Phys 105, Spring 2021, Midterm 1

Problem 1

(a) Eq. (7.34) $\left\{\begin{aligned} x &= X(t) + L\sin\phi \\ y &= -L\cos\phi \end{aligned}\right.$ ("plus const")

$$\dot{x} = at + L\cos\phi\,\dot{\phi}$$

$$\dot{y} = L\sin\phi\,\dot{\phi}$$

$(-1)^2$

$$\boxed{\mathcal{L}(\phi,\dot{\phi},t) =}\ \boxed{\mathcal{L} = \tfrac{1}{2}m\left(a^2t^2 + L^2\dot{\phi}^2 + 2atL\cos\phi\,\dot{\phi}\right) + mgL\cos\phi}$$

(b) $$\frac{\partial\mathcal{L}}{\partial\phi} = -maLt\sin\phi\,\dot{\phi} - mgL\sin\phi$$

equal

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mL^2\dot{\phi} + maLt\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = mL^2\ddot{\phi} - maLt\sin\phi\,\dot{\phi} + maL\cos\phi$$

$$mL^2\ddot{\phi} = -mgL\sin\phi - maL\cos\phi$$

$$\boxed{L\ddot{\phi} = -g\sin\phi - a\cos\phi}$$

Using the hint

(c) $\ddot{\phi} = 0 \Rightarrow a\cos\phi_0 = -g\sin\phi_0 \Rightarrow \boxed{\phi_0 = -\arctan\left(\frac{a}{g}\right)}$

(d) $\phi(t) = \phi_0 + \varepsilon(t) \Rightarrow L\ddot{\varepsilon} = -g\cos\phi_0\,\varepsilon - a(-\sin\phi_0)\,\varepsilon$

$$= -\cos\phi_0\,\varepsilon\left[g - a\tan\phi_0\right]$$

$\sin^2 + \cos^2 = 1$

$\tan^2 + 1 = (\cos)^{-2}$

$((a/g)^2 + 1)^{-1} = \cos^2$

$$= -\varepsilon\left[\frac{1}{\sqrt{a^2g^{-2}+1}}\left(g + a(a/g)\right)\right]$$

$$= -\varepsilon\sqrt{a^2+g^2}$$

$\omega^2 = L^{-1}\sqrt{a^2+g^2}$ $\quad\boxed{\tau = 2\pi\sqrt{\frac{L}{\sqrt{a^2+g^2}}}}$

> Optional Remark "Power = force times velocity"
> dE/dt = rate at which the external agent is doing work on the system.
> Key concepts are $V_{cart} = at$ and computing the tension in the rod.

1(e) $E = T + U$ [For T and U, see part (a)]

$$\frac{dE}{dt} = ma^2 t + mL^2 \dot{\phi}\ddot{\phi} + \underbrace{\frac{d}{dt}\left(maLt\cos\phi\,\dot{\phi}\right)}_{maL\cos\phi\,\dot{\phi} + maLt\cos\phi\,\ddot{\phi} - maLt\sin\phi\,\dot{\phi}^2} + mgL\sin\phi\,\dot{\phi}$$

Main idea: Compute dE/dt, then use EOM to eliminate $\ddot{\phi}$

$$\frac{dE}{dt} = ma^2 t + maLt\left(\cos\phi\,\ddot{\phi} - \sin\phi\,\dot{\phi}^2\right)$$

$$= mat\underbrace{\left[a + \cos\phi\left(-g\sin\phi - a\cos\phi\right) - L\sin\phi\,\dot{\phi}^2\right]}$$

This is not zero. If it were, then $\dot{\phi}$ would be a function of ϕ independent of the initial conditions.

1(f) **NOTE**: The problem was unclear. The "kinetic energy" should be $\frac{1}{2}mL^2\dot{\phi}^2$ which is the kinetic energy relative to the cart.

$$C = \frac{1}{2}mL^2\dot{\phi}^2 + A\cos\phi + B\sin\phi$$

$$\frac{dC}{dt} = mL^2\dot{\phi}\ddot{\phi} - A\sin\phi\,\dot{\phi} + B\cos\phi\,\dot{\phi}$$

$$= mL\dot{\phi}\left[L\ddot{\phi} - \underbrace{\frac{A}{mL}}_{g}\sin\phi + \underbrace{\frac{B}{mL}}_{a}\cos\phi\right]$$

$$\boxed{A = -mLg, \quad B = mLa}$$

Problem 2

(a) Eq (7.34): $\vec{r} = (x, y, z) = (x, y, c_x x + c_y y)$

$$\dot{\vec{r}} = (\dot{x}, \dot{y}, c_x \dot{x} + c_y \dot{y})$$

$$\boxed{T = \frac{1}{2} m \left[\dot{x}^2 + \dot{y}^2 + (c_x \dot{x} + c_y \dot{y})^2 \right] \quad U = mg(c_x x + c_y y), \quad \mathcal{L} = T - U}$$

(b) $\frac{\partial \mathcal{L}}{\partial x} = -mg c_x$, $\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + m(c_x \dot{x} + c_y \dot{y}) c_x$

$\frac{\partial \mathcal{L}}{\partial y} = -mg c_y$, $\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y} + m(c_x \dot{x} + c_y \dot{y}) c_y$

$$\boxed{-g c_x = (1 + c_x^2)\ddot{x} + c_x c_y \ddot{y}, \quad -g c_y = (1 + c_y^2)\ddot{y} + c_x c_y \ddot{x}}$$

(c) Define $\boxed{(a_x, a_y) = -g(c_x, c_y)/(1 + c_x^2 + c_y^2)}$

Then $\boxed{\begin{aligned} x(t) &= x_0 + V_{x0} t + \tfrac{1}{2} a_x t^2 \\ y(t) &= y_0 + V_{y0} t + \tfrac{1}{2} a_y t^2 \end{aligned}}$

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(d) Yes because $\partial \mathcal{L}/\partial t = 0$ and the coordinates are natural.

(e) $dC/dt = (A, B) \cdot (a_x, a_y)$ so choose (A, B) perpendicular to (c_x, c_y). One choice is $\boxed{A = -c_y, \; B = c_x}$

(f) $x = q_1^{1/3} \Rightarrow \dot{x} = \frac{1}{3} q_1^{-2/3} \dot{q}_1$, $y = q_2^{1/5} \Rightarrow \dot{y} = \frac{1}{5} q_2^{-4/5} \dot{q}_2$

$$\boxed{\begin{aligned} \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{m}{2} &\left[\frac{1}{9} q_1^{-\frac{4}{3}} \dot{q}_1^2 + \frac{1}{25} q_2^{-\frac{8}{5}} \dot{q}_2^2 + \left(c_x q_1^{-\frac{2}{3}} \dot{q}_1/3 + c_y q_2^{-\frac{4}{5}} \dot{q}_2/5 \right)^2 \right] \\ &- mg\left(c_x q_1^{1/3} + c_y q_2^{1/5} \right) \end{aligned}}$$