Physics 110A, Spring 2021
Solution to Homework 4
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1. (a) Due to spherical symmetry, without loss of generality, we may assume the observer (field point) is on the z axis at a distance r away from the center. Then

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\frac{\sigma R^2\sin\theta d\phi d\theta}{\sqrt{r^2+R^2-2rR\cos\theta}} = \frac{\sigma}{2\epsilon_0}\frac{R}{r}\sqrt{r^2+R^2-2rR\cos\theta}\bigg|_0^{\pi} \\ &= \frac{\sigma}{2\epsilon_0}\frac{R}{r}\left[(r+R)-|r-R|\right] = \boxed{\begin{cases}\dfrac{\sigma}{\epsilon_0}\dfrac{R^2}{r}, & r>R, \\ \dfrac{\sigma}{\epsilon_0}R, & r<R.\end{cases}} \end{aligned}$$

(b) The electric field is $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}) = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} = \boxed{\begin{cases}\dfrac{\sigma}{\epsilon_0}\dfrac{R^2}{r^2}\hat{\mathbf{r}}, & r>R, \\ \mathbf{0}, & r<R.\end{cases}}$

2. Note that the electric field inside the shell is zero, so the integral over the entire space reduces to that over the region just outside the spherical shell:

$$\begin{aligned} W &= \frac{\epsilon_0}{2}\int_{\text{all space}}E^2 d\tau = \frac{\epsilon_0}{2}\int_{\text{outside}}E^2 d\tau \\ &= \frac{\epsilon_0}{2}\int_R^{\infty}\int_0^{\pi}\int_0^{2\pi}\left(\frac{\sigma}{\epsilon_0}\frac{R^2}{r^2}\right)^2 r^2\sin\theta d\phi d\theta dr = \boxed{\frac{2\pi\sigma^2R^3}{\epsilon_0}}. \end{aligned}$$

3. (a) We can either assume σ is constant or assume the total charge $Q \equiv 4\pi R^2\sigma$ is constant when the shell expands. The former is more complicated because we need to connect the shell to, e.g., a battery to provide charge, and therefore we will have to take into account the electric work done by the battery[1]. So let's assume $Q = 4\pi R^2\sigma$ is constant. In terms of Q, the energy W is expressed as

$$W = \frac{2\pi R^3}{\epsilon_0}\left(\frac{Q}{4\pi R^2}\right)^2 = \frac{Q^2}{8\pi\epsilon_0 R},$$

so the change in W at constant Q is

$$\delta W = \frac{dW}{dR}\delta R = -\frac{Q^2}{8\pi\epsilon_0R^2}\delta R = \boxed{-\frac{2\pi R^2\sigma^2}{\epsilon_0}\delta R.}$$

(b) By letting $\delta W = -f\left(4\pi R^2\right)\delta R$, we obtain

$$f = -\frac{1}{4\pi R^2}\frac{\delta W}{\delta R} = \boxed{\frac{\sigma^2}{2\epsilon_0}}.$$

Note that this is consistent with what we would obtain by using Griffiths equation (2.50):

$$\mathbf{f} = \sigma\frac{\mathbf{E}_{\text{above}}+\mathbf{E}_{\text{below}}}{2} = \sigma\left(\frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}\right).$$

[1] There will be a discussion on this matter in Griffiths section 4.4.4.