Physics 105, Reinsch, Homework 6

Problem 1

(a) Can use Taylor Series or Binomial Series

$$\frac{1}{d} = \frac{1}{d_0}\Big(1 + \underbrace{\frac{2d_0x + x^2+y^2+z^2}{d_0^2}}_{\varepsilon = \frac{2x}{d_0} + \frac{x^2+y^2+z^2}{d_0^2}}\Big)^{-1/2}$$

two different orders in smallness

For ε^2 we need just $(2x/d_0)^2$

Using $(1+\varepsilon)^{-1/2} = 1 - \frac{\varepsilon}{2} + \frac{3}{8}\varepsilon^2 + \dots$, we get

$$\frac{1}{d} = \frac{1}{d_0}\left[1 - \frac{1}{2}\left(\frac{2x}{d_0} + \frac{x^2+y^2+z^2}{d_0^2}\right) + \frac{3}{8}\left(\frac{2x}{d_0}\right)^2 + \dots\right]$$

Thus, $\boxed{U_{tid} = -GM_m m \frac{1}{d_0}\left[1 + \frac{2x^2-y^2-z^2}{2d_0^2} + \dots\right]}$

(b) Note $r^2(Y_{2,-2} + Y_{22}) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}(x^2-y^2)$

$$-\sqrt{6}\, r^2 Y_{20} = +\frac{1}{2}\sqrt{\frac{15}{2\pi}}(x^2+y^2-2z^2)$$

Thus $r^2(3Y_{2,-2} + 3Y_{22} - \sqrt{6}\,Y_{20}) = \sqrt{\frac{15}{2\pi}}(2x^2-y^2-z^2)$

The quadratic term in the U_{tid} expansion in part (a) is

$$\boxed{-GM_m m\left(\frac{1}{d_0}\right)\left(\frac{1}{2d_0^2}\right)\sqrt{\frac{2\pi}{15}}\, r^2\left(3Y_{2,-2} + 3Y_{22} - \sqrt{6}\,Y_{20}\right)}$$

Taylor Ch 9 #22

Frames: $\mathcal{S}_0$ (inertial frame)
$\mathcal{S}$ (rotating frame w/ angular velocity $\vec{\Omega}$ wrt $\mathcal{S}_0$)

In $\mathcal{S}_0$, charge $-q$ orbits Q in weak $\vec{B}$-field.

EoM: $$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{\mathcal{S}_0} = \underbrace{-\frac{kqQ}{r^2}\hat{r}}_{\text{Coulomb Force}} \underbrace{- q\left(\frac{d\vec{r}}{dt}\right)_{\mathcal{S}_0}\times\vec{B}}_{\text{Magnetic Force.}}$$ (elliptical w/ slow precession)

EoM in $\mathcal{S}$: $$m\ddot{\vec{r}} - 2m\dot{\vec{r}}\times\vec{\Omega} - m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$$
$$= -\frac{kqQ}{r^2}\hat{r} - (q\dot{\vec{r}} + \vec{\Omega}\times\vec{r})\times\vec{B}.$$

Chose $\Omega = q\vec{B}/(2m)$ such that $\dot{\vec{r}}$ terms cancel.

Then $\mathcal{S}$ EoM reduces to:
$$m\ddot{\vec{r}} = -\frac{kqQ}{r^2}\hat{r} - \frac{q^2}{4m}(\vec{B}\times\vec{r})\times\vec{B}.$$

Weak $\vec{B}$-field => keep $O(B)$ terms only (ie drop $O(B^2)$)

~~=> $m\ddot{\vec{r}} \approx \frac{q^2}{4m}(\vec{B}\times\vec{r})\times\vec{B}$~~

=> $m\ddot{\vec{r}} \approx \frac{-kqQ^2}{r^2}\hat{r}$ (elliptic motion / hyperbolic)

Taylor 9.23

In the inertial reference frame, the equation of motion is:

$$m\ddot{\vec{r}} = -k\vec{r}$$

Transforming to a rotating reference frame, as in the previous problem:

$$m\ddot{\vec{r}} + 2m\vec{\Omega} \times \dot{\vec{r}} + m\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) = -k\vec{r}$$

Suppose $\vec{\Omega}$ is perpendicular to the plane of motion, then rewriting the centrifugal term:

$$\begin{aligned} m\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) &= m\vec{\Omega}\left(\vec{\Omega} \cdot \vec{r}\right) - m\vec{r}\left(\vec{\Omega} \cdot \vec{\Omega}\right) \\ &= -m\Omega^2\vec{r} \end{aligned}$$

So if $\Omega = \sqrt{\frac{k}{m}}$, then the centrifugal term cancels the Hooke term. In this case, the equation of motion takes a similar form as that of a charge in a magnetic field:

$$m\ddot{\vec{r}} = \dot{\vec{r}} \times 2m\vec{\Omega}$$

Taking the motion to be in the xy-plane, and setting $\eta = x + iy$:

$$\begin{aligned} \ddot{x} &= 2\Omega\dot{y} \\ \ddot{y} &= -2\Omega\dot{x} \\ \ddot{\eta} &= -2i\Omega\dot{\eta} \end{aligned}$$

This has solution $\eta = \eta_1 + \eta_2 e^{-2i\Omega t}$. To transform back to the inertial frame, note that the rotating frame is rotating at Ω relative to the inertial frame, so their angular separation at time t is Ωt. Rotating by that angle is equivalent to multiplying by $e^{i\Omega t}$, thus in the inertial reference frame:

$$\eta = \eta_1 e^{i\Omega t} + \eta_2 e^{-i\Omega t}$$

Express $\eta_j = A_j e^{i\delta_j}$. Then:

$$\begin{aligned} \eta &= A_1 e^{i\delta_1} e^{i\Omega t} + A_2 e^{i\delta_2} e^{-i\Omega t} \\ &= e^{i(\delta_1+\delta_2)/2}\left[A_1 e^{i\left(\Omega t + \frac{1}{2}\delta_1 - \frac{1}{2}\delta_2\right)} + A_2 e^{-i\left(\Omega t + \frac{1}{2}\delta_1 - \frac{1}{2}\delta_2\right)}\right] \\ &= e^{i(\delta_1+\delta_2)/2}\left[(A_1 + A_2)\cos\left(\Omega t + \frac{\delta_1 - \delta_2}{2}\right) + i\left(A_1 - A_2\right)\sin\left(\Omega t + \frac{\delta_1 - \delta_2}{2}\right)\right] \end{aligned}$$

The constant phasor out front simply rotates the solution. The cosine and the sine have different amplitudes and are 90° out of phase (thanks to the i); this exactly describes an ellipse.

Homework 6 Problem 4

Part a

If the merry-go-round isn't rotating, then:

$$x(t) = R - v_0 \cos\alpha\, t$$
$$z(t) = 0 + v_0 \sin\alpha\, t - \frac{1}{2} g t^2$$

Note that $z = 0$ at $t = 0$ (the firing time) and $t = \frac{2v_0 \sin\alpha}{g}$ (the landing time), at which $x = \frac{1}{2}R$. Substituting:

$$v_0 = \sqrt{\frac{gR}{4\cos\alpha\sin\alpha}}$$

Next, if the merry-go-round is rotating, the velocity of the cannon (which is attached) should be added to the velocity of the projectile relative to the cannon. Thus:

$$\begin{aligned} \vec{v}_{S_0}(t=0) &= v_0\left(-\cos\alpha, 0, \sin\alpha\right) + (0,0,\Omega) \times (R,0,0) \\ &= \left(-v_0\cos\alpha, \Omega R, v_0 \sin\alpha\right) \end{aligned}$$

The trajectory is:

$$\begin{aligned} \vec{r}_{S_0}(t) &= (R,0,0) + \left(-v_0\cos\alpha, \Omega R, v_0\sin\alpha\right) t + \frac{1}{2}(0,0,-g)\, t^2 \\ &= \left(R - v_0\cos\alpha\, t, \Omega R t, v_0 \sin\alpha\, t - \frac{1}{2} g t^2\right) \end{aligned}$$

At the landing time:

$$\vec{r}_{S_0} = \left(\frac{R}{2}, \frac{2\Omega R v_0 \sin\alpha}{g}, 0\right)$$

Transforming to the rotating frame, which has rotated through by an angle of $\Omega t_f = \frac{2\Omega v_0 \sin\alpha}{g}$:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\Omega t_f & \sin\Omega t_f & 0 \\ -\sin\Omega t_f & \cos\Omega t_f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then:

$$\vec{r}_S = \left(R\left[\frac{1}{2}\cos\Omega t_f + \Omega t_f \sin\Omega t_f\right], R\left[-\frac{1}{2}\sin\Omega t_f + \Omega t_f \cos\Omega t_f\right], 0\right)$$

Part b

The equation of motion in the rotating frame is:

$$\begin{aligned} m\ddot{\vec{r}} &= m\vec{g} + \vec{F}_{\text{cf}} + \vec{F}_{\text{cor}} \\ m\left(\ddot{x}, \ddot{y}, \ddot{z}\right) &= m\,(0,0,-g) + m\left[(0,0,\Omega) \times (x,y,z)\right] \times (0,0,\Omega) + 2m\left(\dot{x},\dot{y},\dot{z}\right) \times (0,0,\Omega) \end{aligned}$$

For convenience, subscripts indicating the rotating frame have been dropped, as there is only one reference frame in this part of the problem. Taking components:

$$\begin{aligned}
\ddot{x} &= \Omega^2 x + 2\Omega\dot{y} \\
\ddot{y} &= \Omega^2 y - 2\Omega\dot{x} \\
\ddot{z} &= -g
\end{aligned}$$

If Ω is taken to be small, then the differential equations approximate to $\ddot{x} \approx 0$ and $\ddot{y} \approx 0$, so from initial conditions:

$$\begin{aligned}
x &\approx R - v_0 \cos\alpha\, t \\
y &\approx 0 \\
z &= v_0 \sin\alpha\, t - \frac{1}{2} g t^2
\end{aligned}$$

This is accurate up to zeroth order in Ω, and substituting into the right-hand sides of the differential equations would give solutions accurate up to first order in Ω, yielding:

$$\begin{aligned}
\ddot{x} &\approx 0 \\
\ddot{y} &\approx 2\Omega v_0 \cos\alpha \\
\ddot{z} &= 0
\end{aligned}$$

Only the y-equation changes; from initial conditions, y has an initial position and velocity of zero in the rotating frame:

$$y \approx \Omega v_0 \cos\alpha\, t^2$$

At $t = t_f$, $v_0 \cos\alpha\, t \approx \frac{1}{2}R$, thus $y \approx \frac{1}{2}\Omega R t_f$. Also, $x \approx \frac{1}{2}R$ and $z = 0$. On the other hand, taking the first-order terms in the solution from part a:

$$\begin{aligned}
x &\approx R\left(\frac{1}{2} + 0\right) \\
&= \frac{1}{2}R \\
y &\approx R\left(-\frac{1}{2}\Omega t_f + \Omega t_f\,(1)\right) \\
&= \frac{1}{2}\Omega R t_f \\
z &= 0
\end{aligned}$$

As expected, these match. (In principle, the first-order solutions can be substituted into the differential equation to get second-order solutions, and so forth, yielding a power series solution that matches the exact solution.)