Problem Set 5

Physics 110A, UC Berkeley, Spring 2021

Due Monday, 3/1, at 11:59PM

Problem 1

Suppose we have a charge distribution $\rho_1(\boldsymbol{r})$ and a potential $V_2(\boldsymbol{r})$. The potential energy of this system is given by

$$\int_{\text{all space}} \rho_1 V_2 \, d\tau. \tag{1}$$

Recall that for localized charge distributions, and with the boundary condition $V \to 0$ at infinity, one can write the potential created by a charge distribution as

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r'})}{\mathscr{r}} \, d\tau'. \tag{2}$$

Show that the potential energy of ρ_1 in the potential V_2 produced by ρ_2 is equal to the potential energy of ρ_2 in the potential V_1 produced by ρ_1. That is,

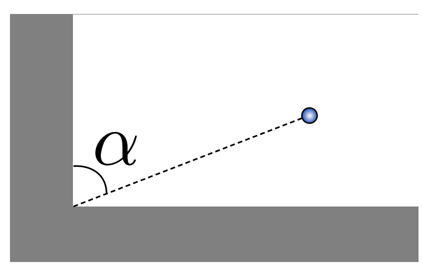
$$\int_{\text{all space}} \rho_1(\boldsymbol{r}) V_2(\boldsymbol{r}) \, d\tau = \int_{\text{all space}} \rho_2(\boldsymbol{r'}) V_1(\boldsymbol{r'}) \, d\tau'. \tag{3}$$

The is called the **Green's reciprocity relation**. In addition, using this relation, prove the "*mean value theorem*" that we discussed in class: the average of $V(\boldsymbol{r})$ over a spherical surface $\mathcal{S}$ that encloses a charge-free volume is equal to the potential at the center of the sphere:

$$\frac{1}{4\pi R^2} \int_{\mathcal{S}} V(\boldsymbol{r}) \, da = V(0) \tag{4}$$

Problem 2

As shown in the right figure, we have a point charge Q being placed in front of two semi-infinite conducing plates that are perpendicular to each other, and are extended infinitely into the page. The line between the vertex and the charge form an angle α with respect to the vertical plate. The conducting plates are grounded so that they have zero electric potential. Find the net charge induced on each plate.



Problem 3

For a given conductor of volume $\mathcal{V}$ with a total charge Q, one can show that the system is at the minimum energy state – if we "perturb" the charge distribution, the energy of the system always goes up. The following steps will guide you through the proof.

1. Let $\rho^*(\mathbf{r})$ be the charge density in the conductor, and $V^*(\mathbf{r})$ be its potential. If we perturb the charge distribution to $\rho^*(\mathbf{r})+\delta\rho(\mathbf{r})$ while keeping the total charge fixed, the potential becomes $V^*(\mathbf{r})+\delta V(\mathbf{r})$. Use conservation of charge to show that

$$\int_{\mathcal{V}} \nabla^2 \delta V \, d\tau = 0$$

   Similarly, because there is no charge outside $\mathcal{V}$, show that

$$\nabla^2 \delta V = 0 \text{ outside } \mathcal{V}.$$

2. The energy of the pertubed system is

$$\begin{aligned} W &= \frac{\epsilon_0}{2}\int_{\text{all space}} \nabla\left(V^*+\delta V\right)\cdot\nabla\left(V^*+\delta V\right)d\tau \\ &= \frac{\epsilon_0}{2}\int_{\text{all space}} \nabla V^*\cdot\nabla V^* d\tau + \epsilon_0\int_{\text{all space}} \nabla V^*\cdot\nabla\delta V\, d\tau + \frac{\epsilon_0}{2}\int_{\text{all space}} \left|\nabla\delta V\right|^2 d\tau. \end{aligned}$$

   The first term corresponds to the energy of the original state, and the extra two terms are due to the perturbation of the charge distribution. Use integration by parts to show that the second term vanishes. You may use the fact that $V^*(\mathbf{r})$ is a constant in $\mathcal{V}$, and the two results from (a).

3. Observe the integrand of the third term. Why can we conclude that any perturbation to the original charge distribution always increases the system's energy?

Below are selected optional problems from Griffiths. We do not collect your work, but you are encouraged to do as many practice problems as you can.

- Problem 3.5
- Problem 3.6
- Problem 3.8
- Problem 3.9
- Problem 3.14
- Problem 3.15