Physics 110A, Spring 2021
Solution to Homework 5
GSI: Yi-Chuan Lu

1. Using divergence theorem,

$$\int_{\text{all space}} \left(V_1\nabla^2 V_2 - V_2\nabla^2 V_1\right) d\tau = \oint_{\text{surface at }\infty} \left(V_1\boldsymbol{\nabla}V_2 - V_2\boldsymbol{\nabla}V_1\right)\cdot d\mathbf{a} = 0.$$

Replacing $\nabla^2 V_1$ with $-\rho_1/\epsilon_0$ and $\nabla^2 V_2$ with $-\rho_2/\epsilon_0$, we get the Green's reciprocity relation

$$\boxed{\int_{\text{all space}} V_1\rho_2 d\tau = \int_{\text{all space}} V_2\rho_1 d\tau.}$$

To prove the mean value theorem, we let ρ_1 and V_1 be the given charge density ρ and its potential V. Note that $\rho = 0$ within the spherical surface of radius R by the assumption of the mean value theorem. To construct a surface integral of V over that spherical surface, we let the second system (ρ_2, V_2) consist of a uniform surface charge density $\sigma = q/4\pi R^2$ distributed exactly on that surface, with its potential being

$$V_2 = \frac{1}{4\pi\epsilon_0}\begin{cases} \dfrac{q}{r}, & r > R, \\ \dfrac{q}{R}, & r < R. \end{cases}$$

With these setups, the left hand side of Green's reciprocity relation becomes

$$\int_{\text{all space}} V_1\rho_2 d\tau = \oint_{\mathcal{S}} V\sigma da = \frac{q}{4\pi R^2}\oint_{\mathcal{S}} V da,$$

and the right hand side becomes

$$\begin{aligned}\int_{\text{all space}} V_2\rho_1 d\tau &= \int_{r<R}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{R}\right)\rho d\tau + \int_{r>R}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right)\rho d\tau \\ &= \int_{r>R}\frac{1}{4\pi\epsilon_0}\frac{q}{r}\rho d\tau = q\int_{r>R}\frac{1}{4\pi\epsilon_0}\frac{\rho(\mathbf{r})}{|\mathbf{0}-\mathbf{r}|}d\tau = qV(\mathbf{0}).\end{aligned}$$

Comparing the two results, one gets $\boxed{\dfrac{1}{4\pi R^2}\oint_{\mathcal{S}} V da = V(\mathbf{0}).}$

2. Assume the coordinates of Q is $(x_0, y_0, 0)$, then $\tan\alpha = x_0/y_0$. To make the potential zero on each plate, we need to place three image charges: $-Q\ (x_0, -y_0, 0)$, $-Q\ (-x_0, y_0, 0)$, $Q\ (-x_0, -y_0, 0)$. With these four charges (1 real and 3 virtual), the potential in the first quadrant is

$$\begin{aligned} V = \frac{Q}{4\pi\epsilon_0}\Bigg[&\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}} - \frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2+z^2}} \\ &- \frac{1}{\sqrt{(x+x_0)^2+(y-y_0)^2+z^2}} + \frac{1}{\sqrt{(x+x_0)^2+(y+y_0)^2+z^2}}\Bigg].\end{aligned}$$

The total induced surface charge on the bottom plate is

$$\begin{aligned}
Q_{\text{bottom}} &= \int_0^\infty \int_{-\infty}^\infty -\epsilon_0 \left.\frac{\partial V}{\partial y}\right|_{y=0} dz dx \\
&= -\frac{Q y_0}{2\pi} \int_0^\infty \int_{-\infty}^\infty \left\{ \frac{1}{\left[(x-x_0)^2 + y_0^2 + z^2\right]^{3/2}} - \frac{1}{\left[(x+x_0)^2 + y_0^2 + z^2\right]^{3/2}} \right\} dz dx \\
&= -\frac{Q y_0}{2\pi} \int_0^\infty \left[\frac{2}{(x-x_0)^2 + y_0^2} - \frac{2}{(x+x_0)^2 + y_0^2} \right] dx \\
&= -\frac{2Q}{\pi} \tan^{-1}\left(\frac{x_0}{y_0}\right) = \boxed{-\frac{2\alpha}{\pi} Q.}
\end{aligned}$$

Similarly, the total charge induced on the vertical plate is

$$\begin{aligned}
Q_{\text{vertical}} &= \int_0^\infty \int_{-\infty}^\infty -\epsilon_0 \left.\frac{\partial V}{\partial x}\right|_{y=x} dz dy \\
&= -\frac{Q x_0}{2\pi} \int_0^\infty \int_{-\infty}^\infty \left\{ \frac{1}{\left[x_0^2 + (y-y_0)^2 + z^2\right]^{3/2}} - \frac{1}{\left[x_0^2 + (y+y_0)^2 + z^2\right]^{3/2}} \right\} dz dy \\
&= -\frac{Q x_0}{2\pi} \int_0^\infty \left[\frac{2}{x_0^2 + (y-y_0)^2} - \frac{2}{x_0^2 + (y+y_0)^2} \right] dy \\
&= -\frac{2Q}{\pi} \tan^{-1}\left(\frac{y_0}{x_0}\right) = -\frac{2Q}{\pi}\left(\frac{\pi}{2} - \alpha\right) = \boxed{\frac{2\alpha}{\pi} Q - Q.}
\end{aligned}$$

Note that the sum of the induced charge on both plate is $-Q$, which is equal to the total amount of the image charge.

3. (a) When we perturb the charge distribution, we fix the total charge $\int_{\mathcal{V}} \rho d\tau = \int_{\mathcal{V}} (\rho + \delta\rho)\, d\tau = Q$, so $\int_{\mathcal{V}} \delta\rho d\tau = 0$. Since $\nabla^2 V = -\rho/\epsilon_0$ and $\nabla^2 (V + \delta V) = -(\rho + \delta\rho)/\epsilon_0$, we also have $\nabla^2 \delta V = -\delta\rho/\epsilon_0$. Therefore, in terms of the potential, the equation $\int_{\mathcal{V}} \delta\rho d\tau = 0$ can be expressed as $\boxed{\int_{\mathcal{V}} \nabla^2 \delta V\, d\tau = 0.}$ Similarly, since $\delta\rho \equiv 0$ outside $\mathcal{V}$, in terms of potential, this can be expressed as $\boxed{\nabla^2 \delta V \equiv 0 \text{ outside } \mathcal{V}.}$

   (b)

$$\begin{aligned}
\int_{\text{all space}} \nabla V^* \cdot \nabla \delta V\, d\tau &= \int_{\text{all space}} \left[\nabla \cdot (V^* \nabla \delta V) - V^* \nabla^2 \delta V\right] d\tau = -\int_{\text{all space}} V^* \nabla^2 \delta V\, d\tau \\
&= -\int_{\mathcal{V}} V^* \nabla^2 \delta V\, d\tau - \int_{\text{outside } \mathcal{V}} V^* \nabla^2 \delta V\, d\tau \\
&= -V^* \int_{\mathcal{V}} \nabla^2 \delta V\, d\tau - \int_{\text{outside } \mathcal{V}} V^* \nabla^2 \delta V\, d\tau = -0 - 0 = 0.
\end{aligned}$$

   (c) The perturbed system has new energy $W = W^* + 0 + \int_{\text{all space}} |\nabla \delta V|^2\, d\tau$. The last term is positive definite, so any perturbation δV will make W larger than W^*.