Homework 5 Solutions

1) Griffiths 6.17

$$E^1_{rel} = \frac{-(E_n)^2}{2mc^2}\left(\frac{4n}{l+1/2} - 3\right) \quad \text{and} \quad E^1_{spin\,orbit} = \frac{+(E_n)^2}{mc^2}\,\frac{n\left(j(j+1) - l(l+1) - \frac{3}{4}\right)}{l(l+1/2)(l+1)}$$

Now either $j = l+1/2$ or $j = l - 1/2$

$$E^1_{rel} + E^1_{spin\,orbit} = \frac{-(E_n)^2}{mc^2}\left(\frac{2n}{l+1/2} - \frac{3}{2} - \frac{n\left((l+1/2)(l+3/2) - l(l+1) - \frac{3}{4}\right)}{l(l+1/2)(l+1)}\right)$$

$$= \frac{-(E_n)^2}{mc^2}\left(\frac{2n}{l+1/2} - \frac{3}{2} - n\,\frac{\cancel{l^2} + 2l + \cancel{3/4} - \cancel{l^2} - l - \cancel{3/4}}{l(l+1/2)(l+1)}\right)$$

$$= \frac{-(E_n)^2}{mc^2}\left(\frac{2n}{l+1/2} - \frac{3}{2} - \frac{n}{(l+1/2)(l+1)}\right)$$

$$= \frac{-(E_n)^2}{mc^2}\left(-\frac{3}{2} + \frac{2nl + 2n - n}{(l+1/2)(l+1)}\right)$$

$$= \frac{-(E_n)^2}{mc^2}\left(-\frac{3}{2} + \frac{2n\cancel{(l+1/2)}}{\cancel{(l+1/2)}(l+1)}\right)$$

$$= \boxed{\frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right)} \quad \text{from } l+1 = \left(l+\tfrac{1}{2}\right) + \tfrac{1}{2} = j + \tfrac{1}{2}$$

Now do $j = l - \frac{1}{2}$

$$E^1_{rel} + E^1_{spin\,orbit} = \frac{-(E_n)^2}{mc^2}\left(\frac{2n}{l+1/2} - \frac{3}{2} - n\,\frac{(l+1/2)(l-1/2) - l(l+1) - 3/4}{l(l+1/2)(l+1)}\right)$$

$$= -\frac{(E_n)^2}{mc^2}\left[\frac{2n}{l+1/2} - \frac{3}{2} - n\cdot\frac{-\cancel{(l+1)}}{l(l+1/2)\cancel{(l+1)}}\right]$$

$$= -\frac{(E_n)^2}{mc^2}\left[-\frac{3}{2} + \frac{2nl+n}{l(l+1/2)}\right]$$

$$= -\frac{(E_n)^2}{mc^2}\left[-\frac{3}{2} + \frac{2n\cancel{(l+1/2)}}{l\cancel{(l+1/2)}}\right] = \frac{(E_n)^2}{2mc^2}\left[3 - \frac{4n}{j+1/2}\right]$$

Using $l = (l-1/2)+1/2 = j+1/2$

2. (Griffiths 6.18)

$$\Delta E = \frac{(-13.6\text{ eV})}{3^2} - \frac{(-13.6\text{ eV})}{2^2} = \underline{-1.89\text{ eV}}$$

$$E = \hbar\omega = h\nu \Rightarrow \nu = \frac{E}{h} =$$

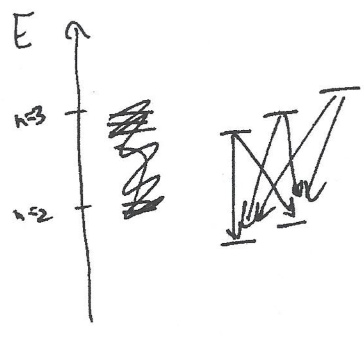
Recall that including fine structure,

$$E_{nj} = -\frac{13.6\text{ eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right]$$

$$\Delta E = E_{3j} - E_{2j'} = \frac{-13.6\text{eV}}{9}\left[1 + \frac{\alpha^2}{9}\left(\frac{3}{j+1/2} - \frac{3}{4}\right)\right] + \frac{13.6\text{ eV}}{4}\left[1 + \frac{\alpha^2}{4}\left(\frac{2}{j'+1/2} - \frac{3}{4}\right)\right]$$

$$= 13.6\text{eV}\cdot\left[\frac{1}{4} - \frac{1}{9} + \frac{\alpha^2}{8(j'+1/2)} - \frac{3\alpha^2}{64} - \frac{\alpha^2}{27(j+1/2)} + \frac{\alpha^2}{108}\right]$$

$$\Delta E_{fs} = 13.6\text{eV}\cdot\alpha^2\cdot\left[\frac{1}{8(j'+1/2)} - \frac{1}{27(j+1/2)} + \frac{1}{108} - \frac{3}{64}\right]$$



Recall $j = |l \pm 1/2|$

For $n=2$, $l=0,1 \Rightarrow j' = 1/2, 3/2$

For $n=3$, $l=0,1,2 \Rightarrow j = 1/2, 3/2, 5/2$

Therefore, we expect to get 6 different Balmer Lines.

$$\frac{1}{8} - \frac{1}{27} + \frac{1}{108} - \frac{3}{64}$$

$\Delta E_{fs} =$

$j \backslash j'$	$1/2$	$3/2$
$1/2$	3.6×10^{-5} eV	-8.8×10^{-6} eV
$3/2$	5.0×10^{-5} eV	4.6×10^{-6} eV
$5/2$	~~[illegible]~~ 5.4×10^{-5} eV	9.1×10^{-6} eV

Order of lines:

$j \to j'$

- $5/2 \to 1/2$
- $3/2 \to 1/2$
- $1/2 \to 1/2$
- $5/2 \to 3/2$
- $3/2 \to 3/2$
- $1/2 \to 3/2$

~~Frequencies in order~~

Frequency spacings between lines:

$\nu_2 - \nu_1 = 3.23\times10^{9}$ Hz

$\nu_3 - \nu_2 = 1.08\times10^{9}$ Hz

$\nu_4 - \nu_3 = 6.60\times10^{9}$ Hz

$\nu_5 - \nu_4 = 3.23\times10^{9}$ Hz

$\nu_6 - \nu_5 = 1.08\times10^{9}$ Hz

3. (Griffiths 6.21)

8 states ; $|2\ell\, j\, m_j\rangle$

$|2\,0\,\tfrac12\,\pm\tfrac12\rangle$, $|2\,1\,\tfrac12\,\pm\tfrac12\rangle$: $E^0_{nj} = \frac{-13{,}6\,eV}{4}\left(1+\frac{\alpha^2}{4}\left(2-\frac34\right)\right) = -3{,}4\,eV\left(1+\frac{5\alpha^2}{16}\right)$

$|2\,1\,\tfrac32\,\pm\tfrac12\rangle$ $|2\,1\,\tfrac32\,\pm\tfrac32\rangle$: $E^0_{nj} = -\frac{13{,}6\,eV}{4}\left(1+\frac{\alpha^2}{4}\left(\frac22-\frac34\right)\right) = -3{,}4\,eV\left(1+\frac{\alpha^2}{16}\right)$

We should first calculate the Landé g-factor for each state

$\ell=0,\ j=\tfrac12$

$$g_J = 1 + \frac{\frac12(\frac32)+\frac34}{2\cdot\frac12(\frac32)} = 1 + \frac{3/2}{3/2} = \boxed{2}$$

$\ell=1,\ j=\tfrac12$

$$g_J = 1 + \frac{\frac12(\frac32) - 1(2) + \frac34}{2\cdot\cancel{\frac12}\cdot(\frac32)} = 1 + \frac{3/2-2}{\cancel{2}\ 3/2} = 1 - \frac{1/2}{3/2}$$

$$= \boxed{\frac23}$$

$\ell=1,\ j=\tfrac32$

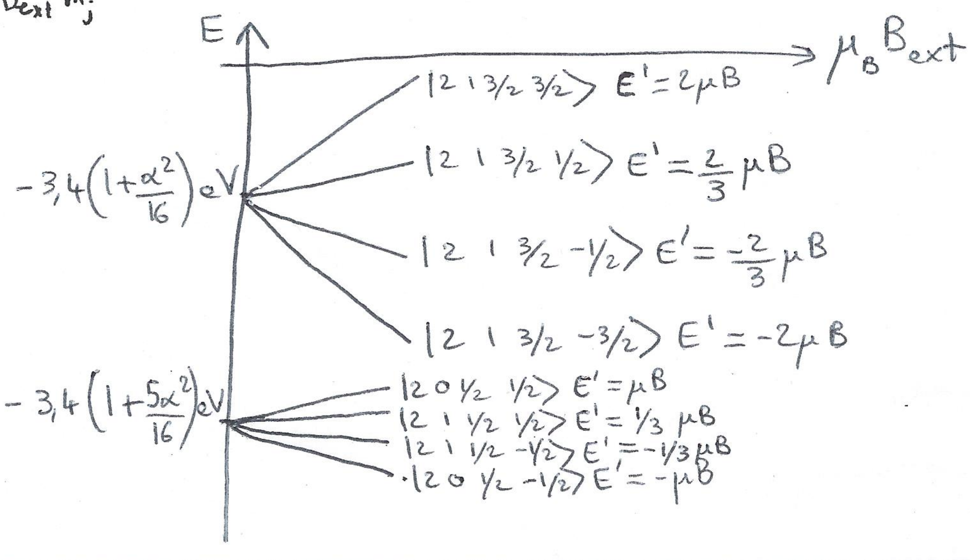
$$g_J = 1 + \frac{(\frac32)(\frac52) - 1(2) + \frac34}{2\cdot\frac32\cdot\frac52}$$

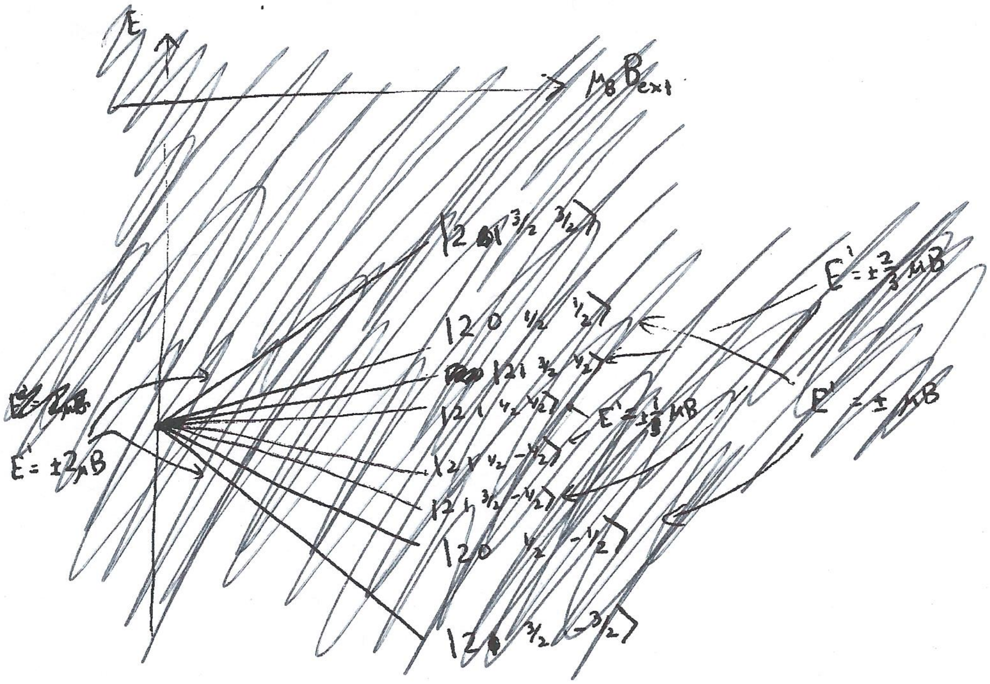
$$= 1 + \frac{\frac{15}{4} - 2 + \frac34}{15/2} = \cancel{\frac{\frac{30-8+3}{4}}{15/2} = \frac{25}{4}\cdot\frac{2}{15} = \frac53}$$

$$= 1 + \frac{\frac{18}{4}-\frac84}{15/2} = 1 + \frac{10}{4}\cdot\frac{2}{15} = 1 + \tfrac13 = \boxed{\tfrac43}$$

Recall,

$E^1_{Zeeman} = \mu_B g_J B_{ext} m_j$





4. (Griffiths 6.24)

$\ell = 0$

$j = s$

$m_j = m_s$

then $\quad E_Z^1 = \frac{e}{2m} \vec{B}_{ext} \cdot \langle \vec{L} + 2\vec{S} \rangle = \frac{e}{m} \vec{B}_{ext} \langle \vec{S} \rangle$ $\quad\Big\}$ Zeeman Part

Take $\quad \vec{B}_{ext} = B_{ext}\hat{z}$, $\quad$ then $\quad E_Z^1 = \frac{e}{m} B_{ext} \langle S_z \rangle = \frac{e}{m} B_{ext}\, m_s \hbar$

Fine Structure Part,

$$E_{nj} = -\frac{13.6\text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right]$$

Note: this is the total energy, including $E^{(0)}$.

$j = s = 1/2$

$$= -\frac{13.6\text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(n - \frac{3}{4}\right)\right]$$

$$\Rightarrow E_{fs}^1 = -\frac{13.6\text{eV}\cdot\alpha^2}{n^4}\cdot\left(n - \frac{3}{4}\right) = +\frac{13.6\text{eV}\cdot\alpha^2}{n^3}\left(\frac{3}{4n} - 1\right)$$

In agreement w/eq. [6.82]

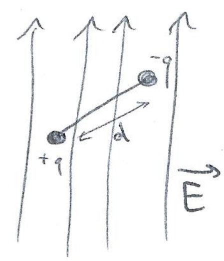
Note that since $[S_z, H] = 0$, the weak and strong bases are the same, so $E_z' = \frac{e B_{ext} m_s \hbar}{m}$ is the correct Zeeman contribution regardless of the strength of the perturbation.

5) Liboff 13.14

$H^\circ = \frac{L^2}{2I}$ $\qquad E_\ell = \frac{\hbar^2 \ell(\ell+1)}{2I}$ are $(2\ell+1)$-Fold degenerate since E_ℓ does not depend on m_ℓ.

$H' = -\vec{d} \cdot \vec{E}$ $\qquad$ Take $\vec{E} = (0, 0, E_0)$

Then $H' = -q\,|\vec{d}|\cos(\theta) E_0 = -q \underbrace{d}_{\text{constant } d \equiv |\vec{d}|} \cos(\theta) E_0$



Now we want to compute $\langle \ell' m' | H' | \ell m \rangle$ to get the matrix elements. However note that L_z commutes with both H' and H° so we can use eigenstates of L_z as the "good states" and only need nondegenerate perturbation theory.

($[L^2, L_z] = 0$ is a standard result, so $[H^\circ, L_z] = 0$.
But why is $[H', L_z] = 0$?? Well, $L_z = \frac{\hbar}{i}\frac{d}{d\phi}$ and H' is independent of the variable ϕ, so we are done!)

So in fact we only need $\langle \ell m | H' | \ell m \rangle$ $\quad$ ($\ell' = \ell$ and $m' = m$)

But $\langle \ell m | \cos(\theta) | \ell m \rangle = 0$! So the first order correction is 0.

You can argue this by a parity argument or pick for instance $\ell = 1$ to get a feel for why this is 0. Then generalize to arbitrary ℓ...

(In fact, the only non-zero matrix elements happen to be $\langle \ell \pm 1 \; m | \cos(\theta) | \ell \; m \rangle$. Proving this for general ℓ is abit delicate as it requires the use of recursion relations on the spherical harmonics $Y_{\ell,m}(\theta,\phi)$:

$$\cos(\theta)\, Y_{\ell,m} = \sqrt{\frac{(\ell+1)^2 - m^2}{4(\ell+1)^2 - 1}}\; Y_{\ell+1,m} + \sqrt{\frac{\ell^2 - m^2}{4\ell^2 - 1}}\; Y_{\ell-1,m}$$

You can ask me for details in office hours...

6) Liboff 13.15

a) $H' = \frac{-e}{2mc} B_0 L_z$ since $\vec{B} = (0, 0, B_0)$ and $\vec{L} = (L_x, L_y, L_z)$

Note L_z commutes with H' (obviously...) and L_z still commutes with H_0 (same H_0 as previous problem)

So use nondegenerate perturbation theory with "good states" $|\ell \; m\rangle$:

$$\boxed{\langle \ell m | H' | \ell m \rangle = \frac{-e B_0 \hbar m}{2mc}} \longrightarrow$$ the degeneracy is completely lifted! ($2\ell+1$ states with $2\ell+1$ distinct energies)

See part b) for the $\ell=1$ example...

b) For simplicity, neglect the electron spin.

P orbital $\Longleftrightarrow$ $\ell = 1$.

Consider states, $|n\, l\, m\rangle$ where $m = 0, \pm 1$ w/ n fixed.

As above,

$$\langle n\, l\, m' | \hat{H}' | n\, l\, m \rangle = \delta_{m_\ell m_\ell'} \left(-\frac{eB}{2mc} \right) m_\ell \hbar \Leftarrow \text{diagonal as in (a).}$$

$$\Rightarrow \boxed{E^{(1)}_{m_\ell} = -\frac{eB}{2mc} m_\ell}$$

More explicitly

$$E^{(1)}_{(1)} = -\frac{eB\hbar}{2mc}$$

$$E^{(1)}_{(0)} = 0$$

$$E^{(1)}_{(-1)} = \frac{eB\hbar}{2mc}$$