Physics 110A, Spring 2021
Solution to Homework 11
GSI: Yi-Chuan Lu

1. (a) The magnetic field due to a single plate can be found by Griffiths Example 5.8, and by using superposition principle, we find

$$\boxed{\mathbf{B}=\begin{cases}-\mu_0 kt\hat{\mathbf{y}}, & |z|<d/2,\\ \mathbf{0}, & |z|>d/2.\end{cases}}$$

(b) Using $\oint \mathbf{E}\cdot d\mathbf{l}=-d\Phi/dt$, between the plates,

$$2El=-\frac{d}{dt}\left(\mu_0 kt\times 2zl\right)\Rightarrow \boxed{\mathbf{E}=\mu_0 kz\hat{\mathbf{x}},\ |z|<d/2,}$$

and outside the plates,

$$2El=-\frac{d}{dt}\left(\mu_0 kt\times dl\right)\Rightarrow \boxed{\mathbf{E}=\begin{cases}(\mu_0 kd/2)\,\hat{\mathbf{x}}, & z>d/2,\\ -(\mu_0 kd/2)\,\hat{\mathbf{x}}, & z<-d/2.\end{cases}}$$

2. As the cylinder rotates, it creates a free current $\mathbf{J}=\rho s\omega\hat{\boldsymbol{\phi}}=a\omega\hat{\boldsymbol{\phi}}$. Using Ampére's law for $\mathbf{H}$ with a square loop as in Griffiths Fig.5.37,

$$HL=L\int_s^R Jds=La\omega\left(R-s\right)\Rightarrow \mathbf{H}=\begin{cases}\omega a\left(R-s\right)\hat{\mathbf{z}}, & s<R,\\ \mathbf{0}, & s>R.\end{cases}$$

Since the material is linear,

$$\mathbf{B}=\mu\mathbf{H}=\begin{cases}\mu\omega a\left(R-s\right)\hat{\mathbf{z}}, & s<R,\\ \mathbf{0}, & s>R.\end{cases}$$

When ω decreases to 0, the B field decreases to 0 too, and thus creates an induced E field to rotate the circular loop. The induced E on the circular loop can be found by $\oint \mathbf{E}\cdot d\mathbf{l}=-d\Phi/dt$:

$$E2\pi L=-\frac{d}{dt}\int_0^R \mu\omega a\left(R-s\right)2\pi s ds=-\frac{d\omega}{dt}\frac{1}{3}\pi\mu aR^3\Rightarrow E=-\frac{d\omega}{dt}\frac{\mu aR^3}{6L}.$$

The torque acting on a wire segment dl is $d\boldsymbol{\tau}=L\left(\lambda dl\right)E\hat{\mathbf{z}}$, so the total torque acting on the entire wire is

$$\boldsymbol{\tau}=L\left(\lambda 2\pi L\right)E\hat{\mathbf{z}}=-\frac{d\omega}{dt}\frac{1}{3}\lambda\pi\mu aLR^3\hat{\mathbf{z}},$$

so

$$\mathbf{L}=\int_0^\infty \boldsymbol{\tau}dt=\frac{1}{3}\lambda\pi\mu aLR^3\hat{\mathbf{z}}\int_\omega^0 -d\omega=\boxed{\frac{1}{3}\lambda\pi\mu\omega aLR^3\hat{\mathbf{z}}.}$$

(a) We rewrite Ohm's law as

$$\frac{\mathbf{J}}{\sigma} = \mathbf{E} + \mathbf{v} \times \mathbf{B}.$$

If $\sigma \to \infty$ while $\mathbf{J}$ remains finite, the left hand side should be zero, and therefore $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}$. Taking the curl of this equation, and apply Faraday's law $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, we obtain

$$-\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) = \mathbf{0}.$$

(b) You may use the argument we used in the discussion, or follow Griffiths. The point is of this theorem is that in a fluid with $\sigma = \infty$,

$$\frac{d\Phi}{dt} = 0,$$

which means if we follow any fluid element, the change in magnetic flux (or magnetic field line) through that fluid element is zero. This is what Griffiths means by "the field lines are frozen in the fluid."