Solutions - Homework 8

1. (Griffiths 9.13)

Need to show, $\langle n' 0 0 | \vec{r} | n 0 0 \rangle$

We can rewrite,

$$\vec{r} = r\cos\theta\,\hat{z} + r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y}$$

Now note that

$$\psi_{n00} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-1)!}{2n[(n!)^3]}} \cdot e^{-\frac{r}{na}} \cdot L'_{n-1}\left(\frac{2r}{na}\right) \frac{1}{\sqrt{4\pi}}$$

$$\therefore \langle n' 0 0 | \vec{r} | n 0 0 \rangle = \int dr\, r^2 \psi^*_{n'00}(r)\, \psi_{n00}(r) \cdot \left[r\hat{z}\left(\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \cos\theta \right) + r\hat{x}\left(\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \sin\theta\cos\phi \right) + r\hat{y}\left(\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \sin\theta \sin\phi \right) \right]$$

Now we can just focus on the angular integrals,

$$\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \cos\theta = 2\pi \cdot \frac{1}{2}\sin^2\theta \Big|_0^\pi = 0$$

$$\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \sin\theta\cos\phi = \int_0^\pi d\theta \sin\theta\cos\theta \cdot (+\sin\phi)\Big|_0^{2\pi} = 0$$

$$\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \sin\theta \sin\phi = \int_0^\pi d\theta \sin^2\theta \cdot (-\cos\phi)\Big|_0^{2\pi} = 0$$

$$\Rightarrow \boxed{\langle n' 0 0 | \vec{r} | n 0 0 \rangle = 0}$$

2. (Griffiths 9.14)

a)

$$\begin{tikzcd}
 & {|211\rangle} \arrow[dr] & \\
{|300\rangle} \arrow[ur] \arrow[r] \arrow[dr] & {|210\rangle} \arrow[r] & {|100\rangle} \\
 & {|21\text{-}1\rangle} \arrow[ur] &
\end{tikzcd}$$

Note: $|300\rangle$ cannot decay straight to an $n=1$ state since the selection rules tell us $\Delta\ell = \pm 1$ and there are NO $|11m\rangle$ states $(\ell \le n-1)$

b) Need to compute $\langle 300|\vec{r}|211\rangle$, $\langle 300|\vec{r}|210\rangle$, $\langle 300|\vec{r}|21\text{-}1\rangle$

From the selection rules,

$\langle 300|z|211\rangle = 0$ $\qquad$ $\langle 300|x|210\rangle = 0$

$\langle 300|z|21\text{-}1\rangle = 0$ $\qquad$ $\langle 300|y|210\rangle = 0$

~~Need~~ to

Also know, $\begin{cases} \langle 300|x|211\rangle = -i\langle 300|y|211\rangle \\ \langle 300|x|21\text{-}1\rangle = i\langle 300|y|21\text{-}1\rangle \end{cases}$

So need to compute $\langle 300|x|211\rangle$, $\langle 300|x|21\text{-}1\rangle$, and $\langle 300|z|210\rangle$

$$\langle 300|z|210\rangle = \underbrace{\int_0^\infty dr\, r^2 \cdot r\, R_{21} R_{30}^*}_{\text{This part will be the same for all integrals...}} \cdot \underbrace{\int_0^\pi d\theta \sin\theta \cos\theta \underbrace{\int_0^{2\pi} d\phi}_{=2\pi \text{ (trivial)}}^{\text{From "z"}} Y_0^{0*}\, Y_1^0}$$

$$= 2\pi \int_0^\pi d\theta\ \sin\theta\cos\theta\ \frac{1}{\sqrt{4\pi}} \cdot \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$= 2\pi \cdot \frac{\sqrt{3}}{4\pi}\left[-\frac{\cos^3\theta}{3}\right]_0^\pi = \frac{1}{2}\sqrt{3}\cdot\frac{2}{3} = \underline{\frac{1}{\sqrt{3}}}$$

$$\langle 300|x|21\pm1\rangle = \begin{pmatrix}\text{radial}\\ \text{part}\end{pmatrix} \cdot \underbrace{\int_0^{\pi} d\theta \sin\theta \overset{\text{From } x}{\sin\theta} \int_0^{2\pi} d\phi \cdot \frac{1}{\sqrt{4\pi}} \cdot (\mp 1) \cdot \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi} \overset{\text{From } x}{\cos\phi}}$$

$$= \mp\sqrt{\frac{3}{2}} \cdot \frac{1}{4\pi} \int_0^{\pi} \sin^3\theta \cdot \int_0^{2\pi} d\phi\, e^{\pm i\phi} \frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right)$$

$$= \pi$$

$$= \mp\sqrt{\frac{3}{2}} \cdot \frac{1}{4\cancel{\pi}} \int_0^{\pi} \sin^3\theta \cdot \cancel{\pi}$$

$$= \mp\sqrt{\frac{3}{2}} \cdot \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{3} = \underbrace{\mp \frac{1}{\sqrt{6}}}$$

The relevant matrix elements

are given by,

$$|\langle 300|\vec{r}|210\rangle|^2 = \overbrace{\begin{pmatrix}\text{radial}\\ \text{part}\end{pmatrix}}^{\text{Call this "C"}} \cdot \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} \cdot C$$

$$|\langle 300|\vec{r}|211\rangle|^2 = C \cdot \left(\underbrace{\left(\frac{1}{\sqrt{6}}\right)^2}_{\text{from } x} + \underbrace{\left(\frac{1}{\sqrt{6}}\right)^2}_{\text{from } y}\right) = C \cdot \frac{1}{3}$$

$$|\langle 300|\vec{r}|21{-1}\rangle|^2 = C \cdot \left(\left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2\right) = C \cdot \frac{1}{3}$$

$\therefore$ They all have equal probability, which equals $1/3$

3. (Liboff 13.45)

$$\mathcal{E} = \mathcal{E}_0 e^{-t/\tau} \Rightarrow \quad V = -\mathcal{E}_0 z e^{-t/\tau}$$

$$\underline{U = -q\mathcal{E}_0 z e^{-t/\tau}}$$

$$H_{if} = -q\mathcal{E}_0 e^{-t/\tau} \langle 21m|z|100\rangle$$

↳ <u>Note:</u> $\langle 21\pm1|z|100\rangle = 0$ from the selection rules...

Need to compute

$$\langle 210|z|100\rangle$$

$$= \int_0^\infty dr\, r^2 R_{10} R_{21}^* \underbrace{\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \cdot Y_1^{0*} Y_0^0 \overbrace{r\cos\theta}^{z}}$$

$$= r\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \frac{1}{\sqrt{4\pi}} \cdot \sqrt{\frac{3}{4\pi}}\cos\theta \cdot \cos\theta$$

$$= \frac{\sqrt{3}}{2\cancel{4\pi}} \cancel{2\pi} r \int_0^\pi d\theta \sin\theta \cos^2\theta$$

$$= \frac{\sqrt{3}\,r}{2}\left[-\frac{\cos^3\theta}{3}\right]_0^\pi = \frac{\sqrt{3}\,r}{\cancel{2}} \cdot \frac{\cancel{2}}{3} = \frac{r}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \int_0^\infty dr \cdot r^3\; 2 \cdot \frac{1}{\sqrt{a^3}} e^{-r/a} \cdot \frac{1}{\sqrt{24}} \cdot \frac{1}{\sqrt{a^3}} \cdot \frac{r}{a} e^{-r/2a}$$

$$= \frac{2}{\sqrt{3} \cdot a^4 \cdot \sqrt{24}} \int_0^\infty dr \cdot r^4 e^{-\frac{3r}{2a}} = \frac{2}{6a^4 \cdot \sqrt{2}} \cdot \frac{256 a^5}{81} = \frac{256 a}{243\sqrt{2}}$$

$$\Rightarrow \quad c_{210} = -\frac{i}{\hbar}\int_0^t dt' \, \frac{256a}{243\sqrt{2}} \cdot (-q\mathcal{E}_0)\, e^{-t'/\tau}\, e^{i\omega_{21}t'}$$

$$= \frac{iq\mathcal{E}_0 a \cdot 256}{243\sqrt{2}\,\hbar}\int_0^t dt' \, e^{-t'\left(1/\tau - i\omega_{21}\right)}$$

$$= \left(\quad \right) \cdot \frac{1 - \underbrace{e^{-t(1/\tau - i\omega_{21})}}_{\approx 0 \text{ for } t \gg \tau}}{1/\tau - i\omega_{21}}$$

$$\simeq \frac{256\, i q \mathcal{E}_0 a}{243\sqrt{2}\,\hbar} \cdot \frac{1}{1/\tau - i\omega_{21}}$$

$$|c_{210}|^2 \approx \left(\frac{256}{243}\right)^2 \cdot \frac{q^2\mathcal{E}_0^2 a^2}{2\hbar^2} \cdot \frac{1}{(1/\tau)^2 + \omega_{21}^2}$$

where $\omega_{21}^2 = \left(\frac{3E_{ground}}{4\hbar}\right)^2$

4. (Ohanian 22)

We should use fermi's Golden Rule,

$$W = \frac{2\pi}{\hbar} |H'_{ba}|^2 \rho_b(E)$$

$$H'_{ba} = \langle \vec{k} | \mathcal{E}_0 \; (-zq) |100\rangle = -\mathcal{E}_0 q \quad \langle \vec{k} | z | 100 \rangle$$

$\vec{E} = \mathcal{E}_0 \cos\omega t \, \hat{z}$

$\Rightarrow$ $\vec{E} = -\vec{\nabla}\phi \Rightarrow \phi = -\mathcal{E}_0 z \cos\omega t$

$\Rightarrow H' = -\mathcal{E}_0 z q \cos\omega t$

Now we need to calculate the energy density of the continuum,

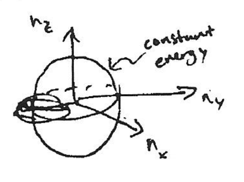
$\rho_b(E)$

$\rho_b(E) = \dfrac{\Delta n}{\Delta E}$ = "# of states" w/ energy between E and E+ΔE

To calculate this, we imagine putting the system in a box w/ sides of length L,

$$E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2)$$

Now lets calculate the # of states w/ energy $\leq E$. Call this $n(E)$. Its useful to draw constant energy surfaces in (n_x, n_y, n_z)-space,



$\therefore$ $n(E) = \frac{1}{8}$ Volume of sphere w/ radius, $\sqrt{\dfrac{2ma^2E}{\pi^2\hbar^2}}$

$\Rightarrow n(E) = \frac{1}{8} \cdot \frac{4}{3}\pi \left(\frac{2ma^2E}{\pi^2\hbar^2}\right)^{3/2}$

Now the energy density is given by,

$$\frac{\Delta n}{\Delta E} = \frac{\partial n}{\partial E} = \frac{1}{\cancel{8}_4} \cdot \frac{\cancel{4}}{\cancel{3}}\pi \cdot \left(\frac{2ma^2}{\pi^2\hbar^2}\right)^{3/2} \cdot \frac{\cancel{3}}{\cancel{2}} E^{1/2}$$

sub in k $\qquad E^{1/2} = \frac{\hbar k}{\sqrt{2m}}$

$$\underbrace{\rho(E) = \frac{\pi}{4}\left(\frac{2ma^2}{\pi^2\hbar^2}\right)^{3/2} E^{1/2}} = \frac{\pi}{4}\left(\frac{2m}{\pi^2\hbar^2}\right)^{3/2} V E^{1/2} = \frac{\cancel{\pi}}{2\cancel{4}} \cdot \frac{\cancel{2}m}{\pi^2\hbar^2} \cdot V \cdot \frac{\cancel{\hbar}k}{\pi \cancel{\hbar}}$$

$$= \underbrace{\frac{mk}{2\pi^2\hbar^2} V}$$

We can assume that the free particle is moving in the $\hat{z}$ direction....

$\vec{k} = k\hat{z}$

$$H'_{ba} = -\mathcal{E}_0 q \cos\omega t \cdot \underbrace{\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, e^{-ikz} \cdot z \frac{1}{\sqrt{4\pi a^3}} e^{-\frac{\sqrt{x^2+y^2+z^2}}{a}}}_{\text{Better to write in spherical coordinates actually,}}$$

$$= \int_0^\infty dr \cdot r^2 \int_0^\pi d\theta \sin\theta \underbrace{\int_0^{2\pi} d\phi}_{=2\pi} \cdot r\cos\theta\, e^{-ikr\cos\theta} \cdot \frac{1}{\sqrt{4\pi a^3}} e^{-\frac{r}{a}}$$

$$= \frac{2\pi}{\sqrt{4\pi a^3}} \int_0^\infty dr\, r^3 e^{-r/a} \int_{-1}^{1} du \cdot u \cdot e^{-ikru}$$

$u = \cos\theta$

$$= \frac{2\pi}{\sqrt{4\pi a^3}} \int_0^\infty dr\, r^{\cancel{3}} e^{-r/a} \frac{2i\left(kr\cos(kr) - \sin(kr)\right)}{k^2 r^{\cancel{2}}}$$

$$= \frac{4\pi i}{\sqrt{4\pi a^3}} \cdot \frac{1}{k^2} \int_0^\infty dr \cdot r\, e^{-r/a} \left(kr\cos(kr) - \sin(kr)\right)$$

$$= \frac{4\pi i}{\sqrt{4\pi a^3}} \frac{1}{\cancel{k^2}} \cdot - \frac{8a^5 k^{\cancel{3}}}{(1 + a^2k^2)^3}$$

$$H'_{ba} = \cdot \frac{32\pi i a^5 k}{(1+a^2k^2)^3 \sqrt{4\pi a^3}}$$

$$\therefore W = \frac{\cancel{2\pi}}{\hbar} \cdot \frac{\varepsilon_0^2 q^2 \cdot \overset{8}{\cancel{32^2}} \cdot \cancel{\pi^2}\, \overset{7}{\cancel{a^{10}}}\, k^2}{(1+a^2k^2)^6 \cdot \cancel{4\pi a^3}} \cdot \frac{mk}{\cancel{2\pi^2}\hbar^2} V$$

$$\boxed{W = \frac{256\, \varepsilon_0^2 q^2 a^7 k^3 m V}{\hbar^3 (1+a^2k^2)^6}}$$

5. (White Light Problem...)

$$R_{a\to b} = \frac{\pi}{3\epsilon_0 \hbar^2} |\wp|^2 \underbrace{\rho(\omega_0)}_{=U_0 \text{ here}}$$

Calculating matrix elements,

$$|\langle 210|\vec{r}|100\rangle|^2, \quad |\langle 21\pm1|\vec{r}|100\rangle|^2, \quad \cancel{|\langle 200|\vec{r}|100\rangle|^2}$$

$=0$ from transition rules...

We computed the relevant angular integrals in problem 2, so just need to compute radial parts

$$\int_0^\infty dr\, r^2 R_{21}^* R_{10} \cdot \underset{(\text{from } \vec{r})}{r} = \int_0^\infty dr \cdot r^3 \cdot \frac{2}{\sqrt{a^3}} e^{-r/a} \cdot \frac{1}{\sqrt{24}} \cdot \frac{1}{\sqrt{a^3}} \cdot \frac{r}{a} e^{-\frac{r}{2a}}$$

$$= \frac{1}{\sqrt{6}\cdot a^3 \cdot a} \int_0^\infty dr \cdot r^4 e^{-\frac{3r}{2a}}$$

$$= \frac{1}{\sqrt{6}\cdot \cancel{a^4}} \cdot \frac{256\, a^{\cancel{5}}}{81} = \frac{256\, a}{81\sqrt{6}}$$

$$\therefore |\langle 210|\vec{r}|100\rangle|^2 = |\langle 21\pm1|\vec{r}|100\rangle|^2 = \left|\frac{256}{81\sqrt{6}}\right|^2 \cdot \left|\frac{1}{\sqrt{3}}\right|^2 a^2$$

$$= \frac{32768}{59049} a^2$$

$$\therefore \quad R_{a\to b} = \frac{\pi q^2 U_0 a^2}{3\epsilon_0 \hbar^2} \cdot \frac{32768}{59049}$$