Physics 105, Spring 2021, Reinsch

Homework Assignment 5

Due February 25, 11:59 pm

Problem 1

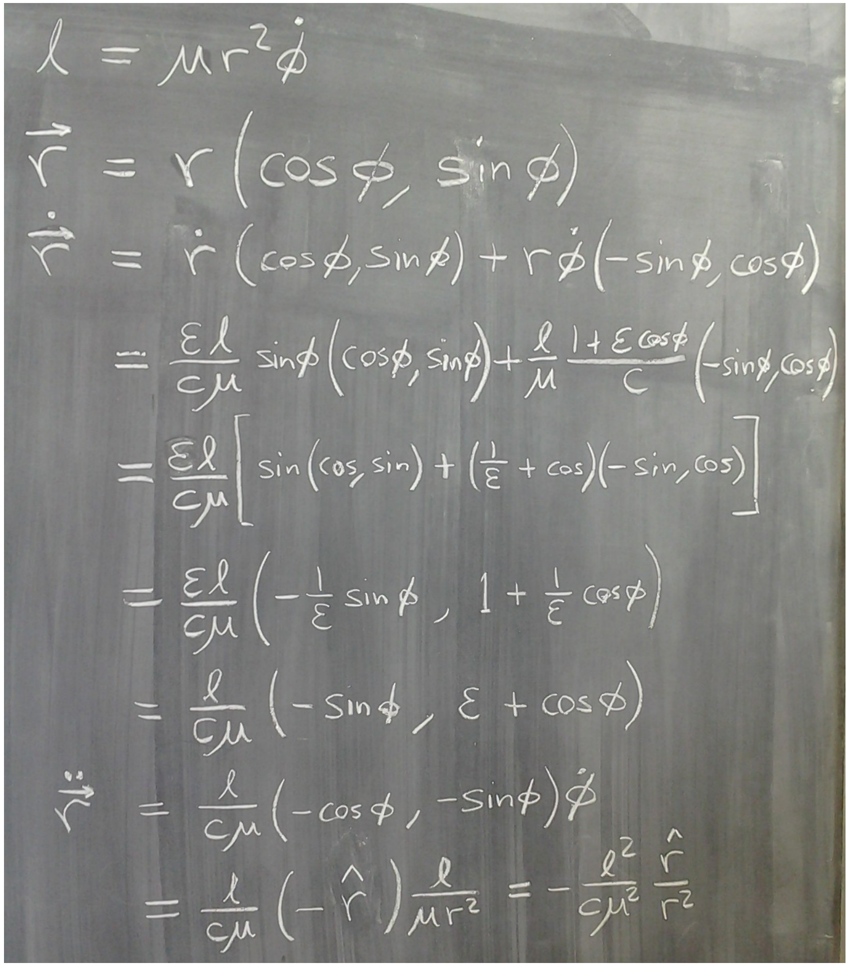
In lecture we discussed the calculations shown at right.

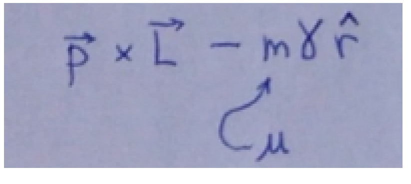
On line 7 it is proven that for a particle moving along an elliptical orbit, the velocity vector traces out a circle in velocity space. The circle is not centered at the origin. On Line 9 we have a direct verification of Newton's Second Law with the appropriate inverse-square force law. What does the prefactor in the last equation simplify to?

Line 1 is from Eq. (8.21) in the text. Line 2 follows from the definition of plane polar coordinates. Line 3 follows from time-differentiation of Line 2. Explain the derivation of Line 4.

The vectors are expressed as ordered pairs, their *x* and *y* components. The *z* components are 0. Treating the vectors as three-component vectors, calculate the second right-hand side of Eq. (8.19). In doing this, use Line 7 for the velocity. Is the result what you expect?

The Laplace-Runge-Lenz vector **A** is defined below. Calculate this vector using the methods of this problem and show that it is a constant of the motion.





Problem 2

Begin by reading (not solving) this problem and its solution on the next page. Our new prob. 2 is on the page after that.

You have discovered a moon (mass M) that is perfectly spherical (radius R), not rotating, and has no atmosphere. Standing on the moon, you throw a stone horizontally with initial speed v_o, releasing it a height h above the surface, and it lands a distance L from your feet, as measured on the surface of the moon. Both h and L are much less than R.

If one were to ignore the curvature of the surface of the moon and assume a uniform gravitational field $\mathbf{g}$ then one would predict $L = v_o\sqrt{2h/g}$, where $g = GM/R^2$. The field is actually not uniform, and the trajectory is actually a section of an ellipse, not a parabola. How large is the error in the formula for L presented above? Compute this using results for Kepler orbits, rather than integrating a $\Delta\mathbf{g}$ along a parabolic trajectory. You can present either the exact result or an approximation; either one will earn full credit.

Note that we have not mentioned the mass of the stone. As discussed in the text, it is useful to understand that a certain limit exists, the limit in which the mass of the stone becomes infinitesimally small compared to the mass of the moon. The trajectory of the stone goes to a well defined limit in this case. For any nonzero value of the mass of the stone, the stone actually accelerates the moon and we can easily treat the problem exactly by going to the CM frame using the methods described in the chapter. However this is cumbersome and it is better to understand this limiting process and the existence of the limit. In this sense, you can use μ for the mass of the stone in your calculations. The answer does not depend on μ.

Hints

- It is easier to compute the exact result than the approximation. Again, either one will earn full credit. The approximate result is more illuminating. The hints below are for the approximation.
- In finding the approximate result, you will be studying a certain arccos(ξ). It is better to study the behavior of the quantity ξ before bringing in a series for the arccos.
- There are two small parameters in the problem. They are $\left(\frac{h}{R}\right)$ and $\left(\frac{v_o^2}{gR}\right)$.
- As you work out a series for ξ, you will see a term proportional to $\left(\frac{h}{R}\right)$ times $\left(\frac{v_o^2}{gR}\right)$. This one will result in the simple formula for L given in the problem statement.
- The next order in smallness has an $\left(\frac{h}{R}\right)^2\left(\frac{v_o^2}{gR}\right)$ term and an $\left(\frac{h}{R}\right)\left(\frac{v_o^2}{gR}\right)^2$ term.
- The series expansion for arccos(ξ) will be tricky because you will be expanding about a very unfortunate point. This can be done by bringing in fractional powers. Alternatively, you can use the fact that if $\cos\beta = 1-\delta$, then $\sin\beta = \sqrt{2\delta-\delta^2}$. This is useful when δ is small.
- As a motivational remark, you may wish to ponder the following question. If you throw one stone with $h = 10$ m and $v_o = 20$ m/s and another stone with $h = 40$ m and $v_o = 10$ m/s, they would land at the same spot according to the simple formula for L. Do they really land at the same spot?

Solution to problem from previous semester

$\gamma = GM\mu$

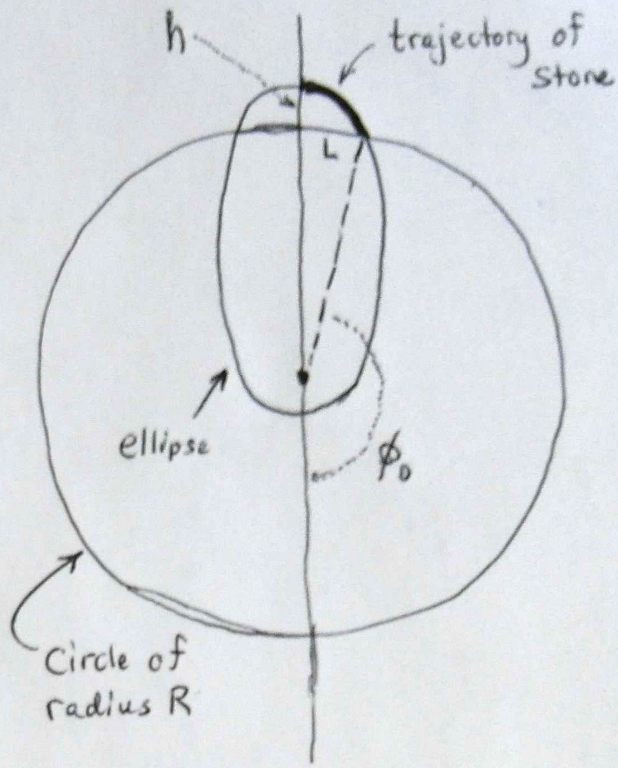
$\boxed{\ell = \mu(R+h)v_0}$

$\boxed{c = \dfrac{\ell^2}{\gamma\mu}}$

$R+h = \dfrac{c}{1-\varepsilon}$ (for $\phi = \pi$)

$1-\varepsilon = \dfrac{c}{R+h}$

$\boxed{\varepsilon = 1 - \dfrac{c}{R+h}}$



At the landing point (where the stone hits the surface)

$R = r(\phi)$

$\Rightarrow \dfrac{c}{R} = 1 + \varepsilon\cos\phi_0 \quad \Rightarrow \quad \boxed{\cos\phi_0 = \dfrac{\frac{c}{R}-1}{\varepsilon}}$

$\boxed{L = (\pi - \phi_0)R}$ exact L

$\boxed{\text{error} = V_0\sqrt{\dfrac{2h}{g}} - L_{exact}}$

<u>Approximation</u> $\cos(\pi-\phi_0) = -\cos\phi_0 = \dfrac{1 - c/R}{1 - c/(R+h)} = 1 - \delta,$

where $\delta = \dfrac{c/R - c/(R+h)}{1 - c/(R+h)}$ small and positive

Note that $\sin(\pi-\phi_0) = \sqrt{2\delta - \delta^2}$. Define $\boxed{\delta_v = \dfrac{V_0^2}{gR}, \ \delta_h = \dfrac{h}{R}}$ (small & positive)

$\delta = \delta_v\delta_h \dfrac{1+\delta_h}{1-\delta_v-\delta_v\delta_h}$. Everything has been exact so far.

$\delta \approx \delta_v\delta_h(1+\delta_v+\delta_h)$. Also use $\arcsin(\lambda) = \lambda + \frac{\lambda^3}{6} + \ldots$

Thus $\pi - \phi_0 = \sqrt{2\delta-\delta^2} + \frac{1}{6}(2\delta-\delta^2)^{3/2} + \ldots = \sqrt{2\delta}\sqrt{1-\delta/2}\left[1 + \frac{1}{6}(2\delta) + \ldots\right]$

$\approx \sqrt{2\delta_v\delta_h}\sqrt{1+\delta_v+\delta_h} \approx \underbrace{\sqrt{2\delta_v\delta_h}}\left(1 + \tfrac{1}{2}\delta_v + \tfrac{1}{2}\delta_h\right)$

This is L_{approx}/R

$\boxed{L - L_{approx} \approx L_{approx}(\delta_v + \delta_h)/2}$

Problem 2, continued

This is the actual statement of the problem, based on the material on the previous two pages.

Draw a copy of the diagram on the previous page. The calculation on the previous page uses the conventions from Chapter 8 of our text, including the plane polar coordinates and the Cartesian axes. The origin of these *xy* axes is at the center of the moon, and the *x* axis points towards the point on the ellipse that is closest to the origin. The y axis points to the right. Indicate these axes in your diagram. Next, we define another coordinate system with axes called *u* and *w*. The *uw* origin is at your feet. The *w* axis points straight up (in the -*x* direction), and the *u* axis is orthogonal to the *w* axis, pointing to the right. Thus the *u* axis is tangent to the surface of the moon at the point where you are standing.

(a) Calculate the exact *u* value where the trajectory crosses the *u* axis.

(b) As on the previous page, calculate the leading terms in the difference between your answer to part (a) and the simple formula that is obtained by assuming a uniform gravitational field (the simple formula is presented in the second paragraph of the original problem statement).

Problem 3

Begin by reading (not solving) this problem and its solution on the next two pages. Our new prob. 3 is on the page after that.

Two particles of mass m_1 and m_2 move in an otherwise empty space and interact via Newtonian gravity. The initial positions (at time $t = 0$) are $\mathbf{r}_{10}$ and $\mathbf{r}_{20}$, and the initial velocities are $\mathbf{v}_{10}$ and $\mathbf{v}_{20}$.

(a) The parameters are such that the masses are NOT gravitationally bound. How would you express this statement mathematically in terms of the four vectors given above?

(b) The parameters are such that the initial value of the time derivative of the inter-particle distance is negative. How would you express this statement mathematically in terms of the four vectors given above?

(c) What is the minimal value of the inter-particle distance?

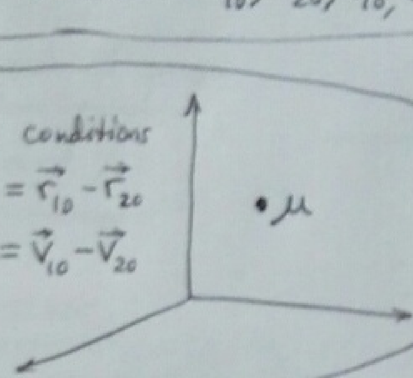
(d) As t goes to infinity, what are the asymptotic values of the velocity vectors?

Prob. 3

(a)

Initial conditions $\vec{r}_{10}, \vec{r}_{20}, \vec{V}_{10}, \vec{V}_{20}$ $\longrightarrow$ $\boxed{6\text{ DOF}}$

Initial conditions
$\vec{r}(0) = \vec{r}_{10} - \vec{r}_{20}$
$\dot{\vec{r}}(0) = \vec{V}_{10} - \vec{V}_{20}$



$\longrightarrow$ $\boxed{3\text{DOF}}$

$\boxed{6\text{ DOF}} \to \boxed{3\text{DOF}} \to \boxed{2\text{DOF}} \to \boxed{1\text{DOF}}$

In Eq. (8.16), $\mathcal{L}_{rel}$ is time independent

$$\Rightarrow \frac{1}{2}\mu\dot{\vec{r}}^{\,2} + U(r) = \text{const}$$

Not bound $\Leftrightarrow$ $\frac{1}{2}\mu\left(\vec{V}_{10} - \vec{V}_{20}\right)^2 + \left(-\frac{Gm_1m_2}{|\vec{r}_{10} - \vec{r}_{20}|}\right) \geq 0$

$$\Leftrightarrow \boxed{\left(\vec{V}_{10} - \vec{V}_{20}\right)^2 \geq \frac{2G(m_1+m_2)}{|\vec{r}_{10} - \vec{r}_{20}|}}$$

(b) $r^2 = (\vec{r}_1 - \vec{r}_2)^2 \Rightarrow 2r\dot{r} = 2(\vec{r}_1 - \vec{r}_2)\cdot(\dot{\vec{r}}_1 - \dot{\vec{r}}_2)$

$$\boxed{\left(\vec{r}_{10} - \vec{r}_{20}\right)\cdot\left(\vec{V}_{10} - \vec{V}_{20}\right) < 0}$$

(c) For 3DOF problem, $\boxed{\vec{\ell} = \mu\left(\vec{r}_{10} - \vec{r}_{20}\right)\times\left(\vec{V}_{10} - \vec{V}_{20}\right)}$

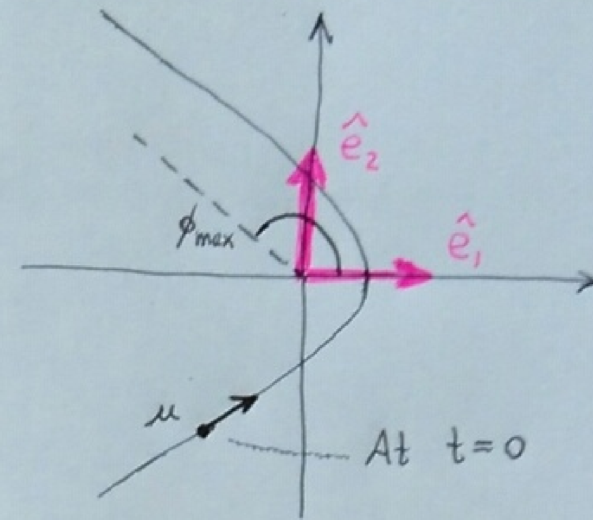
$$\boxed{E = \frac{1}{2}\mu\left(\vec{V}_{10} - \vec{V}_{20}\right)^2 - \frac{Gm_1m_2}{|\vec{r}_{10} - \vec{r}_{20}|}}$$

$\boxed{c = \frac{\ell^2}{\gamma\mu}}$ $\quad E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1) = \frac{\gamma}{2c}(\varepsilon^2 - 1) \Rightarrow \boxed{\varepsilon = \sqrt{\frac{2Ec}{\gamma} + 1}}$

$\boxed{r_{min} = c/(1+\varepsilon)}$

Prob 3(d)

See diagram at right.



Counterclockwise $\Rightarrow$ $\vec{\ell}$ out of page

$$\Rightarrow \boxed{\hat{e}_3 = \vec{\ell}/\ell}$$

Laplace-Runge-Lenz vector

$$\boxed{\vec{A} = \mu(\vec{v}_{10} - \vec{v}_{20}) \times \vec{\ell} - \mu\gamma \frac{\vec{r}_{10} - \vec{r}_{20}}{|\vec{r}_{10} - \vec{r}_{20}|}}$$

$$\boxed{\hat{e}_1 = \vec{A}/A, \quad \hat{e}_2 = \hat{e}_3 \times \hat{e}_1}$$

Eq. $(8.60\tfrac{1}{2})$ $\Rightarrow$ $\boxed{\phi_{max} = \arccos(-1/\varepsilon)}$

For the μ particle, $(t \to \infty)$

$$\boxed{\begin{aligned} v_\infty &= \sqrt{2E/\mu} \\ \vec{v}_\infty &= v_\infty\left(\hat{e}_1 \cos\phi_{max} + \hat{e}_2 \sin\phi_{max}\right) \end{aligned}}$$

$$m_1\vec{v}_{10} + m_2\vec{v}_{20} = m_1\vec{v}_{1\infty} + m_2\vec{v}_{2\infty} = \vec{P}_{total}$$

$$\vec{v}_{1\infty} - \vec{v}_{2\infty} = \vec{v}_\infty$$

$$M\vec{v}_{1\infty} + 0 = \vec{P}_{total} + m_2\vec{v}_\infty$$

$$\boxed{\vec{v}_{1\infty} = \frac{1}{M}\vec{P}_{total} + \frac{m_2}{M}\vec{v}_\infty}$$

$$\boxed{\vec{v}_{2\infty} = \vec{v}_{1\infty} - \vec{v}_\infty}$$

Problem 3

This problem can be solved by entering formulas into a spreadsheet or by writing a computer program in the language of your choosing.   If you are new to programming, see remarks about Python from earlier in the course.

<u>Part (a)</u>: Using the Cassini CSV file provided in the “data” sub-folder, make a plot like the one shown on the next page. Next, make a plot with just one curve which is the sum of the squares of the velocity components. This plot will make it clear that Cassini has gained energy via this Gravity Assist Maneuver (also called “swing by” or “fly by,” unfortunate because it is not flying; it is falling). The axes must be labeled and have units [hours and $(AU/day)^2$]. The Cassini CSV file contains the following columns. We will not use the last three. Each line has a trailing comma.

```
JDTDB    Epoch Julian Date, Barycentric Dynamical Time, decimal days
JDTDB    Epoch Julian Date, Barycentric Dynamical Time, text string
  X      x-component of position vector (AU)
  Y      y-component of position vector (AU)
  Z      z-component of position vector (AU)
  VX     x-component of velocity vector (AU/day)
  VY     y-component of velocity vector (AU/day)
  VZ     z-component of velocity vector (AU/day)
  LT     One-way down-leg Newtonian light-time (day)
  RG     Range; distance from coordinate center (AU)
  RR     Range-rate; radial velocity wrt coord. center (AU/day)
```

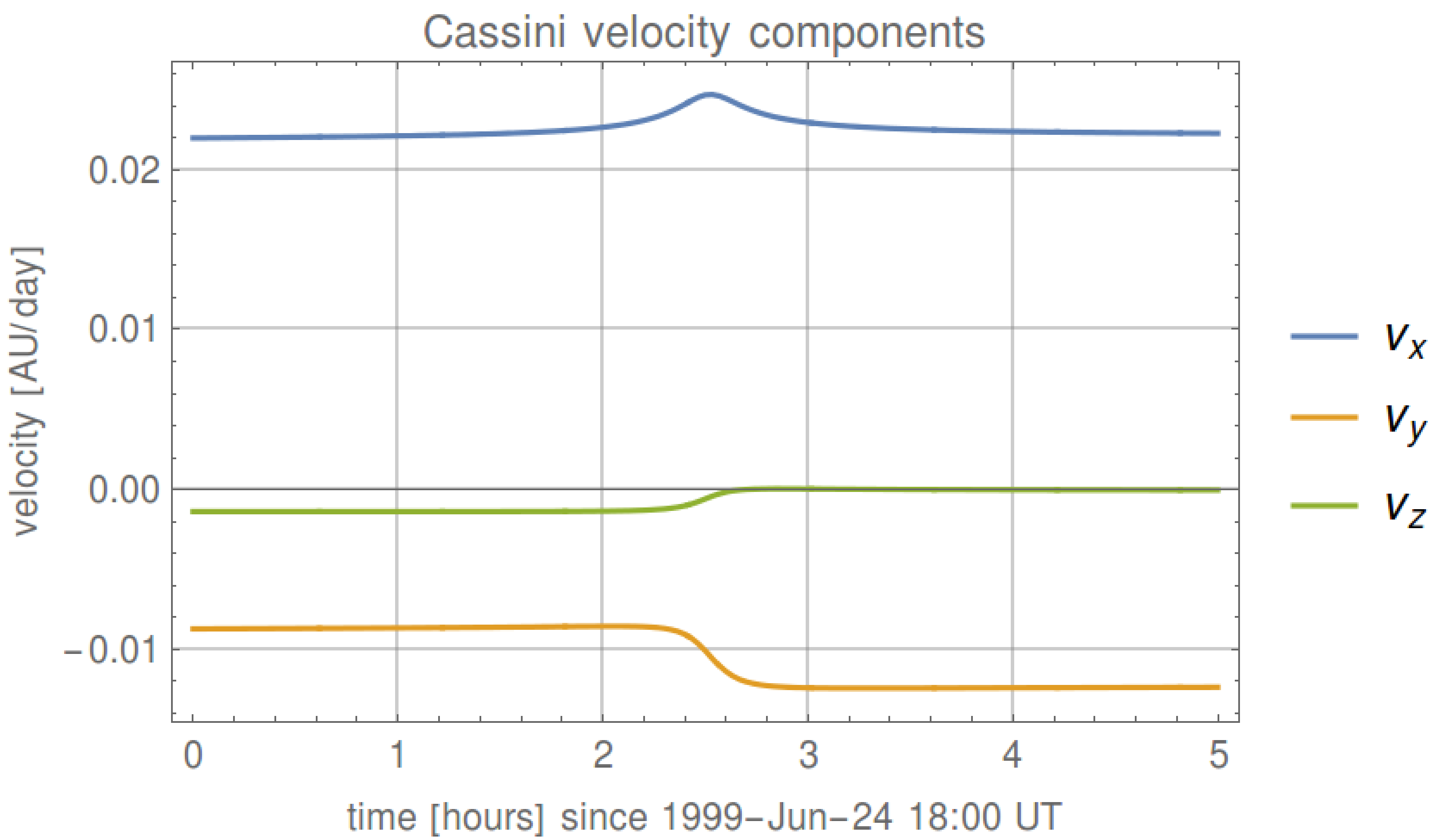
<u>Part (b)</u>: Using both the Cassini and Venus CSV files, create three columns for $\boldsymbol{r}$, the position vector of Cassini relative to the center of Venus. Plot the distance between Cassini and the center of Venus as a function of time. Estimate the time of closest approach. At the time of closest approach, what is $\boldsymbol{r}$, the three-component position vector of Cassini relative to the center of Venus?

<u>Part (c)</u> is on the page after the next page.

Cassini's second pass by Venus, June 24, 1999

https://science.nasa.gov/science-news/science-at-nasa/1999/ast24jun99_1

We use the J2000 Solar System Barycentric coordinate system. It is customary to express distances in units of AU (1 AU = 149597870.700 km) and velocities in units of AU/day (1 day = 86400 s).



Problem 3, continued

Part (c): Along with your data for ***r***, include three columns for ***v***, the relative velocity vector as a function of time. Then calculate three more columns for the angular momentum vector divided by the mass of Cassini. The result is independent of the mass of Cassini, so you don't need the mass of Cassini. (We are working in a regime where the mass of Cassini is much less than the mass of Venus, so the reduced mass is essentially equal to the mass of Cassini). How does the constancy of ***L*** look?

Part (d) Calculate three more columns for the Laplace-Runge-Lenz vector divided by the square of the mass of Cassini. The result is independent of the mass of Cassini, so you don't need the mass of Cassini. You do need the mass of Venus, which is in the files provided. How does the constancy of ***A*** look? How does the direction of this vector compare with the direction of the vector you found in part (b)? Ideally these two would point in the same direction.

Part (e) Normalize the two vectors you found in parts (c) and (d). Compute their cross product so you have a right-handed orthonormal triple. Project the ***r*** data (using scalar products) onto the plane perpendicular to ***L*** and plot this. Your plot should look similar to Fig. 8.11 in the text, with the point of closest approach on the right, as in the figure. What is the eccentricity?