Homework 4 Problem 1

Part a

The position and velocity of the mass are:

$$\vec{r} = r_c \cos(\omega t)\,\hat{x} + r_c \sin(\omega t)\,\hat{y} + R\cos\phi\,\hat{x} + R\sin\phi\,\hat{y}$$
$$\dot{\vec{r}} = -\omega r_c \sin(\omega t)\,\hat{x} + \omega r_c \sin(\omega t)\,\hat{y} - R\dot{\phi}\sin\phi\,\hat{x} + R\dot{\phi}\cos\phi\,\hat{y}$$

Therefore:

$$\begin{aligned} v^2 &= \left(-\omega r_c \sin(\omega t) - R\dot{\phi}\sin\phi\right)^2 + \left(\omega r_c \cos(\omega t) + R\dot{\phi}\cos\phi\right)^2 \\ &= \omega^2 r_c^2 \left(\sin^2(\omega t) + \cos^2(\omega t)\right) + 2\omega r_c R\dot{\phi}\left(\sin(\omega t)\sin\phi + \cos(\omega t)\cos\phi\right) + R^2\dot{\phi}^2\left(\sin^2\phi + \cos^2\phi\right) \\ &= \omega^2 r_c^2 + 2\omega r_c R\dot{\phi}\cos(\phi - \omega t) + R^2\dot{\phi}^2 \end{aligned}$$

Since there are no forces other than the constraint force, then:

$$\mathcal{L} = \frac{1}{2}m\left[\omega^2 r_c^2 + 2\omega r_c R\dot{\phi}\cos(\phi - \omega t) + R^2\dot{\phi}^2\right]$$

The equation of motion is then:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ -m\omega r_c R\dot{\phi}\sin(\phi - \omega t) &= \frac{d}{dt}\left[m\omega r_c R\cos(\phi - \omega t) + mR^2\dot{\phi}\right] \\ &= -m\omega r_c R\left(\dot{\phi} - \omega\right)\sin(\phi - \omega t) + mR^2\ddot{\phi} \end{aligned}$$

Part b

Given $\epsilon = \phi - \omega t$ is small, and $\dot{\epsilon} = \dot{\phi} - \omega$:

$$\begin{aligned} -m\omega r_c R\left(\dot{\epsilon} + \omega\right)\sin\epsilon &= -m\omega r_c R\dot{\epsilon}\sin\epsilon + mR^2\ddot{\epsilon} \\ -m\omega^2 r_c R\epsilon &\approx mR^2\ddot{\epsilon} \\ -\omega^2\frac{r_c}{R}\epsilon &\approx \ddot{\epsilon} \end{aligned}$$

This yields frequency of oscillations as $\omega\sqrt{\frac{r_c}{R}}$.

Problem 2

(a) $x = R\cos\phi$

$y = R\sin\phi$

$z = z$

(b) $$\left.\begin{aligned}\dot{x} &= -R\sin\phi\,\dot{\phi}\\ \dot{y} &= R\cos\phi\,\dot{\phi}\\ \dot{z} &= \dot{z}\end{aligned}\right\} \quad T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{m}{2}\left(R^2\dot{\phi}^2 + \dot{z}^2\right)$$

(c) Use (x_{in}, y_{in}, z_{in}) for inertial frame (tracks, the train station, etc)

$$\left.\begin{aligned}x_{in} &= x &&= R\cos\phi\\ y_{in} &= y + \tfrac{1}{2}at^2 &&= R\sin\phi + \tfrac{1}{2}at^2\\ z_{in} &= z &&= z\end{aligned}\right\} \quad \begin{aligned} T &= \frac{m}{2}\left(\dot{x}_{in}^2 + \dot{y}_{in}^2 + \dot{z}_{in}^2\right)\\ &= \frac{m}{2}\left(R^2\dot{\phi}^2 + \dot{z}^2\right) + \frac{m}{2}\left(a^2t^2 + 2atR\cos\phi\dot{\phi}\right)\end{aligned}$$

Thus $A = \frac{g}{2\omega^2}$, which yields the general solution:

$$x = \frac{g}{2\omega^2}\sin\omega t + C_- e^{-\omega t} + C_+ e^{\omega t}$$

The initial conditions are $x(0) = x_0$ and $\dot{x}(0) = 0$, and substituting:

$$\begin{aligned} x_0 &= C_- + C_+ \\ 0 &= \frac{g}{2\omega} - \omega C_- + \omega C_+ \end{aligned}$$

The solution is $C_+ = \frac{x_0}{2} - \frac{g}{4\omega^2}$ and $C_- = \frac{x_0}{2} + \frac{g}{4\omega^2}$. Therefore:

$$\begin{aligned} x &= \frac{g}{2\omega^2}\sin\omega t + \left(\frac{x_0}{2} + \frac{g}{4\omega^2}\right) e^{-\omega t} + \left(\frac{x_0}{2} - \frac{g}{4\omega^2}\right) e^{\omega t} \\ &= \frac{g}{2\omega^2}\sin\omega t + x_0\cosh\omega t - \frac{g}{2\omega^2}\sinh\omega t \end{aligned}$$

6 Taylor 7.36

6.1 Part a

The position and velocity are:

$$\begin{aligned} \vec{r} &= r\sin\phi\,\hat{x} - r\cos\phi\,\hat{y} \\ \dot{\vec{r}} &= \left(\dot{r}\sin\phi + r\dot{\phi}\cos\phi\right)\hat{x} + \left(-\dot{r}\cos\phi + r\dot{\phi}\sin\phi\right)\hat{y} \end{aligned}$$

Then the kinetic energy is:

$$\begin{aligned} T &= \frac{1}{2}m\left\|\dot{\vec{r}}\right\|^2 \\ &= \frac{1}{2}m\left[\left(\dot{r}^2\sin^2\phi + r\dot{r}\dot{\phi}\cos\phi\sin\phi + r^2\dot{\phi}^2\cos^2\phi\right) + \left(\dot{r}^2\cos^2\phi - r\dot{r}\dot{\phi}\cos\phi\sin\phi + r^2\dot{\phi}^2\sin^2\phi\right)\right] \\ &= \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\right] \end{aligned}$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + mgr\cos\phi - \frac{1}{2}k\left(r - l_0\right)^2$$

6.2 Part b

Lagrange's equations of motion become:

$$\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = mr\dot{\phi}^2 + mg\cos\phi - k\left(r - l_0\right) - \frac{d}{dt}m\dot{r} = mr\dot{\phi}^2 + mg\cos\phi - k\left(r - l_0\right) - m\ddot{r} \\ 0 &= \frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = -mgr\sin\phi - \frac{d}{dt}mr^2\dot{\phi} = -mgr\sin\phi - 2mr\dot{r}\dot{\phi} - mr^2\ddot{\phi} \end{aligned}$$

The radial equation states that the radial forces are from the spring and from some component of gravity. The angular equation states that the torques about the pivot are from some component of gravity.

6.3 Part c

At equilibrium, $\dot{r} = 0$, $\ddot{r} = 0$, $\dot{\phi} = 0$, and $\ddot{\phi} = 0$. Substituting into the equations gives $\phi_{\text{eq}} = 0$ and $l = r_{\text{eq}} = l_0 + \frac{mg}{k}$. Set $r = l + \epsilon$; then the equation of motions become:

$$\begin{aligned} \ddot{\epsilon} &= -\frac{k}{m}\epsilon + \left(l + \epsilon\right)\dot{\phi}^2 + g\left(\cos\phi - 1\right) \\ \frac{d}{dt}\left[\left(l + \epsilon\right)^2\dot{\phi}\right] &= -g\left(l + \epsilon\right)\sin\phi \end{aligned}$$

For small oscillations, any term higher than first-order in ϵ and ϕ can be dropped, yielding:

$$\ddot{\epsilon} = -\frac{k}{m}\epsilon$$
$$\ddot{\phi} = -\frac{g}{l}\phi$$

The solutions are $\epsilon = A_1 \cos\left(\sqrt{\frac{k}{m}}t - \delta_1\right)$ and $\phi = A_2 \cos\left(\sqrt{\frac{g}{l}}t - \delta_2\right)$, which are sinusoidal oscillations both radially and tangentially with possibly different frequencies.

7 Taylor 7.40

7.1 Part a

The position, written in Cartesian coordinates (with the z-axis pointing downward) but with spherical variables, is:

$$\vec{r} = R\cos\phi\sin\theta\,\hat{x} + R\sin\phi\sin\theta\,\hat{y} + R\cos\theta\,\hat{z}$$

Differentiating, the velocity is:

$$\dot{\vec{r}} = \left[-R\sin\phi\sin\theta\,\dot{\phi} + R\cos\phi\cos\theta\,\dot{\theta}\right]\hat{x} + \left[R\cos\phi\sin\theta\,\dot{\phi} + R\sin\phi\cos\theta\,\dot{\theta}\right]\hat{y} - R\sin\theta\,\dot{\theta}\,\hat{z}$$

The magnitude squared of the velocity is thus:

$$\begin{aligned}\left\|\dot{\vec{r}}\right\|^2 &= \left[R^2\sin^2\phi\sin^2\theta\,\dot{\phi}^2 - 2R^2\cos\phi\sin\phi\cos\theta\sin\theta\,\dot{\phi}\dot{\theta} + R^2\cos^2\phi\cos^2\theta\,\dot{\theta}^2\right] \\ &\quad + \left[R^2\cos^2\phi\sin^2\theta\,\dot{\phi}^2 + 2R^2\cos\phi\sin\phi\cos\theta\sin\theta\,\dot{\phi}\dot{\theta} + R^2\sin^2\phi\cos^2\theta\,\dot{\theta}^2\right] + R^2\sin^2\theta\,\dot{\theta}^2 \\ &= R^2\sin^2\theta\,\dot{\phi}^2 + R^2\dot{\theta}^2\end{aligned}$$

Alternatively, note that $R\,d\theta$ is the $\hat{\theta}$ component of displacement, and $R\sin\theta\,d\phi$ is the $\hat{\phi}$ component of displacement. The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}mR^2\left(\sin^2\theta\,\dot{\phi}^2 + \dot{\theta}^2\right) + mgR\cos\theta$$

Lagrange's equations of motion are:

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0 - \frac{d}{dt}mR^2\sin^2\theta\,\dot{\phi}$$
$$0 = \frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = -mgR\sin\theta + mR^2\cos\theta\sin\theta\,\dot{\phi}^2 - \frac{d}{dt}mR^2\dot{\theta}$$

7.2 Part b

The z-component of the angular momentum is:

$$\begin{aligned}\ell_z &= \hat{z}\cdot\left(\vec{r}\times m\dot{\vec{r}}\right) \\ &= mR\cos\phi\sin\theta\left[R\cos\phi\sin\theta\,\dot{\phi} + R\sin\phi\cos\theta\,\dot{\theta}\right] - mR\sin\phi\sin\theta\left[-R\sin\phi\sin\theta\,\dot{\phi} + R\cos\phi\cos\theta\,\dot{\theta}\right] \\ &= mR^2\sin^2\theta\,\dot{\phi}\end{aligned}$$

The ϕ equation immediately gives $\frac{d\ell_z}{dt} = 0$; i.e., ℓ_z is a conserved quantity.

7.3 Part c

If ϕ is a constant, then $\dot{\phi} = 0$, so the θ equation becomes:

$$\ddot{\theta} = -\frac{g}{R}\sin\theta$$

The spherical pendulum oscillates like a simple pendulum.

7.4 Part d

Expressing $\dot{\phi} = \frac{\ell_z}{mR^2 \sin^2\theta}$, and substituting into the θ equation:

$$\ddot{\theta} = -\frac{g}{R}\sin\theta + \left(\frac{\ell_z}{mR^2}\right)^2 \frac{\cos\theta}{\sin^3\theta}$$

For a constant solution $\theta = \theta_0$, then $\ddot{\theta} = 0$. Thus:

$$\sin^3\theta_0 \tan\theta_0 = \left(\frac{\ell_z}{mR^2}\right)^2 \frac{R}{g}$$

The quantity on the right is some finite, nonnegative number. But $\sin^3\theta_0 \tan\theta_0$ is a monotonically increasing function on the interval $\left[0, \frac{\pi}{2}\right)$ with range $[0, +\infty)$. Thus any value is attainable, so there always exists a unique angle for this type of motion. It is called a conical pendulum because the connector to the pivot and the bob traces out the shape of a cone.

7.5 Part e

Rewriting the equation of motion:

$$\frac{\ddot{\theta}}{\sin\theta} = -\frac{g}{R} + \left(\frac{\ell_z}{mR^2}\right)^2 \frac{1}{\sin^3\theta\tan\theta}$$

Suppose $\theta = \theta_0 + \epsilon$ with $\epsilon \ll \theta_0$. Then expanding to the lowest order in ϵ:

$$\begin{aligned}
\frac{1}{\sin^3\theta\tan\theta} &= \frac{1}{\sin^3(\theta_0+\epsilon)\tan(\theta_0+\epsilon)} \\
&\approx \frac{1}{(\sin\theta_0 + \epsilon\cos\theta_0)^3\,(\tan\theta_0 + \epsilon\sec^2\theta_0)} \\
&\approx \frac{1}{\left(\sin^3\theta_0 + 3\epsilon\cos\theta_0\sin^2\theta_0\right)(\tan\theta_0 + \epsilon\sec^2\theta_0)} \\
&\approx \frac{1}{\sin^3\theta_0\tan\theta_0 + \epsilon\sin^3\theta_0\,(3+\sec^2\theta_0)} \\
&= \frac{1}{\sin^3\theta_0\tan\theta_0}\left[\frac{1}{1+\epsilon\cot\theta_0\,(3+\sec^2\theta_0)}\right] \\
&\approx \frac{1}{\sin^3\theta_0\tan\theta_0}\left[1 - \epsilon\cot\theta_0\left(3+\sec^2\theta_0\right)\right]
\end{aligned}$$

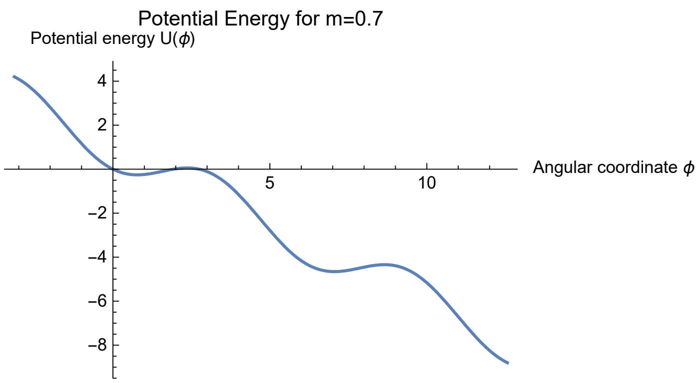
- Specifying parameters

In[1]:=
```
M = 1;
g = 1;
R = 1;
m1 = 0.7;
m2 = 0.8;
```

- Case-1: m=m1=0.7
- Plotting the potential energy.

In[6]:=
```
U1[ϕ_] = M g R (1 - Cos[ϕ]) - m1 g R ϕ;
Plot[U1[ϕ], {ϕ, -Pi, 4 Pi}, PlotRange → All, PlotLabel → "Potential Energy for m=0.7",
 AxesLabel → {"Angular coordinate ϕ", "Potential energy U(ϕ)"}]
```

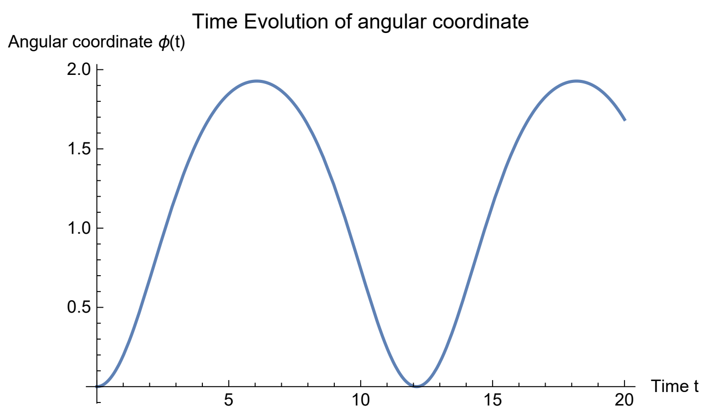
Out[7]=



- Solving the equation of motion and plotting the solution.

In[8]:=
```
Sol1 =
  NDSolve[{(M + m1) R ϕ''[t] == -M g Sin[ϕ[t]] + m1 g, ϕ[0] == 0, ϕ'[0] == 0}, ϕ, {t, 0, 20}];
Plot[Evaluate[ϕ[t] /. Sol1], {t, 0, 20}, PlotRange → All,
 PlotLabel → "Time Evolution of angular coordinate",
 AxesLabel → {"Time t", "Angular coordinate ϕ(t)"}]
```

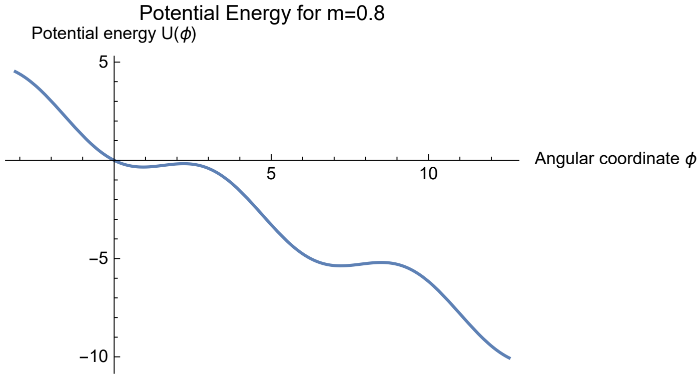
Out[9]=



- The system cannot escape the potential energy well, so the ϕ shows oscillatory behavior.
- Case-2: m=m2=0.8
- Plotting the potential energy.

In[10]:=
```
U1[ϕ_] = M g R (1 - Cos[ϕ]) - m2 g R ϕ;
Plot[U1[ϕ], {ϕ, -Pi, 4 Pi}, PlotRange → All, PlotLabel → "Potential Energy for m=0.8",
 AxesLabel → {"Angular coordinate ϕ", "Potential energy U(ϕ)"}]
```

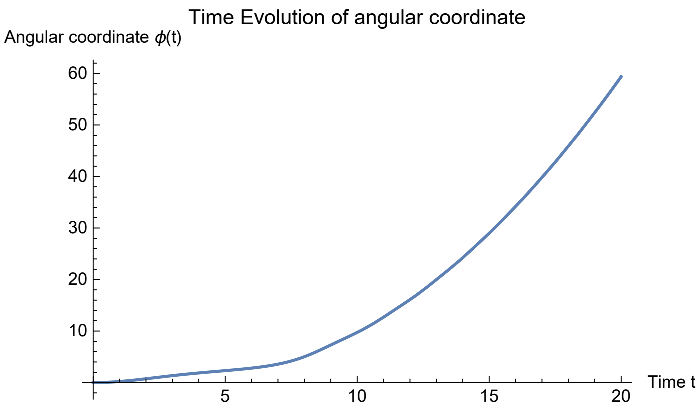
Out[11]=



- Solving the equation of motion and plotting the solution.

In[12]:=
```
Sol2 =
  NDSolve[{(M + m2) R ϕ''[t] == -M g Sin[ϕ[t]] + m2 g, ϕ[0] == 0, ϕ'[0] == 0}, ϕ, {t, 0, 20}];
Plot[Evaluate[ϕ[t] /. Sol2], {t, 0, 20}, PlotRange → All,
 PlotLabel → "Time Evolution of angular coordinate",
 AxesLabel → {"Time t", "Angular coordinate ϕ(t)"}]
```

Out[13]=



- The system escapes the potential energy well, so ϕ runs off to infinity.

7.43

a) $\mathcal{L} = \frac{1}{2}(M+m)R^2\dot{\varphi}^2 - MgR(1-\cos\varphi) + mgR\varphi$

$\Rightarrow$ EoM : $(M+m)R\ddot{\varphi} = -Mg\sin\varphi + mg$

Equilibrium $\Rightarrow \ddot{\varphi} = 0$ ie $\sin\varphi = m/M$

assuming $m < M$, $\exists$ two sol^{n}s:

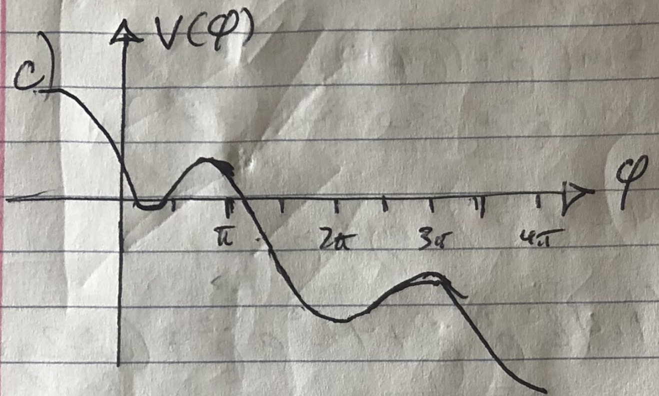
$\varphi = \arcsin(m/M)$ w/ $0 < \varphi < \pi/2$ (stable)

& $\pi/2 < \varphi < \pi$ (unstable)

b) Stable : $\varphi = \pi/6 + 2\pi n$ where $n \in \mathbb{Z}$

Unstable : $\varphi = 5\pi/6 + 2\pi n$

c)



The mass M oscillates indefinitely when released from $\varphi = 0$, swinging past $\pi/2$ & back to 0

d) $m = 0.8 \Rightarrow \max V(\varphi) > \pi/2$ is not high enough to stop the wheel.

7.46

a) Rotation about z axis by ~~χ~~ ε:

$$(r_\alpha, \theta_\alpha, \varphi_\alpha) \longmapsto (r_\alpha, \theta_\alpha, \varphi_\alpha + \varepsilon).$$

Invariance by overall rotation ~~[illegible]~~ means:

$$\mathcal{L}(r_1, \theta_1, \varphi_1, \ldots, r_N, \theta_N, \varphi_N) = \mathcal{L}(r_1, \theta_1, \varphi_1 + \varepsilon, \ldots, r_N, \theta_N, \varphi_N + \varepsilon)$$

$$0 = \mathcal{L}(\{r_\alpha, \theta_\alpha, \varphi_\alpha\}) - \mathcal{L}(\{r_\alpha, \theta_\alpha, \varphi_\alpha + \varepsilon\}) = \sum_\alpha \frac{\partial \mathcal{L}}{\partial \varphi_\alpha} \varepsilon.$$

$$\Rightarrow \sum_\alpha \frac{\partial \mathcal{L}}{\partial \varphi_\alpha} = 0.$$

b) $\frac{\partial \mathcal{L}}{\partial \varphi_\alpha} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_\alpha} = \frac{d l_{\alpha z}}{dt}$ b/c $l_{\alpha z} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_\alpha}$

Invariance $\Rightarrow \frac{d}{dt} \sum_\alpha l_{\alpha z} = 0 \Rightarrow L_z = \sum l_{\alpha z} = \text{const}.$