**Physics 110A, Spring 2021**
**Solution to Homework 2**
**GSI: Yi-Chuan Lu**

1. (a) $\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = \boxed{\begin{cases} \frac{\rho}{\epsilon_0}, & r<R, \\ 0, & r>R, \end{cases}}$, $\boldsymbol{\nabla}\times\mathbf{E} = \frac{1}{r\sin\theta}\frac{\partial E_r}{\partial\phi}\hat{\boldsymbol{\theta}} - \frac{1}{r}\frac{\partial E_r}{\partial\theta}\hat{\boldsymbol{\phi}} = \boxed{\mathbf{0}.}$

(b) $\boldsymbol{\nabla}\cdot\mathbf{B} = \frac{1}{s}\frac{\partial B_\phi}{\partial\phi} = \boxed{0,}$ $\boldsymbol{\nabla}\times\mathbf{B} = -\frac{\partial B_\phi}{\partial z}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial}{\partial s}\left(sB_\phi\right)\hat{\mathbf{z}} = \boxed{\begin{cases} \mu_0 J\hat{\mathbf{z}}, & s<a, \\ \mathbf{0}, & s>a. \end{cases}}$

(c) Using cylindrical coordinates,

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{V} &= \frac{1}{s}\frac{\partial}{\partial s}\left(sV_s\right) + \frac{1}{s}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z} = \boxed{4\cos^2\phi + 6.} \\ \boldsymbol{\nabla}\times\mathbf{V} &= \left[\frac{1}{s}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial V_s}{\partial z} - \frac{\partial V_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}\left(sV_\phi\right) - \frac{\partial V_s}{\partial\phi}\right]\hat{\mathbf{z}} = \boxed{4\sin\phi\cos\phi\hat{\mathbf{z}}.} \end{aligned}$$

(d) $\boldsymbol{\nabla}\cdot\mathbf{V} = 4\cos^2\phi + 6$, $d\tau = sd\phi dsdz$, so

$$\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{V}d\tau = \int_0^2\int_0^1\int_0^{\pi/2}\left(4\cos^2\phi + 6\right)sd\phi dsdz = \boxed{4\pi.}$$

Top: $z=2$, $d\mathbf{a} = sd\phi ds\hat{\mathbf{z}}$, $\mathbf{V} = (\cdots)\,\hat{\mathbf{s}} + (\cdots)\,\hat{\boldsymbol{\phi}} + 6\hat{\mathbf{z}}$, $\int_{top}\mathbf{V}\cdot d\mathbf{a} = \int_0^1\int_0^{\pi/2}6sd\phi ds = \frac{3}{2}\pi$.
Bottom: $z=0$, $d\mathbf{a} = -sd\phi ds\hat{\mathbf{z}}$, $\mathbf{V} = (\cdots)\,\hat{\mathbf{s}} + (\cdots)\,\hat{\boldsymbol{\phi}} + 0\hat{\mathbf{z}}$, $\int_{bottom}\mathbf{V}\cdot d\mathbf{a} = 0$.
Right: $\phi = \pi/2$, $d\mathbf{a} = dsdz\hat{\boldsymbol{\phi}}$, $\mathbf{V} = (\cdots)\,\hat{\mathbf{s}} + 0\hat{\boldsymbol{\phi}} + (\cdots)\,\hat{\mathbf{z}}$, $\int_{right}\mathbf{V}\cdot d\mathbf{a} = 0$.
Left: $\phi = 0$, $d\mathbf{a} = -dsdz\hat{\boldsymbol{\phi}}$, $\mathbf{V} = (\cdots)\,\hat{\mathbf{s}} + 0\hat{\boldsymbol{\phi}} + (\cdots)\,\hat{\mathbf{z}}$, $\int_{left}\mathbf{V}\cdot d\mathbf{a} = 0$.
Front: $s=1$, $d\mathbf{a} = 1d\phi dz\hat{\mathbf{s}}$, $\mathbf{V} = \left(2+\cos^2\phi\right)\hat{\mathbf{s}} + (\cdots)\,\hat{\boldsymbol{\phi}} + (\cdots)\,\hat{\mathbf{z}}$, $\int_{front}\mathbf{V}\cdot d\mathbf{a} = \int_0^2\int_0^{\pi/2}\left(2+\cos^2\phi\right)d\phi dz = \frac{5}{2}\pi$.
Therefore, the total surface integral is

$$\oint_{\mathcal{S}}\mathbf{V}\cdot d\mathbf{a} = \frac{3}{2}\pi + 0 + 0 + 0 + \frac{5}{2}\pi = \boxed{4\pi.}$$

(e) On the top surface, $\boldsymbol{\nabla}\times\mathbf{V} = 4\sin\phi\cos\phi\hat{\mathbf{z}}$, $d\mathbf{a} = sd\phi ds\hat{\mathbf{z}}$, so

$$\int_{\mathcal{S}}\boldsymbol{\nabla}\times\mathbf{V}\cdot d\mathbf{a} = \int_0^1\int_0^{\pi/2}\left(4\sin\phi\cos\phi\right)sd\phi ds = \boxed{1.}$$

Left edge $\phi = 0$, $d\mathbf{l} = ds\hat{\mathbf{s}}$, $\mathbf{V} = 3s\hat{\mathbf{s}} + (\cdots)\,\hat{\boldsymbol{\phi}} + (\cdots)\,\hat{\mathbf{z}}$, $\int_{left}\mathbf{V}\cdot d\mathbf{l} = \int_0^1 3sds = \frac{3}{2}$.
Right edge $\phi = \pi/2$, $d\mathbf{l} = ds\hat{\mathbf{s}}$, $\mathbf{V} = 2s\hat{\mathbf{s}} + (\cdots)\,\hat{\boldsymbol{\phi}} + (\cdots)\,\hat{\mathbf{z}}$, $\int_{right}\mathbf{V}\cdot d\mathbf{l} = \int_1^0 2sds = -1$.
Front edge $s=1$, $d\mathbf{l} = 1d\phi\boldsymbol{\phi}$, $\mathbf{V} = (\cdots)\,\hat{\mathbf{s}} + \sin\phi\cos\phi\hat{\boldsymbol{\phi}} + (\cdots)\,\hat{\mathbf{z}}$, $\int_{front}\mathbf{V}\cdot d\mathbf{l} = \int_0^{\pi/2}\sin\phi\cos\phi d\phi = \frac{1}{2}$.
So, the total path integral is

$$\oint_{\mathcal{P}}\mathbf{V}\cdot d\mathbf{l} = \frac{3}{2} - 1 + \frac{1}{2} = \boxed{1.}$$

2. Since

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{E} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right)-\frac{\partial E_\theta}{\partial\phi}\right]\hat{\mathbf{r}}+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi}-\frac{\partial}{\partial r}\left(rE_\phi\right)\right]\hat{\boldsymbol{\theta}}+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\theta\right)-\frac{\partial E_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \\
&= \frac{1}{r}\left[-\frac{2K}{r^3}\sin\theta+\frac{2K}{r^3}\sin\theta\right]\hat{\boldsymbol{\phi}}=\mathbf{0},
\end{aligned}$$

Helmholtz theorem guarantees that there exits a scalar function $V(\mathbf{r})$ such that $\mathbf{E}=-\boldsymbol{\nabla}V$.

Note that this curl formula above has $r$ in the denominator, so it implicitly assumes $r\neq 0$. Therefore, we actually don't know if $\boldsymbol{\nabla}\times\mathbf{E}$ is zero at the origin or not. However, you can use the formal definition of curl

$$(\boldsymbol{\nabla}\times\mathbf{E})\cdot\hat{\mathbf{n}}=\lim_{\mathcal{S}\to 0}\frac{\oint_{\mathcal{P}}\mathbf{E}\cdot d\mathbf{l}}{\mathcal{S}}$$

that we discussed in the discussion section to prove $\boldsymbol{\nabla}\times\mathbf{E}$ is indeed zero at the origin.

To find out $V$, we write $\mathbf{E}=-\boldsymbol{\nabla}V$ in component form:

$$\begin{aligned}
-\frac{\partial V}{\partial r} &= E_r=\frac{2K}{r^3}\cos\theta, &(1)\\
-\frac{1}{r}\frac{\partial V}{\partial\theta} &= E_\theta=\frac{K}{r^3}\sin\theta, &(2)\\
-\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} &= E_\phi=0. &(3)
\end{aligned}$$

Integrating equation(1) with respect to $r$, we get

$$-V=-\frac{K}{r^2}\cos\theta+C\left(\theta,\phi\right), \tag{4}$$

where $C$ is a constant of $r$, but may contain $\theta$ and $\phi$ dependence. Next we differentiate equation(4) with respect to $\theta$ and $\phi$ separately:

$$\begin{aligned}
-\frac{\partial V}{\partial\theta} &= \frac{K}{r^2}\sin\theta+\frac{\partial C}{\partial\theta},\\
-\frac{\partial V}{\partial\phi} &= \frac{\partial C}{\partial\phi},
\end{aligned}$$

and compare the two results with equations(2)(3). We conclude that $\partial C/\partial\theta=\partial C/\partial\phi=0$, i.e., $C$ actually does not depend on $\theta$ and $\phi$, and is truly a constant. Since we can drop any constant from the potential, equation(4) reduces to

$$\boxed{V\left(\mathbf{r}\right)=\frac{K}{r^2}\cos\theta.}$$

3. (a) Let $\mathbf{F} = T\hat{\mathbf{x}}$, and apply divergence theorem

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot (T\hat{\mathbf{x}})\, d\tau = \oint_{\mathcal{S}} T\hat{\mathbf{x}} \cdot d\mathbf{a}.$$

From product rule, $\boldsymbol{\nabla} \cdot (T\hat{\mathbf{x}}) = \boldsymbol{\nabla} T \cdot \hat{\mathbf{x}} + T\,(\boldsymbol{\nabla} \cdot \hat{\mathbf{x}}) = \boldsymbol{\nabla} T \cdot \hat{\mathbf{x}}$, so the equation above reduces to

$$\left(\int_{\mathcal{V}} \boldsymbol{\nabla} T d\tau\right) \cdot \hat{\mathbf{x}} = \left(\oint_{\mathcal{S}} T d\mathbf{a}\right) \cdot \hat{\mathbf{x}}$$

This means the $x$ component of $\int_{\mathcal{V}} \boldsymbol{\nabla} T d\tau$ equals to the $x$ component of $\oint_{\mathcal{S}} T d\mathbf{a}$. We can repeat our trick by choosing $\mathbf{F} = T\hat{\mathbf{y}}$ and $\mathbf{F} = T\hat{\mathbf{z}}$, then we are able to conclude

$$\boxed{\int_{\mathcal{V}} \boldsymbol{\nabla} T d\tau = \oint_{\mathcal{S}} T d\mathbf{a}}$$

since the two vectors have the same three corresponding components.

(b) Let $\mathbf{F} = \mathbf{V} \times \hat{\mathbf{x}}$ and apply divergence theorem:

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\mathbf{V} \times \hat{\mathbf{x}})\, d\tau = \oint_{\mathcal{S}} (\mathbf{V} \times \hat{\mathbf{x}}) \cdot d\mathbf{a}.$$

From product rule, $\boldsymbol{\nabla} \cdot (\mathbf{V} \times \hat{\mathbf{x}}) = \hat{\mathbf{x}} \cdot (\boldsymbol{\nabla} \times \mathbf{V}) - \mathbf{V} \cdot (\boldsymbol{\nabla} \times \hat{\mathbf{x}}) = \hat{\mathbf{x}} \cdot (\boldsymbol{\nabla} \times \mathbf{V})$, and from the property of triple product, $(\mathbf{V} \times \hat{\mathbf{x}}) \cdot d\mathbf{a} = (d\mathbf{a} \times \mathbf{V}) \cdot \hat{\mathbf{x}}$, so the equation above reduces to

$$\hat{\mathbf{x}} \cdot \int_{\mathcal{V}} (\boldsymbol{\nabla} \times \mathbf{V})\, d\tau = \oint_{\mathcal{S}} (d\mathbf{a} \times \mathbf{V}) \cdot \hat{\mathbf{x}}.$$

Using the same argument as in (a), we conclude that

$$\boxed{\int_{\mathcal{V}} (\boldsymbol{\nabla} \times \mathbf{V})\, d\tau = -\oint_{\mathcal{S}} \mathbf{V} \times d\mathbf{a}.}$$

(c) Let $\mathbf{F} = T\boldsymbol{\nabla} U - U\boldsymbol{\nabla} T$, then

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{F} &= \boldsymbol{\nabla} \cdot (T\boldsymbol{\nabla} U - U\boldsymbol{\nabla} T) \\ &= \boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} U + T\boldsymbol{\nabla}^2 U - \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} T - U\boldsymbol{\nabla}^2 T \\ &= T\boldsymbol{\nabla}^2 U - U\boldsymbol{\nabla}^2 T. \end{aligned}$$

Plug this expression into divergence theorem, and we get

$$\boxed{\int_{\mathcal{V}} \left(T\boldsymbol{\nabla}^2 U - U\boldsymbol{\nabla}^2 T\right) d\tau = \oint_{\mathcal{S}} (T\boldsymbol{\nabla} U - U\boldsymbol{\nabla} T) \cdot d\mathbf{a}.}$$