) & \text{in } H2 \end{cases}} \qquad \boxed{\Gamma_3(t) = \begin{cases} \lambda_3 \dot{\omega}_{H1}(t) & \text{in } H1 \\ 0 & \text{in } H2 \end{cases}}$$

(e) $\vec{\Gamma} = \lambda_3 \dot{\omega}_{H1}(t)\,\hat{e}_3 = \lambda_3 \dot{\omega}_{H1}\,\hat{z}$ <u>in H1</u>

$\vec{\Gamma} = -\lambda_2 \dot{\omega}_{H2}(t)\,\hat{e}_2 = \lambda_2 \dot{\omega}_{H2}\,\hat{x}$ <u>in H2</u>

Agrees with $\vec{\Gamma} = \frac{d\vec{L}}{dt}$

Yes, we have + signs in both cases

Note λ_2 goes with $\hat{x}$ in H2