Prob. 1

No rotation $\Rightarrow$ can treat as point masses

(a) Eq. (7.34): 
$$\begin{cases} \vec{r}_M = x\hat{x} + \text{const} \\ \vec{r}_m = \vec{r}_M + s(\cos\theta, \sin\theta) + \text{const} \end{cases}$$

$$\dot{\vec{r}}_M = (\dot{x}, 0)$$

$$\dot{\vec{r}}_m = (\dot{x}, 0) + \dot{s}(\cos\theta, \sin\theta)$$

$$\boxed{\mathcal{L} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s}\cos\theta\right) - \frac{1}{2}k(s - D_0)^2}$$

(b) $$\frac{\partial \mathcal{L}}{\partial s} = -k(s - D_0)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{s}} = m\dot{s} + m\dot{x}\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = M\dot{x} + m\dot{x} + m\dot{s}\cos\theta$$

$$\boxed{\begin{aligned} m\ddot{s} + m\ddot{x}\cos\theta &= -k(s - D_0) \\ M\dot{x} + m\dot{x} + m\dot{s}\cos\theta &= \text{const} \end{aligned}}$$

(c) The horizontal component of total momentum

$$= (M+m)\,\dot{x} + m\,\dot{s}\cos\theta$$

The total energy

$$= T + U$$ (Look at result in part (a) and reverse sign of $U$)

(d) $$\underbrace{\tfrac{1}{2}k(D-D_0)^2}_{E_0} = \tfrac{1}{2}(M+m)\dot{x}^2 + \tfrac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s}\cos\theta$$

$\dot{x} = -\alpha\dot{s}$, where $\boxed{\alpha = m\cos\theta/(M+m)}$

$$E_0 = \tfrac{1}{2}(M+m)\alpha^2\dot{s}^2 + \tfrac{1}{2}m\dot{s}^2 - m\alpha\dot{s}^2\cos\theta$$

$= \beta\dot{s}^2$, where $\boxed{\beta = \tfrac{1}{2}(M+m)\alpha^2 + \tfrac{1}{2}m - m\alpha\cos\theta}$

$\dot{s} = \sqrt{E_0/\beta}$ ← Not required to plug in and simplify

$$\boxed{\dot{\vec{r}}_m = \left[(-\alpha, 0) + (\cos\theta, \sin\theta)\right]\sqrt{\frac{E_0}{\beta}}}$$

(e) As $M \to \infty$, $\alpha \to 0$ and $\beta \to \frac{1}{2}m$

$$\boxed{\dot{\vec{r}}_m \text{ goes to } \sqrt{\frac{2E_0}{m}}\,(\cos\theta, \sin\theta)}$$

This is expected: The wedge is motionless. The initial energy in the spring goes into kinetic energy of block.

(a) $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$, $U = \frac{1}{2}k(x^2+y^2) \Rightarrow \boxed{\mathcal{L}(x,y,\dot{x},\dot{y}) = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - \frac{k}{2}(x^2+y^2)}$

(b) $\frac{\partial \mathcal{L}}{\partial x} = -kx$, $\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x} \Rightarrow \boxed{m\ddot{x} = -kx}$

$\frac{\partial \mathcal{L}}{\partial y} = -ky$, $\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y}$, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = m\ddot{y} \Rightarrow \boxed{m\ddot{y} = -ky}$

$\boxed{\omega = \sqrt{k/m}}$. The form for the solution that is a linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ is useful here.

$$\boxed{x(t) = x_0 \cos(\omega t) + (V_{x0}/\omega)\sin(\omega t),\ y(t) = y_0\cos(\omega t) + \frac{V_{y0}}{\omega}\sin(\omega t)}$$

(c) $x = r\cos\phi \Rightarrow \dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$
$y = r\sin\phi \Rightarrow \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$ $\Big\} \Rightarrow \dot{x}^2+\dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2$

$$\boxed{\mathcal{L}(r,\phi,\dot{r},\dot{\phi}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{k}{2}r^2}$$

$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 - kr$, $\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = m\ddot{r} \Rightarrow \boxed{m\ddot{r} = mr\dot{\phi}^2 - kr}$

$\frac{\partial \mathcal{L}}{\partial \phi} = 0$, $\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi} \Rightarrow \boxed{\frac{d}{dt}(mr^2\dot{\phi}) = 0}$ simpler in this form.

$\boxed{\ell = mr^2\dot{\phi}}$ is conserved

(d) $U(x,y) = \frac{k}{2}(x^2+y^2) + cy$

$$\boxed{\mathcal{L}(x,y,\dot{x},\dot{y}) = \frac{m}{2}(\dot{x}^2+\dot{y}^2) - \frac{k}{2}(x^2+y^2) - cy}$$

As explained in the Ch.7 summary, since the coordinates are natural, the Hamiltonian is $T+U$.

$$\boxed{\mathcal{H} = \frac{m}{2}(\dot{x}^2+\dot{y}^2) + \frac{k}{2}(x^2+y^2) + cy}$$ This is a conserved quantity.

b is a constant with dimensions of distance. It is not the semiminor axis.

(a) $\vec{R}(0) = (m_1 \vec{r}_1(0) + m_2 \vec{r}_2(0))/M = m_2 b \hat{x}/M$

$\vec{P} = m_1(0, 0, v_0) + m_2(0, v_0, 0) = (0, m_2, m_1) v_0$

$$\boxed{\vec{R}(t) = \frac{m_2 b \hat{x}}{M} + \frac{v_0 t}{M}(0, m_2, m_1)}$$

(b) $\vec{r} = \vec{r}_1 - \vec{r}_2$. $\vec{r}(0) = (-b, 0, 0)$. $\dot{\vec{r}}(0) = (0, -v_0, v_0)$

$\vec{\ell} = \mu(-b\hat{x}) \times (-\hat{y} + \hat{z}) v_0 = \mu b v_0 \, \hat{x} \times (\hat{y} - \hat{z}) = \boxed{\mu b v_0 (0, 1, 1)}$

(c) Total energy in CM frame $E = \frac{1}{2}\mu \dot{\vec{r}}(0)^2 - \frac{\gamma}{b} = \mu v_0^2 - \gamma/b$; $\gamma = G m_1 m_2 = GM\mu$

Eq. (8.52) & Eq. (8.58) $\Rightarrow$ $E = -\gamma/(2a)$

Exact Kepler's 3rd: $\tau = 2\pi\sqrt{a^3 \mu/\gamma} = 2\pi \sqrt{\frac{\mu}{\gamma}}\left(\frac{-\gamma}{2E}\right)^{3/2}$

$$\boxed{\tau = \pi\sqrt{\frac{\mu}{2\gamma}}\left(\frac{\gamma}{\frac{\gamma}{b} - \mu v_0^2}\right)^{3/2}}$$

Note: $\tau \to \infty$ as $v_0^2 \to \frac{\gamma}{b\mu}$

(d) $E = \frac{\gamma}{2c}(\varepsilon^2 - 1) \Rightarrow \varepsilon^2 = 1 + \frac{2c}{\gamma}E = 1 + \frac{2c}{\gamma}\mu v_0^2 - \frac{2c}{b}$, where $c = \frac{\ell^2}{\gamma\mu}$, $\boxed{c = \frac{2b^2 v_0^2}{GM}}$

$= 1 + c\left(\frac{2\mu v_0^2}{\gamma}\right) - \frac{2c}{b} = 1 + \frac{c^2}{b^2} - 2\frac{c}{b}$

$= \left(1 - \frac{c}{b}\right)^2 \Rightarrow \boxed{\varepsilon = \left|1 - \frac{c}{b}\right|}$

Students do not have to get this simple formula for $\varepsilon$

$$\boxed{r_{min, max} = \frac{c}{1 \pm \varepsilon}}$$

where $c$ and $\varepsilon$ are given above

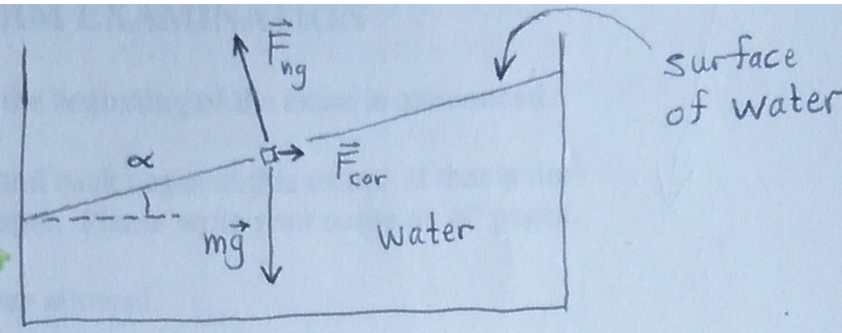
Note one of these is b. If $c > b$ then $\varepsilon = \frac{c}{b} - 1$, $r_{min} = b$

If $c < b$ then $\varepsilon = 1 - \frac{c}{b}$, $r_{max} = b$

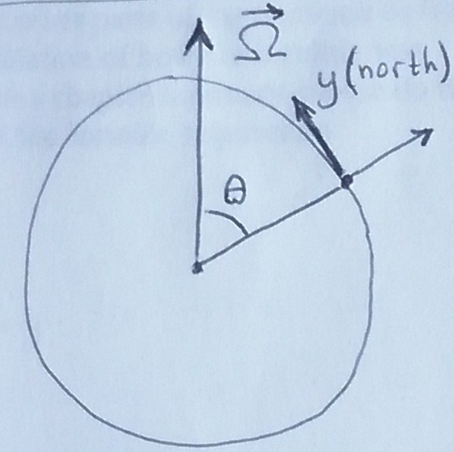
$m$ = mass of small parcel of water

$\vec{F}_{ng}$ = non-grav. force

= buoyant force




This view is looking in the direction the water is flowing




$\hat{x}$ points east

Define x, y, and z axes as in Fig 9.15

$$\vec{\Omega} = \Omega(0, \sin\theta, \cos\theta)$$

$$\vec{v} = (0, v, 0)$$

$$\vec{v} \times \vec{\Omega} = v\Omega\cos\theta\,\hat{x}$$

$$\vec{F}_{cor} = 2mv\Omega\cos\theta\,\hat{x}$$

$$\tan(\alpha) = \frac{|\vec{F}_{cor}|}{mg} = \frac{2v\Omega\cos\theta}{g}$$

height difference = $w\tan\alpha$ = $\boxed{2vw\Omega\cos(\theta)/g}$

$\vec{g}$ includes centrifugal force; see Ch. 9 Summary

East bank is higher because $\vec{F}_{cor}$ points east.

## Problem 5

(A) At $t=0$, $\overleftrightarrow{I} = \frac{1}{2}md^2\begin{pmatrix} 21 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Thus $\boxed{\vec{L} = 9md^2\omega\hat{y}}$

(B) The molecule has been rotated a quarter turn about the y axis:

$$\boxed{\begin{array}{l} O: \left(\frac{d}{2}, 0, 0\right) \\ C: \left(-\frac{d}{2}, 0, 0\right) \\ H: \left(-d, \pm\frac{\sqrt{3}}{2}d, 0\right) \end{array}}$$

oxygen

$$\boxed{\overleftrightarrow{I} = \frac{1}{2}md^2\begin{pmatrix} 3 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 21 \end{pmatrix}}$$

(C) Now it's an eighth of a turn.

$$\boxed{\begin{array}{l} O: \frac{d}{2}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \\ C: -\frac{d}{2}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \\ H: \left(\frac{-d}{\sqrt{2}}, \pm\frac{d\sqrt{3}}{2}, \frac{-d}{\sqrt{2}}\right) \end{array}}$$

The fundamental form $r^2\delta_{ij} - r_i r_j$

$$\left(\frac{d}{2}\right)^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \left(\frac{d}{2}\right)^2\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\left(\frac{7}{4}d^2\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \left(\frac{d}{2}\right)^2 \boxed{\begin{matrix} -\sqrt{2} \\ \sqrt{3} \\ -\sqrt{2} \end{matrix}} \boxed{\begin{matrix} -\sqrt{2} & \sqrt{3} & -\sqrt{2} \end{matrix}}$$

← 1×3 matrix

← 3×1 matrix

for the "+" case

$$= \left(\frac{7}{4}d^2\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{d^2}{4}\begin{pmatrix} 2 & -\sqrt{6} & 2 \\ -\sqrt{6} & 3 & -\sqrt{6} \\ 2 & -\sqrt{6} & 2 \end{pmatrix}$$

For the "−" case there is a sign flip for $\sqrt{6}$

$$\overleftrightarrow{I} = (16m + 12m)\frac{d^2}{8}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} + 2m\left(\frac{d^2}{4}\right)\begin{pmatrix} 5 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{pmatrix}$$

$$\boxed{\overleftrightarrow{I} = \frac{1}{2}md^2\begin{pmatrix} 12 & 0 & -9 \\ 0 & 18 & 0 \\ -9 & 0 & 12 \end{pmatrix}}$$

Same determinant and trace as in (B)

## Problem 6

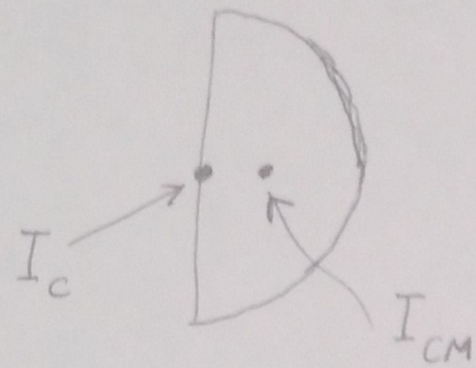
surface mass density

(a) $M = \int_0^R 2\sqrt{R^2-x^2}\,\sigma\,dx$

$h = \frac{1}{M}\int_0^R 2x\sqrt{R^2-x^2}\,\sigma\,dx \Rightarrow \boxed{h = \frac{4R}{3\pi}}$

(b) $I_c = \frac{1}{2}\left(\frac{1}{2}(2M)R^2\right)$

$= \frac{1}{2}MR^2$




$I_c = I_{cm} + Mh^2$

$\boxed{I_{cm} = \frac{1}{2}MR^2 - Mh^2}$

(c) Introduce $\phi_1$ and $\phi_2$ as on p. 432

Compute $T$ for $\phi_1 = \phi_2 = 0$, $\dot\phi_1$ and $\dot\phi_2$ nonzero:

$T = \frac{1}{2}M\left(b\dot\phi_1 + c\dot\phi_2\right)^2 + \frac{1}{2}I_{CM}\dot\phi_2^2$, $\boxed{c = \sqrt{R^2+h^2}}$

$\overleftrightarrow{M} = \begin{pmatrix} Mb^2 & Mbc \\ Mbc & Mc^2 + I_{CM} \end{pmatrix}$, $\overleftrightarrow{K} = \begin{pmatrix} Mgb & 0 \\ 0 & Mgc \end{pmatrix}$ See Section 11.5

$\det\left(\overleftrightarrow{K} - \omega^2\overleftrightarrow{M}\right) = 0 \Rightarrow A\omega^4 + B\omega^2 + C = 0$

with $\boxed{A = bI_{cm},\ B = -gI_{cm} - bcgM - c^2gM,\ C = cg^2M}$

Solutions: $\boxed{\omega^2 = \frac{-B \pm \sqrt{B^2-4AC}}{2A}}$