In[1]:=
```
(* Here we treat both the case that is printed in
    the book and the homework problem with non-uniform rho *)
```

In[2]:=
```
rhoProb1[k_, x_, y_, z_] := k*x^4*y^4*z^4;
```

In[3]:=
```
rhoUniform [k_, x_, y_, z_] := k;
```

In[4]:=
```
rho[which_, k_, x_, y_, z_] := Module[{return},
      If[which == 1, return = rhoProb1[k, x, y, z]];
      If[which == 2, return = rhoUniform [k, x, y, z]];
      return
    ];
```

In[5]:=
```
MomentOfInertia [which_, k_, x0_, x1_, y0_, y1_, z0_, z1_] :=
    Module[{r, i, j, return = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}},
      For[i = 1, i ≤ 3, i++,
        For[j = 1, j ≤ 3, j ++,
            return[[i]][[j]] = Integrate[rho[which, k, x, y, z] *
                        (KroneckerDelta[i, j] * (x^2 + y^2 + z^2) - x_i * x_j) /.
                    {x_1 → x, x_2 → y, x_3 → z}, {x, x0, x1}, {y, y0, y1}, {z, z0, z1}];
          ];
      ];
      return
    ];
```

In[6]:=
```
Ibook = MomentOfInertia [2, M / a^3, 0, a, 0, a, 0, a]; (* See Eq. (10.49) *)
```

In[7]:=
```
Ibook // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix} \frac{2a^2 M}{3} & -\frac{a^2 M}{4} & -\frac{a^2 M}{4} \\ -\frac{a^2 M}{4} & \frac{2a^2 M}{3} & -\frac{a^2 M}{4} \\ -\frac{a^2 M}{4} & -\frac{a^2 M}{4} & \frac{2a^2 M}{3} \end{pmatrix}$$

In[8]:=
```
(* Here we verify that we get the same result as in Eq. (10.72) *)
μbook = M a^2 / 12;
(1 / μbook) Ibook // MatrixForm
```

Out[8]//MatrixForm=

$$\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

In[9]:=
```
(* This is the moment of inertia tensor for the homework problem *)
Ihomework = MomentOfInertia [1, k, 0, a, 0, a, 0, a];
```

In[10]:=
```
Ihomework // MatrixForm
```

Out[10]//MatrixForm=

$$\begin{pmatrix} \frac{2a^{17} k}{175} & -\frac{a^{17} k}{180} & -\frac{a^{17} k}{180} \\ -\frac{a^{17} k}{180} & \frac{2a^{17} k}{175} & -\frac{a^{17} k}{180} \\ -\frac{a^{17} k}{180} & -\frac{a^{17} k}{180} & \frac{2a^{17} k}{175} \end{pmatrix}$$

In[11]:=
```
μhomework = k a^17 / 5; (* It is convenient to define this quantity *)
```

In[12]:= (1/μhomework) Ihomework //MatrixForm

Out[12]//MatrixForm=

$$\begin{pmatrix} \frac{2}{35} & -\frac{1}{36} & -\frac{1}{36} \\ -\frac{1}{36} & \frac{2}{35} & -\frac{1}{36} \\ -\frac{1}{36} & -\frac{1}{36} & \frac{2}{35} \end{pmatrix}$$

```
In[13]:= (* We see that this tensor is a multiple of the one printed in the book
  plus a multiple of the identity matrix . Therefore the eigenvectors
  are the same . Students should either explain this or calculate the
  eigenvectors following the steps in the book. The eigenvalues are then
  gotten by computing the product of the matrix and each eigenvector. *)

In[14]:=

In[15]:= (* Mathematica can find eigenvalues and eigenvectors automatically . However
  this is not an acceptable solution for this homework assignment . *)

In[16]:= Print["μbook", Eigenvalues[Ibook]/μbook]
  μbook{11, 11, 2}

In[17]:= Eigenvectors[Ibook]

Out[17]= {{-1, 0, 1}, {-1, 1, 0}, {1, 1, 1}}

In[18]:= Print[μhomework /10, 10 Eigenvalues[Ihomework ]/μhomework ]
```

$$\frac{a^{17}k}{50}\left\{\frac{107}{126}, \frac{107}{126}, \frac{1}{63}\right\}$$

```
In[19]:= Eigenvectors[Ihomework ]

Out[19]= {{-1, 0, 1}, {-1, 1, 0}, {1, 1, 1}}
```

2 Question 2

Suppose that our sphere is rolling along the x-axis in the positive x-direction, so that the axis of rotation is parallel to $\hat{\mathbf{y}}$ (see Fig. 1):

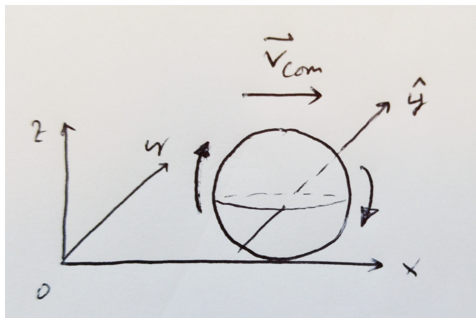


Figure 1: Rolling sphere

Let's first determine the relevant moment of inertia about the sphere's centre. For a sphere V with uniform density ρ, this is

$$I_{yy} = \int\int\int_V dx\, dy\, dz\, \rho(x^2 + z^2) \tag{2.1}$$

Take spherical polar coordinates through the centre of the sphere and parallel to $\hat{\mathbf{y}}$. This yields

$$\begin{aligned} I_{yy} &= \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\theta\, dr\, d\theta\, d\phi\, \rho(r\sin\theta)^2 \\ &= \rho \int_0^R \int_0^\pi \int_0^{2\pi} dr\, d\theta\, d\phi\, r^4 \sin^3\theta \\ &= \rho \left(\int_0^R dr\, r^4\right) \left(\int_0^\pi d\theta\, \sin^3\theta\right) \left(\int_0^{2\pi} d\phi\right) \end{aligned} \tag{2.2}$$

The only non-trivial part is the θ integral. I usually work this out with a $\sin^3\theta$ identity, but as was pointed out in discussion, a nice trick is to notice that

$$\int_0^\pi d\theta \sin^3\theta = \int_0^\pi d\theta \left[\sin\theta - \sin\theta\cos^2\theta\right]$$
$$= \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]\Big|_0^\pi$$
$$= 2 - \frac{2}{3}$$
$$= \frac{4}{3} \tag{2.3}$$

Hence

$$I_{yy} = \rho\left(\frac{R^5}{5}\right)\left(\frac{4}{3}\right)(2\pi)$$
$$= \frac{8\pi R^5\rho}{15}$$
$$= \frac{2MR^2}{5} \tag{2.4}$$

where M denotes the total mass of the sphere. The quickest way to determine the moment of inertia about the point of contact is by using the parallel axis theorem, which yields

$$I_{yy} = \frac{2MR^2}{5} + MR^2 = \frac{7MR^2}{5}. \tag{2.5}$$

Alternatively, you can derive this directly to test your understanding of multiple integrals. Let's now compare the two means of deriving the kinetic energy. First summarize the above results as

$$I_{\text{CoM}} = \frac{2MR^2}{5}, \quad I_{\text{PoC}} = \frac{7MR^2}{5}. \tag{2.6}$$

Decomposing the motion into CoM velocity and rotation about the CoM yields

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}I_{\text{CoM}}\omega^2 = \frac{1}{2}M(R\omega)^2 + \frac{1}{2}\frac{2MR^2}{5}\omega^2$$
$$= MR^2\omega^2\left[\frac{1}{2} + \frac{1}{5}\right]$$
$$= \frac{7}{10}MR^2\omega^2 \tag{2.7}$$

whilst considering rotational motion about the PoC alone yields

$$T = \frac{1}{2} I_{\mathrm{PoC}} \omega^2 = \frac{7MR^2\omega^2}{10}, \tag{2.8}$$

in agreement with (2.7).

3 Question 3

Let $\rho = k_1 + k_2 z$ denote the density of the hemisphere and M its total mass. We assume that the hemisphere lies in the region $z \geq 0$ and take spherical polars along $\hat{z}$. First we need to evaluate the total mass. This is given by

$$\begin{aligned} M &= \int_0^{\pi/2} \int_0^R \int_0^{2\pi} r^2 \sin\theta \, d\theta \, dr \, d\phi \, (k_1 + k_2 r \cos\theta) \\ &= \int_0^{\pi/2} \int_0^R \int_0^{2\pi} d\theta \, dr \, d\phi \left[k_1 r^2 \sin\theta + k_2 r^3 \sin\theta \cos\theta \right] \\ &= 2\pi \left[\frac{k_1 R^3}{3} \cos\theta \Big|_0^{\pi/2} + \frac{k_2 R^4}{4} \frac{1}{2} \sin^2\theta \Big|_0^{\pi/2} \right] \\ &= 2\pi \left[\frac{k_1 R^3}{3} + \frac{k_2 R^4}{8} \right] \\ &= \frac{\pi R^3}{12} \left[8k_1 + 3k_2 R \right] \end{aligned} \tag{3.1}$$

Now consider the centre of mass. By rotational symmetry, its x- and y-coordinates vanish; otherwise we could rotate the space about the z axis to obtain the same hemisphere but a different centre of mass, contradiction. Alternatively, you may argue that the integrals $\int_0^{2\pi} d\phi \, \cos\phi$ and $\int_0^{2\pi} d\phi \, \sin\phi$ vanish. Meanwhile the z coordinate satisfies

$$\begin{aligned} M z_{\mathrm{CoM}} &= \int_0^{\pi/2} \int_0^R \int_0^{2\pi} r^2 \sin\theta d\theta \, dr \, d\phi \, (k_1 + k_2 r \cos\theta) r \cos\theta \\ &= \int_0^{\pi/2} \int_0^R \int_0^{2\pi} d\theta \, dr \, d\phi \, \left[k_1 r^3 \sin\theta \cos\theta + k_2 r^4 \sin\theta \cos^2\theta \right] \\ &= 2\pi \left[\frac{k_1 R^4}{4} \frac{1}{2} \sin^2\theta \Big|_0^{\pi/2} - \frac{k_2 R^5}{5} \frac{1}{3} \cos^3\theta \Big|_0^{\pi/2} \right] \\ &= 2\pi \left[\frac{k_1 R^4}{8} + \frac{k_2 R^5}{15} \right] \\ &= \frac{\pi R^4}{60} \left[15k_1 + 8k_2 R \right] \end{aligned} \tag{3.2}$$

and thus

$$z_{\rm CoM} = \frac{R}{5}\frac{15k_1 + 8k_2 R}{8k_1 + 3k_2 R}. \tag{3.3}$$

In summary, the centre of mass of the hemisphere is at

$$\mathbf{R}_{\rm CoM} = \frac{R}{5}\frac{15k_1 + 8k_2 R}{8k_1 + 3k_2 R}\hat{z} \tag{3.4}$$

When $k_2 = 0$ we get

$$\mathbf{R}_{\rm CoM} = \frac{R}{5}\frac{15k_1}{8k_1}\hat{z} = \frac{3R}{8}\hat{z} \tag{3.5}$$

You can check that this matches Taylor's result for a hemisphere with uniform density.

4 Question 4

When $z = 0$, it is clear that $I_{xz} = I_{zx} = 0$ and $I_{yz} = I_{zy} = 0$. Moreover, letting A denote the lamina and $\sigma(x, y)$ its mass density, we have

$$I_{xx} = \int\int_A dx\, dy\, \sigma(x,y)(y^2 + z^2) = \int\int_A dx\, dy\, \sigma(x,y) y^2 \tag{4.1}$$

$$I_{yy} = \int\int_A dx\, dy\, \sigma(x,y)(x^2 + z^2) = \int\int_A dx\, dy\, \sigma(x,y) x^2 \tag{4.2}$$

and consequently,

$$I_{zz} = \int\int_A dx\, dy\, \sigma(x,y)(x^2 + y^2) = I_{xx} + I_{yy}. \tag{4.3}$$

5 Question 5

a) Observe that the centre of mass lies at the centre of the cuboid. Let's take our origin there. Then off-diagonal elements vanish by reflection symmetry in the planes $x = 0$, $y = 0$ and $z = 0$. You can prove this using tensor methods or by observing that they involve integrals of odd functions over intervals symmetric about zero, e.g. I_{xy} and I_{xz} are proportional to $\int_{-a}^{a} dx\, x = \frac{a^2}{2} - \frac{a^2}{2} = 0$.
Meanwhile the diagonal elements can be evaluated in Cartesian coordinates.

(a) Define z_0 = z coord. of the oxygen atom.

Then $z_C = z_0 - d$, $z_H = z_0 - \frac{3}{2}d$.

$$0 = z_{CM} \Rightarrow 0 = 16 z_0 + 12 z_C + 2 z_H$$
$$= 30 z_0 - 15d$$

Thus $z_0 = d/2$

$$\boxed{\begin{array}{l} O:\ (0, 0, d/2) \\ C:\ (0, 0, -d/2) \\ H:\ (0, \pm\frac{\sqrt{3}}{2}d, -d) \end{array}}$$

(b) The fundamental form $r^2\delta_{ij} - r_i r_j$ results in the following:

particle at $(0, 0, z_0)$ $\rightarrow$ $z_0^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&0\\0&0&z_0^2\end{pmatrix}$

particle at $(\rho c, \rho s, z_0)$ $\rightarrow$ $(\rho^2+z_0^2)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}\rho^2c^2 & \rho^2 cs & \rho c z_0\\ \rho^2 cs & \rho^2 s^2 & \rho s z_0\\ \rho c z_0 & \rho s z_0 & z_0^2\end{pmatrix}$

$(c = \cos(\omega t'), s = \sin(\omega t'))$

$\boxed{t' = t + \frac{\pi}{2\omega}}$

For a particle at $(-\rho c, -\rho s, z_0)$, these terms get a sign flip.

$$\overleftrightarrow{I}(t) \text{ for part (c)} = 16m\left(\frac{d}{2}\right)^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} + 12m\left(\frac{d}{2}\right)^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$
$$+ 2m\left[\left(\frac{3}{4}d^2 + d^2\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - d^2\begin{pmatrix}\frac{3}{4}c^2 & \frac{3}{4}cs & 0\\ (3/4)cs & (3/4)s^2 & 0\\ 0&0&1\end{pmatrix}\right]$$

$\overleftrightarrow{Q} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$

$$\frac{\overleftrightarrow{I}(t)}{md^2} = \overleftrightarrow{Q}(4+3) + \left(\frac{3}{2}+2\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}3c^2 & 3cs & 0\\ 3cs & 3s^2 & 0\\ 0&0&4\end{pmatrix}$$

$\boxed{t' = t + \frac{\pi}{2\omega}}$

$$\boxed{\overleftrightarrow{I}(t) = \frac{7}{2}md^2\left[\overleftrightarrow{1} + \begin{pmatrix}2&0&0\\0&2&0\\0&0&0\end{pmatrix}\right] - \frac{1}{2}md^2\begin{pmatrix}3\cos^2(\omega t') & 3\cos(\omega t')\sin(\omega t') & 0\\ 3\cos(\omega t')\sin(\omega t') & 3\sin^2(\omega t') & 0\\ 0&0&4\end{pmatrix} \text{ for part (c)}}$$

$$\overleftrightarrow{I}(0) = \frac{7}{2}md^2\begin{pmatrix}3&0&0\\0&3&0\\0&0&1\end{pmatrix} - \frac{1}{2}md^2\begin{pmatrix}0&0&0\\0&3&0\\0&0&4\end{pmatrix} = \boxed{\frac{1}{2}md^2\begin{pmatrix}21&0&0\\0&18&0\\0&0&3\end{pmatrix}}$$

Compare Taylor Prob 10.23

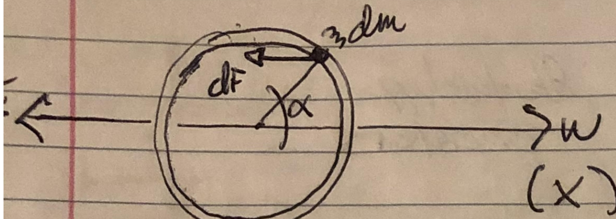
(c) At $t=0$, $\boxed{\vec{L} = \frac{3}{2}md^2\omega\hat{z}}$ $\overleftrightarrow{I}(t)$

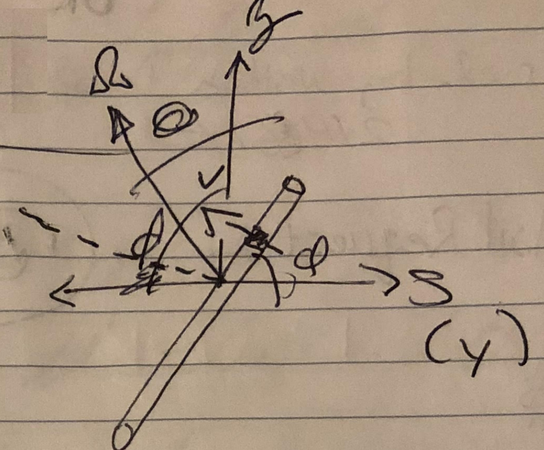
(d) A multiple of $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ does not affect the eigenvectors.

We need to find the eigenvectors of $\begin{pmatrix}3c^2 & 3cs & 0\\ 3cs & 3s^2 & 0\\ 0&0&4\end{pmatrix}$.

We see they rotate with the molecule

These are $\boxed{\hat{e}_1 = (\cos(\omega t'), \sin(\omega t'), 0),\ \hat{e}_2 = (\sin(\omega t'), -\cos(\omega t'), 0),\ \hat{e}_3 = (0,0,1)}$





$dm = \frac{m}{2\pi}\,d\alpha$
(m = total mass of water)

$d\vec{F} \equiv$ Coriolis force on dm (infinitesimal mass on ring)

$v \equiv$ velocity $= \dot{\phi} R \sin\alpha$

$d\vec{F} = 2\,dm\,\vec{v} \times \vec{\Omega} = -2dm\,(\dot{\phi} R \sin\alpha)\,\Omega \sin(\phi - \theta)\,\hat{x}$

que $\equiv d\Gamma = |\vec{r} \times d\vec{F}| = R\,dF \sin\alpha = 2\,dm\,R^2 \dot{\phi}\,\Omega \sin(\phi-\theta) \sin^2\alpha$
(counterclockwise dir)

$\Gamma = \int d\Gamma = \int \frac{d\Gamma}{dm}\,dm = m R^2 \dot{\phi}\,\Omega \sin(\phi - \theta)$

$L \equiv$ tot. ang. momentum $= \int \Gamma\,dt = m R^2 \Omega \int \sin(\phi-\theta)\,\dot{\phi}\,dt$

$= m R^2 \Omega \int_0^{\pi} \sin(\phi - \theta)\,d\phi = 2 m R^2 \Omega \cos\theta$

Let $V \equiv$ final speed of water

$\therefore\ L = mRV \Rightarrow V = 2R\Omega\cos\theta \doteq 0.11$ mm/s