Problem Set 1

Physics 110A, UC Berkeley, Spring 2021

Due Monday, 2/1, at 11:59PM

Problem 1

The electric field of a point charge Q is given by $\boldsymbol{E} = \frac{kQ}{r^2}\hat{\boldsymbol{r}}$, and the magnetic field of an infinitely long wire with current I is $\boldsymbol{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$, where the fields are expressed in spherical and cylindrical coordinate respectively. Express the fields in the Cartesian coordinate and calculate

(a) the divergence and curl of $\boldsymbol{E}$, and

(b) the divergence and curl of $\boldsymbol{B}$.

You can ignore the singularity at $r = 0$ and $s = 0$ in the problem.

Problem 2

Many quantities in vector calculus involves anti-symmetric structures, such as cross products, curl, etc. For these quantities it is easier to derive various properties via Levi-Civita symbol and index notations.

(a) Let $\boldsymbol{S}$ be a symmetric matrix. That is, $S_{ij} = S_{ji}$. Show that

$$\epsilon^{ijk} S_{ij} = 0.$$

(b) Write $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$ in terms of Levi-Civita symbol and components with indices, and then find the relation between $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$ and $\boldsymbol{B} \cdot (\boldsymbol{A} \times \boldsymbol{C})$. *Note that $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$ is the volume of the parallelepiped spanned by the three vectors. In other words, the notion of volume involves anti-symmetric structure.*

(c) Using Levi-Civita symbol, show that

$$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0.$$

(d) Show that

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}).$$

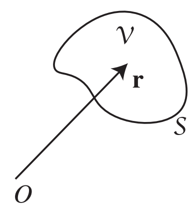
Problem 3

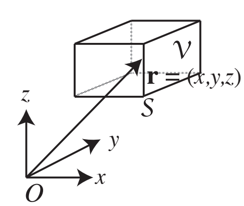
The formal definition of divergence $\mathbf{F}$ at the position $\mathbf{r}$ is given by

$$\nabla \cdot \mathbf{F} = \lim_{\mathcal{V} \to 0} \frac{\oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{a}}{\mathcal{V}}$$

where $\mathcal{S}$ is any surface enclosing $\mathbf{r}$, and $\mathcal{V}$ is the volume bounded by $\mathcal{S}$, as shown in the left figure. Use this definition with a rectangular box enclosing (x, y, z), as shown in the right figure, to prove that in Cartesian coordinates,

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$





Below are selected optional problems from Griffiths. We do not collect your work, but you are encouraged to do as many practice problems as you.

- Problem 1.6
- Problem 1.8
- Problem 1.10
- Problem 1.12
- Problem 1.13
- Problem 1.14
- Problem 1.15
- Problem 1.17
- Problem 1.18
- Problem 1.19
- Problem 1.20
- Problem 1.21
- Problem 1.24
- Problem 1.28

For Problem 1.6, 8, 14, 17, 21, 27 and 28, you can compare the way done in the solution manual with methods based on the discussions in Sec.2 of Lecture Note 01. You might find the expressions in the latter way are more compact and thus easier to see the algebraic structure underneath, even though it takes some time to get used to the ideas and be familiar with doing calculations with indices.