Physics 110A, Spring 2021
Solution to Homework 1
GSI: Yi-Chuan Lu

1. In Cartesian coordinates,

$$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}, \quad \hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}.$$

so

$$\mathbf{E} = \frac{kQ}{r^2}\hat{\mathbf{r}} = \boxed{kQ\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\left(x^2+y^2+z^2\right)^{3/2}}}, \quad \mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} = \boxed{\frac{\mu_0 I}{2\pi}\frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{x^2+y^2}}.$$

(a) Using the expression above,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= kQ\left[\frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} + \frac{\partial}{\partial y}(\cdots) + \frac{\partial}{\partial z}(\cdots)\right] \\
&= kQ\left[\frac{\left(x^2+y^2+z^2\right)^{3/2} - x\frac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}2x}{\left(x^2+y^2+z^2\right)^3} + (\cdots) + (\cdots)\right] \\
&= kQ\left[\frac{\left(x^2+y^2+z^2\right) - 3x^2}{\left(x^2+y^2+z^2\right)^{5/2}} + (\cdots) + (\cdots)\right] \\
&= kQ\frac{3\left(x^2+y^2+z^2\right) - 3x^2 - 3y^2 - 3z^2}{\left(x^2+y^2+z^2\right)^{5/2}} = \boxed{0}.
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{E} &= kQ\begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} & \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} & \frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} \end{pmatrix} \\
&= kQ\left[\hat{\mathbf{x}}\left(\frac{\partial}{\partial y}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} - \frac{\partial}{\partial z}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right) + \hat{\mathbf{y}}(\cdots) + \hat{\mathbf{z}}(\cdots)\right] \\
&= kQ\left[\hat{\mathbf{x}}\left(\frac{-3zy}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}}\right) + \hat{\mathbf{y}}(\cdots) + \hat{\mathbf{z}}(\cdots)\right] = \boxed{\mathbf{0}}.
\end{aligned}$$

(b) Similarly,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \frac{\mu_0 I}{2\pi}\left(\frac{\partial}{\partial x}\frac{-y}{x^2+y^2} + \frac{\partial}{\partial y}\frac{x}{x^2+y^2}\right) = \frac{\mu_0 I}{2\pi}\left[\frac{y2x}{\left(x^2+y^2\right)^2} + \frac{-x2y}{\left(x^2+y^2\right)^2}\right] = \boxed{0}.$$

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{B} &= \frac{\mu_0 I}{2\pi}\begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} = \frac{\mu_0 I}{2\pi}\hat{\mathbf{z}}\left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2} - \frac{\partial}{\partial y}\frac{-y}{x^2+y^2}\right) \\
&= \frac{\mu_0 I}{2\pi}\hat{\mathbf{z}}\left(\frac{\left(x^2+y^2\right) - 2x^2}{\left(x^2+y^2\right)^2} - \frac{-\left(x^2+y^2\right) + 2y^2}{\left(x^2+y^2\right)^2}\right) = \boxed{\mathbf{0}}.
\end{aligned}$$

2. (a) For a symmetric tensor S_{ij},

$$\epsilon^{ijk} S_{ij} \overset{1}{=} \epsilon^{jik} S_{ji} \overset{2}{=} -\epsilon^{ijk} S_{ij}.$$

For equality 1, I simply renamed i as j, and renamed j as i, so nothing has changed. In equality 2, I used the properties $\epsilon^{ijk} = -\epsilon^{jik}$ and $S_{ij} = S_{ji}$. The equation above means $\boxed{\epsilon^{ijk} S_{ij} = 0.}$

(b) The ith component of $\mathbf{B} \times \mathbf{C}$ is $\epsilon_{ijk} B_j C_k$, so $\boxed{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \epsilon_{ijk} A_i B_j C_k.}$ Similarly, $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = \epsilon_{jik} B_j A_i C_k = -\epsilon_{ijk} A_i B_j C_k = \boxed{-\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})}$.

(c) The ith component of $\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B})$ is

$$\begin{aligned}
\left[\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B})\right]_i &= \epsilon_{ijk} \partial_j \left(\mathbf{A} \times \mathbf{B}\right)_k = \epsilon_{ijk} \partial_j \left(\epsilon_{klm} A_l B_m\right) = \epsilon_{ijk} \epsilon_{klm} \partial_j \left(A_l B_m\right) \\
&= \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}\right) \partial_j \left(A_l B_m\right) = \partial_j \left(A_i B_j\right) - \partial_j \left(A_j B_i\right) \\
&= B_j \partial_j A_i + A_i \partial_j B_j - B_i \partial_j A_j - A_j \partial_j B_i \\
&= \left[(\mathbf{B} \cdot \boldsymbol{\nabla}) \mathbf{A} + \mathbf{A} (\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B} (\boldsymbol{\nabla} \cdot \mathbf{A}) - (\mathbf{A} \cdot \boldsymbol{\nabla}) \mathbf{B}\right]_i .
\end{aligned}$$

Since i can stand for x, y or z, the equation above implies

$$\boxed{\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla}) \mathbf{A} + \mathbf{A} (\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B} (\boldsymbol{\nabla} \cdot \mathbf{A}) - (\mathbf{A} \cdot \boldsymbol{\nabla}) \mathbf{B}.}$$

3. Let the length of the rectangular box be Δx, Δy and Δz along the x, y and z direction, respectively, such that (x, y, z) is exactly at the center of the box. The volume bounded by $\mathcal{S}$ is $\Delta x \Delta y \Delta z$, and the flux through the right and left surfaces is

$$\left[F_x \left(x + \frac{\Delta x}{2}, y, z\right) - F_x \left(x - \frac{\Delta x}{2}, y, z\right)\right] \Delta y \Delta z \simeq \frac{\partial F_x}{\partial x} \Delta x \Delta y \Delta z.$$

Similarly, the flux through front/back and top/bottom surfaces is

$$\begin{aligned}
\left[F_y \left(x, y + \frac{\Delta y}{2}, z\right) - F_y \left(x, y - \frac{\Delta y}{2}, z\right)\right] \Delta x \Delta z &\simeq \frac{\partial F_y}{\partial y} \Delta y \Delta x \Delta z, \\
\left[F_z \left(x, y, z + \frac{\Delta z}{2}\right) - F_z \left(x, y, z - \frac{\Delta z}{2}\right)\right] \Delta x \Delta y &\simeq \frac{\partial F_z}{\partial z} \Delta z \Delta x \Delta y.
\end{aligned}$$

So

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \lim_{\substack{\Delta x, \Delta y, \Delta z \\ \to 0}} \frac{\frac{\partial F_x}{\partial x} \Delta x \Delta y \Delta z + \frac{\partial F_y}{\partial y} \Delta y \Delta x \Delta z + \frac{\partial F_z}{\partial z} \Delta z \Delta x \Delta y}{\Delta x \Delta y \Delta z} = \boxed{\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.}$$