IV. TAYLOR 6.12

Similar to the previous problem, we set

$$I = \int_{x_1}^{x_2} f(x; y')dx, \tag{16}$$

with

$$f(x; y') = x\sqrt{1 - y'^2}. \tag{17}$$

The Euler-Lagrange equation gives:

$$\frac{xy'}{\sqrt{1 - y'^2}} = x_0, \tag{18}$$

with x_0 a constant. (Referring to the discussions around Eq. 31.) The above differential equation can be rewritten as

$$\frac{dy}{dx} = \frac{1}{\sqrt{\left(\frac{x}{x_0}\right)^2 + 1}}. \tag{19}$$

Separate the variables and take the integrations, we get

$$y = \int_{x_1}^{x} \frac{1}{\sqrt{\left(\frac{x'}{x_0}\right)^2 + 1}} dx' = x_0 \int_{x_1/x_0}^{x/x_0} \frac{1}{\sqrt{u^2 + 1}} du. \tag{20}$$

Here we picked up an arbitrary constant x_1 as the lower limit in the integral. Take the first substitution of Euler: $\sqrt{1 + u^2} = t - u$, we have

$$y = x_0 \int_{t_1}^{t} \frac{dt'}{t'} = x_0 \log\left(\sqrt{\left(\frac{x}{x_0}\right)^2 + 1} + \frac{x}{x_0}\right) + y_0, \tag{21}$$

yielding us

$$x = x_0 \sinh\left(\frac{y - y_0}{x_0}\right), \tag{22}$$

i. e., $y(x) = y_0 + x_0 \cdot \operatorname{arcsinh}\left(\frac{x}{x_0}\right)$.

Part a

From conservation of energy:

$$\frac{1}{2}mv^2 - \frac{\gamma}{r} = 0 - \frac{\gamma}{r_0}$$
$$v = \sqrt{\frac{2\gamma}{m}\left(\frac{1}{r} - \frac{1}{r_0}\right)}$$

Part b

The path element in polar coordinates is given by:

$$d\vec{s} = dr\,\hat{r} + r\,d\phi\,\hat{\phi}$$
$$ds = \sqrt{dr^2 + r^2\,d\phi^2}$$

Hence, the time it takes to traverse the path $\phi(r)$ is:

$$t = \int_{(r_0,0)}^{(r_2,\phi_2)} \frac{ds}{v}$$
$$= \int_{r_0}^{r_2} \frac{\sqrt{1 + r^2\phi'^2}}{\sqrt{\frac{2\gamma}{m}\left(\frac{1}{r} - \frac{1}{r_0}\right)}}\,dr$$

Taking out constants, this yields:

$$f(\phi, \phi', r) = \sqrt{\frac{1 + r^2\phi'^2}{\frac{1}{r} - \frac{1}{r_0}}}$$

Part c

Since $\frac{\partial f}{\partial \phi} = 0$, then the Euler-Lagrange equation yields $\frac{d}{dr}\frac{\partial f}{\partial \phi'} = 0$, or that $\frac{\partial f}{\partial \phi'}$ is constant in r.

$$C_1 = \frac{1}{\sqrt{\frac{1}{r} - \frac{1}{r_0}}}\frac{r^2\phi'}{\sqrt{1 + r^2\phi'^2}}$$

Rewriting:

$$C_2\left(\frac{1}{r} - \frac{1}{r_0}\right)\left(1 + r^2\phi'^2\right) = r^4\phi'^2$$

Part d

Isolating ϕ':

$$C_2\left(\frac{1}{r} - \frac{1}{r_0}\right) = r^2\left(r^2 - C_2\left(\frac{1}{r} - \frac{1}{r_0}\right)\right)\phi'^2$$
$$\phi'^2 = \frac{C_2\left(\frac{1}{r} - \frac{1}{r_0}\right)}{r^2\left(r^2 - C_2\left(\frac{1}{r} - \frac{1}{r_0}\right)\right)}$$

This is separable; integrating:

$$\phi = \int \frac{1}{r}\left[\frac{C_2\left(\frac{1}{r} - \frac{1}{r_0}\right)}{r^2 - C_2\left(\frac{1}{r} - \frac{1}{r_0}\right)}\right]^{1/2} dr$$

Homework 2 Problem 5

Part a

The path element is given by:

$$
\begin{aligned}
ds &= \sqrt{dx^2 + dy^2 + dz^2} \\
&= \sqrt{dx^2 + dy^2 + \left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy\right)^2} \\
&= \sqrt{\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right] dx^2 + \left[1 + \left(\frac{\partial h}{\partial y}\right)^2\right] dy^2 + 2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y}\, dx\, dy}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
S &= \int ds \\
&= \int_{x_1}^{x_2} \sqrt{\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right] + \left[1 + \left(\frac{\partial h}{\partial y}\right)^2\right] y'^2 + 2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y} y'}\, dx \\
f(y, y', x) &= \sqrt{\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right] + \left[1 + \left(\frac{\partial h}{\partial y}\right)^2\right] y'^2 + 2\frac{\partial h}{\partial x}\frac{\partial h}{\partial y} y'}
\end{aligned}
$$

Part b

For $h(x, y) = A + Bx + Cy$, then:

$$
f(y, y', x) = \sqrt{[1 + B^2] + [1 + C^2]\, y'^2 + 2BCy'}
$$

From the Euler-Lagrange equation:

$$
\begin{aligned}
0 &= \frac{\partial f}{\partial x} - \frac{d}{dx}\frac{\partial f}{\partial y'} \\
&= 0 - \frac{d}{dx}\frac{\left[1 + C^2\right] y' + BC}{\sqrt{[1 + B^2] + [1 + C^2]\, y'^2 + 2BCy'}}
\end{aligned}
$$

Integrating once:

$$
\begin{aligned}
\frac{\left[1 + C^2\right] y' + BC}{\sqrt{[1 + B^2] + [1 + C^2]\, y'^2 + 2BCy'}} &= K_1 \\
\left[1 + C^2\right]^2 y'^2 + 2BC\left[1 + C^2\right] y' + B^2C^2 &= K_2\left[1 + C^2\right] y'^2 + K_2 2BCy' + K_2\left[1 + B^2\right]
\end{aligned}
$$

This is a quadratic equation with constant coefficients. That means the solution is y' is some other constant, call it m. Then:

$$
\begin{aligned}
y' &= m \\
y &= mx + b
\end{aligned}
$$

$$\frac{\partial h}{\partial x} = -\frac{x}{h}, \quad \frac{\partial h}{\partial y} = -\frac{y}{h}$$

$$f = \left(1 + x^2 h^{-2} + (1 + y^2 h^{-2}) y'^2 + 2xyh^{-2}y'\right)^{1/2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2f}\left(-x^2 h^{-4}(-2y) + (2yh^{-2} + 2y^3 h^{-4})y'^2 + 2xh^{-2}y' + 4xy^2 h^{-4} y'\right)$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2f}\left(2y'(1 + y^2 h^{-2}) + 2xyh^{-2}\right) \rightarrow \boxed{\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}}$$

__Now plug in__ $\quad y(x) = \frac{1}{2}\left(\sqrt{2R^2 - 3x^2} - x\right)$

$$h(x, y(x)) = \sqrt{R^2 - x^2 - \frac{1}{4}\left(2R^2 - 3x^2 + x^2 - 2x\sqrt{2R^2 - 3x^2}\right)}$$

$$\boxed{h(x, y(x)) = x + y(x)}$$

Positive root for most of the interval. There is a separate discussion for the negative root.

$$f(y(x), y'(x), x) = \frac{\sqrt{6(x^2 + xy + y^2)}}{x + 2y}, \quad y' = -\frac{2x + y}{x + 2y}$$

$$\frac{\partial f}{\partial y'}(y(x), y'(x), x) = -\frac{\sqrt{\frac{2}{3}(x^2 + xy + y^2)}}{x + y}$$

$$\frac{\partial f}{\partial y}(y(x), y'(x), x) = \frac{(y - x)\sqrt{\frac{2}{3}(x^2 + xy + y^2)}}{(x + y)^2 (x + 2y)}$$

Then differentiate this using this to get this

__Great Circle__

$$h(x, y(x)) = x + y(x) \quad \Rightarrow \quad \boxed{(1, 1, -1) \cdot (x, y, h) = 0}$$

This is the equation for a plane perpendicular to $(1, 1, -1)$.

The intersection of the plane and the sphere $x^2 + y^2 + h^2 = R^2$ is a great circle

Part c

For $h(x,y) = \sqrt{R^2 - x^2}$:

$$
\begin{aligned}
f(y, y', x) &= \sqrt{\left[1 + \left(\frac{-x}{\sqrt{R^2 - x^2}}\right)^2\right] + \left[1 + 0^2\right] y'^2 + 2\left(\frac{-x}{\sqrt{R^2 - x^2}}\right)(0)\, y'} \\
&= \sqrt{\frac{R^2}{R^2 - x^2} + y'^2}
\end{aligned}
$$

From the Euler-Lagrange equation:

$$
\begin{aligned}
0 &= \frac{\partial f}{\partial x} - \frac{d}{dx}\frac{\partial f}{\partial y'} \\
&= 0 - \frac{d}{dx}\frac{y'}{\sqrt{\frac{R^2}{R^2 - x^2} + y'^2}}
\end{aligned}
$$

Integrating once:

$$
\begin{aligned}
\frac{y'}{\sqrt{\frac{R^2}{R^2 - x^2} + y'^2}} &= C_1 \\
C_1^2 \frac{R^2}{R^2 - x^2} + \left(C_1^2 - 1\right) y'^2 &= 0 \\
\frac{R^2}{R^2 - x^2} + C_2 y'^2 &= 0
\end{aligned}
$$

To check the consistency of the solution $x = R\sin(\alpha y + \beta)$:

$$
\begin{aligned}
y &= n\pi \pm \left[-\frac{\beta}{\alpha} + \frac{1}{\alpha}\arcsin\frac{x}{R}\right] \\
y' &= \pm\frac{1}{\alpha}\frac{1}{\sqrt{R^2 - x^2}} \\
y'^2 &= \frac{1}{\alpha^2}\frac{1}{R^2 - x^2}
\end{aligned}
$$

The solution works if $R^2 + \frac{C_2}{\alpha^2} = 0$; since C_2 is arbitrary, this lets α be chosen to match the initial conditions.

Part d

For $h(x,y) = \sqrt{R^2 - y^2}$:

$$
\begin{aligned}
f(y, y', x) &= \sqrt{\left[1 + 0^2\right] + \left[1 + \left(\frac{-y^2}{\sqrt{R^2 - y^2}}\right)^2\right] y'^2 + 2\,(0)\left(\frac{-y^2}{\sqrt{R^2 - y^2}}\right) y'} \\
&= \sqrt{1 + \frac{R^2}{R^2 - y^2} y'^2}
\end{aligned}
$$

Differentiating:

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\frac{-2R^2 y}{(R^2 - y^2)^2} y'^2}{2\sqrt{1 + \frac{R^2}{R^2 - y^2} y'^2}} \\
\frac{\partial f}{\partial y'} &= \frac{2\frac{R^2}{R^2 - y^2} y'}{2\sqrt{1 + \frac{R^2}{R^2 - y^2} y'^2}}
\end{aligned}
$$

Note that the function is independent of x. Therefore:

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{dy}{dx}\frac{\partial f}{\partial y} + \frac{dy'}{dx}\frac{\partial f}{\partial y'} \\
&= 0 + y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right) + \left(\frac{d}{dx}y'\right)\frac{\partial f}{\partial y'} \\
&= \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)
\end{aligned}$$

Integrating once:

$$f - y'\frac{\partial f}{\partial y'} = C$$

Substituting:

$$\sqrt{1 + \frac{R^2}{R^2 - y^2}y'^2} - \frac{\frac{R^2}{R^2-y^2}y'^2}{\sqrt{1 + \frac{R^2}{R^2-y^2}y'^2}} = C_1$$

$$\begin{aligned}
1 + \frac{R^2}{R^2 - y^2}y'^2 - \frac{R^2}{R^2 - y^2}y'^2 &= C_1\sqrt{1 + \frac{R^2}{R^2 - y^2}y'^2} \\
1 &= C_1^2\left(1 + \frac{R^2}{R^2 - y^2}y'^2\right) \\
\frac{R^2}{R^2 - y^2}y'^2 &= C_2
\end{aligned}$$

To check the consistency of the solution $y = R\sin(\gamma x + \delta)$:

$$\begin{aligned}
y' &= \gamma R\cos(\gamma x + \delta) \\
\frac{R^2}{R^2 - y^2} &= \sec^2(\gamma x + \delta)
\end{aligned}$$

The solution works if $\gamma^2 R^2 = C_2$; since C_2 is arbitrary, this lets γ be chosen to match the initial conditions.

(Note that the resulting first-order equations in parts c and d are separable and can be solved with appropriate trigonometric substitutions.)

HW2

① (a) $\frac{\partial f}{\partial y} = 3ay^2y' + 2by$

$\frac{\partial f}{\partial y'} = 2y' + ay^3 \xrightarrow{\frac{d}{dt}} 2y'' + 3ay^2y'$

$2y'' = 2by \Rightarrow \boxed{y'' = by}$

(b) $\boxed{y(x) = B_1 \cosh(\sqrt{b}\,x) + B_2 \sinh(\sqrt{b}\,x)}$

(c) $y'' = -(-b)y$ $\boxed{y(x) = B_1 \cos(\sqrt{-b}\,x) + B_2 \sin(\sqrt{-b}\,x)}$

② (a) $\left.\begin{array}{l} \frac{\partial f}{\partial y} = ax^3 \\ \frac{\partial f}{\partial y'} = 2y' \xrightarrow{\frac{d}{dt}} 2y'' \end{array}\right\}$ $\boxed{2y'' = ax^3}$

$y' = \left(\frac{a}{2}\right)\left(\frac{x^4}{4}\right) + C_1$

$y = \frac{a}{8}\,\frac{x^5}{5} + C_1x + C_2$

(b) $\longrightarrow$ $\boxed{y(x) = \frac{1}{40}ax^5 + C_1x + C_2}$