Solutions - Homework 2

1. (Griffiths 4.34)

a) $$\hat{S}_-|10\rangle = (\hat{S}_{-1} + \hat{S}_{-2})\left(\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\hbar|\downarrow\downarrow\rangle + \hbar|\downarrow\downarrow\rangle\right) = \frac{2\hbar}{\sqrt{2}}|\downarrow\downarrow\rangle = \sqrt{2}\hbar\,|1\,-1\rangle \quad \blacksquare$$

b) $$\hat{S}_\pm|00\rangle = (\hat{S}_{\pm 1} + \hat{S}_{\pm 2})\left(\frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle\right)$$

Choose raising

$$(\hat{S}_{+1} + \hat{S}_{+2})\left(\frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[\hbar|\uparrow\uparrow\rangle - \hbar|\uparrow\uparrow\rangle\right] = 0$$

$$(\hat{S}_{-1} + \hat{S}_{-2})\left(\frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle\right)$$

$$= \frac{\hbar}{\sqrt{2}}|\downarrow\downarrow\rangle - \frac{\hbar}{\sqrt{2}}|\downarrow\downarrow\rangle = 0 \quad \blacksquare$$

c) $$S^2 = (\vec{S}_1 + \vec{S}_2)\cdot(\vec{S}_1\,\vec{S}_2) = S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2$$

$$= S_1^2 + S_2^2 + 2S_{1z}S_{2z} + \underbrace{2S_{1x}S_{2x} + 2S_{1y}S_{2y}}$$

Can rewrite as

$$S_{1+}S_{2-} + S_{1-}S_{2+}$$

$$= (S_{x_1} + iS_{y_1})(S_{x_2} - iS_{y_2})$$

$$+ (S_{x_1} - iS_{y_1})(S_{x_2} + iS_{y_2}) \quad \blacksquare$$

$$\therefore\ \underline{S^2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}}$$

$$S^2|1\,1\rangle = \left(S_1^2 + S_2^2 + 2S_{z_1}S_{z_2} + S_{1+}S_{2-} + S_{1-}S_{2+}\right)|\uparrow\uparrow\rangle$$

$$= \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + 2\cdot\left(\frac{\hbar}{2}\right)^2 + 0 + 0\right)|\uparrow\uparrow\rangle$$

$$= \hbar^2\left(\frac{3}{4} + \frac{3}{4} + \frac{2}{4}\right)|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle$$

$\Updownarrow$

Spin 1 since

~~$S^2 S_{spin}$~~ $S^2|\ell\ m\rangle = \hbar^2\ell(\ell+1)|\ell\ m\rangle$ ∎

Spin ℓ

$$S^2|1\,{-1}\rangle = \left(S_1^2 + S_2^2 + 2S_{z_1}S_{z_2} + S_{1+}S_{2-} + S_{1-}S_{2+}\right)|\downarrow\downarrow\rangle$$

$$= \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + 2\left(-\frac{\hbar}{2}\right)^2 + 0 + 0\right)|\downarrow\downarrow\rangle$$

$$= 2\hbar^2|\downarrow\downarrow\rangle$$ ∎

2. (Griffiths 4.35)

a) Two spin $\frac{1}{2}$ $\longrightarrow$ $\frac{1}{2}+\frac{1}{2} = 1$, $\frac{1}{2}-\frac{1}{2} = 0$ $\xrightarrow{\text{Now introduce a third spin } \frac{1}{2}}$ $1+\frac{1}{2} = \frac{3}{2}$, $1-\frac{1}{2} = \frac{1}{2}$, $0+\frac{1}{2} = \frac{1}{2}$

So Baryons can only have spins $\frac{3}{2}$ and $\frac{1}{2}$.

b) From above, taking two spin $\frac{1}{2}$s gives spins 0 and 1.

3. (Griffiths 5.10)

We'll ignore the coulomb repulsion between electrons in Helium, so that the available energy levels are,

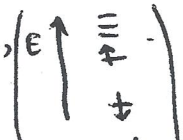
$$E_n = \frac{Z^2 E_1}{n^2} = \frac{2^2\cdot(-13.6)\text{eV}}{n^2} = \frac{-54.4\text{ eV}}{n^2} \qquad n=1,2,\ldots$$

For "bosonic" electrons, both electrons would be in the single-particle ground state. Since

Ground State

both particles have the same spatial wavefunction, ~~they~~ they must have a symmetric spatial wavefunc., ~~which is~~ (since an antisymmetric combination of the same states would ~~vanish~~ = 0). But since the electrons are "bosons" here, this means the spin part of the wavefunction must also be symmetric, in order to make the whole ~~...~~ wavefunction symmetric. The triplet is symmetric and has 3 states, so the ground state has degeneracy = 3.

The first excited state looks like,



excited state

We can make the spatial wavefunction symmetric or antisymmetric.

If its symmetric, then the spin part must be symmetric. If the spatial part is anti-symmetric, then the spin part must be antisymmetric also. This gives a total degeneracy of 4 for the first excited state. See the table on the next page for a summary.

4. (Ohanian 8)

Possible j_{TOT}'s	# of $m_{j_{TOT}}$'s corresponding to each j_{TOT}
j_1+j_2	$2(j_1+j_2)+1$
j_1+j_2-1	$2(j_1+j_2-1)+1$
$\vdots$	$\vdots$
$\lvert j_1-j_2\rvert$	$2(\lvert j_1-j_2\rvert)+1$

The total # of states is given by adding up all these states

The Helium Energy Spectrum for Bosonic Electrons

State	Energy Picture	Spatial Wavefunction	Spatial Symmetry	Spin Wavefunction	Spin Symmetry	Degeneracy
Ground		$\psi = \psi_1(x_1)\psi_1(x_2)$	Symmetric	$\lvert\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\rangle)$, $\lvert\downarrow\downarrow\rangle$	Symmetric	3
1st excited		$\psi = \frac{1}{\sqrt{2}}(\psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1))$	Antisymmetric	$\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle - \lvert\downarrow\uparrow\rangle)$	Antisym	1
		OR				+ 3
		$\psi = \frac{1}{\sqrt{2}}(\psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1))$	Symmetric	$\lvert\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle + \lvert\downarrow\uparrow\rangle)$, $\lvert\downarrow\downarrow\rangle$	Sym	= 4

The spectrum for distinguishable particles looks similar, except that the degeneracies will be larger since we need to specify "which" particle has spin up or down

Helium Spectrum for Distinguishable Particles

State	Energy Picture	Spatial Wavefunc	Spin Wavefunc.	Degeneracy
Ground		$\psi = \psi_1(x_1)\psi_1(x_2)$	$\lvert\uparrow\uparrow\rangle, \lvert\uparrow\downarrow\rangle, \lvert\downarrow\uparrow\rangle, \lvert\downarrow\downarrow\rangle$	4
1st excited		$\psi = \psi_1(x_1)\psi_2(x_2)$ OR $\psi = \psi_2(x_1)\psi_1(x_2)$	$\lvert\uparrow\uparrow\rangle, \lvert\uparrow\downarrow\rangle, \lvert\downarrow\uparrow\rangle, \lvert\downarrow\downarrow\rangle$	$= \underbrace{2}_{\text{From Spatial Part}} \cdot \underbrace{4}_{\text{From Spin Part}} = 8$

4. (Ohanian 8)

Possible j_{TOT}'s	# of $m_{j_{TOT}}$'s corresponding to each j_{TOT}
j_1+j_2	$2(j_1+j_2)+1$
j_1+j_2-1	$2(j_1+j_2-1)+1$
$\vdots$	
$\lvert j_1-j_2\rvert$	$2(\lvert j_1-j_2\rvert)+1$

The total # of states is given by adding up all these states

8. (Ohanian 8 Continued)

$$\text{Total \# of States} = \sum_{k=0}^{2j_2} \left(2(j_1+j_2-k)+1\right) = 2\left(\sum_{k=0}^{2j_2}(j_1+j_2-k)\right) + 2j_2+1$$

Assume $j_1 > j_2$ (if not can just flip $1 \leftrightarrow 2$)

$$= \frac{\cancel{2}(2j_2+1)(2j_1)}{\cancel{2}} + 2j_2+1$$

$$= (2j_2+1)(2j_1+1)$$

which agrees with another way of calculating this #. ∎

5. (Ohanian 14)

a) $\psi_A(\vec{x}) := e^{i\vec{p}_A\cdot\vec{x}}$

$\psi_B(\vec{x}) := e^{i\vec{p}_B\cdot\vec{x}}$

$$\Psi(x_1,x_2) = \underbrace{\frac{1}{\sqrt{2}}\left(e^{i\vec{p}_A\cdot\vec{x}_1}e^{i\vec{p}_B\cdot\vec{x}_2} + e^{i\vec{p}_A\cdot\vec{x}_2}e^{i\vec{p}_B\cdot\vec{x}_1}\right)}_{\text{Symmetric}} \cdot \underbrace{\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}_{\text{Spin 0} \Updownarrow \text{Antisymmetric}}$$

b)

$$\Psi(x_1,x_2) = \frac{1}{\sqrt{2}}\left(e^{i\vec{p}_A\cdot\vec{x}_1}e^{i\vec{p}_B\cdot\vec{x}_2} - e^{i\vec{p}_A\cdot\vec{x}_2}e^{i\vec{p}_B\cdot\vec{x}_1}\right)\cdot|\uparrow\uparrow\rangle$$

6. (Ohanian 13)

$$|\psi\rangle = \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle + \sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,0\rangle$$

$$J_{TOT}^2 = (\vec{J}_1+\vec{J}_2)\cdot(\vec{J}_1+\vec{J}_2) = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

Using the result from question 1

$$J_{TOT}^2|\psi\rangle = \underbrace{\left(J_1^2+J_2^2+2J_{1z}J_{2z}+J_{1+}J_{2-}+J_{1-}J_{2+}\right)\cdot\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle}_{\text{Part A}} + \underbrace{\left(J_1^2+J_2^2+2J_{1z}J_{2z}+J_{1+}J_{2-}+J_{1-}J_{2+}\right)\sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,0\rangle}_{\text{Part B}}$$

Part A $= \left[\hbar^2\tfrac{3}{2}(\tfrac{3}{2}+1) + \hbar^2\cdot1\cdot(1+1) + 2(-\tfrac{\hbar}{2})(-\hbar)\right]\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle$

$+ \hbar\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-(-\tfrac{1}{2})(-\tfrac{1}{2}+1)}\cdot\hbar\sqrt{1(1+1)-\underbrace{(-1)(-1+1)}_{=0}}\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,0\rangle$

$+ \cancel{\hbar\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)}}$ (other term $=0$ since $J_{-2}|1,-1\rangle = 0$) +

Part B Next Page

Using the fact that

$$J_\pm|j\;m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\;|j\;m\pm1\rangle$$

6. (Ohanian 13)

a)

$$J_{TOT}^2 = (\vec{J}_1+\vec{J}_2)\cdot(\vec{J}_1+\vec{J}_2) = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

$$J_{TOT}^2|\psi\rangle = \underbrace{\left(\hbar^2 \tfrac{3}{2}(\tfrac{3}{2}+1) + \hbar^2(1)(1+1)\right)|\psi\rangle}_{J_1^2+J_2^2 \text{ term}} + \underbrace{2\cdot(-\tfrac{\hbar}{2})(-\hbar)\cdot\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle}_{2J_{1z}J_{2z}}$$

$$+ \underbrace{\sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle\cdot\hbar\sqrt{\tfrac{15}{4}-\tfrac{3}{4}}\cdot\hbar\sqrt{2-0}}_{J_{1+}J_{2-}}$$

$$+ \underbrace{\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1\,0\rangle\cdot\hbar\sqrt{\tfrac{15}{4}-\tfrac{3}{4}}\cdot\hbar\sqrt{2-0}}_{J_{1-}J_{2+}}$$

$$= \hbar^2\left(\tfrac{15}{4}+2\right)|\psi\rangle + \hbar^2\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle + \hbar^2\sqrt{6}\cdot\sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle$$

$$+\hbar^2\sqrt{6}\cdot\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,0\rangle$$

$$= \hbar^2\left(\tfrac{23}{4}\right)|\psi\rangle + \hbar^2\cdot 3\cdot\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle + \hbar^2\sqrt{6}\cdot\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,0\rangle$$

$$= \hbar^2\left(\tfrac{23}{4}\right)|\psi\rangle + \hbar^2 3\underbrace{\left(\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle + \sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,0\rangle\right)}_{=|\psi\rangle} = \hbar^2\left(\tfrac{35}{4}\right)|\psi\rangle$$

$$\tfrac{35}{4} = \tfrac{5}{2}\cdot(\tfrac{5}{2}+1) \Rightarrow j = \tfrac{5}{2}$$

$$J_{z_{TOT}} = J_{z_1} + J_{z_2}$$

$$(J_{z_1}+J_{z_2})|\psi\rangle = \left(-\tfrac{\hbar}{2}-\hbar\right)\sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle + \left(-\tfrac{3}{2}\hbar\right)\sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,0\rangle$$

$$= -\tfrac{3\hbar}{2}|\psi\rangle \quad \blacksquare \qquad m_j = -\tfrac{3}{2}$$

(b) $|\psi\rangle = \sqrt{\frac{3}{5}}\ |\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,-1\rangle + \sqrt{\frac{2}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,0\rangle$

$J_{TOT_+} = J_{1_+} + J_{2_+}$

$$J_{TOT_+}|\psi\rangle = J_{1_+}|\psi\rangle + J_{2_+}|\psi\rangle$$

$$= \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle|1,-1\rangle\cdot\hbar\underbrace{\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-(-\tfrac{1}{2})(-\tfrac{1}{2}+1)}}_{=\sqrt{\frac{15}{4}+\frac{1}{4}}\ =2}$$

$$+ \sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,0\rangle\ \hbar\underbrace{\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-(-\tfrac{3}{2})(-\tfrac{3}{2}+1)}}_{=\sqrt{\frac{15}{4}-\frac{3}{4}}\ =\sqrt{3}}$$

$$+ \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1\,0\rangle\ \hbar\underbrace{\sqrt{1(1+1)-(-1)\underbrace{(-1+1)}_{=0}}}_{=\sqrt{2}}$$

$$+ \sqrt{\tfrac{2}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1\,+1\rangle\ \hbar\underbrace{\sqrt{1(1+1)-0(0+1)}}_{=\sqrt{2}}$$

$$= \hbar\Big[\sqrt{\tfrac{12}{5}}\ |\tfrac{3}{2},\tfrac{1}{2}\rangle|1,-1\rangle + \sqrt{\tfrac{6}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,0\rangle$$
$$+ \sqrt{\tfrac{6}{5}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,0\rangle + \sqrt{\tfrac{4}{5}}\ |\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,+1\rangle\Big]$$

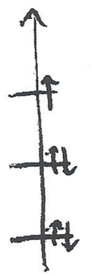
$$|\psi'\rangle = \Rightarrow A\Big(\sqrt{\tfrac{12}{5}}\ |\tfrac{3}{2},\tfrac{1}{2}\rangle|1,-1\rangle + \sqrt{\tfrac{24}{5}}\ |\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,0\rangle$$
$$+ \sqrt{\tfrac{4}{5}}\ |\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,+1\rangle\Big)$$

$$1 = \langle\psi'|\psi'\rangle = A^2\cdot\Big(\tfrac{12}{5}+\tfrac{24}{5}+\tfrac{4}{5}\Big) = A^2\cdot\tfrac{40}{5} = A^2\cdot 8$$

$$A = \tfrac{1}{\sqrt{8}} = \tfrac{1}{2\sqrt{2}}$$

$$\therefore\ |\psi'\rangle = \frac{1}{2\sqrt{2}}\Big(\sqrt{\tfrac{12}{5}}\ |\tfrac{3}{2},\tfrac{1}{2}\rangle|1,-1\rangle + \sqrt{\tfrac{24}{5}}\ |\tfrac{3}{2},-\tfrac{1}{2}\rangle|1,0\rangle$$
$$+ \sqrt{\tfrac{4}{5}}\,|\tfrac{3}{2},-\tfrac{3}{2}\rangle|1,+1\rangle\Big)$$

7. (Ohanian 16)



For one particle

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2ma^2}$$

So the five-particle ground state energy is given by,

$$= 2 \cdot \frac{\pi^2\hbar^2}{2ma^2} + 2 \cdot \frac{\pi^2\hbar^2 \cdot 2^2}{2ma^2} + \frac{\pi^2\hbar^2 3^2}{2ma^2}$$

$$= \frac{\pi^2\hbar^2}{2ma^2}(2+8+9)$$

$$= \frac{19\pi^2\hbar^2}{2ma^2}$$

If you assume the ground state is given by,

Slater determinant

$$\left\|\begin{matrix} \psi_1(x_1)|\uparrow\rangle & \psi_1(x_2)|\uparrow\rangle & \psi_1(x_3)|\uparrow\rangle & \psi_1(x_4)|\uparrow\rangle & \psi_1(x_5)|\uparrow\rangle \\ \psi_1(x_1)|\downarrow\rangle & \psi_1(x_2)|\downarrow\rangle & \psi_1(x_3)|\downarrow\rangle & \psi_1(x_4)|\downarrow\rangle & \psi_1(x_5)|\downarrow\rangle \\ \psi_2(x_1)|\uparrow\rangle & & \cdots & & \\ \psi_2(x_1)|\downarrow\rangle & & \cdots & & \\ \psi_3(x_1)|\uparrow\rangle & & \cdots & & \end{matrix}\right\|$$

or $\psi_3(x_1)|\downarrow\rangle$

then $\langle S_z \rangle = \frac{\hbar}{2}$ since $S_z = S_{z_1} + S_{z_2} + S_{z_3} + S_{z_4} + S_{z_5}$.

S_z acting on any of the terms in the Slater determinant gives,

$$\langle S_z \rangle = \frac{\hbar}{2} + \frac{\hbar}{2} - \frac{\hbar}{2} - \frac{\hbar}{2} \pm \frac{\hbar}{2} = \boxed{\pm\frac{\hbar}{2}}$$

Note that $\langle S_z \rangle$ depends on which state you chose. There is some ambiguity since the ground state has a twofold degeneracy.