Problem Set 9

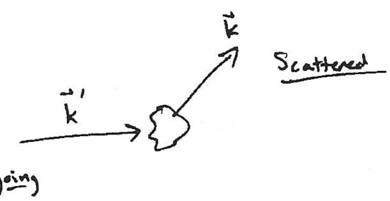
1. (Griffiths 11.12)

Yukawa potential: $V(r) = \beta \frac{e^{-\mu r}}{r}$

Since there's spherical symmetry

$$\Rightarrow \quad f(\theta) \cong -\frac{2m}{\hbar^2 \kappa} \int_0^{\infty} r \cdot \beta \frac{e^{-\mu r}}{r} \sin(\kappa r)\, dr = -\frac{2m\beta}{\hbar^2(\mu^2 + \kappa^2)}$$

where $\kappa = \vec{k}' - \vec{k}$



Further, $\kappa = 2k\sin\left(\frac{\theta}{2}\right)$

Now we need to calculate the total cross-section

$$\sigma = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi\, |f(\theta,\phi)|^2 = \int_0^{\pi} d\theta \sin\theta \cdot 2\pi \cdot \frac{4m^2\beta^2}{\hbar^4\left(\mu^2 + 4k^2\sin^2\left(\frac{\theta}{2}\right)\right)^2}$$

Note: $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos\theta)$

$$= \frac{8\pi m^2\beta^2}{\hbar^4} \int_0^{\pi} d\theta \sin\theta \cdot \left(\mu^2 + 2k^2 - 2k^2\cos\theta\right)^{-2}$$

$$= \frac{8\pi m^2\beta^2}{\hbar^4} \cdot -\frac{1}{2k^2} \cdot \frac{1}{(\mu^2 + 2k^2 - 2k^2\cos\theta)}\Bigg|_0^{\pi}$$

$$= \frac{8\pi m^2\beta^2}{\hbar^4} \cdot \frac{-1}{2k^2}\left[\frac{1}{\mu^2 + 2k^2 + 2k^2} - \frac{1}{\mu^2}\right]$$

$$= \frac{4\pi m^2\beta^2}{\hbar^4 k^2}\left(\frac{1}{\mu^2} - \frac{1}{\mu^2 + 4k^2}\right)$$

~~Finally recall that~~

$$\cancel{E = \frac{\hbar^2 k^2}{2ma^2} \Rightarrow k^2 = 2ma^2} \leftarrow$$

Recall that

$$E = \frac{\hbar^2 k^2}{2m} \Rightarrow k^2 = \frac{2mE}{\hbar^2}$$

$$\sigma = \frac{4\pi m^2 \beta^2}{\hbar^4 \cancel{k^2}} \cdot \frac{4\cancel{k^2}}{\mu^2 \cdot (\mu^2 + 4k^2)}$$

$$= \frac{4\pi m^2 \beta^2 \cdot 4}{\hbar^4 \mu^2} \cdot \frac{1}{\mu^2 + \frac{8mE}{\hbar^2}}$$

$$= \pi \cdot \left(\frac{4m\beta}{\hbar\mu}\right)^2 \cdot \frac{1}{\hbar^2\mu^2 + 8mE}$$

2. ~~(Griffiths 11.13 a, b)~~

~~Soft sphere scattering~~ $\Rightarrow V(r) = \begin{cases} V_0 & \text{for } r \le a \\ 0 & \text{for } r > a \end{cases}$

~~a)~~

b) For all energies, using spherical symmetry,

$$f(\theta) \cong -\frac{2m}{\hbar^2 K}\int_0^\infty rV(r)\sin(Kr)\,dr = -\frac{2mV_0}{\hbar^2 K}\int_0^a r\sin(Kr)\,dr$$

$$= -\frac{2mV_0}{\hbar^2 K} \cdot \frac{\partial}{\partial K}\left[\int_0^a -\cos(Kr)\,dr\right]$$

$$\cancel{= -\frac{2mV_0}{\hbar^2 K} \cdot \frac{\partial}{\partial K}\left(-\frac{\sin(Kr)}{K}\Big|_0^a\right)}$$

$$= -\frac{2mV_0}{\hbar^2 K} \cdot \frac{\partial}{\partial K}\left(-\frac{\sin(Ka)}{K}\right)$$

2. (Griffiths 11.13a,b)

a) In the low-energy approximation

$$f(\theta) \cong -\frac{m}{2\pi\hbar^2}\int V(r)\,d^3r$$

$$= -\frac{m}{2\pi\hbar^2}\int dr\cdot r^2\,\alpha\,\delta(r-a)\cdot 4\pi = -\frac{2m\alpha a^2}{\hbar^2}$$

$$\Rightarrow D = |f|^2 = \frac{4m^2\alpha^2 a^4}{\hbar^4}$$

$$\Rightarrow \sigma = 4\pi D = \frac{16\pi m^2\alpha^2 a^4}{\hbar^4}$$

b) for all energies, using spherical symmetry,

$$f(\theta) \cong -\frac{2m}{\hbar^2\kappa}\int_0^\infty rV(r)\sin(\kappa r)\,dr = -\frac{2m}{\hbar^2\kappa}\int_0^\infty r\cdot\alpha\,\delta(r-a)\sin(\kappa a)\,dr$$

$$= \boxed{-\frac{2m\alpha a}{\hbar^2\kappa}\sin(\kappa a)}$$

3. (Liboff 14.6)

This is identical to Griffiths Example 11-4, except that we're <u>not</u> assuming low energy scattering,

$$f(\theta) \cong -\frac{2m}{\hbar^2 \kappa}\int_0^\infty rV(r)\sin(\kappa r)\,dr$$

$$= -\frac{2mV_0}{\hbar^2 \kappa}\int_0^a r\sin(\kappa r)\,dr$$

$$= -\frac{2mV_0}{\hbar^2 \kappa}\frac{\partial}{\partial \kappa}\int_0^a -\cos(\kappa r)\,dr$$

$$= -\frac{2mV_0}{\hbar^2 \kappa}\frac{\partial}{\partial \kappa}\left(-\frac{\sin(\kappa r)}{\kappa}\Big|_0^a\right)$$

$$= -\frac{2mV_0}{\hbar^2 \kappa}\frac{\partial}{\partial \kappa}\left(-\frac{\sin(\kappa a)}{\kappa}\right)$$

$$= -\frac{2mV_0}{\hbar^2 \kappa}\left(\frac{\sin(\kappa a)}{\kappa^2} - \frac{a\cos(\kappa a)}{\kappa}\right)$$

$$E = \frac{\hbar^2 k^2}{2m} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

Use that

$$\kappa = 2k\sin\left(\frac{\theta}{2}\right)$$

$=$

$$= -\frac{2mV_0}{\hbar^2 \kappa^3}\left(\sin(\kappa a) - a\kappa\cos(\kappa a)\right)$$

$$\Rightarrow f(\theta) = -\frac{\cancel{2}mV_0}{\cancel{\hbar}} \cdot \frac{\hbar^{\cancel{3}}}{(2mE)^{3/2}\sin^3\left(\frac{\theta}{2}\right)\cdot \cancel{8}_4}\left(\sin\left(2a\sin\left(\frac{\theta}{2}\right)\cdot\frac{\sqrt{2mE}}{\hbar}\right) - \frac{2a\sin\left(\frac{\theta}{2}\right)\cdot\sqrt{2mE}}{\hbar}\cos\left(2a\sin\left(\frac{\theta}{2}\right)\cdot\frac{\sqrt{2mE}}{\hbar}\right)\right)$$

$$f(\theta) = -\frac{mV_0\hbar}{4(2mE)^{3/2}\sin^3\left(\frac{\theta}{2}\right)}\left[\sin\left(2a\sin\left(\frac{\theta}{2}\right)\cdot\frac{\sqrt{2mE}}{\hbar}\right) - \frac{2a\sqrt{2mE}\sin\left(\frac{\theta}{2}\right)}{\hbar}\cos\left(2a\sin\left(\frac{\theta}{2}\right)\cdot\frac{\sqrt{2mE}}{\hbar}\right)\right]$$

4. (14.7)

Use spherical symmetry again,

$$f(\theta) = +\frac{2mV_0}{\hbar^2 K}\int_0^\infty dr\, r\sin(Kr)\, e^{-\frac{r^2}{a^2}}$$