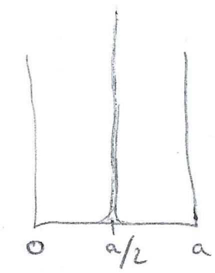
Homework 3 SOLUTIONS

1) Griffiths 6.1



$$H = H_0 + H' = \frac{p^2}{2m} + \alpha\,\delta\!\left(x - \frac{a}{2}\right)$$

Eigenstates of H_0 are $\psi_n^0(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

Eigenvalues of H_0 are $E_n^0 = \frac{n^2\pi^2\hbar^2}{2Ma^2}$

To first order,

$$\Delta E_n = \langle \psi_n^0 | H' | \psi_n^0 \rangle = \int_0^a dx\, \frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right)\alpha\,\delta\!\left(x - \frac{a}{2}\right)$$

$$= \frac{2}{a}\alpha \sin^2\left(\frac{n\pi \cancel{a}}{\cancel{a}\cdot 2}\right)$$

$$= \frac{2\alpha}{a}\sin^2\left(\frac{n\pi}{2}\right) = \boxed{\begin{cases} \frac{2\alpha}{a} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}}$$

And

$$\psi_1^1 = \sum_{m\neq 1} \frac{\langle \psi_m^0 | H' | \psi_1^0 \rangle}{E_1^0 - E_m^0}\psi_m^0$$

$$= \sum_{m>1} \frac{\int d\cancel{x}\, \sqrt{\frac{2}{a}}\sin\left(\frac{m\pi \cancel{x}}{a}\right)\alpha\,\delta\!\left(\cancel{x} - \frac{a}{2}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{\pi \cancel{x}}{a}\right)}{\frac{\pi^2\hbar^2}{2Ma^2}\left(1 - m^2\right)}\sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right)$$

$$= \frac{2Ma^2}{\pi^2\hbar^2}\frac{2\alpha}{a}\sum_{m>1}\frac{\sin\left(\frac{m\pi}{2}\right)\cancel{\sin\left(\frac{\pi}{2}\right)}^{\,1}}{1-m^2}\sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right)$$

$$= \boxed{\frac{4Ma\alpha}{\pi^2\hbar^2}\sum_{\substack{m \text{ odd} \\ m>1}}\frac{(-1)^{\frac{m-1}{2}}}{1-m^2}\sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right)}$$

$$\sin\left(\frac{m\pi}{2}\right) = \begin{cases} 0 & \text{if } m \text{ even} \\ 1 & \text{if } m = 1,5,9,\ldots \\ -1 & \text{if } m = 3,7,11,\ldots \end{cases} \;=\; \begin{cases} 0 & \text{if } m \text{ even} \\ (-1)^{\frac{m-1}{2}} & \text{if } m \text{ odd} \end{cases}$$

$$= \frac{4Ma\alpha}{\pi^2\hbar^2}\sqrt{\frac{2}{a}}\left(\frac{1}{8}\sin\left(\frac{3\pi x}{a}\right) - \frac{1}{24}\sin\left(\frac{5\pi x}{a}\right) + \frac{1}{48}\sin\left(\frac{7\pi x}{a}\right) - \cdots\right)$$

2. (Griffiths 6.2)

a) $E_n' = (n+\frac{1}{2})\hbar\omega' = (n+\frac{1}{2})\hbar\omega\sqrt{1+\epsilon}$

$= (n+\frac{1}{2})\hbar\omega\left(1+\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2\right) \Rightarrow \cancel{\Delta E = \frac{\epsilon\hbar\omega}{2}(n+\frac{1}{2})}$

$\Delta E = \hbar\omega(n+\frac{1}{2})\left(\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2\right)$

(b) $H_0 = \frac{p^2}{2m} + \frac{1}{2}kx^2$

$H_{new} = \frac{p^2}{2m} + \frac{1}{2}k(1+\epsilon)x^2$

$H_{new} = H_0 + H' \Rightarrow H' = \frac{1}{2}k\epsilon x^2$

$\Delta E_n = \langle n|H'|n\rangle = \frac{1}{2}k\epsilon\langle n|x^2|n\rangle \leftarrow$ Can now use $x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$

$= \frac{k\epsilon}{2}\cdot\frac{\hbar}{2m\omega}\langle n|(a^\dagger+a)^2|n\rangle$

$= \frac{k\epsilon\hbar}{4m\omega}\langle n|a^\dagger a + aa^\dagger|n\rangle \leftarrow$ using $\langle n|(a^\dagger)^2|n\rangle = 0$, $\langle n|a^2|n\rangle = 0$

$= \frac{k\epsilon\hbar}{4m\omega}\langle n|n\rangle\left(\sqrt{n}\cdot\sqrt{n} + \sqrt{n+1}\sqrt{n+1}\right)$

$= \frac{k\hbar\epsilon}{4m\omega}\cdot(2n+1) = \frac{\hbar\omega\epsilon}{2}(n+\frac{1}{2})$

$\uparrow$ using $\frac{k}{m} = \omega^2$

Agrees w/ above to first order in ϵ

3. (Griffiths 6.3)

a) Assuming no interactions for now,

For one particle, energy eigenstates are $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ w/ energies $E_n = \frac{\pi^2 n^2\hbar^2}{2ma^2}$

For two bosons, the ground state is,

$\psi(x_1,x_2) = \frac{1}{2}\left(\psi_1(x_1)\psi_1(x_2) + \psi_1(x_2)\psi_1(x_1)\right) = \psi_1(x_1)\psi_1(x_2)$

$= \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$

$E_{ground} = \frac{\pi^2\hbar^2}{2ma^2}\cdot 2 = \frac{\pi^2\hbar^2}{ma^2}$

First excited state,

$\psi(x_1,x_2) = \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1)\right)$

$= \frac{\sqrt{2}}{a}\left(\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{2\pi x_1}{a}\right)\right)$

$E_{excited} = \frac{\pi^2\hbar^2}{2ma^2} + \frac{4\pi^2\hbar^2}{2ma^2} = \frac{5\pi^2\hbar^2}{2ma^2}$

b) Now we turn on interaction and look @ how the energies and eigenstates change,

$$\Delta E_{ground} = \langle \Psi_{ground} | -aV_0\,\delta(x_1-x_2) | \Psi_{ground} \rangle$$

$$= \int_0^a dx_1 \int_0^a dx_2 \left(-\cancel{a}V_0\,\delta(x_1-x_2)\right) \cdot \frac{2}{\cancel{a}} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) \cdot \frac{2}{a} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right)$$

$$= -2V_0 \frac{2}{a} \int_0^a dx_1 \sin^4\left(\frac{\pi x_1}{a}\right) \cancel{-2V_0 \cdot \frac{a}{2}} = -\frac{4V_0}{\cancel{a}} \cdot \frac{3\cancel{a}}{8} = -\frac{3V_0}{2}$$

$$\Delta E_{excited} = \langle \Psi_{excited} | -aV_0\,\delta(x_1-x_2) | \Psi_{excited} \rangle$$

$$= \cancel{\int_0^a dx_1 \int_0^a dx_2 \left(-aV_0\,\delta(x_1-x_2)\right) \frac{2}{a} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{a}\right) \cdot \frac{2}{a} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{2\pi}{}\right.}$$

$$= \int_0^a dx_1 \int_0^a dx_2 \left(-aV_0\,\delta(x_1-x_2)\right) \left(\frac{\sqrt{2}}{a}\right)^2 \left(\sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{\pi x_2}{a}\right) \sin\left(\frac{2\pi x_1}{a}\right)\right)^2$$

$$= (-aV_0)\left(\frac{2}{a^2}\right) \int_0^a dx_1 \cdot 4 \cdot \sin^2\left(\frac{\pi x_1}{a}\right) \sin^2\left(\frac{2\pi x_1}{a}\right)$$

$$= (-\cancel{a}V_0)\left(\frac{2}{\cancel{a^2}}\right) \cdot \cancel{4} \cdot \frac{\cancel{a}}{\cancel{4}} = -2V_0$$

4. (Griffiths 6.4)

a) $H' = \alpha\,\delta\left(x - \frac{a}{2}\right)$

$$\langle \psi_m^0 | H' | \psi_n^0 \rangle = \int dx \cdot \frac{2}{a} \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n x}{a}\right) \cdot \alpha\,\delta\left(x - \frac{a}{2}\right)$$

$$= \frac{2\alpha}{a} \sin\left(\frac{\pi m}{2}\right) \sin\left(\frac{\pi n}{2}\right)$$

For n even $\Rightarrow \langle \psi_m^0 | H' | \psi_n^0 \rangle = 0$

So the energy levels to second order are NOT perturbed for n even!

$$E_n^2 = 0 \text{ for } n \text{ even}$$

For n odd, $\langle \psi_m^0 | H' | \psi_n^0 \rangle = \frac{2}{a}(-1)^{\frac{n-1}{2}}\alpha \cdot \begin{cases} 0 & \text{if } m \text{ even} \\ (-1)^{\frac{m-1}{2}} & \text{if } m \text{ odd} \end{cases}$

For n odd

$$\Rightarrow E_n^{(2)} = \sum_{m\neq n} \frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0}$$

$$= \sum_{\substack{m\neq n\\ m \text{ odd}}} \frac{\left|(-1)^{\frac{m-1}{2}}(-1)^{\frac{n-1}{2}}\alpha\frac{2}{a}\right|^2}{\frac{\pi^2\hbar^2}{2ma^2}(n^2-m^2)}$$

$$= \frac{2m\cancel{a^2}}{\pi^2\hbar^2}\cdot\frac{4\alpha^2}{\cancel{a^2}}\cdot\underbrace{\sum_{\substack{m\neq n\\ m\text{odd}}}\frac{1}{n^2-m^2}}$$

$$= \sum_{\substack{m\neq n\\ m\text{odd}}}\frac{1}{(m+n)(n-m)}$$

let $m=2j+1$, $n=2k+1$

$$= \sum_{\substack{j\neq k\\ j=0,..\infty\\ k=0,..\infty}}\frac{1}{(2j+2k+2)(2)(k-j)} = \frac{1}{4}\sum_{j\neq k}\frac{1}{(j+k+1)(k-j)} = -\frac{1}{4}\sum_{\substack{a\neq 0\\ a=-k}}^{\infty}\left(\frac{1}{(a+2k+1)(a)}\right)$$

$a=j-k$

Using partial fractions

$$= -\frac{1}{4}\sum_{\substack{a\neq 0\\ a=-k}}^{\infty}\left(\frac{1}{a}-\frac{1}{2k+1+a}\right)\cdot\frac{1}{2k+1}$$

$$= -\frac{1}{4(2k+1)}\sum_{\substack{a\neq 0\\ a=-k}}^{\infty}\frac{1}{a} + \frac{1}{4(2k+1)}\sum_{\substack{b\neq 2k+1,\\ b=k+1}}^{\infty}\frac{1}{b}$$

let $b=a+2k+1$

$$= -\frac{1}{4(2k+1)}\Bigg(\underbrace{\sum_{\substack{a\neq 0\\ a=-k}}^{k}\frac{1}{a} + \cancel{\sum_{a=k+1}^{2k}\frac{1}{a}}}_{\text{Breaking up the sum}} + \frac{1}{2k+1} + \sum_{a=2k+2}^{\infty}\frac{1}{a} - \cancel{\sum_{b=k+1}^{2k}\frac{1}{b}} - \sum_{b=2k+2}^{\infty}\frac{1}{b}$$

These pairs cancel

$$= -\frac{1}{4(2k+1)}\underbrace{\left(\sum_{\substack{a\neq 0\\ a=-k}}^{k}\frac{1}{a}\right)} - \frac{1}{4(2k+1)^2} = -\frac{1}{4(2k+1)^2} = \underline{-\frac{1}{4n^2}}$$

$$= \frac{1}{-k}+\frac{1}{(-k+1)}+\dots+\frac{1}{-1}+\frac{1}{1}+\dots+\frac{1}{k-1}+\frac{1}{k}$$

$$= 0$$

$$\therefore \; E_n^{(2)} = \frac{8m\alpha^2}{\pi^2\hbar^2}\cdot\left(-\frac{1}{4n^2}\right)$$

$$= -\frac{2m\alpha^2}{\pi^2\hbar^2 n^2} \quad \text{for odd } n$$

$$E_n^{(2)} = 0 \quad \text{for even}$$

(b) Checking 2nd order correction for problem 6.2

Expect: $E_n^{(2)} = -\frac{\hbar\omega}{8}(n+\tfrac{1}{2})$

$$\langle m|H'|n\rangle = \tfrac{1}{2}k\langle m|x^2|n\rangle$$

$$= \tfrac{1}{2}k\langle m|(a^\dagger+a)^2|n\rangle\cdot\frac{\hbar}{2m\omega}$$

$$= \tfrac{1}{2}k\langle m|\Big(\sqrt{(n+1)(n+2)}\,|n+2\rangle + (2n+1)|n\rangle + \sqrt{n(n-1)}\,|n-2\rangle\Big)\cdot\frac{\hbar}{2m\omega}$$

$$= \frac{\hbar}{2m\omega}\cdot\tfrac{1}{2}k\cdot\begin{cases}\sqrt{(n+1)(n+2)} & \text{if } m=n+2\\ 2n+1 & \text{if } m=n\\ \sqrt{n(n-1)} & \text{if } m=n-2\\ 0 & \text{otherwise}\end{cases}$$

$$\therefore\; E_n^{(2)} = \sum_{m\neq n}\frac{|\langle m|H'|n\rangle|^2}{E_n^{(0)}-E_m^{(0)}}$$

$$= \frac{\left|\sqrt{(n+1)(n+2)}\right|^2\cdot\frac{1}{4}k^2\cdot\frac{\hbar^2}{4m^2\omega^2}}{\hbar\omega\left(n+\cancel{\tfrac{1}{2}}-(n+2)-\cancel{\tfrac{1}{2}}\right)} + \frac{\left|\sqrt{n(n-1)}\right|^2\cdot\frac{1}{4}k^2\cdot\frac{\hbar^2}{4m^2\omega^2}}{\hbar\omega\left(n+\cancel{\tfrac{1}{2}}-(n-2)-\cancel{\tfrac{1}{2}}\right)}$$

$$= \frac{1}{4}\frac{k^2}{\hbar\omega}\cdot\left(-\frac{(n+1)(n+2)}{2}+\frac{n(n-1)}{2}\right)\cdot\frac{\hbar^2}{4m^2\omega^2}$$

$$= \frac{k^2}{8\hbar\omega}\left(-\cancel{n^2}-3n-2+\cancel{n^2}-n\right)\cdot\frac{\hbar^2}{4m^2\omega^2} = -\frac{4k^2}{8\hbar\omega}\left(n+\tfrac{1}{2}\right)\cdot\frac{\hbar^2}{4m^2\omega^2}$$

$$= -\frac{1}{8}\cdot\frac{\hbar^{\cancel{2}}}{\cancel{\hbar}}\cdot\frac{k^2(n+\frac{1}{2})}{m^2\omega^3} = -\frac{1}{8}\cdot\frac{\hbar\cdot m^2\omega^4}{m^2\omega^3}(n+\tfrac{1}{2}) = -\frac{\hbar\omega}{8}(n+\tfrac{1}{2})$$

5. (Ohanian Question 6)

a)

$$H_0 = \frac{p^2}{2m}$$

$$H_{new} = \sqrt{p^2c^2 + m^2c^4} - mc^2$$

$$H_{new} = H_0 + H' \quad \Rightarrow \quad H' = \sqrt{p^2c^2 + m^2c^4} - mc^2 - \frac{p^2}{2m}$$

To first order in perturbation theory,

$$\Delta E = \left\langle \psi_n \middle| \sqrt{\hat{p}^2c^2 + m^2c^4} - mc^2 - \frac{\hat{p}^2}{2m} \middle| \psi_n \right\rangle$$

We'll use the fact that

$$\hat{p}^2 |\psi_n\rangle = \frac{\pi^2\hbar^2n^2}{a^2} |\psi_n\rangle$$

$$\therefore \Delta E = \left\langle \psi_n \middle| \left(\sqrt{\frac{\pi^2\hbar^2n^2c^2}{a^2} + m^2c^4} - mc^2 - \frac{\pi^2\hbar^2n^2}{2ma^2} \right) \middle| \psi_n \right\rangle$$

$$= \sqrt{\frac{\pi^2\hbar^2n^2c^2}{a^2} + m^2c^4} - mc^2 - \frac{\pi^2\hbar^2n^2}{2ma^2}$$

$$\therefore \underset{\text{(first order)}}{E_{new}} = \frac{\pi^2\hbar^2n^2}{2ma^2} + \Delta E = \sqrt{\frac{\pi^2\hbar^2n^2c^2}{a^2} + m^2c^4} - mc^2$$

(b) Why this is exact:

Note that $[H_0, H'] = 0 \Rightarrow$ That H_0 and H' are simultaneously diagonalizable

In fact,

$$H' = \sqrt{H_0 \cdot 2mc^2 + m^2c^4} - mc^2 - H_0$$

If $|\psi\rangle$ is an eigenstate of H_0,

$\therefore H_0|\psi\rangle = E|\psi\rangle$ its also an eigenstate of H', $H'|\psi\rangle = \Big(\underbrace{\sqrt{(E_0 \cdot 2mc^2) + m^2c^4} - mc^2 - E_0}_{\text{Call this } \Delta E}\Big)|\psi\rangle$

$$\therefore H|\psi\rangle = (H_0 + H')|\psi\rangle = (E_0 + \Delta E)|\psi\rangle$$

But, first order perturbation theory gives,

$$\langle \psi_n | H' | \psi_n \rangle = \langle \psi_n | \Delta E | \psi_n \rangle = \Delta E$$

$\therefore$ First order perturbation is exact here

6. (Ohanian Question 12)

(a) $$\Delta E = \langle \psi_{n00} | A\delta(\vec{r}) | \psi_{n00} \rangle = \int dx\,dy\,dz \cdot A\,\delta(\vec{r}) \cdot \left(\frac{2}{\sqrt{4\pi}} (na_0)^{-3/2} \right)^2$$

$$= A \cdot \frac{4}{4\pi} \cdot (na_0)^{-3} = \frac{A\,(na_0)^{-3}}{\pi}$$

(b) $$\Delta \psi_n^{(1)} = \sum_{\substack{n' \neq n \\ \ell' m'}} \frac{\langle \psi^0_{n'\ell'm'} | H' | \psi^0_{n00} \rangle}{E_{n00} - E_{n'\ell'm'}} \psi^0_{n'\ell'm'}$$

$$= \sum_{(n',\ell',m') \neq (n,0,0)} \frac{\langle \psi^0_{n'\ell'm'} | H' | \psi^0_{n00} \rangle}{E_1 \left(\frac{1}{n^2} - \frac{1}{n'^2} \right)} \psi^0_{n'\ell'm'}$$

Note that $\langle \psi^{(0)}_{n'\ell'm'} | H' | \psi^{(0)}_{n00} \rangle = \int d\vec{x}\, \psi^{(0)}_{n00}(\vec{x}) \psi^{*(0)}_{n'\ell'm'}(\vec{x}) \cdot A\delta(\vec{r}) = A\,\psi^{(0)}_{n00}(0)\, \psi^{(0)}_{n'\ell'm'}(0)$

But note that $\psi^{(0)}_{n'\ell'm'}(0) = 0$ if $\ell' \neq 0$

$\therefore$ the only terms contributing to the sum are $(n',\ell',m') = (n',0,0)$

$$\therefore \langle \psi_{n'00} | H' | \psi_{n00} \rangle = A \cdot \frac{2}{\sqrt{4\pi}} (n'a_0)^{-3/2} \cdot \frac{2}{\sqrt{4\pi}} (na_0)^{-3/2}$$

$$= \frac{A a_0^{-3}}{\pi} (nn')^{-3/2}$$

$$\therefore \Delta\psi_n^{(1)} = \sum_{n' \neq n} \frac{A a_0^{-3}}{\pi E_1} \frac{(n n')^{-3/2}}{\frac{1}{n^2} - \frac{1}{n'^2}} \cdot \psi_{n'00}(x)$$

$$= \frac{A a_0^{-3}}{\pi E_1} \sum_{n' \neq n} \frac{(n' n)^{1/2}}{n'^2 - n^2} \psi_{n'00}(\vec{x})$$