Physics 110A, Spring 2021
Solution to Homework 12
GSI: Yi-Chuan Lu

1. To find the inductance, we run a current I along one wire, and run an opposite current $-I$ along the other. Imagine the two wires are *connected at infinity* to form a large closed loop. The magnetic flux through this large loop is

$$\Phi = 2\left(l\int_a^{d-a}\frac{\mu_0 I}{2\pi s}ds\right) = \frac{\mu_0 I}{\pi}l\ln\left(\frac{d}{a}-1\right).$$

By the definition of inductance $\Phi = LI$, we get

$$\boxed{\frac{L}{l}=\frac{\mu_0}{\pi}\ln\left(\frac{d}{a}-1\right)\simeq\frac{\mu_0}{\pi}\ln\left(\frac{d}{a}\right)}.$$

To find the capacitance, we put stationary charge $+Q$ on one wire and $-Q$ on the other one. The electric potential between the two wires is

$$V = \frac{\lambda}{2\pi\epsilon_0}\int_a^{d-a}\left(\frac{1}{s}+\frac{1}{d-s}\right)ds = \frac{\lambda}{\pi\epsilon_0}\ln\left(\frac{d}{a}-1\right) = \frac{Q}{\pi\epsilon_0 l}\ln\left(\frac{d}{a}-1\right).$$

By the definition of capacitance $C = Q/V$, we get

$$\boxed{\frac{C}{l} = \frac{\pi\epsilon_0}{\ln\left(\frac{d}{a}-1\right)}\simeq\frac{\pi\epsilon_0}{\ln\left(\frac{d}{a}\right)}}.$$

2. (a) Initially there is only a radial current I_0, and therefore the charge is flowing with a radial velocity $\mathbf{v}$. By definition, the corresponding current density J_s is I_0 divided by its cross-sectional area $2\pi st$, and is also equal to nev :

$$J_s = \frac{I_0}{2\pi st} = nev,$$

so the radial velocity is $v = \dfrac{I_0}{2\pi stne}$. Using Lorentz force law, the charge feels a magnetic force in the circular direction, and using Ohm's law, this circular current density J_ϕ is

$$J_\phi = \sigma vB = \frac{\sigma I_0 B}{2\pi stne}.$$

Therefore the total *circular* current is

$$I_\phi = \int J_\phi da = \int_{R_1}^{R_2}\frac{\sigma I_0 B}{2\pi stne}tds = \boxed{\frac{\sigma I_0 B}{2\pi ne}\ln\left(\frac{R_2}{R_1}\right)}.$$

(b) Note that there are two currents in this problem: radial current $J_s\hat{\mathbf{s}}$ and circular current $J_\phi\hat{\boldsymbol{\phi}}$. The $J_s\hat{\mathbf{s}}$ is driven by some electric field $\mathbf{E}$ that is not given explicitly by the problem, and the $J_\phi\hat{\boldsymbol{\phi}}$ is driven by the magnetic field $\mathbf{B}$:

$$\mathbf{J} = \sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \sigma\mathbf{E}+\sigma\mathbf{v}\times\mathbf{B} = J_s\hat{\mathbf{s}}+J_\phi\hat{\boldsymbol{\phi}}.$$

So $\mathbf{E}$ is in the radial direction, and the equipotential curves should be perpendicular to it, i.e., they should be in the $\boxed{\text{circular direction.}}$

3. (a) Consider a line segment $d\mathbf{l} = dl\hat{\boldsymbol{\phi}}$ on the ring. The magnetic force acting on this segment is $d\mathbf{F} = \lambda dl\mathbf{v} \times \mathbf{B}$, where $\lambda \equiv Q/2\pi R$ is the line charge density of the ring. The circular component of $d\mathbf{F}$, which is responsible for rotating the ring, is $dF_\phi = d\mathbf{F} \cdot \hat{\boldsymbol{\phi}} = \lambda (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$, and the torque is $d\tau = RdF_\phi = \lambda R (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$. So the total torque acting on the entire ring is

$$\begin{aligned}\tau &= RdF_\phi = \lambda R \oint_{\mathcal{P}} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = \lambda R \int_{\mathcal{S}} \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{a} \\ &= \lambda R \int_{\mathcal{S}} [\mathbf{v} (\boldsymbol{\nabla} \cdot \mathbf{B}) - (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{B}] \cdot d\mathbf{a} \simeq -\lambda R v \frac{\partial \mathbf{B}}{\partial z} \cdot (\pi R^2 \hat{\mathbf{z}}) = -\frac{QR^2}{2} \frac{dB_z}{dt}.\end{aligned}$$

Here, we have assumed the ring is small enough so that $\int_{\mathcal{S}} (\cdots) \cdot d\mathbf{a}$ is approximated as $(\cdots) \cdot (\pi R^2 \hat{\mathbf{z}})$. Note that the magnetic field is static, and the time derivative is understood as

$$\frac{dB_z}{dt} = \frac{\partial B_z(s,z)}{\partial z} \frac{dz}{dt} = \frac{\partial B_z(s,z)}{\partial z} v_z.$$

(b) The final angular momentum is

$$\mathbf{L} = \int_0^\infty \boldsymbol{\tau} dt = -\frac{QR^2}{2} \int_0^\infty \frac{dB_z}{dt} dt \hat{\mathbf{z}} = -\frac{QR^2}{2} \hat{\mathbf{z}} \int_{B_z(h)}^{B_z(0)} dB_z = \boxed{\frac{QR^2}{2} [B_z(h) - B_z(0)] \hat{\mathbf{z}}.}$$

Since the ring has moment of inertia MR^2, its angular velocity is $\omega = \dfrac{L}{MR^2} = \boxed{\dfrac{Q}{2M} [B_z(h) - B_z(0)],}$ and the rotational kinetic energy is $K = \frac{1}{2} L\omega = \boxed{\dfrac{Q^2 R^2}{8M} [B_z(h) - B_z(0)]^2.}$

(c) When the ring has angular velocity ω, its current is $I = \lambda v = \dfrac{Q}{2\pi R} R\omega = \dfrac{Q\omega}{2\pi}$, and so the dipole moment is $\mathbf{m} = I\pi R^2 = \boxed{\dfrac{QR^2}{2} \omega \hat{\mathbf{z}}.}$

(d) In part (b), if we only integrated from h to some intermediate height z, we would get

$$\mathbf{L} = -\frac{QR^2}{2} \hat{\mathbf{z}} \int_{B_z(h)}^{B_z(z)} dB_z = \frac{QR^2}{2} [B_z(z) - B_z(0)] \hat{\mathbf{z}},$$

and so the angular velocity at height z is $\omega = \dfrac{L}{MR^2} = \dfrac{Q}{2M} [B_z(z) - B_z(0)]$, and the corresponding dipole moment is $\mathbf{m} = \dfrac{QR^2}{2} \omega \hat{\mathbf{z}} = \dfrac{Q^2 R^2}{4M} [B_z(z) - B_z(0)] \hat{\mathbf{z}}$. So, at height z, the force acting on the ring is

$$\mathbf{F} = \boldsymbol{\nabla} (\mathbf{m} \cdot \mathbf{B}) = (\mathbf{m} \cdot \boldsymbol{\nabla}) \mathbf{B} = \frac{Q^2 R^2}{4M} [B_z(z) - B_z(0)] \frac{dB_z}{dz} \hat{\mathbf{z}},$$

and the work done on the falling ring is

$$\begin{aligned}W &= \int_h^0 \mathbf{F} \cdot dz\hat{\mathbf{z}} = \frac{Q^2 R^2}{4M} \int_h^0 [B_z(z) - B_z(0)] \frac{dB_z}{dz} dz \\ &= \frac{Q^2 R^2}{4M} \int_{B_z(h)}^{B_z(0)} [B_z(z) - B_z(0)] dB_z = \frac{Q^2 R^2}{4M} \left[\frac{B_z(z)^2}{2} - B_z(0) B_z(z) \right]_{B_z(h)}^{B_z(0)} \\ &= \boxed{-\frac{Q^2 R^2}{8M} [B_z(0) - B_z(h)]^2.}\end{aligned}$$