Homework 10 Problem 1

Tuesday, April 17, 2018 9:04 PM

(a) The kinetic energy

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m(\dot{x}^2 + h'(x)^2\dot{x}^2)$$

Potential energy

$$V = mgh(x)$$

The Lagrangian

$$L = T - V = \frac{1}{2}m(1+h'(x)^2)\dot{x}^2 - mgh(x)$$

The canonical momentum

$$p = \frac{\partial L}{\partial \dot{x}} = m(1+h'(x)^2)\dot{x}$$

The Hamiltonian is of the form.

$$H = p\dot{x} - L = \frac{p^2}{2m(1+h'(x)^2)} + mgh(x)$$

(b) Hamilton's canonical equations

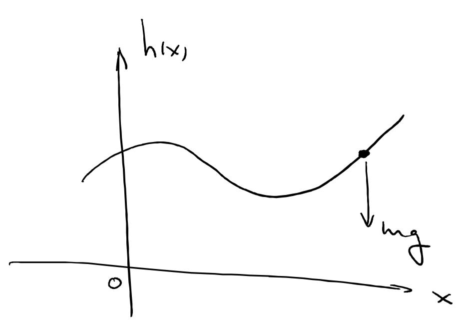
$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m(1+h'(x)^2)} \quad \cdots (1)$$

$$-\dot{p} = \frac{\partial H}{\partial x} = -\frac{p^2}{m(1+h'(x)^2)^2}h'(x)h''(x) + mgh'(x) \quad \cdots (2)$$

Using Newton's 2nd law

$$F_{tang} = m\dot{v}$$

where v is the velocity of the roller coaster along the tangential direction.



... along the tangential direction.

$$v = \pm\sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{1+h'(x)^2}\ \dot{x}$$

$$\dot{v} = \sqrt{1+h'(x)^2}\ \ddot{x} + \frac{h'(x)h''(x)}{\sqrt{1+h'(x)^2}}\ \dot{x}^2$$

$$F_{tang} = -mg\sin\theta \qquad \text{where} \qquad \tan\theta = h'(x)$$

$$\therefore -mg\frac{h'(x)}{\sqrt{1+h'(x)^2}} = m\sqrt{1+h'(x)^2}\ \ddot{x} + \frac{m h''(x) h'(x)}{\sqrt{1+h'(x)^2}}\ \dot{x}^2$$

$$\Rightarrow \quad -mgh'(x) = m\left(1+h'(x)^2\right)\ddot{x} + m h''(x) h'(x)\dot{x}^2 \qquad \cdots (3)$$

From (1) & (2)

$$-\frac{d}{dt}\left(m(1+h'(x)^2)\ \dot{x}\right) = -m\dot{x}^2 h'(x)h''(x) + mgh'(x)$$

$$-m(1+h'(x)^2)\ddot{x} - 2mh'(x)h''(x)\dot{x}^2 = -m\dot{x}^2 h'(x)h''(x) + mgh'(x)$$

$$\Rightarrow \quad m(1+h'(x)^2)\ddot{x} + mh'(x)h''(x)\dot{x}^2 + mgh'(x) = 0$$

which is equivalent to eq (3).

Thus, the Hamilton's eq and Newton's 2nd law give the same result.

Phys. 105, Homework 10, Prob. 2

$$\vec{p} = m\overleftrightarrow{N}\dot{\vec{q}}$$

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = m \underbrace{\begin{pmatrix} 1+c_x^2 & c_x c_y \\ c_x c_y & 1+c_y^2 \end{pmatrix}}_{\text{Define this to be } \overleftrightarrow{N}} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Note: $\det \overleftrightarrow{N} = 1 + c_x^2 + c_y^2$

$$\overleftrightarrow{N}^{-1} = \frac{1}{\det \overleftrightarrow{N}} \begin{pmatrix} 1+c_y^2 & -c_x c_y \\ -c_x c_y & 1+c_x^2 \end{pmatrix}$$

Eq. (7.97) $\Rightarrow T = \frac{1}{2}\vec{p}\cdot\dot{\vec{q}}$

MT solutions

Eq. (7.92) $\Rightarrow$ $\boxed{\mathcal{H}(\vec{q},\vec{p}) = \frac{1}{2m}\vec{p}\cdot\left(\overleftrightarrow{N}^{-1}\vec{p}\right) + U(\vec{q})}$ $\boxed{\vec{q} = \begin{pmatrix} x \\ y \end{pmatrix}}$

$$\frac{\partial \mathcal{H}}{\partial \vec{p}} = \frac{1}{m}\overleftrightarrow{N}^{-1}\vec{p}$$

general principle explained in lecture in the context of Eq. (11.55) $\Rightarrow$ Eq. (11.60)

$$\frac{\partial \mathcal{H}}{\partial \vec{q}} = \vec{\nabla}_q U = \frac{\partial U}{\partial \vec{q}} = mg\begin{pmatrix} c_x \\ c_y \end{pmatrix} = mg\vec{c}$$

$\boxed{\text{Hamilton's Eq.s}}$

$$\begin{cases} \dot{\vec{q}} = \dfrac{\partial \mathcal{H}}{\partial \vec{p}} & \checkmark \\ \dot{\vec{p}} = -\dfrac{\partial \mathcal{H}}{\partial \vec{q}} \Rightarrow m\overleftrightarrow{N}\ddot{\vec{q}} = -mg\vec{c} & \checkmark \end{cases}$$

Note $\overleftrightarrow{N}^{-1}\vec{c} = \frac{1}{\det \overleftrightarrow{N}}\vec{c}$

Optional note: $\overleftrightarrow{N} = \overleftrightarrow{1} + \vec{c}\,\vec{c}^{\,T} \Rightarrow \overleftrightarrow{N}\vec{c} = \overbrace{(1+\vec{c}\cdot\vec{c})}^{\det \overleftrightarrow{N}}\vec{c}$

$\vec{c}$ is an eigenvector of $\overleftrightarrow{N}$

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L = (1 / 2) (M + m) q2dot^2 + (1 / 2) m (q1dot^2 + 2 q1dot q2dot Cos[α]) + m g q1 Sin[α]
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$$\frac{1}{2}(m+M)\,\text{q2dot}^2+\frac{1}{2}m\left(\text{q1dot}^2+2\,\text{q1dot}\,\text{q2dot}\,\text{Cos}[\alpha]\right)+g\,m\,\text{q1}\,\text{Sin}[\alpha]$$

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(* part (a) *)

p1expression = D[L, q1dot]
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$$\frac{1}{2}m\,(2\,\text{q1dot}+2\,\text{q2dot}\,\text{Cos}[\alpha])$$

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p2expression = D[L, q2dot]
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$$(m+M)\,\text{q2dot}+m\,\text{q1dot}\,\text{Cos}[\alpha]$$

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(* part (b) *)

sol = Solve[(p1 == p1expression) && (p2 == p2expression), {q1dot, q2dot}]
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$$\left\{\left\{\text{q1dot}\to-\frac{m\,\text{p1}+M\,\text{p1}-m\,\text{p2}\,\text{Cos}[\alpha]}{m\left(-m-M+m\,\text{Cos}[\alpha]^2\right)},\ \text{q2dot}\to-\frac{\text{Sec}[\alpha]\,(\text{p1}-\text{p2}\,\text{Sec}[\alpha])}{-m+m\,\text{Sec}[\alpha]^2+M\,\text{Sec}[\alpha]^2}\right\}\right\}$$

```
H = (p1 q1dot + p2 q2dot - L) /. sol // First // Simplify
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$$-\frac{m\,\text{p1}^2+M\,\text{p1}^2+m\,\text{p2}^2-2\,m\,\text{p1}\,\text{p2}\,\text{Cos}[\alpha]-2\,g\,m^2\,(m+M)\,\text{q1}\,\text{Sin}[\alpha]+2\,g\,m^3\,\text{q1}\,\text{Cos}[\alpha]^2\,\text{Sin}[\alpha]}{m\,(-m-2\,M+m\,\text{Cos}[2\,\alpha])}$$

```
(* Yes H = T+U, and H is conserved, as per the theorems in Chapter 7 *)

(* part (c) *)

p1dot = -D[H, q1]
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$$\frac{-2\,g\,m^2\,(m+M)\,\text{Sin}[\alpha]+2\,g\,m^3\,\text{Cos}[\alpha]^2\,\text{Sin}[\alpha]}{m\,(-m-2\,M+m\,\text{Cos}[2\,\alpha])}$$

```
p2dot = -D[H, q2]
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$$0$$

```
q1dotExpression = D[H, p1] // FullSimplify
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$$\frac{2\,((m+M)\,\text{p1}-m\,\text{p2}\,\text{Cos}[\alpha])}{m\,(m+2\,M-m\,\text{Cos}[2\,\alpha])}$$

```
q2dotExpression = D[H, p2] // FullSimplify
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$$\frac{2\,(\text{p2}-\text{p1}\,\text{Cos}[\alpha])}{m+2\,M-m\,\text{Cos}[2\,\alpha]}$$

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(* part (d) *)

q1doubledot =
 Factor[q1dotExpression /. {p1 → p1dot, p2 → p2dot} /. Cos[2 α] -> 2 Cos[α]^2 - 1]
 (* agrees with Eq. (7.67) *)
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$$-\frac{g\,(m+M)\,\text{Sin}[\alpha]}{-m-M+m\,\text{Cos}[\alpha]^2}$$

```
q2doubledot = q2dotExpression /. {p1 → p1dot, p2 → p2dot} // FullSimplify
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$$\frac{g\ m\ \mathrm{Sin}[2\ \alpha]}{-m - 2\ M + m\ \mathrm{Cos}[2\ \alpha]}$$

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q2doubledot / q1doubledot // FullSimplify (* agrees with Eq. (7.66) *)
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$$-\frac{m\ \mathrm{Cos}[\alpha]}{m + M}$$

Phys 105, Homework 10, Problem 4

(a) $p = \dfrac{\partial \mathcal{L}}{\partial \dot\theta} = mR^2\dot\theta$

(b) $\mathcal{H} = p\dot\theta - \mathcal{L} = \dfrac{p^2}{mR^2} - \left[\dfrac{1}{2}\dfrac{p^2}{mR^2} + \dfrac{1}{2}mR^2\omega^2\sin^2\theta - mgR(1-\cos\theta)\right]$

$$\boxed{\mathcal{H}(\theta, p) = \frac{p^2}{2mR^2} - \frac{1}{2}mR^2\omega^2\sin^2\theta + mgR(1-\cos\theta)}$$

This is not $T+U$. The coordinate is not "natural" – see discussion of Eq. (7.92). $\mathcal{H}$ is conserved because $\partial\mathcal{L}/\partial t = 0$ – see gray box on page 270. For this problem $T+U$ is not conserved because the motor does work on the system.

(c)
$$\boxed{\begin{aligned} \dot\theta &= \partial\mathcal{H}/\partial p = p/(mR^2) \\ \dot p &= -\partial\mathcal{H}/\partial\theta = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta \end{aligned}}$$

(d)
$$\boxed{\begin{aligned} \theta(t) &= \theta_0 + A\cos(\Omega' t - \delta) \\ p(t) &= -mR^2\Omega' A\sin(\Omega' t - \delta) \end{aligned}}$$

A is small compared to θ_0

The $\dot\theta = \partial\mathcal{H}/\partial p$ equation is satisfied exactly. For the other equation, we have

$\dot p = -mR^2\Omega'^2 A\cos(\Omega' t - \delta)$

and the right-hand side is $mR^2\left[\omega^2\cos(\theta_0+\varepsilon) - \frac{g}{R}\right]\sin(\theta_0+\varepsilon)$ where $\varepsilon = A\cos(\Omega' t - \delta)$. We expand to first order as in Eq. (7.78) and thereafter. Thus the right-hand side is $mR^2(-\Omega'^2\varepsilon)$, which agrees with the left-hand side.

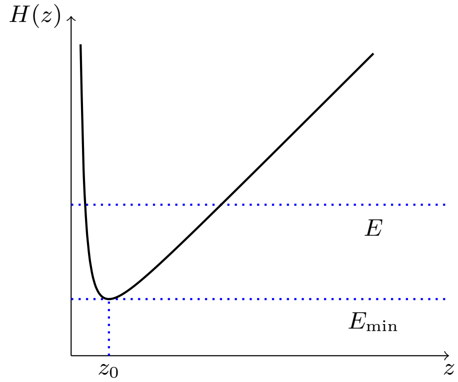

FIG. 1: There are two values of z for given energy E.

V. [TAYLOR] 13.14: MASS STRAYING ON A CONIC SURFACE

The Hamiltonian is given by [Taylor] (13.33),

$$H = \frac{1}{2m}\left[\frac{p_z^2}{(c^2+1)} + \frac{p_\phi^2}{c^2 z^2}\right] + mgz. \tag{29}$$

The Hamilton's equations for ϕ and p_ϕ are

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mc^2 z^2}, \qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0. \tag{30}$$

The second equation indicates that p_ϕ is a constant. The Hamilton's equation for z is

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m(c^2+1)}, \tag{31}$$

$\dot{z} = 0$ happens if and only if $p_z = 0$. Now take $p_z = 0$, we get

$$E \equiv H(z) = \frac{p_\phi^2}{2mc^2 z^2} + mgz. \tag{32}$$

$H(z)$ reaches its minimum when $H'(z)|_{z=z_0} = 0$, which solves

$$z_0 = \left(\frac{p_\phi^2}{gc^2 m^2}\right)^{\frac{1}{3}}. \tag{33}$$

Therefore,

$$E_{\min} = H(z_0) = \frac{3}{2}\left(\frac{mg^2 p_\phi^2}{c^2}\right)^{\frac{1}{3}} \tag{34}$$

$H(z)$ has been graphed in Fig. 1, which indicates that there usually (whenever $E > E_{\min}$) exist two values of z, one minimum and another maximum.

VI. [TAYLOR] 13.23: MODIFIED ATWOOD MACHINE

(a) Assume that the distance from the pulley to M is y_M. The total potential energy of the system is

$$U = \frac{1}{2}k(\ell_e + x - \ell_0)^2 - Mgy_M - mgy - mg(y + \ell_e + x). \tag{35}$$

Meanwhile, ℓ_0 (the original length of the spring), $\ell_e = \ell_0 + \frac{mg}{k}$ (the equilibrium length of the spring), and $y + y_M = \ell$ (the length of the rope joining the upper m and M), are all constants. Considering that $M = 2m$, we have

$$U = \left(\frac{1}{2}kx^2 + mgx + \frac{m^2g^2}{2k}\right) - Mg\ell - mg\ell_e - mgx = \frac{1}{2}kx^2 + \left(\frac{m^2g^2}{2k} - Mg\ell - mg\ell_e\right). \tag{36}$$

Discarding the constants in the parentheses, we obtain

$$U = \frac{1}{2}kx^2 \tag{37}$$

(b) The kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2}M\dot{y}_M^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m(\dot{x}+\dot{y})^2 \\ &= 2m\dot{y}^2 + m\dot{x}\dot{y} + \frac{1}{2}m\dot{x}^2. \end{aligned}$$

The Lagrangian is

$$L = T - U = \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{y} + 2m\dot{y}^2 - \frac{1}{2}kx^2. \tag{38}$$

The generalized momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m(\dot{x}+\dot{y}), \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m(\dot{x}+4\dot{y}),$$

or, equivalently,

$$\dot{x} = \frac{4p_x - p_y}{3m}, \qquad \dot{y} = -\frac{p_x - p_y}{3m}.$$

The Hamiltonian is

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} - L \\ &= \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{y} + 2m\dot{y}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2m}\left[\frac{1}{3}(p_x - p_y)^2 + p_x^2\right] + \frac{1}{2}kx^2. \end{aligned}$$

(c) The Hamilton's equations are

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{4p_x - 3p_y}{3m}, & \dot{p}_x &= -\frac{\partial H}{\partial x} = -kx; \\ \dot{y} &= \frac{\partial H}{\partial p_y} = -\frac{p_x - p_y}{3m}, & \dot{p}_y &= -\frac{\partial H}{\partial y} = 0. \end{aligned}$$

The equation of motions are

$$\ddot{x} = -\frac{4k}{3m}x, \qquad \ddot{y} = \frac{k}{3m}x. \tag{39}$$

The general solution for $x(t)$ and $y(t)$ read

$$x(t) = A\cos(\omega t + \varphi), \qquad y(t) = -\frac{1}{4}A\cos(\omega t + \varphi) + B + Ct, \tag{40}$$

with $\omega = \sqrt{\frac{4k}{3m}}$. Taking into account of the initial conditions,

$$x(t=0) = x_0, \qquad \left.\frac{dx}{dt}\right|_{t=0} = 0, \tag{41}$$

$$y(t=0) = y_0, \qquad \left.\frac{dy}{dt}\right|_{t=0} = 0, \tag{42}$$

$x(t)$ and $y(t)$ solve as

$$x(t) = x_0 \cos(\omega t), \qquad y(t) = y_0 + \frac{x_0}{4} \left[1 - \cos(\omega t)\right]. \tag{43}$$