Problem Set n°10

Solutions

1) Griffiths 11.8

$$\left(\vec{\nabla}^2+k^2\right)\left(-\frac{e^{ikr}}{4\pi r}\right)=\vec{\nabla}\cdot\left(-\frac{e^{ikr}}{4\pi}\vec{\nabla}\left(\frac{1}{r}\right)-\frac{1}{4\pi r}\vec{\nabla}\left(e^{ikr}\right)\right)-\frac{k^2e^{ikr}}{4\pi r}$$

$$=-\frac{\vec{\nabla}\left(e^{ikr}\right)}{4\pi}\cdot\vec{\nabla}\left(\frac{1}{r}\right)-\frac{e^{ikr}}{4\pi}\vec{\nabla}^2\left(\frac{1}{r}\right)-\frac{1}{4\pi r}\vec{\nabla}\left(\frac{1}{r}\right)\cdot\vec{\nabla}\left(e^{ikr}\right)$$

$$-\frac{1}{4\pi r}\vec{\nabla}^2\left(e^{ikr}\right)-\frac{k^2}{4\pi r}$$

$$=-\frac{e^{ikr}}{4\pi}\left(-4\pi\delta^3(\vec{r})\right)-\frac{1}{2\pi}(ik)e^{ikr}\left(\frac{-1}{r^2}\right)$$

$$-\frac{1}{4\pi r}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}e^{ikr}\right)-\frac{k^2e^{ikr}}{4\pi r}$$

$$=e^{ikr}\delta^{(3)}(\vec{r})+\cancel{\frac{ik}{2\pi r^2}e^{ikr}}\underbrace{-\frac{1}{4\pi r^3}\frac{\partial}{\partial r}\left(ikr^2e^{ikr}\right)}-\cancel{\frac{k^2e^{ikr}}{4\pi r}}$$

$$=\frac{-1}{4\pi r^3}\left(2ikre^{ikr}-k^2r^2e^{ikr}\right)$$

$$=\cancel{\frac{-ik}{2\pi r^2}e^{ikr}}+\cancel{\frac{k^2e^{ikr}}{4\pi r}}$$

$$\Rightarrow\left(\vec{\nabla}^2+k^2\right)\left(\frac{-e^{ikr}}{4\pi r}\right)=e^{ikr}\delta^{(3)}(\vec{r})=e^{ik\cdot 0}\delta^{(3)}(\vec{r})=\boxed{\delta^{(3)}(\vec{r})}$$

2. (Griffiths 11.9)

$$\int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \cdot V(\vec{r}_0)\,\psi(\vec{r}_0)\,d^3\vec{r}_0$$

$$= \int_0^\infty r_0^2 dr_0 \int_0^\pi d\theta_0 \sin\theta_0 \underbrace{\int_0^{2\pi} d\phi_0}_{\substack{\text{Can do this integral}\\ \text{easily since}\\ \phi_0 \text{ never appears}\\ \text{in the integrand}}} \frac{e^{ik\sqrt{r^2+r_0^2-2rr_0\cos\theta_0}}}{\sqrt{r^2+r_0^2-2rr_0\cos\theta_0}} \cdot \underbrace{\frac{-e^2}{4\pi\varepsilon_0 r_0}}_{\substack{\text{Coulomb}\\ \text{Potential}}} \cdot \underbrace{\frac{1}{\sqrt{\pi a^3}}\, e^{-r_0/a}}_{\substack{\text{Ground State}\\ \text{Wavefunction}}}$$

$$\left(a = \frac{4\pi\epsilon_0 \hbar^2}{me^2} \right)$$

$$= 2\pi \int_0^\infty r_0^{\cancel{2}} dr_0 \frac{1}{\sqrt{\pi a^3}} e^{-r_0/a} \cdot \frac{-e^2}{4\pi \cancel{r_0} \varepsilon_0} \underbrace{\int_0^\pi d\theta_0 \sin\theta_0 \frac{e^{ik\sqrt{r^2+r_0^2-2rr_0\cos\theta_0}}}{\sqrt{r^2+r_0^2-2rr_0\cos\theta_0}}}_{\frac{i}{rr_0k}\left[e^{ik\sqrt{(r-r_0)^2}} - e^{ik\sqrt{(r+r_0)^2}}\right]}$$

$\therefore =$

$$= \cancel{2\pi} \cdot \frac{-e^2}{\cancel{4\pi}_{2}\,\varepsilon_0} \cdot \frac{1}{\sqrt{\pi a^3}} \cdot \int_0^\infty \cancel{r_0} dr_0\, e^{-r_0/a} \cdot \frac{i}{r\cancel{r_0}k}\left[e^{-K|r-r_0|} - e^{-K(r+r_0)}\right]$$

$$= \frac{-e^2}{2\sqrt{\pi a^3}\cdot\varepsilon_0} \cdot \frac{i}{rK}\left[\underbrace{\int_0^\infty dr_0\, e^{-\frac{r_0}{a}-K|r-r_0|}}_{\mathrm{I}} - \underbrace{\int_0^\infty dr_0\, e^{-Kr} \cdot e^{-r_0(+K+1/a)}}_{\mathrm{II}}\right]$$

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$$\mathrm{II} = -\frac{1}{K+1/a} e^{-Kr} e^{-r_0(K+1/a)} \Big|_0^\infty$$

$$= \frac{e^{-Kr}}{K+1/a}$$

At this point its helpful to find the explicit relationship between K and a

$$a = \frac{4\pi\varepsilon_0 \hbar^2}{me^2} \qquad K = \frac{\sqrt{-2mE}}{\hbar} = \frac{\sqrt{+\cancel{2}m \cdot \frac{m}{\cancel{2}\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2}}{\hbar} = \frac{m}{\hbar^2}\,\frac{e^2}{4\pi\varepsilon_0} = \frac{1}{a}$$

Now we break up the first integral, I,

$$= \int_0^r dr_0\, e^{-\frac{r_0}{a} - Kr + Kr_0} + \int_r^\infty dr_0\, e^{-\frac{r_0}{a} + Kr - Kr_0}$$

Using $K = 1/a$

$$= \int_0^r dr_0\, e^{-Kr} + \int_r^\infty dr_0\, e^{Kr} \cdot e^{-\frac{2r_0}{a}}$$

$$= re^{-Kr} + e^{Kr} \cdot \left(-\frac{a}{2}\right) e^{-\frac{2r_0}{a}} \Big|_r^\infty$$

$$= re^{-Kr} + \frac{a}{2} e^{\frac{r}{a}} e^{-\frac{2r}{a}}$$

$$= \underbrace{re^{-\frac{r}{a}} + \frac{a}{2} e^{-\frac{r}{a}}}$$

Combining everything together,

$$= -\frac{e^2}{2\sqrt{\pi a^3}\cdot\varepsilon_0} \cdot \frac{a}{r} \cdot \Big[\underbrace{re^{-\frac{r}{a}} + \cancel{\frac{a}{2} e^{-\frac{r}{a}}}}_{\text{I}} - \underbrace{\cancel{\frac{a}{2} e^{-r/a}}}_{\text{II}} \Big]$$

$$= - \frac{e^2}{2\sqrt{\pi a^3}} \, a \, e^{-\frac{r}{a}}$$

$\Rightarrow$ Plugging this integral back into the integral Schrödinger's Eqn on the R.H.S. gives,

$$= \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2} \cdot \frac{-e^2}{2\sqrt{\pi a^3} \cdot \varepsilon_0} \cdot a \, e^{-\frac{r}{a}}$$

$$= \psi_0(\vec{r}) + \underbrace{\frac{me^2}{4\pi\varepsilon_0\hbar^2}}_{= \frac{1}{a}} \cdot \frac{1}{\sqrt{\pi a^3}} \cdot a \, e^{-\frac{r}{a}}$$

$$\therefore \; = \cancel{\psi_0(\vec{r})} + \frac{1}{\sqrt{\pi a^3}} e^{-r/a} = \psi_{\text{ground state}}(\vec{r}) \quad \blacksquare$$

Set this to zero using boundary conditions

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3. (Park)

$$\frac{d\sigma}{d\Omega} = 4\cos^2\left(\tfrac{1}{2}\overbrace{(\vec{k}-\vec{k}')}^{\text{Call this } K}\cdot\vec{\ell}\right)|f(\theta)|^2$$

Need to average over the orientation of the "molecule". $\longrightarrow$ We'll do this first.

~~We should change~~

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle_{\text{avg over molecule orientation}} = \frac{1}{4\pi}\int_0^{\pi} d\tilde{\theta}\sin\tilde{\theta}\int_0^{2\pi} d\tilde{\phi} \; 4\cos^2\left(\tfrac{1}{2}K\ell\cos\tilde{\theta}\right)|f(\theta)|^2$$

since we're averaging...

$$= \frac{1}{4\pi} \cdot 2\pi \cdot 4|f(\theta)|^2 \cdot \int_{-1}^{1} du \, \cos^2\left(\tfrac{1}{2} X\ell \cdot u\right)$$

From ϕ integral

Define $u := \cos\theta$

$$= 2|f(\theta)|^2 \cdot \left(1 + \frac{\sin(X\ell)}{2 \cdot \frac{1}{2} X\ell}\right)$$

$$\boxed{\left\langle \frac{d\sigma}{d\Omega} \right\rangle_\ell = 2|f(\theta)|^2 \left(1 + \frac{\sin(X\ell)}{X\ell}\right)}$$

Does this make sense?

Taking $\ell \to 0$ (More precisely $X\ell \to 0$)

is like putting two scattering centers on top of each other. This would lead to $f(\theta)_{\text{one center}} \to 2 \cdot f(\theta)_{\text{one center}}$

$$\Rightarrow \left\langle \frac{d\sigma}{d\Omega} \right\rangle = |2f(\theta)|^2 = 4|f(\theta)|^2$$

But we see that as $X\ell \to 0$

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle_\ell \approx 2|f(\theta)|^2 \left(1 + \frac{X\ell}{X\ell}\right) = 4|f(\theta)|^2$$ as expected ∎

To get the total cross-section, we simply integrate over θ, ϕ to give

From trivial ϕ-integral

$$\boxed{\sigma_{TOT} = 2\pi \int_0^\pi d\theta \sin\theta \cdot 2|f(\theta)|^2 \cdot \left(1 + \frac{\sin\left(2k\ell \sin\left(\frac{\theta}{2}\right)\right)}{2k\ell \sin\left(\frac{\theta}{2}\right)}\right)}$$