Physics 105, Spring 2021, Reinsch

Homework Assignment 2

Due Thursday, February 4, 11:59 pm

Problem 1

In this problem we will use the notation of Chapter 6, with x being the independent variable and y being the dependent variable. The function $f(y, y', x)$ is defined to be

$$f(y, y', x) = (y')^2 + a\, y^3\, y' + b\, y^2 \tag{1}$$

where a and b are positive constants.

(a) Use the Euler-Lagrange Equation to get a differential equation for $y(x)$.

(b) Find the general solution to the differential equation you found in part (a). Re-write if necessary so that it only involves real numbers.

(c) If the constant b is negative, find the general solution to the differential equation you found in part (a). Again we would like formulas that involve only real numbers. [For example, no $\sin(\omega t)$ where ω is a complex number]

Problem 2

Repeat Problem 1, parts (a) and (b), for the following

$$f(y, y', x) = (y')^2 + a\, x^3\, y \tag{2}$$

where a can be either positive or negative.

Problem 3

Taylor, Problem 6.12

Problem 4

In this problem we study the brachistochrone in a non-uniform gravitational field. We are building a roller coaster on a small uniform spherical asteroid that does not rotate, so we do not have to worry about the Coriolis force or the centrifugal force. We use plane polar coordinates (r, ϕ) with the origin at the center of the asteroid. The gravitational potential energy of the roller coaster car (see Example 6.2) is now $-\gamma/r$, where γ is a constant. The car starts from rest at $r = r_o$ and $\phi = 0$.

(a) What is the speed of the car as a function of its location (not as a function of time)?

(b) We will work with the function $\phi(r)$. Write an integral for the time it takes to get to (r_2, ϕ_2) along a given path $\phi(r)$, similar to Eq. (6.19). Write out the definition of your function $f(\phi, \phi', r)$.

(c) Your function $f(\phi, \phi', r)$ should be independent of one of its arguments. Use this fact to obtain a first-order differential equation for $\phi(r)$.

(d) Using your result from part (c), write an integral for $\phi(r)$. You do not have to solve the integral.

Problem 5

In this problem we use a function $z = h(x, y)$ to describe an arbitrary surface above the xy-plane. The point P_1 is $(x_1, y_1, h(x_1, y_1))$, and the point P_2 is $(x_2, y_2, h(x_2, y_2))$.

(a) Given a function $y(x)$ with $y(x_1) = y_1$ and $y(x_2) = y_2$ write an integral for S, the length of the path defined on the surface, going from P_1 to P_2. Write out the definition of your function $f(y, y', x)$. We are interested in finding curves that make S stationary.

(b) For this part of the problem we treat the case $h(x, y) = A + Bx + Cy$, where A, B and C are constants. The surface defined in this way is an inclined plane. Write out the Euler-Lagrange equation. Find the general solution for $y(x)$.

(c) For this part of the problem we treat the case $h(x, y) = \sqrt{R^2 - x^2}$, where R is a positive constant, and we consider only the region that has $|x| < R$. Write out the Euler-Lagrange equation. You should be able to find a first integral. You do not have to solve the differential equation. Instead, we use our knowledge of geodesics on a cylinder to write $x = R\sin(\alpha y + \beta)$ for the inverse of the function we want. Solve this to get $y(x)$ at least locally and plug this into your differential equation to check its validity. α and β are constants.

(d) For this part of the problem we treat the case $h(x, y) = \sqrt{R^2 - y^2}$, where R is a positive constant, and we consider only the region that has $|y| < R$. Write out the Euler-Lagrange equation for $y(x)$. You should be able to find a first integral [using a different method than in part (c)]. Verify that $y = R\sin(\gamma x + \delta)$ is a solution. γ and δ are constants.

(e) For this part of the problem we treat the case $h(x, y) = \sqrt{R^2 - x^2 - y^2}$, where R is a positive constant, and we consider only the region that has $R^2 > x^2 + y^2$. Write out the Euler-Lagrange equation for $y(x)$. You do not have to find the general solution. Verify $y = \frac{1}{2}\left(\sqrt{2R^2 - 3x^2} - x\right)$ is a solution. Is this solution the projection of a part of a great circle onto the plane?