Problem Set 7 Solutions

1) Griffiths 9.18

$$c_b(t) = \frac{-i}{\hbar}\int_0^t dt'\, H_{ab}\, e^{i\omega_{ab}t'} = \frac{-i}{\hbar}\,\frac{e^{i\omega_{ab}t}-1}{i\omega_{ab}}\, H_{ab}$$

$$\Rightarrow |c_b(t)|^2 = \frac{|H_{ab}|^2}{\hbar^2}\,\frac{4\sin^2(\omega_{ab}t/2)}{\omega_{ab}^2} \qquad \left(\omega_{ab} \equiv \frac{3\hbar\pi^2}{2ma^2}\right)$$

$$= \frac{|H_{ab}|^2}{\hbar^2}\,\frac{4\sin^2\left(\frac{3\hbar t\pi^2}{4ma^2}\right)}{\frac{9\hbar^2\pi^4}{4m^2a^4}}$$

$$H_{ab} = V_0\int_0^{a/2} dx\,\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right) = \frac{2V_0}{\cancel{a}}\,\frac{2\cancel{a}}{3\pi} = \frac{4V_0}{3\pi}$$

$$\Rightarrow |c_b(t)|^2 = \frac{16V_0^2}{9\pi^2}\,\frac{1}{\hbar^2}\,\frac{4\sin^2\left(\frac{3\hbar\pi^2 t}{4ma^2}\right)}{\frac{9\hbar^2\pi^4}{4m^2a^4}}$$

$$= \frac{256\,V_0^2 m^2 a^4}{81\pi^6\hbar^4}\sin^2\left(\frac{3\hbar\pi^2 t}{4ma^2}\right) \qquad 0<t<T$$

So for $t>T$, $|c_b(t)|^2$ gets frozen at its value $|c_b(t=T)|^2$

$$\Rightarrow |c_b(t)|^2 \text{ for } t>T = \frac{256\,V_0^2 m^2 a^4}{81\pi^6\hbar^4}\sin^2\left(\frac{3\hbar\pi^2 T}{4ma^2}\right)$$

2. (Saxon #15)

a) $U = -qEx = -eEx$ in this problem...

$$H_{n0} = \langle n | -qEx | 0 \rangle = -qE \langle n | \cdot \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) | 0 \rangle$$

$$= -qE \cdot \sqrt{\frac{\hbar}{2m\omega}} \, \delta_{n1}$$

$$\therefore \quad |c_1(t)|^2 = \frac{4 |H_{a0}|^2}{\hbar^2} \frac{\sin^2\left(\frac{\omega_{ab} t}{2}\right)}{\omega_{ab}^2}$$

$$= \frac{4^2}{\hbar^2} \cdot q^2 E^2 \cdot \frac{\hbar}{2m\omega} \frac{\sin^2\left(\frac{\omega t}{2}\right)}{\omega^2}$$

For $0 < t < \tau$
$$= \begin{cases} \dfrac{2q^2E^2 \sin^2\left(\frac{\omega t}{2}\right)}{m\omega^3 \hbar} & \text{for } n=1 \\ 0 & \text{for any other } n \end{cases}$$

For $t > \tau$
$$= \begin{cases} \dfrac{2q^2E^2 \sin^2\left(\frac{\omega \tau}{2}\right)}{m\omega^3 \hbar} & \text{for } n=1 \\ 0 & \text{other } n \end{cases}$$

3. (Ohanian 19)

a) ~~$\vec{E} = \mathcal{E}_0 \cos\omega t$~~

$$\vec{E} = \mathcal{E}_0 \cos\omega t \,\hat{x}$$

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{x} \Rightarrow V = +\mathcal{E}_0 x \cos\omega t$$

$$\Rightarrow H' = \mathcal{E}_0 q x \cos\omega t$$

$$\therefore \quad P_{0\to n}(t) = \frac{|\langle 0|\mathcal{E}_0 q x|n\rangle|^2}{\hbar^2} \frac{\sin^2\left[\frac{(\tilde{\omega}_0-\omega)t}{2}\right]}{(\tilde{\omega}_0-\omega)^2} \qquad \tilde{\omega}_0 = \frac{\Delta E}{\hbar} = \frac{n\hbar\omega_0}{\hbar} = n\omega_0$$

$$\langle 0|\mathcal{E}_0 q x|n\rangle = \mathcal{E}_0 q \cdot \sqrt{\frac{\hbar}{2m\omega}} \langle 0|(a^\dagger + a)|n\rangle$$

$$= \mathcal{E}_0 q \sqrt{\frac{\hbar}{2m\omega}} \,\delta_{n,1}$$

$$\therefore \quad P_{0\to 1} = \frac{\mathcal{E}_0^2 q^2 \frac{\cancel{\hbar}}{2m\omega}}{\hbar^{\cancel{2}}} \cdot \frac{\sin^2\left[\frac{(n\omega_0-\omega)t}{2}\right]}{(n\omega_0-\omega)^2}$$

$$P_{0\to 1} = \frac{\mathcal{E}_0^2 q^2}{2m\omega\hbar} \frac{\sin^2\left[\frac{(\omega_0-\omega)t}{2}\right]}{(\omega_0-\omega)^2}$$

q = charge of the electron

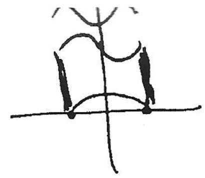
$$P_{0\to n} = 0 \qquad \text{for } n > 1$$

b) Perturbation theory breaks down when $P_{0\to 1} > 1$

$$\lim_{\omega_0\to\omega} P_{0\to 1} \approx \frac{\mathcal{E}_0^2 q^2}{2m\omega\hbar} \cdot \frac{\cancel{(\omega_0-\omega)^2}\, t^2}{4\,\cancel{(\omega_0-\omega)^2}} = \frac{\mathcal{E}_0^2 q^2}{2m\omega\hbar} \cdot \frac{t^2}{4} = 1$$

$$\Rightarrow \frac{t}{2} = \sqrt{\frac{2m\omega\hbar}{\mathcal{E}_0^2 q^2}} \qquad t = \frac{2}{\mathcal{E}_0 q}\sqrt{2m\omega\hbar}$$

(Farry) 4. Notice that the well goes from $-a/2 < x < a/2$,



which means the eigenstates are,

For n odd,

$$\psi_n(x) = \sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right)$$

For n even,

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

H_{n3} for n even is,

$$H_{n3} = \int_{-a/2}^{a/2} dx\, \sqrt{\frac{2}{a}}\cdot\cos\left(\frac{3\pi x}{a}\right)\cdot\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\cdot x$$

$$= \frac{2}{a}\int_{-a/2}^{a/2} dx\, \cos\left(\frac{3\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\cdot x$$

$$= \frac{2}{a}\cdot\frac{a^2\cdot n}{(n^2-9)\pi} = \frac{2an}{(n^2-9)\pi}$$

H_{n3} for n odd is,

$$H_{n3} = \int_{-a/2}^{a/2} dx'\, \underbrace{\sqrt{\frac{2}{a}}\cos\left(\frac{3\pi x}{a}\right)\cdot\sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right)\cdot x}_{\text{odd } \Rightarrow} = 0$$

Prob to measure the system in state n @ some time $t>\tau$ for n even is,

$$\Delta\omega = \frac{\hbar\pi^2}{2ma^2}(9-n^2)$$

$$= \frac{4|H_{n3}|^2}{\hbar^2}\,\frac{\sin^2\left(\frac{(\Delta\omega)t}{2}\right)}{\Delta\omega^2}$$

$$= \frac{4}{\hbar^2}\cdot\frac{4a^2n^2}{(n^2-9)^2\pi^2}\;\frac{\sin^2\left(\frac{\hbar\pi^2 t}{4ma^2}(9-n^2)\right)}{\frac{\hbar^2\pi^2}{4m^2a^4}(9-n^2)^2}$$

$$= \frac{64\, a^6 n^2 m^2}{\hbar^4 (n^2-9)^4 \pi^4} \sin^2\left(\frac{\hbar^2 \pi^2 t}{4 m a^2} (9-n^2) \right)$$

Proba of state n with n odd is 0.

5. (Griffiths 1st edition 9.18)

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Let $\Delta E = E_{1^{st}\ \text{excited}} - E_{\text{ground}}$

1st excited

$$\frac{dc_b}{dt} = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} \quad \Big\} \text{ 1st order Pert. Theory}$$

$$\Downarrow$$

$$c_b = -\frac{i}{\hbar}\int_0^t dt'\, H'_{ba}\, e^{i\omega_0 t'} = -\frac{iH'_{ba}}{\hbar}\int_0^t dt'\, e^{i\omega_0 t'} = -\frac{iH'_{ba}}{\hbar}\frac{\left(e^{i\omega_0 t} - 1\right)}{i\omega_0}$$

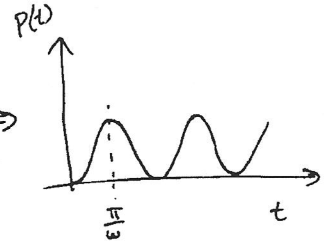
In this case H'_{ba} is constant in time

$$= -\frac{iH'_{ba}}{\hbar} e^{\frac{i\omega_0 t}{2}} \cdot 2i \cdot \frac{\sin\left(\frac{\omega_0 t}{2}\right)}{i\omega_0}$$

$$= \frac{2H'_{ba}}{\hbar} e^{\frac{i\omega_0 t}{2}} \frac{\sin\left(\frac{\omega_0 t}{2}\right)}{i\omega_0}$$

$$\Rightarrow P(t) = |c_b|^2 = \frac{4|H'_{ba}|^2}{\hbar^2}\frac{\sin^2\left(\frac{\omega_0 t}{2}\right)}{\omega_0^2}$$

$\therefore$ Prob of transition $\Rightarrow$



We'll say that a transition has occurred when $P(t)$ reaches its first peak

$$\Rightarrow \frac{\omega \Delta t}{2} = \frac{\pi}{2} \Rightarrow \Delta t = \frac{\pi}{\omega} \qquad \text{Recall } \omega = \frac{\Delta E}{\hbar}$$

$$\therefore \Delta E \Delta t = \cancel{\Delta E}\left(\frac{\pi\hbar}{\cancel{\Delta E}}\right) = \pi\hbar \qquad \therefore \boxed{\Delta E \Delta t \sim \pi\hbar = \frac{h}{2}}$$