**Physics 110A, Spring 2021**
**Solution to Homework 9**
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1. (a) The system is just a special case of a toroid with zero inner radius, so you can either use the argument shown in the discussion, or Griffiths Example 5.10.

   (b) Apply Ampere's law to a circle of radius $s$ about the axis of the toroid as in Griffiths Example 5.10, and we will get

$$\boxed{\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{for points inside the coil,} \\ \mathbf{0}, & \text{for points outside the coil.} \end{cases}}$$

   (c) Top cap: $\mathbf{K} = \dfrac{I}{2\pi s}\hat{\mathbf{s}}$, $\mu_0\mathbf{K}\times\hat{\mathbf{n}} = \mu_0\mathbf{K}\times\hat{\mathbf{z}} = -\dfrac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$. $\mathbf{B}_{\text{above}} = \mathbf{0}$, $\mathbf{B}_{\text{below}} = \dfrac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$.
   Side wall: $\mathbf{K} = \dfrac{I}{2\pi R}\hat{\mathbf{z}}$, $\mu_0\mathbf{K}\times\hat{\mathbf{n}} = \mu_0\mathbf{K}\times\hat{\mathbf{s}} = \dfrac{\mu_0 I}{2\pi R}\hat{\boldsymbol{\phi}}$. $\mathbf{B}_{\text{above}} = \mathbf{0}$, $\mathbf{B}_{\text{below}} = \dfrac{\mu_0 I}{2\pi R}\hat{\boldsymbol{\phi}}$.
   For both cases, $\boxed{\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0\mathbf{K}\times\hat{\mathbf{n}}.}$

2. Assume the strength of magnetic field $|\mathbf{B}|$ has a local maximum at $P$, then so does $B^2$ (mathematically it is easier to deal with squares than absolute values). Now we choose a small spherical surface $\mathcal{S}$ centered at $P$. Since the gradient operator $\nabla B^2$ always points to the direction in which $B^2$ increases most rapidly, on the surface of $\mathcal{S}$, the area vector $d\mathbf{a}$ and $\nabla B^2$ must be opposite in direction: $\nabla B^2 \cdot d\mathbf{a} < 0$, so

$$\oint_{\mathcal{S}} \nabla B^2 \cdot d\mathbf{a} < 0.$$

However, according to divergence theorem, $\oint_{\mathcal{S}} \nabla B^2 \cdot d\mathbf{a} = \int_{\mathcal{V}} \nabla^2 B^2 d\tau$, and

$$\begin{aligned} \nabla^2 B^2 &= \partial_i\partial_i B_j B_j = 2\partial_i\left[(\partial_i B_j)B_j\right] \\ &= 2\left[B_j(\partial_i\partial_i B_j) + (\partial_i B_j)(\partial_i B_j)\right] = 2\left[\mathbf{B}\cdot\nabla^2\mathbf{B} + (\partial_i B_j)^2\right]. \end{aligned}$$

However, according to vector identity,

$$\nabla\times(\nabla\times\mathbf{B}) = \nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B},$$

and in a current-free region, $\nabla\cdot\mathbf{B} = 0$, $\nabla\times\mathbf{B} = \mathbf{0}$, so $\nabla^2\mathbf{B} = \mathbf{0}$. Therefore

$$\oint_{\mathcal{S}} \nabla B^2 \cdot d\mathbf{a} = \int_{\mathcal{V}} 2(\partial_i B_j)^2 d\tau \geq 0,$$

which contradicts to our assumption.

3. The spinning sphere creates a current density

$$\mathbf{J} = \rho\mathbf{v} = \rho(\boldsymbol{\omega}\times\mathbf{r}) = \rho\omega r\sin\theta\hat{\boldsymbol{\phi}}$$

in the sphere, where $\rho = Q/\left(\frac{4}{3}\pi R^3\right)$.

(a) Consider a loop shown in the left figure, the infinitesimal current flowing through it is

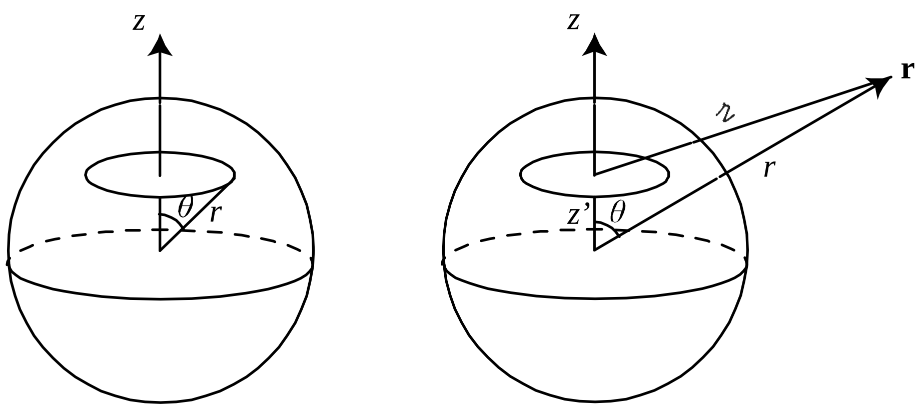
$$dI = \mathbf{J} \cdot d\mathbf{a} = \left(\rho\omega r \sin\theta \hat{\boldsymbol{\phi}}\right) \cdot \left(r d\theta dr \hat{\boldsymbol{\phi}}\right) = \rho\omega r^2 \sin\theta d\theta dr,$$

and the dipole of this current loop is

$$d\mathbf{m} = A dI = \pi (r \sin\theta)^2 \times \rho\omega r^2 \sin\theta d\theta dr \hat{\mathbf{z}} = \pi\rho\omega r^4 \sin^3\theta d\theta dr \hat{\mathbf{z}}.$$

Note that there are two different areas involved in this calculation, one is the cross sectional area $d\mathbf{a} = r d\theta dr \hat{\boldsymbol{\phi}}$ supporting the current, and the other is the area $A = \pi (r\sin\theta)^2$ bounded by the current loop. The total dipole of the system is

$$\mathbf{m} = \int d\mathbf{m} = \int_0^R \int_0^\pi \pi\rho\omega r^4 \sin^3\theta d\theta dr \hat{\mathbf{z}} = \frac{4}{15}\pi\rho\omega R^5 \hat{\mathbf{z}} = \boxed{\frac{1}{5}\omega Q R^2 \hat{\mathbf{z}}.}$$




(b) If $r \gg R$, we can approximate the system as a single dipole $\mathbf{m}$ located at the origin, instead of many current loops in a finite sphere as we analyzed in (a). So the vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 \rho\omega R^5}{15 r^2}\sin\theta\hat{\boldsymbol{\phi}} = \boxed{\frac{\mu_0 \omega Q R^2}{20\pi r^2}\sin\theta\hat{\boldsymbol{\phi}}.}$$

(c) Consider the same current loop we analyzed in (a). Its dipole moment is $d\mathbf{m} = \pi\rho\omega r'^4 \sin^3\theta' d\theta' dr' \hat{\mathbf{z}} \equiv dm\hat{\mathbf{z}}$. Here I use the primed variables $(r', \theta')$ to denote the position of the current loop, since the field point $\mathbf{r}$ $(r, \theta)$ will be involved in the following calculation. The infinitesimal vector potential $d\mathbf{A}$ due to this dipole is (see the right figure)

$$\begin{aligned} d\mathbf{A} &= \frac{\mu_0}{4\pi}\frac{d\mathbf{m} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi}\frac{d\mathbf{m} \times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0}{4\pi}\frac{dm\hat{\mathbf{z}} \times (\mathbf{r} - z'\hat{\mathbf{z}})}{(r^2 + z'^2 - 2rz'\cos\theta)^{3/2}} \\ &= \frac{\mu_0}{4\pi}\frac{dm\hat{\mathbf{z}} \times \mathbf{r}}{(r^2 + z'^2 - 2rz'\cos\theta)^{3/2}} = \frac{\mu_0}{4\pi}\frac{\pi\rho\omega r'^4 \sin^3\theta' d\theta' dr'}{(r^2 + z'^2 - 2rz'\cos\theta)^{3/2}} r\sin\theta\hat{\boldsymbol{\phi}}. \end{aligned}$$

So

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_0^R \int_0^\pi \frac{\pi\rho\omega r'^4 \sin^3\theta' d\theta' dr'}{(r^2 + z'^2 - 2rz'\cos\theta)^{3/2}} r\sin\theta\hat{\boldsymbol{\phi}} = \frac{\mu_0\rho\omega R^5}{15r^2}\sin\theta\hat{\boldsymbol{\phi}} = \boxed{\frac{\mu_0\omega Q R^2}{20\pi r^2}\sin\theta\hat{\boldsymbol{\phi}},}$$

which is exactly the same as the approximate solution (b). The integration in the last step is pretty involved, so you can also use the results of Griffiths Example 5.11, by treating the solid sphere as multiple thin shells with surface charge density $\rho dr$.