**Physics 110A, Spring 2021**
**Solution to Homework 7**
**GSI: Yi-Chuan Lu**

1. In the discussion, we have shown

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int_{\mathcal{V}}\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) r'^2 \rho\left(\mathbf{r}'\right) d\tau' = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{Q}\cdot\hat{\mathbf{r}}}{r^3}, \tag{1}$$

where

$$\mathbf{Q} = \int_{\mathcal{V}}\left[\frac{3}{2}\begin{pmatrix} | \\ \mathbf{r} \\ | \end{pmatrix}\left(-\mathbf{r}-\right) - \frac{r^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\rho\left(\mathbf{r}\right) d\tau. \tag{2}$$

In component form, equation (1) reads

$$V_{\text{quad}} = \boxed{\frac{1}{4\pi\epsilon_0}\sum_{i,j=1}^{3}\frac{\hat{r}_i Q_{ij}\hat{r}_j}{r^3},}$$

and if we express every element in equation (2) explicitly, we get

$$\begin{aligned}
\mathbf{Q} &= \int_{\mathcal{V}}\left[\frac{3}{2}\begin{pmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{pmatrix} - \frac{x^2+y^2+z^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\rho\left(\mathbf{r}\right) d\tau \\
&= \int_{\mathcal{V}}\frac{1}{2}\left[\begin{pmatrix} 2x^2-y^2-z^2 & 3xy & 3xz \\ 3yx & 2y^2-x^2-z^2 & 3yz \\ 3zx & 3zy & 2z^2-x^2-y^2 \end{pmatrix}\right]\rho\left(\mathbf{r}\right) d\tau.
\end{aligned}$$

From the first line, the $(i,j)$ component is $Q_{ij} = \boxed{\int_{\mathcal{V}}\left(\frac{3}{2}r_i r_j - \frac{r^2}{2}\delta_{ij}\right)\rho\left(\mathbf{r}\right) d\tau,}$ and from the second line, the trace is clearly $\boxed{\text{zero.}}$

2. When the origin shifts by a distance $\mathbf{a}$, the position $\mathbf{r}$ changes to $\bar{\mathbf{r}} \equiv \mathbf{r} - \mathbf{a}$. The functional form of the charge density $\rho$ changes to $\bar{\rho}$ in such a way that $\rho\left(\mathbf{r}\right) = \bar{\rho}\left(\bar{\mathbf{r}}\right)$, and $d\tau = d\bar{\tau}$. Therefore the quadrupole changes as following:

$$\begin{aligned}
\bar{\mathbf{Q}} &= \int_{\mathcal{V}}\left(\frac{3}{2}\bar{\mathbf{r}}\bar{\mathbf{r}} - \frac{\bar{r}^2}{2}\mathbf{1}\right)\bar{\rho}\left(\bar{\mathbf{r}}\right) d\bar{\tau} \\
&= \int_{\mathcal{V}}\left[\frac{3}{2}\left(\mathbf{r}-\mathbf{a}\right)\left(\mathbf{r}-\mathbf{a}\right) - \frac{r^2 - 2\mathbf{r}\cdot\mathbf{a} + a^2}{2}\mathbf{1}\right]\rho\left(\mathbf{r}\right) d\tau \\
&= \int_{\mathcal{V}}\left(\frac{3}{2}\mathbf{r}\mathbf{r} - \frac{r^2}{2}\mathbf{1}\right)\rho d\tau + \int_{\mathcal{V}}\left[-\frac{3}{2}\left(\mathbf{r}\mathbf{a}+\mathbf{a}\mathbf{r}\right) + \left(\mathbf{r}\cdot\mathbf{a}\right)\mathbf{1}\right]\rho d\tau + \int_{\mathcal{V}}\left(\frac{3}{2}\mathbf{a}\mathbf{a} - \frac{a^2}{2}\mathbf{1}\right)\rho d\tau \\
&= \mathbf{Q} + \left[-\frac{3}{2}\left(\mathbf{p}\mathbf{a}+\mathbf{a}\mathbf{p}\right) + \left(\mathbf{p}\cdot\mathbf{a}\right)\mathbf{1}\right] + \left(\frac{3}{2}\mathbf{a}\mathbf{a} - \frac{a^2}{2}\mathbf{1}\right) q,
\end{aligned}$$

so if both the dipole $\mathbf{p}$ and the monopole $q$ vanish, $\boxed{\bar{\mathbf{Q}} = \mathbf{Q}.}$ Notation: in the derivation, when two vectors are put together such as $\mathbf{ra}$, the left vector $\mathbf{r}$ is a row vector and the right $\mathbf{a}$ is a column, and the product $\mathbf{ra}$ is a 3 by 3 matrix; on the contrary, $\mathbf{r}\cdot\mathbf{a}$ is the ordinary dot product and is a scalar. $\mathbf{1}$ stands for the identity matrix.

3. The monopole is simply the total charge on the disk $q = \sigma\pi R^2$. Since the charge is confined on the two-dimensional disk, any source point vector can be written as $\mathbf{r} = s(\cos\phi, \sin\phi, 0)$ for $0 \le s \le R$ and $0 \le \phi \le 2\pi$. Therefore, the dipole is

$$\mathbf{p} = \int_{\text{disk}} \mathbf{r}\sigma da = \sigma \int_0^R \int_0^{2\pi} s(\cos\phi, \sin\phi, 0)\, s d\phi ds = (0,0,0) = \mathbf{0},$$

and the quadrupole is

$$\begin{aligned}
\mathbf{Q} &= \int_{\text{disk}} \left(\frac{3}{2}\mathbf{rr} - \frac{r^2}{2}\mathbf{1}\right)\sigma da \\
&= \sigma \int_0^R \int_0^{2\pi} \left[\frac{3}{2}s^2 \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \end{pmatrix} - \frac{s^2}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] s d\phi ds \\
&= \sigma \int_0^{2\pi} \left[\frac{1}{2}\begin{pmatrix} 3\cos^2\phi - 1 & 3\sin\phi\cos\phi & 0 \\ 3\sin\phi\cos\phi & 3\sin^2\phi - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right] d\phi \int_0^R s^3 ds \\
&= \frac{\sigma\pi R^4}{8}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned}$$

Notice that the trace of $Q$ is indeed zero. Once we have $\mathbf{Q}$, we use $\hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ to compute $\hat{\mathbf{r}} \cdot \mathbf{Q} \cdot \hat{\mathbf{r}}$ :

$$\begin{aligned}
\hat{\mathbf{r}} \cdot \mathbf{Q} \cdot \hat{\mathbf{r}} &= \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \frac{\sigma\pi R^4}{8}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \\
&= -\frac{\sigma\pi R^4}{4}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right).
\end{aligned}$$

So our final answer is

$$\begin{aligned}
V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{r^2} + \frac{\hat{\mathbf{r}} \cdot \mathbf{Q} \cdot \hat{\mathbf{r}}}{r^3} + \cdots\right) \\
&= \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{\sigma\pi R^2}{r} + \frac{0}{r^2} - \frac{\sigma\pi R^4}{4r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots\right]}.
\end{aligned}$$