# Homework 5 Problem 1

$$
\begin{aligned}
\ell &= \mu r^2 \dot\phi \\
\vec r &= r(\cos\phi, \sin\phi) \\
\dot{\vec r} &= \dot r(\cos\phi, \sin\phi) + r\dot\phi(-\sin\phi, \cos\phi)
\end{aligned}
$$

Make sure to understand the next line

$$
\begin{aligned}
r\dot\phi &= \frac{\ell}{\mu}\frac{1}{r} = \frac{\ell}{\mu}\frac{1+\epsilon\cos\phi}{c} \\
\dot r &= \frac{dr}{d\phi}\dot\phi = \frac{c\epsilon\sin\phi}{(1+\epsilon\cos\phi)^2}\dot\phi \\
&= \frac{\epsilon\sin\phi}{c}r^2\dot\phi = \frac{\epsilon\sin\phi}{c}\frac{\ell}{\mu}
\end{aligned}
$$

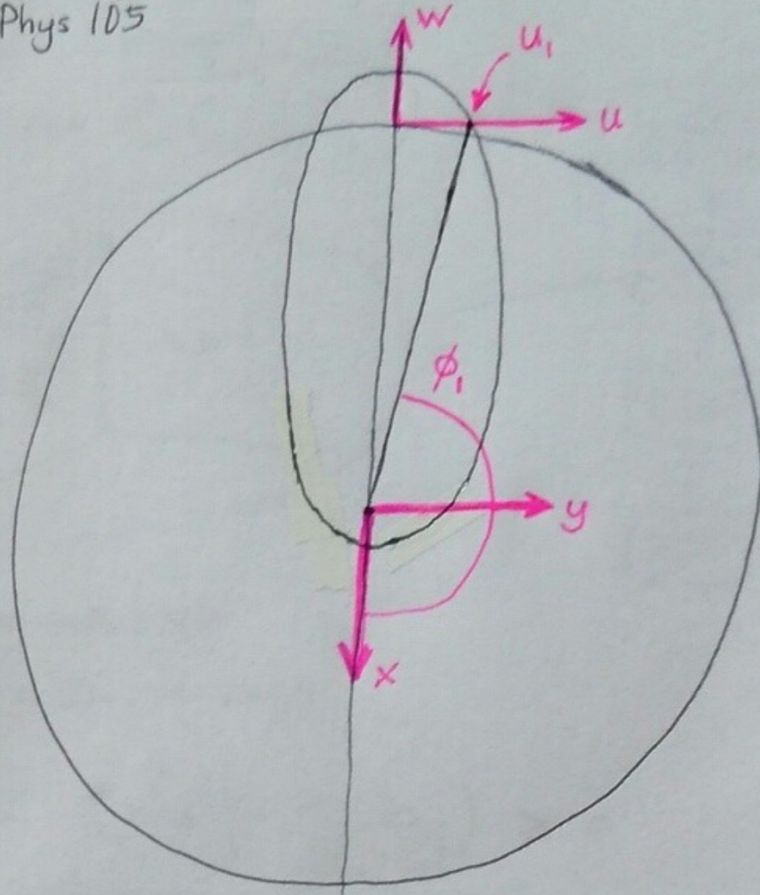
The angular momentum vector as compared to the magnitude $\ell$

$$
\begin{aligned}
\vec r &= r(\cos\phi, \sin\phi, 0) \\
\dot{\vec r} &= \frac{\ell}{c\mu}(-\sin\phi, \epsilon+\cos\phi, 0) \\
\vec r \times \mu\dot{\vec r} &= \frac{r\ell}{c}\Big(0\sin\phi - 0(\epsilon+\cos\phi), 0*\sin\phi - 0*\cos\phi, \\
&\quad \cos\phi(\epsilon+\cos\phi) + \sin^2\phi\Big) \\
&= \frac{r\ell}{c}\left(0, 0, \cos^2\phi + \epsilon\cos\phi + \sin^2\phi\right) \\
&= \frac{r\ell}{c}\left(0, 0, \frac{c}{r}\right) = \ell\hat z
\end{aligned}
$$

The Runge-Lenz vector

$$
\begin{aligned}
A &= p \times L - \mu\gamma\hat{r} \\
\vec{p} \times \vec{L} &= \frac{\ell}{c\mu}\mu\ell(\epsilon + \cos\phi, \sin\phi, 0) \\
\mu\gamma\hat{r} &= \mu\gamma(\cos\phi, \sin\phi, 0) \\
A &= \left(\frac{\ell^2\mu}{c\mu}(\epsilon + \cos\phi) - \mu\gamma\cos\phi,\right. \\
&\qquad \left.\frac{\ell^2\mu}{c\mu}\sin\phi - \mu\gamma\sin\phi, 0\right) \\
c &= \frac{\ell^2}{\gamma\mu} \\
A &= (\gamma\mu\epsilon, 0, 0)
\end{aligned}
$$

Homework 5, Problem 2, Phys 105




(a) $r(\phi_1)\cos\phi_1 = -R$

$$\frac{c\cos\phi_1}{1+\varepsilon\cos\phi_1} = -R$$

$$\cos\phi_1 = \frac{-1}{\varepsilon + \frac{c}{R}}$$

$$u_1 = R\tan(\pi - \phi_1)$$

$$= -R\tan\phi_1$$

$$= R\frac{\sin\phi_1}{(-\cos\phi_1)}$$

$$= R\sqrt{\sec^2\phi_1 - 1}$$

$$u_1 = R\sqrt{\left(\frac{c}{R}+\varepsilon\right)^2 - 1} \qquad \boxed{u_1 = \sqrt{(c+\varepsilon R)^2 - R^2}}$$

$\varepsilon, c$ same as before

(b) $c = \frac{\ell^2}{\gamma\mu} = \frac{[\mu(R+h)v_0]^2}{GM\mu^2} = R\delta_v(1+\delta_h)^2$ exactly

$\varepsilon = 1 - \frac{c}{R+h}$ from previous solution

$= 1 - \delta_v(1+\delta_h)$ exactly

$$u_1 = R\sqrt{\left(\delta_v(1+\delta_h)^2 + 1 - \delta_v(1+\delta_h)\right)^2 - 1}$$

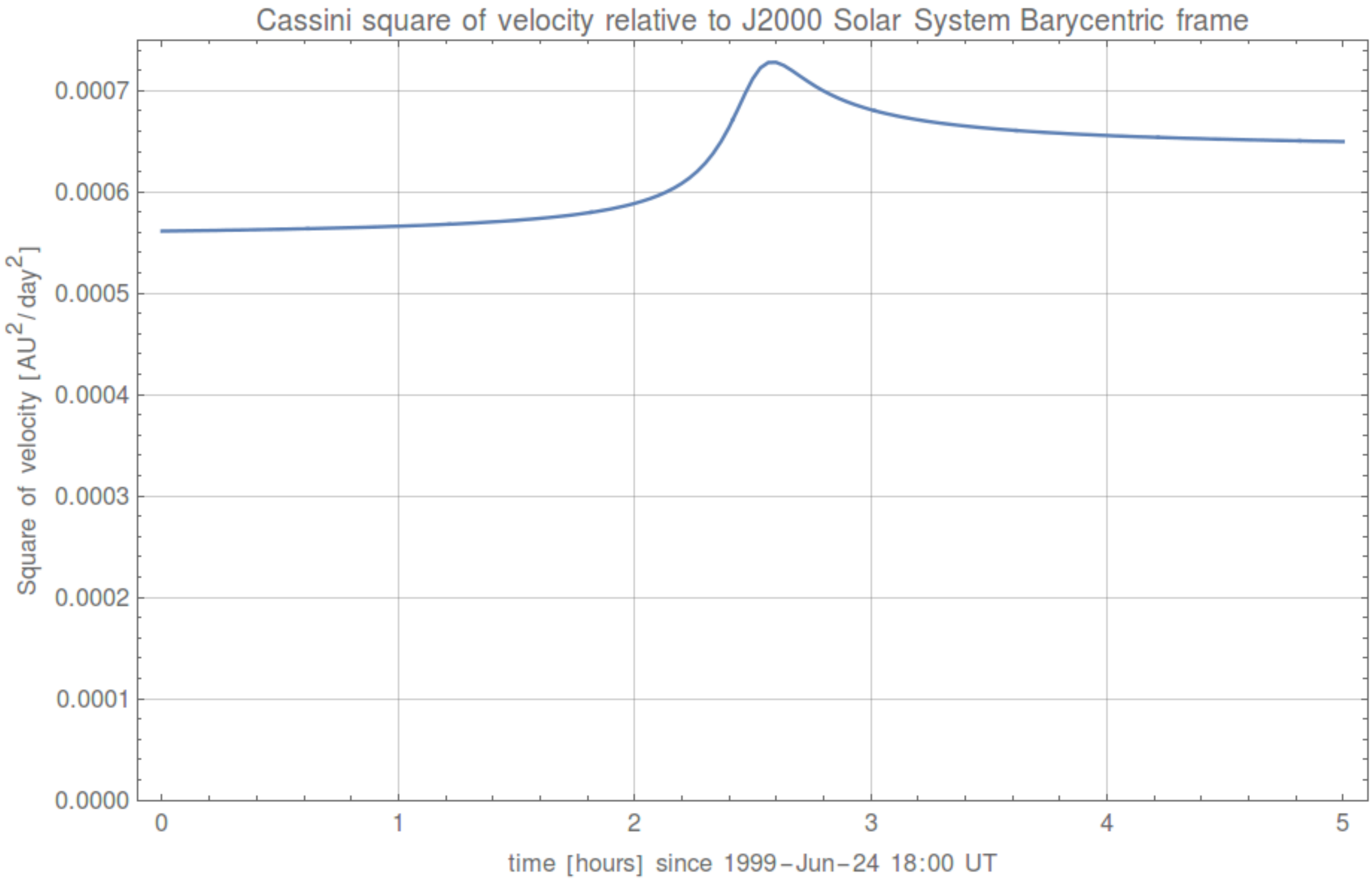
$= R\sqrt{\left(1+\delta_v\delta_h(1+\delta_h)\right)^2 - 1}$ exact

$= R\sqrt{2\delta_v\delta_h(1+\delta_h) + \delta_v^2\delta_h^2 \text{ terms and smaller}}$

$$\approx L_{approx}\sqrt{1+\delta_h} \approx L_{approx}\left(1+\frac{\delta_h}{2}\right)$$

$$\boxed{u_1 - L_{approx} = L_{approx}\,\delta_h/2}$$

Part (a): A plot of the sum of the squares of the velocity components relative to the J2000 Solar System Barycentric frame. This plot will make it clear that Cassini has gained energy via this Gravity Assist Maneuver. The axes must be labeled and have units [hours and $(AU/day)^2$].



Part (b): Estimate the time of closest approach. At the time of closest approach, what is ***r***, the three-component position vector of Cassini relative to the center of Venus?

The closest approach occurs between data points number 76 and 77. NOTE: If you use “zero-based indices” as in Python, then you would say “75 and 76” here.

(-7.90871*10^-6, 0.0000412517, -0.0000153559) AU          2.5 hours after 1999-Jun-24 18:00 UT

(2.95177*10^-6, 0.000042222, -0.0000144914) AU          2.53333 hours after 1999-Jun-24 18:00 UT

Students will earn full credit for either one of these. However a better approximation can be calculated as follows. We compute a linear interpolation for both the position and the velocity vectors. Then compute the dot product as a function of time. We look for the zero of this function of time. In this way, we arrive at the following more accurate result:

(-2.5621*10^-6, 0.0000417294, -0.0000149303) AU          2.51641 hours after 1999-Jun-24 18:00 UT

This will be useful later in the problem when we compare the direction of this position vector with the direction of the Laplace-Runge-Lenz vector.

If we assume Venus is a sphere with radius equal to the published mean radius, then Cassini passed about 600 km above the surface. In actuality Venus has some oblateness, so this number should be reduced.

Part (c): Angular momentum vector divided by the mass of Cassini. How does the constancy of ***L*** look?

Plot not required. When plotted, each of the three components look extremely constant.

Part (d) The Laplace-Runge-Lenz vector divided by the square of the mass of Cassini. How does the constancy of ***A*** look? How does the direction of this vector compare with the direction of the vector you found in part (b)?

Plot not required. When plotted, each of the three components look extremely constant.

For data point number 76 (near closest approach) we get [That would be “75” with zero-based indices.]

***A***/mu^2 = (-1.17612*10^-10, 1.91574*10^-9, -6.85428*10^-10) AU^3/day^2

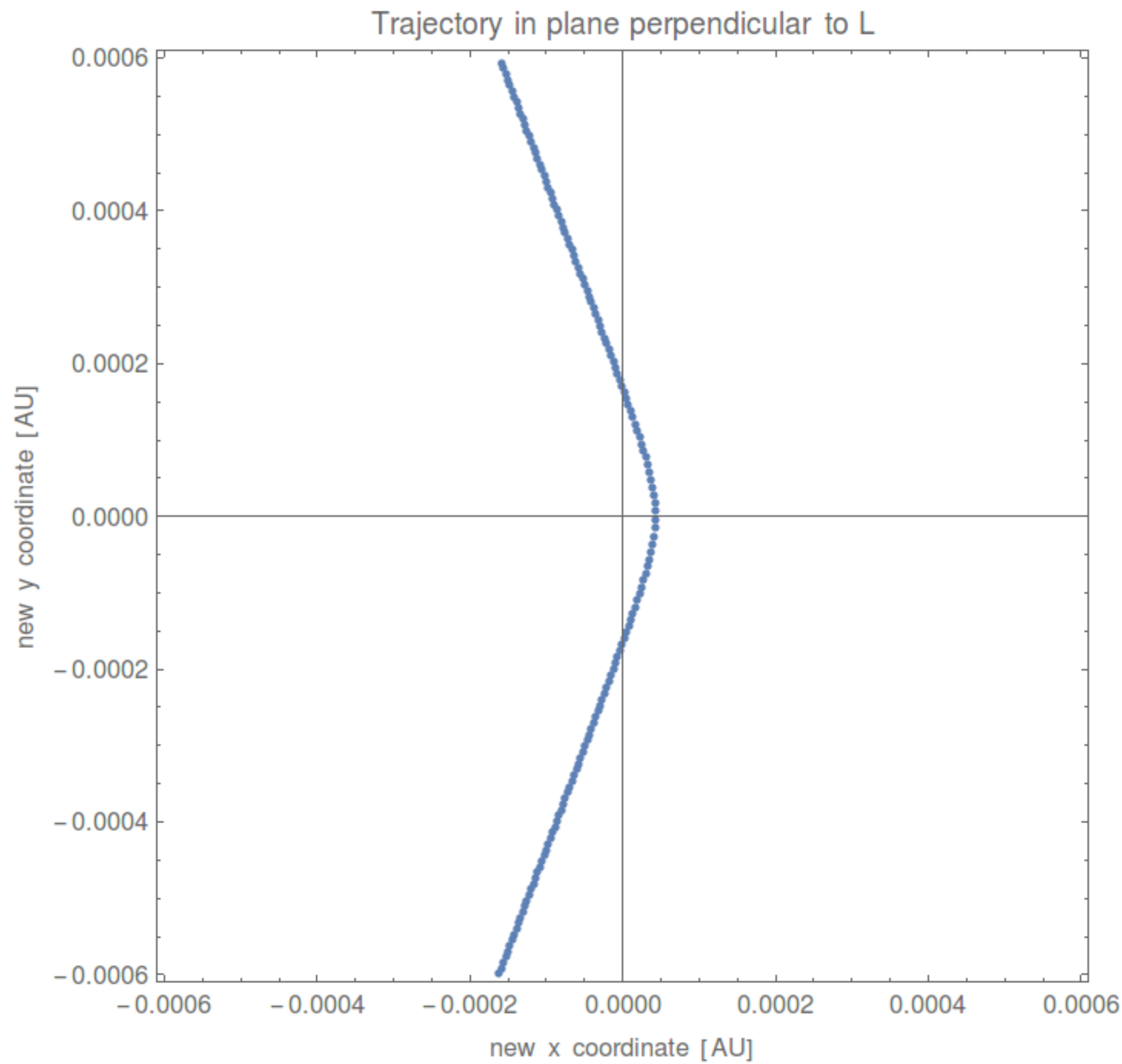
The angle between this vector and the position vector at closest approach is about 5 microradians.

Part (e) Normalize the two vectors you found in parts (c) and (d).
Compute their cross product so you have a right-handed orthonormal triple.
Project the ***r*** data (using scalar products) onto the plane perpendicular to ***L*** and plot this.
Your plot should look similar to Fig. 8.11 in the text, with the point of closest approach on the right, as in the figure.
What is the eccentricity?

| | | |
|---|---|---|
| Unit vector in direction of ***L***: | (0.104129, -0.32937, -0.938441) | (this is the new z-hat) |
| Unit vector in direction of ***A***: | (-0.0577076, 0.93998, -0.336314) | (this is the new x-hat) |
| New y-hat (from cross product): | (0.992888, 0.0891753, 0.0788725) | |

The plot is on the next page.

The eccentricity is 2.81. This can be gotten from the magnitude of the A vector, as explained in Problem 1. Alternatively, the eccentricity can be gotten from the asymptotic direction of the velocity vectors in the plot on the next page. Another alternative is to compute the total mechanical energy and the angular momentum, and then proceed as explained in the Problem 2 discussion.



Trajectory in plane perpendicular to L
new y coordinate [AU]
new x coordinate [AU]
0.0006
0.0004
0.0002
0.0000
−0.0002
−0.0004
−0.0006