Homework $\boxed{4}$ SOLUTIONS

1) Griffiths 6.7

$$\hat{H} = \frac{\hat{p}^2}{2m} \qquad \hat{H}\psi(x) = E\,\psi(x) \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - E\psi = 0 \Rightarrow \frac{d^2\psi}{dx^2} + k^2\psi = 0$$

$$\left(\text{with } k^2 = \frac{2mE}{\hbar^2}\right)$$

So $\psi(x) = Ae^{ikx} + Be^{-ikx}$

Now $\psi(x) = \psi(x+L)$. In particular $\psi(0) = \psi(L) \Rightarrow Ae^{ikL} + Be^{-ikL} = A+B$

Also for $x = \frac{\pi}{2k}$, $\psi\left(\frac{\pi}{2k}\right) = \left(\frac{\pi}{2k}+L\right) \Rightarrow iAe^{ikL} - iBe^{-ikL} = iA - iB$

Add the two above equations to get $2Ae^{ikL} = 2A$

So $A=0$ or $e^{ikL} = 1$

$\Rightarrow kL = 2n\pi \quad (n = 0, \pm1, \pm2, \pm3, \ldots)$

If $A=0$, $Be^{-ikL} = B$, so once again $kL = 2n\pi$ $(n \in \mathbb{Z})$ since both A and B cannot both be 0.

So $\psi_n(x) = A\,e^{\frac{2\pi i n x}{L}} \quad (n \in \mathbb{Z})$

Normalize: $1 = \int_0^L dx\,|\psi_n(x)|^2 = \int_0^L dx\,|A|^2\,\cancel{e^{\frac{2\pi i n x}{L}}}\,\cancel{e^{-\frac{2\pi i n x}{L}}} = |A|^2 L$

So $\boxed{\psi_n(x) = \frac{1}{\sqrt{L}}\,e^{\frac{2\pi i n x}{L}}}$ $(n \in \mathbb{Z})$

And $E_n = \frac{\hbar^2 k^2}{2m} = \left(\frac{2n\pi}{L}\right)^2 \frac{\hbar^2}{2m} = \boxed{\frac{2}{m}\left(\frac{n\pi\hbar}{L}\right)^2}$

Note: Except for $n=0$, all states all doubly degenerate ($\pm n$ states have the same energy)

(b) $H' = -V_0 e^{-\frac{x^2}{a^2}}$ where $a \ll L$

$$\langle n|H'|n\rangle = -V_0 \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, \cancel{e^{-\frac{2\pi i n x}{L}}}\, \cancel{e^{\frac{2\pi i n x}{L}}}\, e^{-\frac{x^2}{a^2}}$$

$$\approx \frac{-V_0}{L}\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{a^2}} = -V_0 \frac{1}{L}\cdot \sqrt{\pi}\, a = \frac{-V_0\sqrt{\pi}\, a}{L}$$

$$\langle -n|H'|-n\rangle = \frac{-V_0\sqrt{\pi}\, a}{L}$$ By the same argument...

$$\langle n|H'|-n\rangle = -V_0\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, e^{-\frac{2\pi i n x}{L}}\cdot e^{-\frac{2\pi i n x}{L}}\, e^{-\frac{x^2}{a^2}}$$

$$\approx -V_0\frac{1}{L}\int_{-\infty}^{\infty} dx\, e^{-\frac{4\pi i n x}{L}}\, e^{-\frac{x^2}{a^2}}$$

$$= -V_0\frac{1}{L}\int_{-\infty}^{\infty} dx\cdot e^{-\frac{\left(x+\frac{2\pi i n a^2}{L}\right)^2}{a^2}}\cdot e^{-\frac{4\pi^2 n^2 a^2}{L^2}}$$

$$= -V_0\frac{1}{L}\cdot\sqrt{\pi}\, a\cdot e^{-\frac{4\pi^2 n^2 a^2}{L^2}}$$

$$\Rightarrow \langle -n|H'|n\rangle = -V_0\frac{1}{L}\sqrt{\pi}\, a\, e^{-\frac{4\pi^2 n^2 a^2}{L^2}} \quad \therefore\; W = -V_0\frac{\sqrt{\pi}\, a}{L}\begin{pmatrix} 1 & e^{-\frac{4\pi^2 n^2 a^2}{L^2}} \\ e^{-\frac{4\pi^2 n^2 a^2}{L^2}} & 1 \end{pmatrix}$$

Eigenvalues: $\left(-V_0\frac{\sqrt{\pi}a}{L} - \lambda\right)^2 - \left(\frac{\sqrt{\pi}\, a V_0}{L}\right)^2 e^{-\frac{8\pi^2 n^2 a^2}{L^2}} = 0$

$$\Downarrow$$

$$\lambda_\pm = \frac{1}{2}\left[-V_0\frac{2\sqrt{\pi}\, a}{L} \pm \sqrt{\underbrace{\left(\frac{\sqrt{\pi}a}{L} - \frac{\sqrt{\pi}a}{L}\right)^2}_{=0} + 4\cdot\left(\frac{\sqrt{\pi}a}{L}\right)^2 e^{-\frac{8\pi^2 n^2 a^2}{L^2}}}\; V_0\right]$$

$$= -V_0\frac{\sqrt{\pi}\, a}{L} \pm \cancel{\frac{V_0}{2}} \cdot \cancel{2} \cdot \frac{a\sqrt{\pi}}{L}\, e^{-\frac{4\pi^2 n^2 a^2}{L^2}}\, V_0$$

$$E_{n_\pm} = -V_0 \frac{a\sqrt{\pi}}{L}\left(1 \pm e^{-\frac{4\pi^2 n^2 a^2}{L^2}}\right)$$

(c) We can work out the good states ~~by~~ from the above matrix, but we can also figure out the good basis by using the Parity operator, P.

$$P : x \rightarrow -x$$

$$[P, H'] = 0$$
$$[P, H] = 0$$

$\therefore$ The good basis consists of eigenstates of P,

$$|n_S\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle + |-n\rangle\right) \qquad P|n_S\rangle = |n_S\rangle$$

$$|n_A\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle - |-n\rangle\right) \qquad P|n_A\rangle = -|n_A\rangle$$

$$\psi_{n_S}(x) = \left(e^{\frac{2\pi i n x}{L}} + e^{-\frac{2\pi i n x}{L}}\right) \cdot \frac{1}{\sqrt{2L}}$$

$$\psi_{n_A}(x) = \frac{1}{\sqrt{2L}}\left(e^{\frac{2\pi i n x}{L}} - e^{-\frac{2\pi i n x}{L}}\right)$$

We can check from the matrix W that these states are actually a good basis

$$\frac{\sqrt{\pi}\, a}{L}\begin{pmatrix} 1 & e^{-\frac{4\pi^2 n^2 a^2}{L^2}} \\ e^{-\frac{4\pi^2 n^2 a^2}{L^2}} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\sqrt{\pi}\, a}{L}\left(1 + e^{-\frac{4\pi^2 n^2 a^2}{L^2}}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\frac{\sqrt{\pi}\, a}{L}\begin{pmatrix} 1 & e^{-\frac{4\pi^2 n^2 a^2}{L^2}} \\ e^{-\frac{4\pi^2 n^2 a^2}{L^2}} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\sqrt{\pi}\, a}{L}\left(1 - e^{-\frac{4\pi^2 n^2 a^2}{L^2}}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2. (Griffiths 6.8)

$$H' = a^3 V_0\, \delta(x - \tfrac{a}{4})\, \delta(y - \tfrac{a}{2})\, \delta(z - \tfrac{3a}{4})$$

<u>Ground State:</u>

$$E_0^{(1)} = \langle \psi_{111} | H' | \psi_{111} \rangle = \int dx\,dy\,dz \left(\sqrt{\tfrac{2}{a}}\right)^6 \sin^2\left(\tfrac{\pi x}{a}\right) \sin^2\left(\tfrac{\pi y}{a}\right) \sin^2\left(\tfrac{\pi z}{a}\right) \cdot a^3 V_0\, \delta(x - \tfrac{a}{4})\, \delta(y - \tfrac{a}{2})\, \delta(z - \tfrac{3a}{4})$$

$$= \left(\tfrac{2}{a}\right)^3 \cdot a^3 V_0 \underbrace{\sin^2\left(\tfrac{\pi}{4}\right)}_{=1/2} \underbrace{\sin^2\left(\tfrac{\pi}{2}\right)}_{=1} \underbrace{\sin^2\left(\tfrac{3\pi}{4}\right)}_{=1/2}$$

$$= \frac{8}{\cancel{a^3}} \cancel{a^3}\, V_0 \cdot 1 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 2V_0$$

<u>1st Excited State</u>

We'll guess that the good states are simply, $|\psi_{112}\rangle, |\psi_{121}\rangle, |\psi_{211}\rangle$

call them: $|a\rangle \quad |b\rangle \quad |c\rangle$

$$\cancel{\langle a | H' | b\rangle = \langle \psi_{112} | H' | \psi_{121} \rangle = \int dx\,dy\,dz \left(\sqrt{\tfrac{2}{a}}\right)^6 \sin^2\left(\tfrac{\pi x}{a}\right) \sin\left(\tfrac{2\pi y}{a}\right) \sin\left(\tfrac{\pi y}{a}\right) \sin\left(\tfrac{2\pi z}{a}\right) \sin\left(\tfrac{\pi z}{a}\right) \cdot V_0 a^3 \delta(x - \tfrac{a}{4}) \delta(y - \tfrac{a}{2}) \delta(z - \tfrac{3a}{4})}$$

$$\cancel{= \frac{8}{a^3} a^3 V_0 \sin^2\left(\tfrac{\pi}{4}\right) \sin(\pi) \sin(}$$

Note that,

$$\langle \psi_{n_1 n_2 n_3} | H' | \psi_{m_1 m_2 m_3} \rangle = \int dx\,dy\,dz \left(\sqrt{\tfrac{2}{a}}\right)^6 \sin\left(\tfrac{n_1 \pi x}{a}\right) \sin\left(\tfrac{m_1 \pi x}{a}\right) \cdot \sin\left(\tfrac{n_2 \pi y}{a}\right) \sin\left(\tfrac{m_2 \pi y}{a}\right) \sin\left(\tfrac{n_3 \pi z}{a}\right) \sin\left(\tfrac{m_3 \pi z}{a}\right)$$
$$\cdot V_0 a^3 \delta(x - \tfrac{a}{4}) \delta(y - \tfrac{a}{2}) \delta(z - \tfrac{3a}{4})$$

$$= \frac{8}{\cancel{a^3}} \cdot V_0 \cancel{a^3} \sin\left(\tfrac{n_1 \pi}{4}\right) \sin\left(\tfrac{m_1 \pi}{4}\right) \sin\left(\tfrac{n_2 \pi}{2}\right) \sin\left(\tfrac{m_2 \pi}{2}\right)$$
$$\cdot \sin\left(\tfrac{3 n_3 \pi}{4}\right) \sin\left(\tfrac{3 m_3 \pi}{4}\right)$$

Note that if $n_2 = 2$ <u>or</u> $m_2 = 2$ $\Rightarrow$ $\langle \psi_{n_1 n_2 n_3} | H' | \psi_{m_1 m_2 m_3} \rangle = 0$

$$\therefore\ W = \begin{array}{c} \begin{pmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & * \end{pmatrix} \\ \begin{array}{ccc} |\psi_{112}\rangle & |\psi_{121}\rangle & |\psi_{211}\rangle \end{array} \end{array}$$

Need to calculate the other values

$$\langle 112|H'|112\rangle = \sin\left(\tfrac{\pi}{4}\right)\sin\left(\tfrac{\pi}{4}\right)\cdot\sin\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{3\pi}{2}\right)\sin\left(\tfrac{3\pi}{2}\right) 8V_0$$

$$= \left(\tfrac{1}{\sqrt{2}}\right)^2\cdot 1^2\cdot(-1)^2 = \tfrac{1}{2}\cdot 8V_0$$

$$\langle 112|H'|211\rangle = \sin\left(\tfrac{\pi}{4}\right)\sin\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{3\pi}{2}\right)\sin\left(\tfrac{3\pi}{4}\right)\cdot 8V_0$$

$$= \tfrac{1}{\sqrt{2}}\cdot 1\cdot 1\cdot 1\cdot(-1)\cdot\left(\tfrac{1}{\sqrt{2}}\right)\cdot 8V_0$$

$$= \underline{-\tfrac{1}{2}}\cdot 8V_0$$

$$\langle 211|H'|112\rangle = -\tfrac{1}{2}\cdot 8V_0 \quad \checkmark \text{ from symmetry...}$$

$$\langle 211|H'|211\rangle = \sin\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{\pi}{2}\right)\sin\left(\tfrac{3\pi}{4}\right)\sin\left(\tfrac{3\pi}{4}\right)\cdot 8V_0$$

$$= 1\cdot 1\cdot 1\cdot 1\cdot\left(\tfrac{1}{\sqrt{2}}\right)^2 = \tfrac{1}{2}\cdot 8V_0$$

$$\therefore W = \begin{pmatrix} \tfrac{1}{2} & 0 & -\tfrac{1}{2} \\ 0 & 0 & 0 \\ -\tfrac{1}{2} & 0 & \tfrac{1}{2} \end{pmatrix}\cdot 8V_0 \qquad \cancel{\det(W-\lambda I) = \left|\begin{pmatrix} \tfrac{1}{2}-\lambda & 0 & -\tfrac{1}{2} \\ 0 & -\lambda & 0 \\ -\tfrac{1}{2} & 0 & \tfrac{1}{2}-\lambda \end{pmatrix}\right| = -\lambda\left(\tfrac{1}{2}-\lambda\right)^2 + \tfrac{1}{4}\lambda = \lambda\left[\tfrac{1}{4} - \tfrac{1}{4} - \lambda^2 + \lambda\right]}$$

$$\Rightarrow \det(W-\lambda I) = \left|\begin{pmatrix} 4V_0-\lambda & 0 & -4V_0 \\ 0 & -\lambda & 0 \\ -4V_0 & 0 & 4V_0-\lambda \end{pmatrix}\right| = -\lambda(4V_0-\lambda)^2 + (4V_0)^2\lambda$$

$$= -\lambda(4V_0)^2 + 2\lambda^2(4V_0) - \lambda^3 + \cancel{\lambda(4V_0)^2}$$

$$= \lambda^2(8V_0 - \lambda)$$

$$\therefore\ \Delta E_1^{(1)} = 0 \qquad \Delta E_2^{(1)} = 0 \qquad \Delta E_3^{(1)} = 8V_0$$

3) Griffiths 6.12

$$\boxed{2\langle T\rangle = \langle \vec{r}\cdot\vec{\nabla}V\rangle}$$ Virial theorem

$$\langle \vec{r}\cdot\vec{\nabla}V\rangle = \left\langle r\frac{e^2}{4\pi\varepsilon_0 r^2}\right\rangle = \frac{e^2}{4\pi\varepsilon_0}\left\langle\frac{1}{r}\right\rangle$$

But from Griffiths 4.40, $\langle T\rangle = -E_n = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{2an^2}$

$a \equiv \frac{4\pi\varepsilon_0\hbar^2}{me^2}$ (Bohr radius)

So $\frac{\cancel{e^2}}{\cancel{4\pi\varepsilon_0}}\frac{1}{an^2} = \frac{\cancel{e^2}}{\cancel{4\pi\varepsilon_0}}\left\langle\frac{1}{r}\right\rangle \Rightarrow \boxed{\left\langle\frac{1}{r}\right\rangle = \frac{1}{an^2}}$

OR Use Feynman-Hellmann as I did in section! $\frac{dE_n}{d\lambda} = \langle\psi_n|\frac{\partial\hat{H}}{\partial\lambda}|\psi_n\rangle$

Choose for instance $\lambda = \varepsilon_0$.

Recall $E_n = \frac{-m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\frac{1}{n^2}$. So $\frac{dE_n}{d\varepsilon_0} = \frac{me^4}{\hbar^2(4\pi)^2\varepsilon_0^3}\frac{1}{n^2}$

And $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\varepsilon_0 r}$. So $\langle\psi_n|\frac{\partial\hat{H}}{\partial\varepsilon_0}|\psi_n\rangle = \left\langle\frac{e^2}{4\pi\varepsilon_0^2 r}\right\rangle = \frac{e^2}{4\pi\varepsilon_0^2}\left\langle\frac{1}{r}\right\rangle$

So $\left\langle\frac{1}{r}\right\rangle = \frac{4\pi\varepsilon_0^2}{e^2}\frac{me^4}{\hbar^2(4\pi)^2\varepsilon_0^3}\frac{1}{n^2} = \frac{me^2}{4\pi\varepsilon_0\hbar^2}\frac{1}{n^2} = \boxed{\frac{1}{an^2}}$ as promised!

Feynman-Hellmann theorem

4. (Griffiths 6.16)

a) We'll do everything using index notation since its much easier.

$\vec{L}\cdot\vec{S} = L_j S_j$ ← We'll use "Einstein Summation" notation. This means that whenever you see an index twice (e.g. j), that means that its summed over.

i.e. $L_j S_j = \sum_{j=1}^{3} L_j S_j$

(unit vector, i.e. $\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$, $\hat{e}_3 = \hat{z}$)

$$[\vec{L}\cdot\vec{S}, \vec{L}] = [L_j S_j, L_k \hat{e}_k]$$

$$= \hat{e}_k S_j [L_j, L_k] \qquad \text{since } [S_j, L_k] = 0,\ [\hat{e}_k, L_j] = 0,\ [e_k, S_j] = 0$$

$$= \hat{e}_k S_j \epsilon_{jkl} \hbar L_l$$

Here we've used $[L_x, L_y] = \hbar L_z$, $[L_y, L_z] = \hbar L_x$, etc.

This is expressed in index notation as,

$$[L_i, L_j] = \epsilon_{ijk} \hbar L_k$$

where $\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312 \\ -1 & \text{if } ijk = 132, 321, 213 \\ 0 & \text{otherwise} \end{cases}$

Note also that if

$\vec{A}\times\vec{B} = \vec{C}$

this can be written as,

$\cancel{A_i B_j \epsilon_{ijk} = \epsilon_k}$

$$\vec{A}\times\vec{B} = \epsilon_{ijk} A_i B_j \hat{e}_k$$

$$= \hbar S_j L_l \hat{e}_k \underbrace{\epsilon_{jkl}}_{=\epsilon_{ljk}}$$

$$= \hbar L_l S_j \hat{e}_k \epsilon_{ljk}$$

$$= \hbar \vec{L}\times\vec{S}$$

b) $$[\vec{L}\cdot\vec{S}, \vec{S}] = [L_j S_j, S_k \hat{e}_k]$$

$$= L_j \hat{e}_k [S_j, S_k]$$

$$= L_j \hat{e}_k \hbar \epsilon_{jkl} S_l$$

$$= \hbar L_j S_l \hat{e}_k \epsilon_{jkl}$$

$$= \hbar L_j S_l \hat{e}_k \epsilon_{ljk}$$

$$= \hbar \vec{S}\times\vec{L}$$

(c) $= [\vec{L}\cdot\vec{S}, \vec{J}] = [\vec{L}\cdot\vec{S}, \vec{L}+\vec{S}]$

$$= [\vec{L}\cdot\vec{S}, \vec{L}] + [\vec{L}\cdot\vec{S}, \vec{S}]$$

$$= \hbar\, \vec{L}\times\vec{S} + \hbar\,\vec{S}\times\vec{L}$$

$$= \hbar\,\vec{L}\times\vec{S} - \hbar\,\vec{L}\times\vec{S} = 0$$

(d)

$$[\vec{L}\cdot\vec{S}, L^2] = [L_j S_j, L^2] = S_j \underbrace{[L_j, L^2]}_{=0} = 0$$

(e) $[\vec{L}\cdot\vec{S}, S^2] = [L_j S_j, S^2] = L_j [S_j, S^2] = 0$

(f) $[\vec{L}\cdot\vec{S}, J^2] = [\vec{L}\cdot\vec{S}, (\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S})]$

$$= [\vec{L}\cdot\vec{S}, L^2 + S^2 + 2\vec{S}\cdot\vec{L}]$$

$$= \underbrace{[\vec{L}\cdot\vec{S}, L^2]}_{=0} + \underbrace{[\vec{L}\cdot\vec{S}, S^2]}_{=0} + 2\underbrace{[\vec{L}\cdot\vec{S}, \vec{S}\cdot\vec{L}]}_{=0}$$

$$= 0$$

5. (Liboff #13.12)

$$E = 3\hbar\omega_0 = \hbar\omega_0(n_x + n_y + 1)$$

<u>Threefold degeneracy comes from</u>: $(n_x, n_y) = (2,0)$, $(0,2)$, $(1,1)$

Need to calculate the matrix,

$$W_{ab} = \langle a | H' | b \rangle$$

To do so, it helps to write $H' = K'xy$ in terms of raising and lowering operators,

$$H' = K' \cdot \sqrt{\frac{\hbar}{2m\omega}} \underbrace{(a^\dagger + a)}_{\text{raising and lowering for x direction}} \cdot \sqrt{\frac{\hbar}{2m\omega}} \underbrace{(b^\dagger + b)}_{\text{raising and lowering for y direction}}$$

$$= \frac{K'\hbar}{2m\omega}(a^\dagger + a)(b^\dagger + b) = \frac{K'\hbar}{2m\omega}(a^\dagger b^\dagger + a^\dagger b + a b^\dagger + ab)$$

$$H'|20\rangle = \frac{K'\hbar}{2m\omega}\left(a^\dagger b^\dagger|20\rangle + a^\dagger b|20\rangle + ab^\dagger|20\rangle + ab|20\rangle\right)$$

$$= \frac{K'\hbar}{2m\omega}\left(\sqrt{3}\,|31\rangle + 0 + \sqrt{2}\,|11\rangle + |00\rangle\right)$$

$\therefore$ $\underline{\langle 20|H'|20\rangle = 0}$

$\underline{\langle 11|H'|20\rangle = \sqrt{2}\cdot\frac{K'\hbar}{2m\omega}}$

$\underline{\langle 02|H'|20\rangle = 0}$

$$H'|02\rangle = \frac{K'\hbar}{2m\omega}\left(a^\dagger b^\dagger|02\rangle + a^\dagger b|02\rangle + ab^\dagger|02\rangle + ab|02\rangle\right)$$

$$= \frac{K'\hbar}{2m\omega}\left(\sqrt{3}\,|13\rangle + \sqrt{2}\,|11\rangle + 0 + 0\right)$$

$\therefore$ $\underline{\langle 20|H'|02\rangle = 0}$

$\underline{\langle 11|H'|02\rangle = \sqrt{2}\cdot\frac{K'\hbar}{2m\omega}}$

$\underline{\langle 02|H'|02\rangle = 0}$

$$H'|11\rangle = \frac{K'\hbar}{2m\omega}\left(a^\dagger b^\dagger|11\rangle + a^\dagger b|11\rangle + ab^\dagger|11\rangle + ab|11\rangle\right)$$

$$= \frac{K'\hbar}{2m\omega}\left(2|22\rangle + \sqrt{2}\,|20\rangle + \sqrt{2}\,|02\rangle + |00\rangle\right)$$

$\therefore$ $\langle 11|H'|11\rangle = 0$

$\langle 20|H'|11\rangle = \frac{\sqrt{2}K'\hbar}{2m\omega}$

$\langle 02|H'|11\rangle = \frac{\sqrt{2}K'\hbar}{2m\omega}$

$\therefore$ The W matrix is given by,

$$W = \begin{array}{c} \begin{array}{ccc} |20\rangle & |11\rangle & |02\rangle \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \cdot \sqrt{2}\cdot\frac{K'\hbar}{2m\omega}$$

Need to solve for the eigenvalues...

$$\det(W-\lambda\mathbb{1}) = \det\begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix} = -\lambda^3 + 2\lambda$$

$$= \lambda(2-\lambda^2) = (\lambda-\sqrt{2})(\lambda+\sqrt{2})\cdot -\lambda = 0$$

$\therefore$ The eigenvalues are $\left\{ \begin{array}{l} \lambda = 0 \\ \lambda = \sqrt{2} \\ \lambda = -\sqrt{2} \end{array} \right\}$

$\therefore$

$$E_1^{(1)} = 0$$

$$E_2^{(1)} = \sqrt{2}\cdot\frac{\sqrt{2}k'\hbar}{2m\omega} = \frac{k'\hbar}{m\omega}$$

$$E_3^{(1)} = -\sqrt{2}\cdot\frac{\sqrt{2}k'\hbar}{2m\omega} = -\frac{k'\hbar}{m\omega}$$

$\blacksquare$