Physics 110A, Spring 2021
Solution to Homework 6
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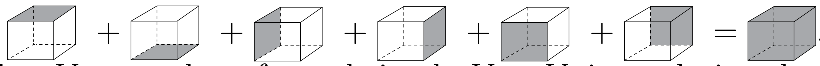
1. . For the composite cube which has V_0 on each surface, obviously $V \equiv V_0$ is a solution that satisfies the Laplace equation and all the boundary conditions. According to the uniqueness theorem, this is the solution. By symmetry, each individual cube must contribute $\boxed{V_0/6}$ to the center.

2. $\rho = \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = -V_0/R^2 - V_0/R^2 + 2V_0/R^2 = \boxed{0.}$ So the charge only resides on the surface. If we express $\mathbf{E} = -\boldsymbol{\nabla} V$ in component form,

$$-\frac{\partial V}{\partial x} = -\frac{V_0 x}{R^2}, \quad -\frac{\partial V}{\partial y} = -\frac{V_0 y}{R^2}, \quad -\frac{\partial V}{\partial z} = \frac{2V_0 z}{R^2},$$

then we can integrate these equations to get V as we did in Problem Set 2:

$$V(x, y, z) = \frac{V_0 x^2}{2R^2} + \frac{V_0 y^2}{2R^2} - \frac{V_0 z^2}{R^2} + C,$$

or in terms of spherical coordinates,

$$V(r, \theta) = -\frac{V_0}{R^2} r^2 \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + C. \tag{1}$$

Note that we keep the constant term C for now. It turns out that the constant term corresponds to some physical charge on the surface.

Since the potential inside a sphere has a general solution of the form

$$V_{\text{in}}(r, \theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta) = a_0 + a_1 r\cos\theta + a_2 r^2 \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots, \tag{2}$$

by comparing equations (1) and (2), we have $a_0 = C$, $a_2 = -V_0/R^2$, and $a_l \equiv 0$ otherwise. Similarly, the potential outside the sphere has a general solution

$$V_{\text{out}}(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) = \frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta + \frac{B_2}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots, \tag{3}$$

and since $V_{\text{in}} = V_{\text{out}}$ at the surface $r = R$, by matching the coeficients in (2) and (3), we obtain $B_0/R = C$, $B_2/R^3 = -V_0$, and $B_l \equiv 0$ otherwise. Hence,

$$V(r, \theta) = \begin{cases} V_{\text{in}} = & C - V_0 \dfrac{r^2}{R^2}\left(\dfrac{3}{2}\cos^2\theta - \dfrac{1}{2}\right), \\ V_{\text{out}} = & C\dfrac{R}{r} - V_0 \dfrac{R^3}{r^3}\left(\dfrac{3}{2}\cos^2\theta - \dfrac{1}{2}\right). \end{cases}$$

Using $\partial V_{\text{in}}/\partial r - \partial V_{\text{out}}/\partial r = \sigma/\epsilon_0$, one can find that

$$\sigma(\theta) = \boxed{\frac{\epsilon_0 C}{R} - 5\frac{\epsilon_0 V_0}{R}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right).}$$

The surface charge $\boxed{\sigma \text{ is not unique,}}$ as different C gives different σ. Physically, C corresponds to the total charge on the surface: $Q = \int \sigma da = \int_0^{\pi}\int_0^{2\pi} \sigma R^2 \sin\theta d\phi d\theta = 4\pi\epsilon_0 RC$.

3. (a) The system would look the same if we rotate it (change ϕ to $\phi + \delta\phi$), so the system has azimuthal symmetry. Similarly, since the cone and plate are already infinite, they would look the same if we magnify them together (change r to λr) , so the system has scaling symmetry. This is actually how we define symmetry in physics: we apply some action to the system, and if the system looks exactly the same, then the system is said to have the symmetry corresponding to the variable we changed (r and ϕ in our example). The potential thus will not depend on r or ϕ.

   (b) Without the r and ϕ dependence, the Laplace equation reduces to

$$\nabla^2 V = \frac{1}{r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0.$$

So

$$\sin\theta \frac{dV}{d\theta} = C \Rightarrow V = \int \frac{C}{\sin\theta} d\theta = C \ln\left(\tan\frac{\theta}{2}\right) + K.$$

Here C and K are integration constants. Imposing the boundary conditions $V(\pi/4) = V_0$ and $V(\pi/2) = 0$, we can fix those two constants, and obtain

$$V = \boxed{V_0 \frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\pi}{8}\right)}}.$$