Physics 110A, Spring 2021
Solution to Homework 8
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1. (a) Since there is no electric field outside the parallel plates,

$$\sigma_b = -\frac{\chi_e}{1+\chi_e}\sigma_f + \frac{\chi_e}{1+\chi_e}\left(\mathbf{E}_{\text{above}} \cdot \hat{\mathbf{n}}\right) = -\frac{\chi_e}{1+\chi_e}\sigma_f$$

(prove this relation yourself if you haven't done so.) If we integrate all the surface charge, we will get $Q_b = -\frac{\chi_e}{1+\chi_e}Q_f$. (Left plate has positive free charge, so the bound charge there is negative; similar for the charges on the right.) However, in this problem, the notation is that Q_b stands for the positive bound charge on the right, so $\boxed{Q_b = \frac{\chi_e}{1+\chi_e}Q_f.}$

(b) The system has plane symmetry, so we can use Gauss's law and find $\mathbf{D} = \frac{Q_f}{A}\hat{\mathbf{x}}$, and use $\mathbf{D} = \epsilon\mathbf{E}$ to find the electric field $\boxed{\mathbf{E} = \frac{Q_f}{\epsilon A}\hat{\mathbf{x}}.}$

(c) At some intermediate step when the plates have free charge $\pm q_f$, the electric field in the dielectric material is $\mathbf{E} = \frac{q_f}{\epsilon A}\hat{\mathbf{x}}$, and therefore to bring an additional charge dq_f to the left plate, the work needed is $dW = \left(\frac{q_f d}{A\epsilon}\right) dq_f$. Integrate this and we get the total work needed

$$W = \int_0^{Q_f} \left(\frac{q_f d}{\epsilon A}\right) dq_f = \boxed{\frac{Q_f^2 d}{2\epsilon A}.}$$

(d) Using $\mathbf{D}$ and $\mathbf{E}$ we found in (b), we have $W = \frac{1}{2}\left(\frac{Q_f}{A}\right)\left(\frac{Q_f}{\epsilon A}\right) Ad = \boxed{\frac{Q_f^2 d}{2\epsilon A}.}$

(e) Imagine we fix the left plate (with charge Q_f), and move the right one. The field felt by the right plate is $\frac{Q_f}{2\epsilon_0 A}\hat{\mathbf{x}}$, so to move this plate by a distance d, the work we need to do is $\boxed{W_1 = \frac{Q_f^2 d}{2\epsilon_0 A}.}$

(f) Similarly, we can fix the left bound charge ($-Q_b$) and move right bound charge (Q_b), which experiences an electric field of $\frac{Q_f}{\epsilon_0 A}\hat{\mathbf{x}} + \frac{-Q_b}{2\epsilon_0 A}\hat{\mathbf{x}}$. So the work needed to move Q_b by a distance d is $W_2 = -Q_b\left(\frac{Q_f - Q_b/2}{\epsilon_0 A}\right) d$. If we express Q_b in terms of Q_f via (a), we get $\boxed{W_2 = \left(\frac{\epsilon_0^2}{\epsilon^2} - 1\right)\frac{Q_f^2 d}{2\epsilon_0 A}.}$

(g) Using the results from (b), $W = \frac{\epsilon_0}{2}\left(\frac{Q_f}{\epsilon A}\right)^2 Ad = \boxed{\frac{\epsilon_0 Q_f^2 d}{2\epsilon^2 A}.}$

2. Since the system has spherical symmetry, we can use Gauss's law for $\mathbf{D}$ and find $\mathbf{D} = \mathbf{0}$ everywhere. Using $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$, we get

$$\boxed{\mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0} = \begin{cases} -(k/r)\,\hat{\mathbf{r}}, & a < r < b, \\ \mathbf{0}, & \text{otherwise.} \end{cases}}$$

3. The inductor must have a constant potential, which we assume to be V_0 for now. Thus, at $r = a$, the boundary condition is

$$V = V_0 \text{ for } r = a.$$

At $r = b$, the bound charge is $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{r}} = P_0 \cos\theta$, and therefore the boundary conditions are

$$\begin{cases} V_{\text{above}} = V_{\text{below}}, \\ -\dfrac{\partial V_{\text{above}}}{\partial r} + \dfrac{\partial V_{\text{below}}}{\partial r} = \dfrac{P_0}{\epsilon_0}\cos\theta, \end{cases} \quad \text{for } r = b.$$

Note that there is also some bound charge at $r = a$, but we don't know the (free) surface charge on the conductor, and therefore it is not possible to write down a similar Neumann boundary condition there.

Since the system has azimuthal symmetry, the general solution can be written as

$$V = \begin{cases} V_0, & r < a, \\ \displaystyle\sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta), & a < r < b, \\ \displaystyle\sum_{l=0}^{\infty} \frac{C_l}{r^{l+1}} P_l(\cos\theta), & b < r. \end{cases}$$

The boundary condition at $r = a$ gives

$$\sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}} \right) P_l(\cos\theta) = V_0,$$

which implies

$$A_0 + \frac{B_0}{a} = V_0 \text{ and } A_l a^l + \frac{B_l}{a^{l+1}} = 0 \text{ for } l = 1, 2, 3 \cdots.$$

The boundary condition at $r = b$ gives

$$\begin{aligned} \sum_{l=0}^{\infty} \left(A_l b^l + \frac{B_l}{b^{l+1}} - \frac{C_l}{b^{l+1}} \right) P_l(\cos\theta) &= 0 \\ \sum_{l=0}^{\infty} \left(l A_l b^{l-1} - (l+1)\frac{B_l}{b^{l+2}} + (l+1)\frac{C_l}{b^{l+2}} \right) P_l(\cos\theta) &= \frac{P_0}{\epsilon_0}\cos\theta, \end{aligned}$$

which implies

$$\begin{aligned} A_l b^l + \frac{B_l}{b^{l+1}} - \frac{C_l}{b^{l+1}} &= 0 \text{ for } l = 0, 1, 2, 3 \cdots, \\ A_1 - 2\frac{B_1}{b^3} + 2\frac{C_1}{b^3} &= \frac{P_0}{\epsilon_0} \text{ and } l A_l b^{l-1} - (l+1)\frac{B_l}{b^{l+2}} + (l+1)\frac{C_l}{b^{l+2}} = 0 \text{ for } l = 0, 2, 3, \cdots. \end{aligned}$$

We focus on (A_0, B_0, C_0) now. The equations they satisfy are

$$A_0 + \frac{B_0}{a} = V_0,\ A_0 + \frac{B_0}{b} - \frac{C_0}{b} = 0,\quad -\frac{B_0}{b^2} + \frac{C_0}{b^2} = 0,$$

which give the solution $A_0 = 0$, $B_0 = C_0 = aV_0$. Next we focus on (A_1, B_1, C_1), which are governed by

$$A_1 a + \frac{B_1}{a^2} = 0,\ A_1 b + \frac{B_1}{b^2} - \frac{C_1}{b^2} = 0,\ A_1 - 2\frac{B_1}{b^3} + 2\frac{C_1}{b^3} = \frac{P_0}{\epsilon_0}.$$

The equations give $A_1 = \dfrac{P_0}{3\epsilon_0}$, $B_1 = -\dfrac{a^3 P_0}{3\epsilon_0}$, $C_1 = (b^3 - a^3)\dfrac{P_0}{3\epsilon_0}$. For $l \geq 2$, the coefficients satisfy

$$\begin{aligned} A_l a^l + \frac{B_l}{a^{l+1}} &= 0, \\ A_l b^l + \frac{B_l}{b^{l+1}} - \frac{C_l}{b^{l+1}} &= 0, \\ l A_l b^{l-1} - (l+1)\frac{B_l}{b^{l+2}} + (l+1)\frac{C_l}{b^{l+2}} &= 0, \end{aligned}$$

which are homogeneous in A_l, B_l and C_l. Thus $A_l = B_l = C_l = 0$. So the potential is

$$V = \begin{cases} V_0, & r < a, \\ \dfrac{aV_0}{r} + \dfrac{P_0}{3\epsilon_0}\left(r - \dfrac{a^3}{r^2}\right)\cos\theta, & a < r < b, \\ \dfrac{aV_0}{r} + \dfrac{P_0}{3\epsilon_0}\dfrac{b^3 - a^3}{r^2}\cos\theta, & b < r. \end{cases}$$

To find V_0, we recognize that aV_0/r from the last line should be the monopole potential, and since the total charge of the system is zero, V_0 should be zero too. Thus our final answer is

$$\boxed{V = \begin{cases} 0, & r < a, \\ \dfrac{P_0}{3\epsilon_0}\left(r - \dfrac{a^3}{r^2}\right)\cos\theta, & a < r < b, \\ \dfrac{P_0}{3\epsilon_0}\dfrac{b^3 - a^3}{r^2}\cos\theta, & b < r. \end{cases}}$$