Problem Set 11 - Solutions

1. $$\operatorname{Im} f(0) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\operatorname{Im}\left(e^{i\delta_l}\right)\sin(\delta_l)\,P_l(\cos 0)$$

$$= \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l)\,P_l(1)$$

Now we use the fact that $P_l(1) = 1 \ \forall\, l$

$$= \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l)$$

$$= \frac{k}{4\pi}\sigma_{TOT}$$

$$\therefore \quad \sigma_{TOT} = \frac{4\pi}{k}\operatorname{Im} f(0)$$

(or conservation of probability)

It follows from $\boxed{\text{conservation of flux}}$ ~~Here's another derivation that makes this physical connection more clear.~~

See Recitation notes for proof

$$\cancel{\psi(\vec r) = \underbrace{e^{ikz}}_{\psi_0(\vec r)} + \underbrace{f(\theta)\frac{e^{ikr}}{r}}_{=:\phi(\vec r)}}$$

~~The probability current is,~~

$$\cancel{\vec j(x,t) = \frac{\hbar}{4m}\operatorname{Im}\left(\psi^* \vec\nabla\psi\right)}$$

~~Since the above is a stationary solution, by conservation of flux, we know~~

$$\cancel{0 = \int_{\text{large sphere}} r^2\, d\Omega\, \hat r\cdot\vec j = \frac{\hbar r^2}{4m}\operatorname{Im}\int d\Omega\, \psi^* \frac{\partial\psi}{\partial r}}$$

(Saxon 9) 2. First, converting to radians gives,

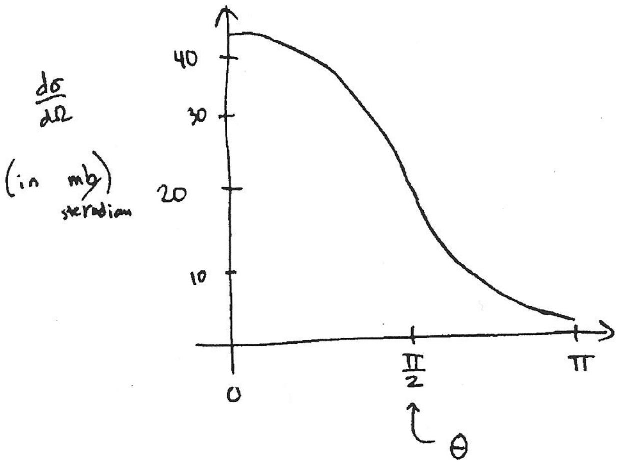
a)

$$\delta_0 = 32.5 \cdot \frac{2\pi}{360} \approx 0.567$$

$$\delta_1 = 8.6 \cdot \frac{2\pi}{360} \approx 0.150$$

$$\delta_2 = 0.4 \cdot \frac{2\pi}{360} \approx 0.00698$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left|\left[e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1 \cos\theta + 5e^{i\delta_2}\sin\delta_2\left(\tfrac{1}{2}\right)\left(3\cdot\cos^2\theta - 1\right)\right]\right|^2$$



$$k^2 = \frac{2mE}{\hbar^2}$$

$$= 2 \cdot \frac{(5\,\text{MeV}) \cdot (1.67262 \times 10^{-27}\,\text{kg})}{(1.05457 \times 10^{-34}\,\text{J s})^2} \cdot \frac{10^6 \cdot 1.602 \times 10^{-19}\,\text{J}}{1\,\text{MeV}}$$

$$\approx 24.094 \times 10^{-27} \cdot 10^{6} \cdot 10^{-19} \cdot 10^{68}$$

$$\approx 24.094 \times 10^{28}$$

$$1/k^2 \approx 4.15 \times 10^{-30}\ \text{m}^2$$

recall, $10^{-31}\,\text{m}^2$ = mb = millibarn

$\left(1\ \text{barn} = 10^{-28}\ \text{m}^2\right)$

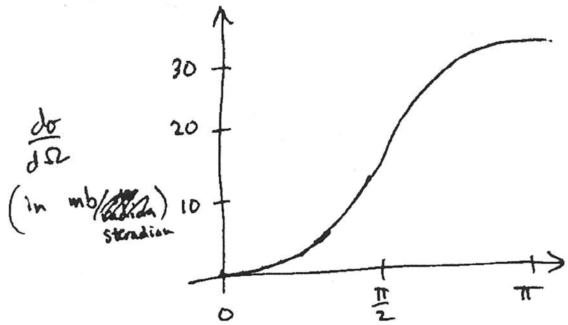
Integrating gives

$$\sigma_{TOT} \approx 186\ \text{mb} = 0.186\ \text{b}$$

$$= 1.86 \times 10^{-29}\ \text{m}^2$$

This is reasonable since ~~1 barn is defined to be a~~ 1 barn is approximately the cross-sectional area of Uranium.

c) With the sign of δ_0 reversed,



$\boxed{\sigma_{TOT} = 186 \text{ mb}}$

b) Reversing all phases does nothing since

$$\frac{d\sigma}{d\Omega} = |f(\theta_{rev})|^2 \quad \text{and} \quad f(\theta_{rev}) = -f(\theta)^*$$

(d) ~~Total Scattered / sec = # beam neutrons / cm² · s · (Cross-Sec Area of beam) · # nuclei / cm² · σ TOT of one nucleus~~

~~$= 10^{10} \cdot 2 \cdot 10^{21} \cdot (1.86 \times 10^{-29} \text{ m}^2 \cdot \frac{(100)^2 \text{ cm}^2}{1 \text{ m}^2})$~~

~~$= 3.7 \times 10^{31} \cdot 10^{-29} \cdot 10^{4}$~~

~~$= 3.7 \times 10^{6}$ Neutrons Scattered/sec~~

$$\frac{\text{Total Scattered}}{\text{sec}} = \frac{\#\text{ in beam}}{\text{cm}^2 \cdot \text{s}} \cdot \underbrace{\text{Total Cross Section}}_{= (\text{Cross-Section of beam}) \cdot \frac{\#\text{ nuclei}}{\text{cm}^2} \cdot \sigma_{\text{nucleus}}}$$

$$= 10^{10} \cdot 2 \cdot 10^{21} \cdot \left(1.86 \times 10^{-29} \text{ m}^2 \cdot \frac{(100)^2 \text{ cm}^2}{1 \text{ m}^2}\right)$$

$$= 3.7 \times 10^{31} \cdot 10^{-29} \cdot 10^{4}$$

$$= \boxed{3.7 \times 10^{6} \text{ Neutrons Scattered/sec}}$$

Scattered in the region $= \left(\frac{\#\text{ in beam}}{cm^2 \cdot s}\right) \cdot \left(\text{beam cross-section}\right) \cdot \frac{\#\text{ nuclei}}{cm^2} \cdot \left.\frac{d\sigma}{d\Omega}\right|_{\theta=\frac{\pi}{2}} \cdot 2\times10^{-5}$

$\approx 10^{10} \cdot 2 \cdot 10^{21} \cdot 11.3\times10^{-31} \cdot 2\times10^{-5}$

$= 44 \cdot \cancel{10^{31}} \cdot 10^{-5} \cdot \cancel{10^{-31}}$

$= 44\times10^{-5} = 4.4\times10^{-4}$ Neutrons Scattered into the region / sec

3. (Liboff 14.3)

$$\delta_\ell = \sin^{-1}\left[\frac{(iak)^\ell}{\sqrt{(2\ell+1)\,\ell!}}\right]$$

a) $$\sigma_{TOT} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\left(\sin^{-1}\left(\frac{(iak)^\ell}{\sqrt{(2\ell+1)\,\ell!}}\right)\right)$$

$$= \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}\cancel{(2\ell+1)}\cdot\frac{(iak)^{2\ell}}{\cancel{2\ell+1}\cdot\ell!}$$

$$= \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}\frac{(-a^2k^2)^\ell}{\ell!} = \underline{\frac{4\pi}{k^2}e^{-a^2k^2}}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \Rightarrow k^2 = \frac{2mE}{\hbar^2}$$

$$\Rightarrow \sigma_{TOT} = \frac{4\pi\hbar^2}{2mE}e^{-a^2\cdot\frac{2mE}{\hbar^2}} = \frac{2\pi\hbar^2}{mE}e^{-\frac{2mEa^2}{\hbar^2}}$$

(b) S-Wave scattering $\Longleftrightarrow \ell=0$

$$\sigma_{S\text{-Wave}}^{(\ell=0)} = \frac{4\pi}{k^2}\cdot 1\cdot 1 = \frac{2\pi\hbar^2}{mE}$$

This gives a good estimate when $\boxed{\frac{2mEa^2}{\hbar^2} \ll 1}$

$$\Updownarrow$$

$$\boxed{E \ll \frac{\hbar^2}{2ma^2}}$$

So it gives a good estimate for low-energy scattering, This makes sense intuitively, because at low energies the particle does not have enough energy to overcome the centripetal barrier that exists for $\ell > 0$.

4. (Ohanian 9)

Should be ~~ ~~ 2.55

$$\frac{d\sigma}{d\Omega}_{\text{experiment}} = \frac{1}{k^2}\left(0.86 + \underset{2.55}{\cancel{\quad}}\cos\theta + 2.77\cos^2\theta\right)$$

$$\frac{d\sigma}{d\Omega}_{\text{theory}} = \frac{1}{k^2}\left(\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin(\delta_\ell)P_\ell(\cos\theta)\right)\left(\sum_{\ell'=0}^{\infty}(2\ell'+1)e^{-i\delta_{\ell'}}\sin(\delta_{\ell'})P_\ell(\cos\theta)\right)$$

Since the highest power appearing in the experimental data is $\cos^2\theta$ this implies that $\delta_\ell = 0$ for $\ell > 1$. This is because

$P_\ell(\cos\theta)$ contains a $\cos^2\theta$ term for $\ell > 1$

which would give $\cos^3\theta$ and $\cos^4\theta$ terms when squared.

$$\therefore \frac{d\sigma}{d\Omega}_{\text{theory}} = \frac{1}{k^2}\left(e^{i\delta_0}\sin(\delta_0) + 3e^{i\delta_1}\sin(\delta_1)\cos\theta\right)\left(e^{-i\delta_0}\sin(\delta_0) + 3e^{-i\delta_1}\sin(\delta_1)\cos\theta\right)$$

$$= \frac{1}{k^2}\left[\sin^2(\delta_0) + 6\sin(\delta_1)\sin(\delta_0)\cos(\delta_1-\delta_0)\cos\theta + 9\sin^2(\delta_1)\cos^2\theta\right.$$

Check:

$$0.86 = \sin^2(\delta_0) \Rightarrow \pm 0.93 = \sin(\delta_0)$$

$\delta_0 = 1.1873$ or $\pi - 1.1873$ or $\pi + 1.1873$ or $2\pi - 1.1873$

$$2.77 = 9\sin^2(\delta_1) \Rightarrow \delta_1 = 0.59 \text{ or } \pi - 0.59,\ \pi + 0.59,\ 2\pi - 0.59$$

Now need to check the middle value,

$$6\sin(\delta_1)\sin(\delta_0)\cos(\delta_1-\delta_0) = \cancel{\quad}\ 2.55 \Rightarrow$$

Only contributions come from

$\delta_1 = 0.59 \leftarrow \ell=0$

$\delta_0 = 1.19 \leftarrow \ell=1$

$$\sigma = 2\pi \cdot \int_0^{\pi} d\theta \sin\theta \frac{d\Omega}{d\theta}$$

$$= \frac{2\pi}{k^2} \cdot \int_0^{\pi} d\theta \sin\theta \left(0.86 + 2.55 \cos\theta + 2.77 \cos^2\theta\right)$$

$$= \frac{22.41}{k^2}$$