Homework 12 - Solutions

1. (Griffiths 8.1)

$$\pi n\hbar = \int_0^a p(x)\,dx = \int_0^a dx\,\sqrt{2m(E-V(x))}$$

$$= \int_0^{a/2} dx\,\sqrt{2m(E-V_0)} + \int_{a/2}^{a} dx\,\sqrt{2mE}$$

$$\pi n\hbar = \frac{a}{2}\sqrt{2m(E-V_0)} + \frac{a}{2}\sqrt{2mE}$$

$$\frac{4\pi^2 n^2\hbar^2}{a^2} = 2mE - 2mV_0 + 2mE + 2\cdot 2m\sqrt{E(E-V_0)}$$

$$= 4mE - 2mV_0 + 4mE\sqrt{1-\frac{V_0}{E}}$$

$$\cancel{= 4mE - 2mV_0 + 4mE - 2mV_0}$$

$$\cancel{\frac{4\pi^2 n^2\hbar^2}{a^2} = 8mE - 4mV_0}$$

$$\cancel{E = \frac{1}{2}\left(\frac{\pi^2\hbar^2 n^2}{ma^2} + V_0\right)}$$

$$\left(\frac{\pi^2 n^2\hbar^2}{ma^2 E} - 1 + \frac{V_0}{2E}\right)^2 = \left(\sqrt{1-\frac{V_0}{E}}\right)^2$$

$$\left(\frac{2E_n^0}{E} - 1 + \frac{V_0}{2E}\right)^2 = 1 - \frac{V_0}{E}$$

$$\frac{4(E_n^0)^2}{E^2} + \cancel{1} + \frac{V_0^2}{4E^2} - \cancel{\frac{V_0}{E}} + \frac{2E_n^0 V_0}{E^2} - \frac{4E_n^0}{E} = \cancel{1} - \cancel{\frac{V_0}{E}}$$

$$4(E_n^0)^2 + \frac{V_0^2}{4} + 2E_n^0 V_0 - 4E_n^0 E = 0$$

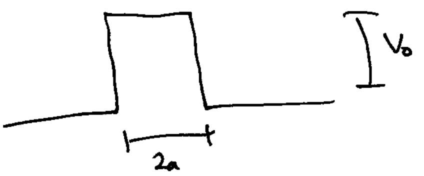
$$E = \frac{4(E_n^0)^2 + \frac{V_0^2}{4} + 2E_n^0 V_0}{4E_n^0}$$

$$E = E_n^0 + \frac{V_0^2}{16E_n^0} + \frac{1}{2}V_0$$

From 6.1, we obtained $E = E_n^0 + \frac{V_0}{2}$

$\therefore$ This agrees w/ the above assuming $\frac{V_0}{E_n^0} \ll 1$

2. (Griffiths 8.3)



$$T \cong e^{-2\gamma} \qquad \gamma = \frac{1}{\hbar}\int_0^{2a} dx\, |p(x)|$$

$$= \frac{1}{\hbar}\cdot 2a \cdot \sqrt{2m(V_0 - E)}$$

$$\therefore \; T \approx e^{-\frac{4a}{\hbar}\sqrt{2m(V_0-E)}}$$

Exact result:

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0-E)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)}$$

~~Assume $V_0 \gg E$~~

$\approx$

Assume

$$\frac{2a}{\hbar}\sqrt{2m(V_0-E)} \gg 1$$

$$\Rightarrow \quad T \approx \frac{1}{\frac{V_0^2}{4E(V_0-E)} e^{\frac{4a}{\hbar}\sqrt{2m(V_0-E)}}}$$

$$= \frac{4E(V_0-E)}{V_0^2} e^{-\frac{4a}{\hbar}\sqrt{2m(V_0-E)}} \propto e^{-\frac{4a}{\hbar}\sqrt{2m(V_0-E)}}$$ as derived from the WKB approximation.

8.7. $$\int_{x_1}^{x_2} p(x)\,dx = \left(n - \tfrac{1}{2}\right)\pi\hbar \qquad n=1,2,3,\ldots$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$p = \sqrt{2m\left(E - \tfrac{1}{2}m\omega^2x^2\right)} \qquad \left\{ \begin{aligned} x_1 &= -\sqrt{\frac{2E}{m\omega^2}} \\ x_2 &= +\sqrt{\frac{2E}{m\omega^2}} \end{aligned} \right\}$$

$$\therefore \int_{-\sqrt{\frac{2E}{m\omega^2}}}^{\sqrt{\frac{2E}{m\omega^2}}} \sqrt{2m\left(E - \tfrac{1}{2}m\omega^2x^2\right)}\,dx = \int_{-1}^{1}\sqrt{2m\left(E - Ey^2\right)}\cdot\sqrt{\frac{2E}{m\omega^2}}\,dy = \frac{2E}{\omega}\int_{-1}^{1}\sqrt{1-y^2}\,dy$$

$$\underbrace{x = \sqrt{\frac{2E}{m\omega^2}}\,y}$$

$$y = \sin\theta \qquad dy = \cos\theta\,d\theta$$

$$= \frac{2E}{\omega}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\,\cos^2\theta = \frac{E\pi}{\omega}$$

$$\therefore \quad \frac{E\cancel{\pi}}{\omega} = \left(n + \tfrac{1}{2}\right)\cancel{\pi}\hbar \qquad n = 0,1,2,\ldots$$

$$E = \hbar\omega\left(n + \tfrac{1}{2}\right) \qquad n = 0,1,2,\ldots$$

4. (Liboff 7.62)

a)

$$\int_0^{\Phi/e\mathcal{E}} p(x)\,dx = \int_0^{\frac{\Phi}{e\mathcal{E}}} dx\,\sqrt{2m\left(\Phi + \cancel{E_F} - e\mathcal{E}x - \cancel{E_F}\right)}$$

$$= \sqrt{2m}\cdot\left(\Phi - e\mathcal{E}x\right)^{3/2}\cdot\left(-\frac{2}{3}\cdot\frac{1}{e\mathcal{E}}\right)\Bigg|_0^{\frac{\Phi}{e\mathcal{E}}}$$

$$= -\frac{2\sqrt{2m}}{3e\mathcal{E}}\left(0 - \Phi^{3/2}\right)$$

$$= \frac{2\sqrt{2m}}{3e\mathcal{E}}\Phi^{3/2}$$

$$\therefore \quad T \cong e^{-\frac{4\sqrt{2m}}{3e\mathcal{E}\hbar}\Phi^{3/2}}$$

b)

$$J_{trans} = 1\ ^{mA}/_{cm^2}\cdot\frac{1\,A}{10^3\,mA}\cdot\frac{(100)^2\,cm^2}{1\,m^2}$$

$$= 10\ ^{A}/_{m^2}$$

$$\left|\frac{J_{trans}}{J_{inc}}\right| = e^{-\frac{4\sqrt{2m}}{3e\mathcal{E}\hbar}\Phi^{3/2}} \qquad \Rightarrow \log\left|\frac{J_{inc}}{J_{trans}}\right| = \frac{4\sqrt{2m}}{3e\mathcal{E}\hbar}\Phi^{3/2}$$

$$\mathcal{E} = \frac{\frac{4\sqrt{2m}}{3e\hbar}\Phi^{3/2}}{\log\left|\frac{J_{inc}}{J_{trans}}\right|}$$

$$J_{inc} = env = en\sqrt{\frac{2E_f}{m}}$$

Also, $E_f = \frac{\hbar^2}{2m}\left(\frac{3n}{8\pi}\right)^{2/3} \Rightarrow \left(\frac{2mE_f}{\hbar^2}\right)^{3/2} = \frac{3n}{8\pi} \Rightarrow \underbrace{n = \frac{8\pi}{3}\left(\frac{2mE_f}{\hbar^2}\right)^{3/2}}$

$E = \frac{1}{2}mv^2$

$$\therefore\ J_{inc} = \frac{8\pi e}{3\hbar^3 m} \cdot (2mE_f)^2$$

$$= \boxed{\frac{32\pi m e E_f^2}{3\hbar^3}}$$

$$= \log\left|\frac{J_{inc}}{J_{trans}}\right| = \log\left(\frac{4.78\times10^{17}}{10}\right)$$

$$\approx 38.406$$

$$\therefore\ \mathcal{E} = \frac{2.088\times10^{10}}{38.406} \approx 5.4\times10^{8}\ \mathrm{V/m}$$