Prob 1 $\quad \vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin(\theta)\dot{\phi}\,\hat{\phi}$

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 + (r\dot{\phi}\sin\theta)^2$$

$$\boxed{\mathcal{L}(r,\theta,\phi,\dot{r},\dot{\theta},\dot{\phi}) = T - U = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) - U(r)}$$

The r Equation

$$\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 + mr\dot{\phi}^2\sin^2\theta - U'(r)$$

(For this prob, U is a function of r only)

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$

$$mr\dot{\theta}^2 + mr\dot{\phi}^2\sin^2\theta - U'(r) = m\ddot{r}$$

$$\boxed{F_r = m\left(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right)}$$

$$\boxed{F_r = m a_r}$$

radial acceleration $= \vec{a}\cdot\hat{r}$

The θ Equation

$$\frac{\partial \mathcal{L}}{\partial \theta} = mr^2\dot{\phi}^2\sin\theta\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$mr^2\dot{\phi}^2\sin\theta\cos\theta = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$\boxed{0 = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta}$$

$$\boxed{0 = \vec{a}\cdot\hat{\theta}}$$

The φ Equation

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta = m(r\sin\theta)^2\dot{\phi}$$

Note: $\hat{z}\cdot(\vec{r}\times\vec{v}) = \vec{v}\cdot(\hat{z}\times\vec{r}) = \vec{v}\cdot(r\sin\theta\,\hat{\phi}) = (r\sin\theta)^2\dot{\phi}$

$$\left.\right\} \boxed{L_z = \text{Const}}$$

(z-component of angular momentum)

2 (a)

$$\left.\begin{aligned} q_1 &= \sqrt{x^2+y^2} \\ q_2 &= \sqrt{(x-a)^2+y^2} \end{aligned}\right\} \qquad x = \frac{q_1^2 - q_2^2 + a^2}{2a}$$

$$y = \sqrt{q_1^2 - \underbrace{x^2}_{x(q_1,q_2)}} = y(q_1, q_2)$$

part (b)

$$\boxed{U(q_1,q_2) = -mgy(q_1,q_2)}$$

$$\dot{x} = \frac{1}{a}\left(q_1 \dot{q}_1 - q_2 \dot{q}_2\right), \quad \dot{y} = \frac{\partial y}{\partial q_1}\dot{q}_1 + \frac{\partial y}{\partial q_2}\dot{q}_2$$

These are functions of (q_1, q_2). Evaluate later.

$$\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{m}{2}\left[\frac{1}{a^2}\left(q_1\dot{q}_1 - q_2\dot{q}_2\right)^2 + \left(\frac{\partial y}{\partial q_1}\dot{q}_1 + \frac{\partial y}{\partial q_2}\dot{q}_2\right)^2\right] - U(q_1, q_2)$$

$$\frac{\partial y}{\partial q_1} = \frac{1}{2}\frac{1}{y}\left(2q_1 - 2x\frac{\partial x}{\partial q_1}\right) = \frac{q_1 - x\overbrace{\frac{\partial x}{\partial q_1}}^{\frac{q_1}{a}}}{y}$$

$$\frac{\partial y}{\partial q_2} = \frac{1}{2}\frac{1}{y}\left(-2x\frac{\partial x}{\partial q_2}\right) = -\frac{x}{y}\overbrace{\frac{\partial x}{\partial q_2}}^{-\frac{q_2}{a}}$$

$$\boxed{\begin{aligned} \frac{\partial y}{\partial q_1} &= \frac{q_1 - x q_1/a}{y} \\ \frac{\partial y}{\partial q_2} &= \frac{x\, q_2}{y\, a} \end{aligned}}$$

Problem 3

(a) $\boxed{T = \frac{1}{8} m a^2 (\cosh^2\mu - \cos^2\nu)(\dot{\mu}^2 + \dot{\nu}^2)}$

(b) $$\boxed{\begin{aligned} q_1 &= \frac{a}{2}(\cosh\mu + \cos\nu) \\ q_2 &= \frac{a}{2}(\cosh\mu - \cos\nu) \end{aligned}}$$

Taylor 7.8

Part a

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k\left(x_1 - x_2 - l\right)^2$$

Part b

Letting $X = \frac{1}{2}\left(x_1 + x_2\right)$ and $x = x_1 - x_2 - l$, then:

$$\begin{aligned}\dot{X}^2 + \frac{1}{4}\dot{x}^2 &= \left[\frac{1}{2}\left(\dot{x}_1 + \dot{x}_2\right)\right]^2 + \frac{1}{4}\left[\dot{x}_1 - \dot{x}_2\right]^2 \\ &= \frac{1}{4}\left[\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2\right] + \frac{1}{4}\left[\dot{x}_1^2 - 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2\right] \\ &= \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2\end{aligned}$$

Substituting:

$$\mathcal{L} = m\dot{X}^2 + \frac{1}{4}m\dot{x}^2 - \frac{1}{2}kx^2$$

Then Lagrange's equations of motion are:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial X} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} = 0 - \frac{d}{dt}2m\dot{X} = -2m\ddot{X} \\
0 &= \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = -kx - \frac{d}{dt}\frac{1}{2}m\dot{x} = -kx - \frac{1}{2}m\ddot{x}
\end{aligned}$$

2.3 Part c

Let $\omega^2 = \frac{2k}{m}$, then:

$$\begin{aligned}
X(t) &= X(0) + \dot{X}(0)\,t \\
x(t) &= A\cos(\omega t - \delta)
\end{aligned}$$

The first equation says that X (the center of mass) moves with constant velocity, while the second equation says that x (the extension of the spring relative to equilibrium) oscillates.

Taylor 7.14

Because the yoyo rolls around the string without slipping, $\dot{x} = \omega R$. The moment of inertia of a cylinder is $I = \frac{1}{2}mR^2$, thus the Lagrangian is:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 - mg\left(-x\right) \\ &= \frac{3}{4}m\dot{x}^2 + mgx\end{aligned}$$

Therefore, Lagrange's equation of motion is:

$$\begin{aligned}0 &= \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\ &= mg - \frac{d}{dt}\frac{3}{2}m\dot{x} \\ &= mg - \frac{3}{2}m\ddot{x}\end{aligned}$$

Hence the downwards acceleration is $\ddot{x} = \frac{2}{3}g$.

Taylor 7.33

Say the motion is in the yz-plane with z vertical, then the position and the velocity are:

$$\begin{aligned}
\vec{r} &= x\cos\omega t\,\hat{y} + x\sin\omega t\,\hat{z} \\
\dot{\vec{r}} &= (\dot{x}\cos\omega t - \omega x\sin\omega t)\,\hat{y} + (\dot{x}\sin\omega t + \omega x\cos\omega t)\,\hat{z}
\end{aligned}$$

Therefore, the kinetic energy is:

$$\begin{aligned}
T &= \frac{1}{2}m\left\|\dot{\vec{r}}\right\|^2 \\
&= \frac{1}{2}m\Big[\left(\dot{x}^2\cos^2\omega t - 2\omega x\dot{x}\cos\omega t\sin\omega t + \omega^2x^2\sin^2\omega t\right) \\
&\qquad + \left(\dot{x}^2\sin^2\omega t + 2\omega x\dot{x}\cos\omega t\sin\omega t + \omega^2x^2\cos^2\omega t\right)\Big] \\
&= \frac{1}{2}m\left[\dot{x}^2 + \omega^2x^2\right]
\end{aligned}$$

Thus the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\left[\dot{x}^2 + \omega^2x^2\right] - mgx\sin\omega t$$

Lagrange's equation of motion become:

$$\begin{aligned}
0 &= \frac{\partial\mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \\
&= \left[m\omega^2x - mg\sin\omega t\right] - \frac{d}{dt}\left[m\dot{x}\right] \\
&= m\omega^2x - mg\sin\omega t - m\ddot{x}
\end{aligned}$$

Or, rewriting:

$$\ddot{x} - \omega^2x = -g\sin\omega t$$

The general solution of the homogeneous differential equation is $x_h = C_-e^{-\omega t} + C_+e^{\omega t}$. For a particular solution, guess the ansatz $x_p = A\sin\omega t$, then:

$$-\omega^2A\sin\omega t - \omega^2A\sin\omega t = -g\sin\omega t$$

Thus $A = \frac{g}{2\omega^2}$, which yields the general solution:

$$x = \frac{g}{2\omega^2}\sin\omega t + C_- e^{-\omega t} + C_+ e^{\omega t}$$

The initial conditions are $x(0) = x_0$ and $\dot{x}(0) = 0$, and substituting:

$$\begin{aligned} x_0 &= C_- + C_+ \\ 0 &= \frac{g}{2\omega} - \omega C_- + \omega C_+ \end{aligned}$$

The solution is $C_+ = \frac{x_0}{2} - \frac{g}{4\omega^2}$ and $C_- = \frac{x_0}{2} + \frac{g}{4\omega^2}$. Therefore:

$$\begin{aligned} x &= \frac{g}{2\omega^2}\sin\omega t + \left(\frac{x_0}{2} + \frac{g}{4\omega^2}\right)e^{-\omega t} + \left(\frac{x_0}{2} - \frac{g}{4\omega^2}\right)e^{\omega t} \\ &= \frac{g}{2\omega^2}\sin\omega t + x_0\cosh\omega t - \frac{g}{2\omega^2}\sinh\omega t \end{aligned}$$