Physics 110A, Spring 2021
Solution to Homework 3
GSI: Yi-Chuan Lu

1. Method 1: Use integration by parts,

$$\begin{aligned} J &= \int_{\mathcal{V}} \frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla f(\mathbf{r})\, d\tau = \int_{\mathcal{V}} \left[\nabla \cdot \left(f(\mathbf{r}) \frac{\hat{\mathbf{r}}}{r^2} \right) - f(\mathbf{r}) \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right] d\tau \\ &= \oint_{\mathcal{S}} f(\mathbf{r}) \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a} - 4\pi \int_{\mathcal{V}} f(\mathbf{r})\, \delta^{(3)}(\mathbf{r})\, d\tau = \boxed{-4\pi f(\mathbf{0}).} \end{aligned}$$

Method 2: Use spherical coordinates. Assume the boundary of $\mathcal{V}$ is defined by $r = r_b(\theta, \phi)$, then

$$\begin{aligned} J &= \int_{\mathcal{V}} \frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla f(\mathbf{r})\, d\tau \\ &= \int_0^{\pi} \int_0^{2\pi} \int_0^{r_b(\theta,\phi)} \frac{\hat{\mathbf{r}}}{r^2} \cdot \left(\frac{\partial f(\mathbf{r})}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f(\mathbf{r})}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \frac{\partial f(\mathbf{r})}{\partial \phi} \hat{\boldsymbol{\phi}} \right) r^2 \sin\theta\, dr\, d\phi\, d\theta \\ &= \int_0^{\pi} \int_0^{2\pi} \int_0^{r_b(\theta,\phi)} \frac{\partial f(\mathbf{r})}{\partial r} \sin\theta\, dr\, d\phi\, d\theta = \int_0^{\pi} \int_0^{2\pi} \left[f(r_b(\theta,\phi), \theta, \phi) - f(\mathbf{0}) \right] \sin\theta\, d\phi\, d\theta \\ &= \boxed{-4\pi f(\mathbf{0}).} \end{aligned}$$

2. Please see the discussion notes. Answer:

$$\mathbf{E} = \boxed{\frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \hat{\mathbf{z}}.}$$

As $R \to \infty$, $\mathbf{E} \to \boxed{\frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}}.}$ If $z \gg R$, then $\frac{z}{\sqrt{z^2+R^2}} = \frac{1}{\sqrt{1+\left(\frac{R}{z}\right)^2}} \simeq 1 - \frac{1}{2}\left(\frac{R}{z}\right)^2$, so

$$\mathbf{E} \simeq \frac{\sigma}{2\epsilon_0} \left[1 - \left(1 - \frac{1}{2} \left(\frac{R}{z} \right)^2 \right) \right] \hat{\mathbf{z}} = \boxed{\frac{\sigma}{4\epsilon_0} \frac{R^2}{z^2} \hat{\mathbf{z}}.}$$

Note that if we write $Q = 4\pi R^2 \sigma$, then $\mathbf{E} \simeq \dfrac{1}{4\pi\epsilon_0} \dfrac{Q}{z^2} \hat{\mathbf{z}}$, which is the same as the field due to a single charge Q at the origin.

3. For $\mathbf{E} = ay\hat{\mathbf{y}}$, the charge density is $\rho = \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \epsilon_0 \partial E_y / \partial y = \boxed{\epsilon_0 a.}$ For $\mathbf{E} = (1/3)\, ar\hat{\mathbf{r}}$, $\rho = \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \epsilon_0 (1/r^2)\, \partial (r^2 E_r) / \partial r = \boxed{\epsilon_0 a.}$ Physically, the charge distribution must be confined in a domain with some shape. If the domain is a sphere, then $\rho = \epsilon_0 a$ would produce $\mathbf{E} = (1/3)\, ar\hat{\mathbf{r}}$. If the domain is a slab (infinite in the x and z directions, but finite in the y direction), then $\rho = \epsilon_0 a$ would produce $\mathbf{E} = ay\hat{\mathbf{y}}$.

However, what if the charge density $\rho = \epsilon_0 a$ really fills up the entire universe? Should the electric field be $\mathbf{E} = ay\hat{\mathbf{y}}$ or $\mathbf{E} = (1/3)\, ar\hat{\mathbf{r}}$? Mathematically, we do not have a unique solution for $\mathbf{E}$ (Helmholtz theorem requires ρ to decay faster than $1/r^2$ to guarantee a unique $\mathbf{E}$). Probably the electric field then will depend on, say, the shape of the universe boundary?