Phys 105, Reinsch, Homework 1

Prob. 5

(a) $\rho^p r^{-p} + z^p h^{-p} = 1, \qquad r, h = \text{const}$

Apply $\frac{d}{d\rho}$ to get

chain rule

$$p\rho^{p-1} r^{-p} + p z^{p-1} z' h^{-p} = 0, \qquad \text{where } z = z(\rho)$$

Apply $\frac{d}{d\rho}$ again, after dividing by p (optional)

$$(p-1)\rho^{p-2} r^{-p} + h^{-p}\left((p-1) z^{p-2} z'^2 + z^{p-1} z''\right) = 0$$

For $p > 2$, we can now say:

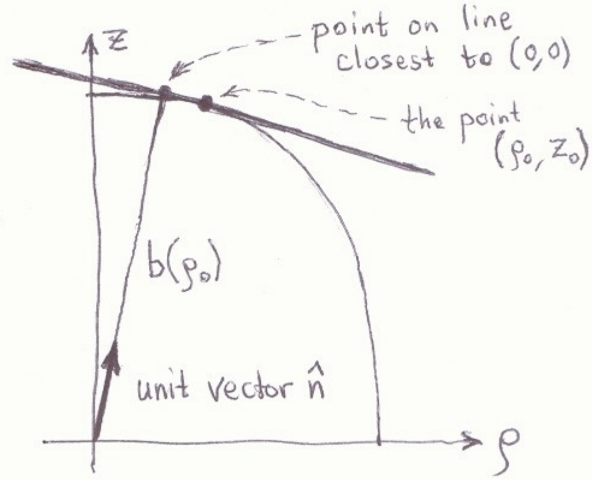
$$\left.\begin{array}{l} \text{First equation} \Rightarrow z'(0) = 0 \\ \text{Second equation} \Rightarrow z''(0) = 0 \end{array}\right\} \text{limit } \rho \to 0$$

(b) $z = h\left(1 - (\rho/r)^p\right)^{1/p} \approx h\left(1 - \frac{1}{p}(\rho/r)^p\right)$ binomial series

Thus z'' goes as ρ^{p-2} for small ρ.

(c) s = slope of tangent line

$$= z'(\rho_0) = -\left(\frac{h}{r}\right)^p \left(\frac{\rho_0}{z_0}\right)^{p-1}$$



The unit vector $\hat{n}$ is perpendicular to this line, so its slope is $-1/s$.

Define $\sigma = -1/s = \left(\frac{r}{h}\right)^p \left(\frac{z_0}{\rho_0}\right)^{p-1}$

$$\hat{n} = \frac{(1, \sigma)}{\sqrt{1+\sigma^2}}$$

$$b = \hat{n} \cdot (\rho_0, z_0) = \frac{\rho_0 + \sigma z_0}{\sqrt{1+\sigma^2}} = \frac{r^p \rho_0^{1-p}}{\sqrt{1+\sigma^2}}$$

Then use $\sigma = \frac{r}{h}\left[\left(\frac{r}{\rho_0}\right)^p - 1\right]^{1-1/p}$ to get

$$\boxed{b(\rho_0) = \left(r^{-2p}\rho_0^{2p-2} + h^{-2}\left[1 - (\rho_0/r)^p\right]^{2-2/p}\right)^{-1/2}}$$

Prob 5

(d) Look at $h^2 b^{-2}$ first

This term is between these two.

$$h^2 b^{-2} = \underbrace{\left[1-(\rho_0/r)^p\right]^{2-2/p}}_{\text{Use binomial series.}} + h^2 r^{-2p} \rho_0^{2p-2}$$

The terms are $1 + \text{Const}\cdot\rho_0^p + \text{const}\,\rho_0^{2p} + \cdots$

Therefore, to leading order we have

$$h^2 b^{-2} \approx 1 - (2-2/p)\left(\frac{\rho_0}{r}\right)^p$$

This leads to

$$\frac{b}{h} \approx \left(1 - (2-2/p)\left(\frac{\rho_0}{r}\right)^p\right)^{-1/2}$$

To leading order we get

$$\frac{b}{h} \approx 1 - \left(-\tfrac{1}{2}\right)\left(2-\tfrac{2}{p}\right)\left(\frac{\rho_0}{r}\right)^p$$

$$b(\rho_0) \approx h - h\left(\tfrac{1}{p}-1\right)\left(\frac{\rho_0}{r}\right)^p$$

$$\boxed{b(\rho_0) \approx h + h\left(\frac{p-1}{p}\right)\left(\frac{\rho_0}{r}\right)^p}$$

For $p = \frac{5}{2}$, this is $\boxed{b(\rho_0) \approx h + \frac{3}{5}h\left(\frac{\rho_0}{r}\right)^{5/2}}$

Optional note: Let θ = angle between axis of superegg and vertical

Then $\theta \propto \rho^{p-1}$

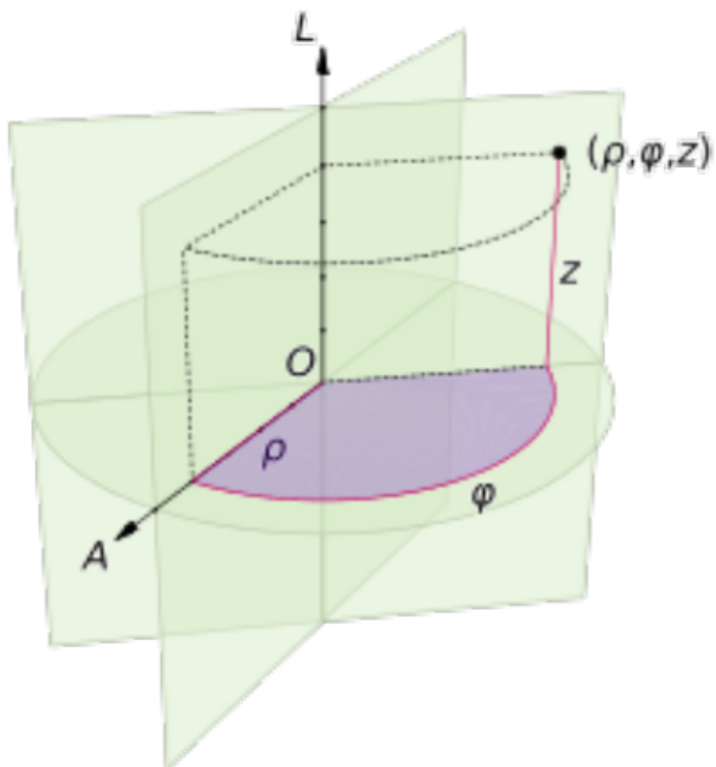
Physics 105, Spring 2021, Reinsch

Homework Assignment 1

Solutions to Problems 1 and 4

1.1 1.47

1.1.1 Part a



$$\begin{aligned} x &= \rho \cos\theta \\ y &= \rho \sin\theta \\ z &= z \end{aligned}$$

1.1.2 Part b

1.1.2 Part b

Write the unit vectors for the cylindical coordinate system in terms of the Cartesian unit vectors.

Use a test function f to describe the associated directional derivatives for each of these new unit vectors.

$$
\begin{aligned}
\frac{\partial f}{\partial \rho} &= \frac{\partial x}{\partial \rho}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \rho}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial \rho}\frac{\partial f}{\partial z} \\
&= \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial f}{\partial z} \\
&= -\rho\sin\theta\frac{\partial f}{\partial x} + \rho\cos\theta\frac{\partial f}{\partial y}
\end{aligned}
$$

We now removing the arbitrary test function, switch back to vector notation and rescale the θ one by a factor of ρ in order to make it a unit vector.

$$\begin{aligned}
\hat{\rho} &= \cos\theta\hat{x} + \sin\theta\hat{y} \\
\hat{\theta} &= -\sin\theta\hat{x} + \cos\theta\hat{y} \\
\hat{z} &= \hat{z}
\end{aligned}$$

This gives the position dependent change of frame as:

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\theta \\ A_z \end{pmatrix}$$

$$\begin{aligned}
\vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\
&= \rho(\cos\theta\hat{x} + \sin\theta\hat{y}) + z\hat{z} \\
&= \rho\hat{\rho} + z\hat{z}
\end{aligned}$$

1.1.3 Part c

$$\begin{aligned}
\dot{\hat{\rho}} &= -\sin\theta\dot{\theta}\hat{x} + \cos\theta\dot{\theta}\hat{y} \\
&= \dot{\theta}\hat{\theta} \\
\dot{\hat{\theta}} &= -\cos\theta\dot{\theta}\hat{x} - \sin\theta\dot{\theta}\hat{y} \\
&= -\dot{\theta}\hat{\rho}
\end{aligned}$$

$$
\begin{aligned}
\dot{\vec{r}} &= \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}} + \dot{z}\hat{z} + z\dot{\hat{z}} \\
&= \dot{\rho}\hat{\rho} + \rho\dot{\theta}\hat{\theta} + \dot{z}\hat{z} + 0 \\
\ddot{\vec{r}} &= \ddot{\rho}\hat{\rho} + \dot{\rho}\dot{\hat{\rho}} \\
&+ \dot{\rho}\dot{\theta}\hat{\theta} + \rho\ddot{\theta}\hat{\theta} + \rho\dot{\theta}\dot{\hat{\theta}} \\
&+ \ddot{z}\hat{z} + \dot{z}\dot{\hat{z}} \\
&= (\ddot{\rho} - \rho\dot{\theta}^2)\hat{\rho} + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\hat{\theta} + \ddot{z}\hat{z}
\end{aligned}
$$

1.2 Modified form of 5.6

We know the amplitude and frequncy of the oscillation to be $A = \frac{5x_0}{4}$ and ω respectively. Therefore we may immediately write down

$$
x(t) = \frac{5x_0}{4}\cos(\omega t - \delta)
$$

for some unknown δ. To solve for that use $x(0) = x_0$ and that it starts out approaching the origin.

$$
\begin{aligned}
x(0) &= \frac{5x_0}{4}\cos(-\delta) \\
\delta &= \pm\cos^{-1}\frac{4}{5} \\
v(t) &= -\frac{5x_0}{4}\omega\sin(\omega t - \delta) \\
v(0) &= -\frac{5x_0}{4}\omega\sin(-\delta) \\
&= \frac{5x_0}{4}\omega\sin\delta
\end{aligned}
$$

$$x(t) = \frac{5x_0}{4}\cos(\omega t - \delta)$$

for some unknown δ. To solve for that use $x(0) = x_0$ and that it starts out approaching the origin.

$$\begin{aligned}
x(0) &= \frac{5x_0}{4}\cos(-\delta) \\
\delta &= \pm\cos^{-1}\frac{4}{5} \\
v(t) &= -\frac{5x_0}{4}\omega\sin(\omega t - \delta) \\
v(0) &= -\frac{5x_0}{4}\omega\sin(-\delta) \\
&= \frac{5x_0}{4}\omega\sin\delta
\end{aligned}$$

so $v(0)$ has the same sign as δ. We want it to be negative so pick the negative solution $-\cos^{-1}\frac{4}{5}$.

$$\begin{aligned}
x(t) &= \frac{5x_0}{4}\cos(\omega t + \cos^{-1}\frac{4}{5}) \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
x(t) &= \frac{5x_0}{4}(\cos\omega t\cdot\cos\cos^{-1}\frac{4}{5} - \sin\omega t\cdot\sin\cos^{-1}\frac{4}{5}) \\
&= \frac{5x_0}{4}(\cos\omega t\frac{4}{5} - \sin\omega t\frac{3}{5}) \\
&= x_0\cos\omega t - \frac{3x_0}{4}\sin\omega t
\end{aligned}$$

Problem 2.

(2a) Taylor 2.7

Netwon's law for the motion of a particle of mass m in one dimension is:

$$F = m\ddot{x}$$

where x is our 1D position coordinate, a dot denotes a time derivative, and the force F is generically a function of the particle's position x and velocity $\dot{x}$. In this problem we consider the special case that F depends only on velocity:

$$F\left(\dot{x}\right) = m\ddot{x}$$

This is a separable equation and we can integrate it directly to find $v(t)$. First, we write it more suggestively using v for $\dot{x}$ and differential notation for the derivatives:

$$F(v) = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

Now we can rearrange and "multiply by $\mathrm{d}t$", then integrate both sides

$$\begin{aligned} \mathrm{d}t &= \frac{m}{F(v)}\,\mathrm{d}v \\ \int_0^t \mathrm{d}t' &= \int_{v(0)}^{v(t)} \frac{m}{F(v')}\,\mathrm{d}v' \end{aligned}$$

where I've added *primes* to the integration variables t and v to distinguish them from the limits of integration.

Also, it is important to remember that this business of "multiplying by $\mathrm{d}t$" is really an *abuse of notation.* It is a shorthand for a more tedious procedure of integrating over t and properly changing variables from t to v, and it will result in nonsense if naively applied to a non-separable equation, such as $\ddot{x} = x$. Yet separable equations appear often enough that it a very useful and common shorthand in physics.

Finally, do the integral on the LHS:

$$t = m\int_{v(0)}^{v(t)} \frac{\mathrm{d}v'}{F(v')}$$

This is a nice formal solution to the problem of motion due to a force $F(v)$. For any force law $F(v)$, we could use this expression to write a computer program which would calculate the velocity $v(t)$ for any particular time t. And, for particular nice force laws, we can solve the above equation *symbolically* for $v(t)$.

Perhaps the simplest force law is the constant force $F(v) = F_0$. This gives:

$$\begin{aligned} t &= m\int_{v(0)}^{v(t)} \frac{\mathrm{d}v'}{F(v')} \\ &= \frac{m}{F_0}\int_{v(0)}^{v(t)} \mathrm{d}v' \\ &= \frac{m}{F_0}\left[v(t) - v(0)\right] \\ \Rightarrow \qquad v(t) &= \frac{F_0}{m}t + v(0) \end{aligned}$$

which you should recognize as a path of constant acceleration F_0/m starting from an initial velocty $v(0)$.

(2b) Taylor 2.8

We have a force law $F(v) = -cv^{3/2}$ and we want to solve for $v(t)$:

$$\begin{aligned}
t &= m \int_{v(0)}^{v(t)} \frac{\mathrm{d}v'}{F(v')} \\
&= \frac{-m}{c} \int_{v(0)}^{v(t)} v^{-3/2}\,\mathrm{d}v \\
&= \frac{-m}{c} (-2)\, v^{-1/2} \Big|_{v_0}^{v(t)} \\
&= \frac{2m}{c} \left[\frac{1}{\sqrt{v(t)}} - \frac{1}{\sqrt{v_0}} \right] \\
\Rightarrow \quad \frac{1}{\sqrt{v(t)}} &= \frac{1}{\sqrt{v_0}} + \frac{c}{2m} t \\
\frac{1}{|v(t)|} &= \left[\frac{1}{\sqrt{v_0}} + \frac{c}{2m} t \right]^2 = \frac{1}{|v_0|} \left[1 + \frac{c\sqrt{v_0}}{2m} t \right]^2
\end{aligned}$$

Let's choose our coordinates so that $v_0 > 0$, and from the above result we see that the velocity never changes sign, so $v(t) > 0$ as well. The final result is then:

$$v(t) = \frac{v_0}{\left[1 + \frac{c\sqrt{v_0}}{2m} t \right]^2}$$

Nominally, this particle never comes to rest as the RHS is always greater than zero. But in practice, once $t \approx 2m/c\sqrt{v_0}$ this particle will have essentially reached rest. This is the time at which the particle has lost $\sim \mathcal{O}(1)$ of its initial velocity.

Problem 3.

(3a) Taylor 4.8

As the puck slides a contact force will act on the puck from the ball. The magnitude of this force can take any positive value, but it must be directed normal to the ball's surface and away from the ball. The magnitude will vary as the puck slides in order to maintain circular motion. The puck will leave the surface if there is ever a moment at which no outward-directed normal force is able to maintain circular motion.

Let's calculate the normal force needed for circular motion as a function of the puck's angular position ϕ, defined in Figure 1. The radial force acting at ϕ is:

$$F_r = N - mg\cos\phi$$

where N is the magnitude of the normal force and positive values indicate an outward force. To produce circular motion, this my be inward and have magnitude of mv^2/R, where v is the velocity at ϕ and R the ball's radius:

$$N - mg\cos\phi = -m\frac{v^2}{R} \quad \text{[for circular motion]}$$

We now need v as a function of ϕ. This can be found by conservation of energy. When the puck reaches ϕ, it has dropped through a height of $\Delta h = R(1-\cos\phi)$ (see Figure 1). So the speed at ϕ is:

$$\frac{1}{2}mv^2 = mgR(1-\cos\phi)$$
$$v^2 = 2gR(1-\cos\phi)$$

Which gives the required normal force:

$$N = mg\cos\phi - m\frac{v^2}{R} = mg\cos\phi - 2mg(1-\cos\phi) = mg[3\cos\phi - 2]$$

This is continually decreasing with ϕ. At $\cos\phi = 2/3$ it vanishes and for any larger ϕ it goes negative. But, the ball cannot produce a negative (*inward*) normal force, so at $\cos\phi = 2/3$ the puck leaves the surface.

By what vertical height has the puck fallen when it reaches $\cos\phi = 2/3$? As above, $\Delta h = R(1-\cos\phi)$ so:

$$\Delta h_{\text{liftoff}} = \frac{R}{3}$$

(3b)

After the puck leaves the ball, it will follow a parabolic trajectory due to the constant acceleration of gravity. This path starts at the point of lift-off with an initial velocity tangent to the ball's surface. From the above calculation we know that the angle of lift off, as defined in Figure 1, is given by:

$$\cos\phi = \frac{2}{3}\,,\ \sin\phi = \sqrt{1-\cos^2\phi} = \frac{\sqrt{5}}{3}$$

And this gives us the initial velocity of the puck:

$$\mathbf{v}_0 = v_0(\cos\phi\hat{\mathbf{x}} - \sin\phi\hat{\mathbf{y}}) = v_0\left(\frac{2}{3}\hat{\mathbf{x}} - \frac{\sqrt{5}}{3}\hat{\mathbf{y}}\right)$$

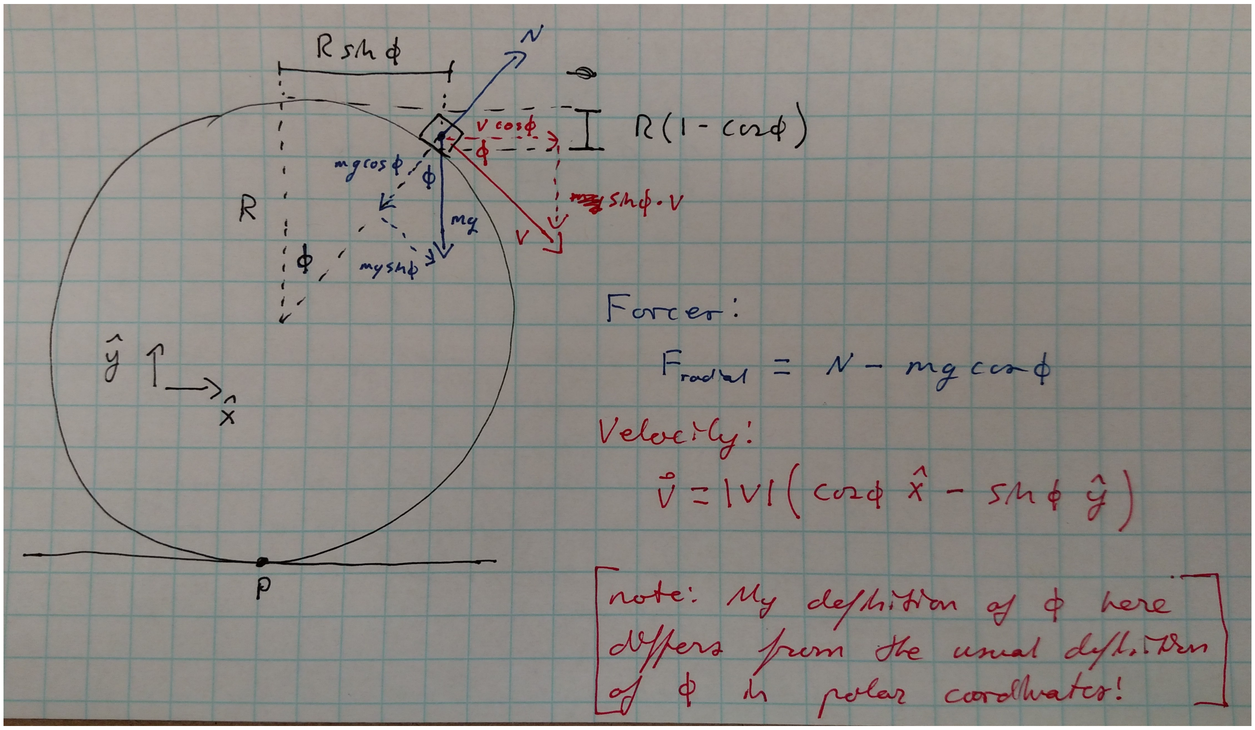


Figure 1: Puck sliding on fixed ball.

where v_0 is the speed of the puck at lift-off, found by conservation of energy:

$$v_0 = \sqrt{2gR(1-\cos\phi)} = \sqrt{\frac{2}{3}gR}$$

To know the horizontal distance traveled, we need to know how much time elapsed before the puck hits the table. We can formulate this in a number of ways - I'll think about the speed. By conservation of energy, we know that when the puck reaches the table it has fallen a distance $2R$ from rest, and so it's final speed must be:

$$v_f^2 = 4Rg$$

We can now ask: at what time does the puck attain this speed? That will be the time of collision with the table. The puck is subject to constant acceleration of $-g\hat{y}$, so it's speed after leaving the ball is:

$$\mathbf{v}(t) = \frac{2}{3}\sqrt{\frac{2}{3}gR}\hat{\mathbf{x}} - \left(\frac{\sqrt{5}}{3}\sqrt{\frac{2}{3}gR} + gt\right)\hat{\mathbf{y}}$$

$$\Rightarrow \quad \|\mathbf{v}(t)\|^2 = \frac{8}{27}gR + \left(\sqrt{\frac{10}{27}gR} + gt\right)^2$$

and so we solve:

$$\begin{aligned}
\frac{8}{27}gR + \left(\sqrt{\frac{10}{27}gR} + gt\right)^2 &= 4Rg \\
\frac{8}{27} + \left(\sqrt{\frac{10}{27}} + \sqrt{\frac{g}{R}}t\right)^2 &= 4 \\
\Rightarrow \quad t = \sqrt{\frac{10}{27}}\left(\sqrt{10} - 1\right)\sqrt{\frac{R}{g}}
\end{aligned}$$

where we taken the physical positive solution and discarded the negative one. Now the horizontal distance traveled in this time is:

$$\Delta x = v_x t = \left[\frac{2}{3}\sqrt{\frac{2}{3}gR}\right]\left[\sqrt{\frac{10}{27}}\left(\sqrt{10} - 1\right)\sqrt{\frac{R}{g}}\right] = \frac{4\sqrt{5}}{27}\left(\sqrt{10} - 1\right)R$$

And since this is just the distance traveled since lift-off, to get the final distance from the point P we must include the horizontal position of the lift-off point, which is $R\sin\phi = \sqrt{5}R/3$. The final position is thus:

$$x_{\text{table}} = \frac{4\sqrt{5}}{27}\left(\sqrt{10} - 1\right)R + \frac{\sqrt{5}}{3}R = \frac{\sqrt{5}}{27}\left[4\sqrt{10} + 5\right]R \approx 1.46R$$