COMP 3711 Course Notes

Design and Analysis of Algorithms

LIN, Xuanyu

ALGORITHMS

COMP 3711 Design and Analysis of Algorithms

THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

October 17, 2023

Contents

1 Asymptotic Notation

Upper Bounds $T(n) = O(f(n))$
if exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, $T(n) \leq c \cdot f(n)$.
Lower Bounds $T(n) = \Omega(f(n))$
if exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, $T(n) \geq c \cdot f(n)$.
Tight Bounds $T(n) = \Theta(f(n))$
if $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$.
Note: Here ”=” means ”is”, not equal.

2 Introduction - The Sorting Problem

2.1 Selection Sort

Algorithm 1: Selection Sort

```
Input: An array A[1...n] of elements
Output: Array A[1...n] of elements in sorted order (asending)
for i ← 1 to n − 1 do
    for j ← i + 1 to n do
        if A[i] > A[j] then
            swap A[i] and A[j]
        end
    end
end
```

Running Time: $\frac{n(n-1)}{2}$
Best-Case = Worst-Case: $T(n) = \Theta(\frac{n(n-1)}{2}) = \Theta(n^2)$

2.2 Insertion Sort

Algorithm 2: Insertion Sort

```
Input: An array A[1...n] of elements
Output: Array A[1...n] of elements in sorted order (asending)
for i ← 2 to n do
    j ← i − 1 while j ≥ 1 and A[j] > A[j + 1] do
        swap A[j] and A[j + 1]
    end
    j ← j − 1
end
```

Running Time: Depends on the input array, ranges between $(n-1)$ and $\frac{n(n-1)}{2}$
Best-Case: $T(n) = n - 1 = \Theta(n)$ (Useless)
Worst-Case: $T(n) = \Theta(\frac{n(n-1)}{2}) = \Theta(n^2)$ (Commonly-Used)
Average-Case: $T(n) = \Theta(\sum_{i=2}^{n} \frac{i-1}{2}) = \Theta(\frac{n(n-1)}{4}) = \Theta(n^2)$ (Sometimes Used)

2.3 Wild-Guess Sort

Running Time: Depends on the random generation, could be faster than the insertion sort.

2.4 Worst-Case Analysis

The algorithm’s worst case running time is $O(f(n)) \implies$ On all inputs of (large) size n, the running time of the algorithm is $\leq c \cdot f(n)$.

Algorithm 3: Wild-Guess Sort

Input: An array $A[1...n]$ of elements
Output: Array $A[1...n]$ of elements in sorted order (asending)
$\pi \leftarrow [4, 7, 1, 3, 8, 11, 5, ...]$ Create random permutation Check if $A[\pi[i]] \leq A[\pi[i+1]]$ for all $i = 1, 2, ..., n-1$ If yes, output A according to π and terminate else $Insertion - Sort(A)$

The algorithm' s worst case running time is $\Omega(f(n)) \implies$ There exists at least one input of (large) size n for which the running time of the algorithm is $\geq c \cdot f(n)$.

Thus, Insertion sort runs in $\Theta(n^2)$ time.

Notice

Selection sort, insertion sort, and wild-guess sort all have worst-case running time $\Theta(n^2)$. How to distinguish between them?

- Closer examination of hidden constants
- Careful analysis of typical expected inputs
- Other factors such as cache efficiency, parallelization are important
- Empirical comparison

Stirling's Formula

Prove that $\log(n!) = \Theta(n \log n)$

First $\log(n!) = O(n \log n)$ since:

$$\log(n!) = \sum_{i=1}^{n} \log i \leq n \times \log n = O(n \log n)$$

Second $\log(n!) = \Omega(n \log n)$ since:

$$\log(n!) = \sum_{i=1}^{n} \log i \geq \sum_{i=n/2}^{n} \log i \geq n/2 \times \log n/2 = n/2(\log n - \log 2) = \Omega(n \log n)$$

Thus, $\log(n!) = \Theta(n \log n)$

3 Divide & Conquer

Main idea of D & C: Solve a problem of size n by breaking it into one or more smaller problems of size less than n. Solve the smaller problems recursively and combine their solutions, to solve the large problem.

3.1 Binary Search

Example: Binary Search

Input: A sorted array $A[1, ..., n]$, and an element x

Output: Return the position of x, if it is in A; otherwise output nil

Idea of the binary search: Set $q \leftarrow$ middle of the array. If $x = A[q]$, return q. If $x < A[q]$, search $A[1, ..., q-1]$, else search $A[q+1, ..., n]$.

Algorithm 4: Binary Search

```
Input: Array A[1...n] of elements in sorted order
BinarySearch(A[], p, r, x)(p, r being the left & right iteration, x being the element being searched)
if p > r then
 | return nil
end
q ← [(p + r)/2]
if x = A[q] then
 | return q
end
if x < A[q] then
 | BinarySearch(A[], p, q − 1, x)
end
else
 | BinarySearch(A[], q + 1, r, x)
end
```

Recurrence of the algorithm, supposing $T(n)$ being the number of the comparisons needed for n elements: $T(n) = T(\frac{n}{2}) + 2$ if $n > 1$, with $T(1) = 2$.

$\implies T(n) = 2\log_2 n + 2 \implies O(\log n)$ algorithm

Example: Binary Search in Rotated Array

Suppose you are given a sorted array A of n distinct numbers that has been rotated k steps, for some unknown integer k between 1 and $n-1$. That is, $A[1...k]$ is sorted in increasing order, and $A[k+1...n]$ is also sorted in increasing order, and $A[n] < A[1]$.

Design an $O(\log n)$-time algorithm that for any given x, finds x in the rotated sorted array, or reports that it does not exist.

Algorithm:

First conduct a $O(\log n)$ algorithm to find the value of k, then search for the target value in either the first part or the second part.

$Find - x(A, x)$

$k \leftarrow Find - k(A, 1, n)$ (First find k)

$if\ x \geq A[1]\ then\ return\ BinarySearch(A, 1, k, x)$

$Else\ return\ BinarySearch(A, k + 1, n, x)$

Example: Finding the last 0

You are given an array $A[1...n]$ that contains a sequence of 0 followed by a sequence of 1 (e.g., 0001111111). A contains k 0(s) ($k > 0$ and $k << n$) and at least one 1.

Design an $O(\log k)$-time algorithm that finds the position k of the last 0.

Algorithm:

$i \leftarrow 1$
$while\ A[i] = 0$
  $i \leftarrow 2i$
$find - k(A[i/2...i])$

3.2 Merge Sort

Principle of the Merge Sort:
- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

Algorithm 5: Merge Sort

```
MergeSort(A, p, r) (p, r being the left & right side of the array to be sorted)
if p = r then
    return
end
q ← [(p + r)/2]
MergeSort(A, p, q)
MergeSort(A, q + 1, r)
Merge(A, p, q, r)
First Call: MergeSort(A, 1, n)
```

Algorithm 6: Merge

```
Input: Two Arrays L ← A[p...q] and R ← A[q + 1...r] of elements in sorted order
Merge(A, p, q, r)
Append ∞ at the end of L and R
i ← 1, j ← 1
for k ← p to r do
    if L[i] ≤ R[j] then
        A[k] ← L[i]
        i ← i + 1
    end
    else
        A[k] ← R[j]
        j ← j + 1
    end
end
```

Let $T(n)$ be the running time of the algorithm on an array of size n.

Merge Sort Recurrence:

$$T(n) \leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n), \quad n > 1, \quad T(1) = O(1)$$

Simplification:

$$\implies T(n) = 2T(n/2) + n, \quad n > 1, \quad T(1) = 1$$

Result:

$$T(n) = n \log_2 n + n = O(n \log n)$$

3.3 Inversion Counting

Definition of the Inversion Numbers: Given array $A[1...n]$, two elements $A[i]$ and $A[j]$ are inverted if $i < j$ but $A[i] > A[j]$. The inversion number of A is the number of inverted pairs.

Theorem:
The number of swaps used by Insertion Sort = Inversion Number (Proved by induction on the size of the array)
Algorithm to Compute Inversion Number:
Algorithm 1: Check all $\Theta(n^2)$ pairs.
Algorithm 2: Run Insertion Sort and count the number of swaps -Also $\Theta(n^2)$ time.
Algorithm 3: Divide and Conquer

3.3.1 Counting Inversions: Divide-and-Conquer

Principle of the Algorithm:
- Divide: divide array into two halves
- Conquer: recursively count inversions in each half
- Conbine: count inversions where a_i and a_j are in different halves, and return sum of three quantities

Inversion counting during the combine step is very similar to the Merge Algorithm (Algorithm 6), by counting the sum of each inversion number of the right array (indicated by $I[j]$) comparing to the left array.

Algorithm 7: Inversion Count during Combination

Input: Two Arrays $L \leftarrow A[p...q]$ and $R \leftarrow A[q+1...r]$ of elements in sorted order
Count(A, p, q, r)
$i \leftarrow 1,\ j \leftarrow 1,\ c \leftarrow 0$
while $(i \leq q - p + 1) \&\& (j \leq r - q)$ **do**
  if $L[i] \leq R[j]$ **then**
    $i \leftarrow i + 1$
  end
  else
    $I[j] = q - p - i + 2$
    $c \leftarrow c + I[j]$
    $j \leftarrow j + 1$
  end
end

The time-complexity of the algorithm is $\Theta(n \log n)$, same as the Merge Sort.

3.3.2 Implementation of the Algorithm

Algorithm 8: Main Algorithm

Sort-and-Count(A, p, r)
if $p = r$ **then**
  return *0*
end
$q \leftarrow \lfloor (p + r)/2 \rfloor$
$c_1 \leftarrow$ Sort-and-Count(A, p, q)
$c_2 \leftarrow$ Sort-and-Count($A, q + 1, r$)
$c_3 \leftarrow$ Merge-and-Count(A, p, q, r)
return $c_1 + c_2 + c_3$
First Call: Sort-and-Count($A, 1, n$)

Algorithm 9: Merge-and-Count

Input: Two Arrays $L \leftarrow A[p...q]$ and $R \leftarrow A[q+1...r]$ of elements in sorted order

```
Merge-and-Count(A, p, q, r)
Append ∞ at the end of L and R
i ← 1, j ← 1, c ← 0
for k ← p to r do
    if L[i] ≤ R[j] then
        A[k] ← L[i]
        i ← i + 1
    end
    else
        A[k] ← R[j]
        j ← j + 1
        c ← c + q − p − i + 2
    end
end
return c
```

3.4 Basic Summary of D&C: Problem Size & Number of Problems

Observations of D&C in Logarithmic Patterns:

- Break up problem of size n into p parts of size n/q.
- Solve parts recursively and combine solutions into overall solution.
- At level i, we break i times and we have p^i problems of size n/q^i.
- When we cannot break up any more, usually when the problem size becomes 1. Usually $i \approx \log_q n$.

The number of problems at (bottom) level $\log_q n$ is $p^i = p^{\log_q n} = n^{\log_q p}$.

Observations of D&C in Non-Logarithmic Patterns:

- Break up problem of size n into $p(\leq 2)$ parts of size $n - q$. (e.g. $q = 1$ for Hanoi Problem)
- Assume that $q = 1$
- At level i, we break i times and we have p^i problems of size $n - i$.
- If we stop when the problem size becomes 1, then $n - i = 1 \implies i = n - 1$.

The number of problems at (bottom) level $n - 1$ is: $p^i = p^{n-1}$.

3.5 Maximum Contiguous Subarray

Example: The Maximum Subarray Problem

Input: An array of numbers $A[1, ..., n]$, both positive and negative

Output: Find the maximum $V(i, j)$, where $V(i, j) = \sum_{k=i}^{j} A[k]$

Brute-Force Algorithm

Idea: Calculate the value of $V(i, j)$ for each pair $i \leq j$ and return the maximum value.

Requires three nested for-loop, time complexity: $\Theta(n^3)$.

A Data-Reuse Algorithm

Idea: $V(i, j) = V(i, j - 1) + A[j]$

Requires two nested for-loop, time complexity: $\Theta(n^2)$.

3.5.1 A D&C Algorithm

Idea: Cut the array into two halves, all subarrays can be classified into three cases: entirely in the first/second half, or crosses the cut.

Compare with the merge sort: Whole algorithm will run in $\Theta(n \log n)$ time if the cross-cut can be solved in $O(n)$ time.

Algorithm 10: Maximum Subarray

```
MaxSubArray(A, p, r)
if p = r then
|  return A[p]
end
q ← ⌊(p + r)/2⌋
M_1 ← MaxSubArray(A, p, q)
M_2 ← MaxSubArray(A, q + 1, r)
L_m, R_m ← −∞
V ← 0
for i ← q to p do
|  V ← V + A[i]
|  if V > L_m then
|  |  L_m ← V
|  end
end
V ← 0
for i ← q + 1 to r do
|  V ← V + A[i]
|  if V > R_m then
|  |  R_m ← V
|  end
end
return max(M_1, M_2, L_m + R_m)
First Call: MaxSubArray(A, 1, n)
```

Recurrence: $T(n) = 2T(n/2) + n \implies T(n) = \Theta(n \log n)$

3.5.2 Kadane's Algorithm

Idea: Based on the principles of **Dynamic Programming**. Let $V[i]$ be the (local) maximum sub-array that ends at $A[i]$, then we let:

•$V[1] = A[1]$

•$V[i] = \max(A[i], A[i] + V[i-1])$

The maximum of $V[i]$, namely V_{max} is the maximum continuous subarray found so far.

Algorithm 11: Kadane's Algorithm

```
V_max ← −∞; V ← 0; start ← 1; end ← 1; temp ← 1 (Note: start & end specify the maximum sub-array)
for i ← 1 to n do
|  V ← V + A[i]
|  if V < A[i] then // Implies V[i − 1] is negative, restart from the current position
|  |  V ← A[i]; temp ← i
|  end
|  if V > V + max then // Found a max sum, update start and end
|  |  V_max ← V; start ← temp; end ← i
|  end
end
```

Time Complexity: $\Theta(n)$

Example: Maximizing Stock Profits

You are presented with an array $p[1 \dots n]$ where $p[i]$ is the price of the stock on day i.

Design an divide-and-conquer algorithm that finds a strategy to make as much money as possible, i.e., it finds a pair i, j with $1 \le i \le j \le n$ such that $p[j] - p[i]$ is maximized over all possible such pairs. Note that you are only allowed to buy the stock once and then sell it later.

Idea 1: Divide and Conquer

- Cut the array into two halves.
- All i, j solutions can be classified into three cases: both i, j are entirely in the first(second) half, or i is in the left half while j is in the right half.
- Maximizing a Case 3 result $p[j] - p[i]$ means finding the smallest value in the first half and the largest in the second half.

Time Complexity: $T(n) = 2T(n/2) + n \implies T(n) = \Theta(n \log n)$

Idea 2: Kadane's Algorithm

- Create a **Profit** array with $Profit[i] = Price[i+1] - Price[i]$.
- Perform the Kadane's Algorithm.

Time Complexity: $O(n)$

3.6 Integer Multiplication

3.6.1 A Simple D&C Algorithm for Integer Multiplication

Goal: Given two n-bit binary integers a and b, compute: $a \cdot b$.
Idea: Multiplication by 2^k can be done in one time unit by a left shift of k bits.
- Rewrite the two numbers as $a = 2^{n/2}a_1 + a_0$, $b = 2^{n/2}b_1 + b_0$.
- The product becomes: $a \cdot b = (2^{n/2}a_1 + a_0)(2^{n/2}b_1 + b_0) = 2^n a_1 b_1 + 2^{n/2}(a_1 b_0 + a_0 b_1) + a_0 b_0$
- The new computation requires 4 products of integers, each with $n/2$ bits.
- Apply D&C by splitting a problem of size n, to 4 problems of size $n/2$.

Algorithm 12: Binary Multiplication

```
Multiply(A, B)
n ← size of A
if n = 1 then
|  return A[1] · B[1]
end
mid ← ⌊n/2⌋
U ← Multiply (A[mid + 1..n], B[mid + 1..n]) // a_1 b_1
V ← Multiply (A[mid + 1..n], B[1..mid]) // a_1 b_0
W ← Multiply (A[1..mid], B[mid + 1..n]) // a_0 b_1
Z ← Multiply (A[1..mid], B[1..mid]) // a_0 b_0
M[1..2n] ← 0
M[1..n] ← Z // a_0 b_0
M[mid + 1..] ← M[mid + 1..] ⊕ V ⊕ W // +[(a_1 b_0 + a_0 b_1) ≪ (left shift) n/2]
M[2mid + 1..] ← M[2mid + 1..] ⊕ U // +[a_1 b_1 ≪ n]
return M
```

Time Complexity: $T(n) = 4T(n/2) + n \implies T(n) = \Theta(n^2)$

3.6.2 Karatsuba Multiplication

Goal: Given two n-bit binary integers a and b, compute: $a \cdot b$.
Idea:
- We've seen that $ab = a_1b_12^n + (a_1b_0 + a_0b_1)2^{n/2} + a_0b_0$, so we only need the result of $a_1b_0 + a_0b_1$.
- Note that $a_1b_0 + a_0b_1 = (a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0$, thus only requires performing 3 multiplications of size $n/2$.

Algorithm 13: Binary Multiplication (Karatsuba's Multiplication Algorithm)

```
Multiply(A, B)
n ← size of A
if n = 1 then
 | return A[1] · B[1]
end
mid ← ⌊n/2⌋
U ← Multiply (A[mid + 1..n], B[mid + 1..n]) // a_1 b_1
Z ← Multiply (A[1..mid], B[1..mid]) // a_0 b_0
A' ← A[mid + 1..n] ⊕ A[1..mid] // a_1 + a_0
B' ← B[mid + 1..n] ⊕ B[1..mid] // b_1 + b_0
Y ← Multiply (A', B') // (a_1 + a_0)(b_! + b_0)
M[1..2n] ← 0
M[1..n] ← Z // a_0 b_0
M[mid + 1..] ← M[mid + 1..] ⊕ Y ⊖ U ⊖ Z // +[(a_1 b_0 + a_0 b_1) ≪ (left shift) n/2]
M[2mid + 1..] ← M[2mid + 1..] ⊕ U // +[a_1 b_1 ≪ n]
return M
```

Time Complexity: $T(n) = 3T(n/2) + n \implies T(n) = \Theta(n^{\log_2 3}) = \Theta(n^{1.585\cdots})$
For recent research, see: Integer Multiplication in $O(n \log n)$ Time (David Harvey & Joris van der Hoeven, 2021)

3.7 Matrix Multiplication

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \qquad c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Brute Force Method: $\Theta(n^3)$ time.

3.7.1 A D&C Solution to Matrix Multiplication

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad \begin{cases} C_{11} = (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} = (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} = (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} = (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{cases}$$

Recursion: $T(n) = 8T(n/2) + O(n^2) \implies T(n) = O(n^3)$

3.7.2 Strassen's Matrix Multiplication Algorithm

Idea: Muliply 2-by-2 block matrices with only 7 multiplications

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\begin{cases} P_1 = A_{11} \times (B_{12} - B_{22}) \\ P_2 = (A_{11} + A_{12}) \times B_{22} \\ P_3 = (A_{21} + A_{22}) \times B_{11} \\ P_4 = A_{22} \times (B_{21} - B_{11}) \\ P_5 = (A_{11} + A_{12}) \times (B_{11} + B_{22}) \\ P_6 = (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 = (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{cases} \qquad \begin{cases} C_{11} = P_5 + P_4 - P_2 + P_6 \\ C_{12} = P_1 + P_2 \\ C_{21} = P_3 + P_4 \\ C_{22} = P_5 + P_1 - P_3 - P_7 \end{cases}$$

Recursion: $T(n) = 7T(n/2) + n^2 \implies T(n) = \Theta(n^{\log_2 7}) = \Theta(n^{2.807\cdots})$
For recent research, see: Powers of Tensors and Fast Matrix Multiplication (Le Gall, 2014)
Conjecture: Close to $\Theta(n^2)$

3.8 Master Theorem

For recurrences of form

$$T(n) = aT(n/b) + f(n) \text{ or } T(n) \le aT(n/b) + f(n), \text{ Let } c \equiv \log_b a$$

where
- $a \ge 1$ and $b > 1$ both being constants
- $f(n)$ is a (asymptotically) positive polynomial function
- n/b could be either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$

3.8.1 Master Theorem for Equalities

(1) Work Increases: $f(n) = O(n^{c-\epsilon})$ for some $\epsilon \implies T(n) = \Theta(n^c)$

(2) Work Remains: $f(n) = \Theta(n^c \log^k n)$ for $k > -1 \implies T(n) = \Theta(n^c \log^{k+1} n)$
Note: For the case $k = -1$, $T(n) = \Theta(n^c \log\log n)$; For the case $k < -1$, $T(n) = \Theta(n^c)$

(3) Work Decreases: $f(n) = \Omega(n^{c+\epsilon})$ for some $\epsilon \implies T(n) = \Theta(f(n))$
Note: Rigorously, the third case requires $af(n/b) \le kf(n)$ for some $k < 1$ and sufficiently large n

(4) For a special case $T(n) = \sum_i T(\alpha_i n) + n$ where $\alpha_i > 0$ with $\sum_i \alpha_i < 1$, we have $T(n) = \Theta(n)$

3.8.2 Master Theorem for Inequalities

(1) Work Increases: $f(n) = O(n^{c-\epsilon})$ for some $\epsilon \implies T(n) = O(n^c)$

(2) Work Remains: $f(n) = O(n^c) \implies T(n) = O(n^c \log n)$

(3) Work Decreases: $f(n) = \Omega(n^{c+\epsilon})$ for some $\epsilon \implies T(n) = O(f(n))$

4 Advanced Sorting Algorithms

4.1 Probability & Statistics, Random Permutation

$$E[X] = \sum i \cdot Pr[X = i]$$

$$E[X + Y] = E[X] + E[Y]$$

For independent random variables X & Y,

$$E[XY] = E[X] \cdot E[Y]$$

Algorithm 14: Random Permutation

```
RandomPermute(A)
n ← A.length
for i ← 1 to n do
  | swap A[i] with A[Random(1, i)]
end
```

4.2 Randomized Algorithm - Quicksort

Idea: Quicksort chooses item as pivot. It partitions array so that all items less than or equal to pivot are on the left and all items greater than pivot on the right. It then recursively Quicksorts left and right sides.

Algorithm 15: Quicksort

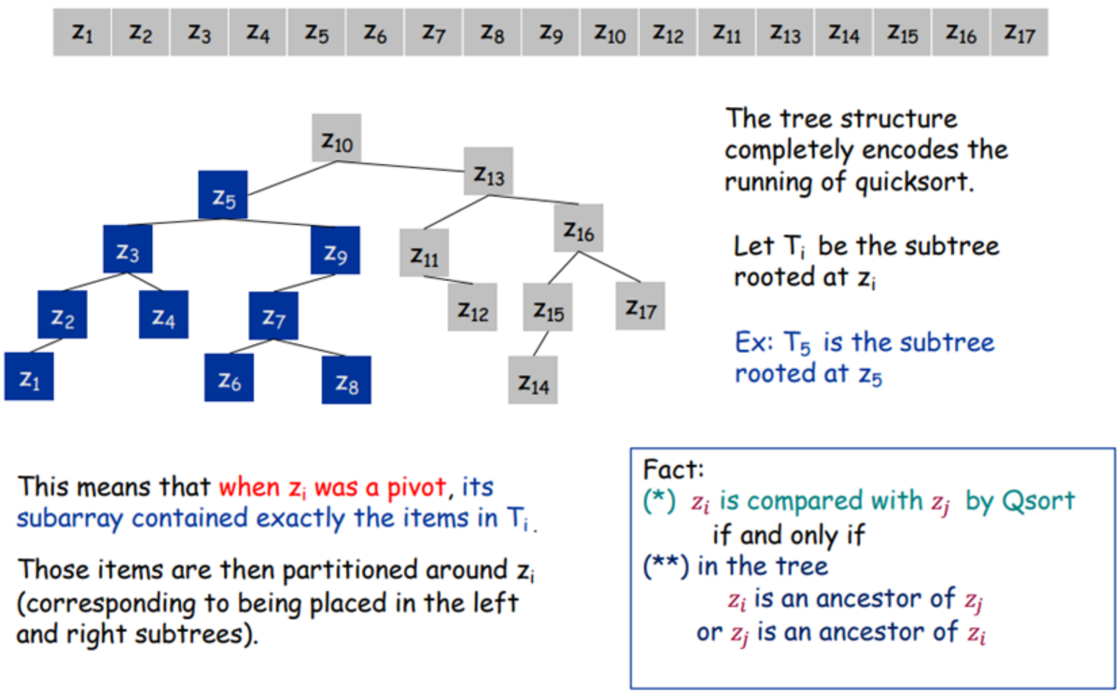
```
Quicksort(A, p, r) // Array from A[p] to A[r]
if p ≥ r then
  | return
end
q =Partition (A, p, r) // Set a new pivot position
Quicksort (A, p, q − 1)
Quicksort (A, q + 1, r)
First Call: MaxSubArray(A, 1, n)

Partition(A, p, r)
x ← A[r] // Set the last item as pivot, or randomly swap away the last item before choosing the pivot
i ← p − 1
for j ← p to r − 1 do
  | if A[j] ≤ x then
  |   | i ← i + 1
  |   | swap A[i] and A[j] // Put all items ≤ A[r] on the left
  | end
end
swap A[i + 1] and A[r]
return i + 1
```

4.2.1 Running Time

Best Case: Always select the median element as the pivot - $\Theta(n \log n)$ time.
Worst Case: Always select the smallest (or the largest) element - $\Theta(n^2)$ time.
To make running time independent of input, we can randomly choose an element as the pivot by swapping it with last item in array before running the partition.

4.2.2 Binary Tree Representation



4.2.3 Expected Running Time for Random-Based Quicksort

- Two elements z_i and z_j are compared at most once, iff z_i or z_j is the first to be chosen among $z_i, \cdots, z_j$
- The probability above (any indicated two elements z_i and z_j are compared) is $\frac{2}{j-1+1}$

$$\implies E_{\text{Num of comparisons made}} = \sum_{i<j} \frac{2}{j-1+1} = O(n \log n)$$

4.2.4 Find the i-th Smallest Element Using Quicksort

Example: Find the i-th Smallest Element
Given an unsorted array $A[1 \ldots n]$ and an integer i, return the i-th smallest element of $A[1 \ldots n]$.

Idea:
1. Choose a Pivot x from $A[p \ldots r]$
2. Partition A around x. (linear time)
3. After partitioning, pivot x will be at known location q
   If $i = q - p + 1$, then x is the actual solution
   If $i < q - p + 1$, then the i-th element of $A[p \ldots r]$ is the i-th element of $A[p \ldots q-1]$, solve recursively
   If $i > q - p + 1$, then the i-th element of $A[p \ldots r]$ is the $j = (i - q + p - 1)$-th element of $A[q+1 \ldots r]$, solve recursively

Algorithm 16: i-th Smallest Element

```
Select(A, p, r, i)
if p = r then
  | return A[p]
end
Randomly choose an element in A[p ... r] as the pivot and swap it with A[r]
q ←Partition(A, p, r)
k ←q − p + 1
if i = k then
  | return A[q]
end
else if i < k then
  | return Select(A, p, q − 1, i)
end
else
  | return Select(A, q + 1, r, i − k)
end
First Call: Select(A, 1, n, i)
```

4.2.5 Expected Running Time for Finding the i-th Smallest Element

• A pivot is "good" if it's between the 25%- and 75%-percentile of sorted A, eliminating at least 1/4 of the array. The probability for such "good" pivot is 1/2.

• Let i-th stage be the time between the i-th good pivot (not including) and the $(i+1)$-st good pivot (including), $i = 0, 1, 2, \cdots$, then the expected pivots selected within a stage is 2.

• Let Y_i = the running time of i-th stage, X_i = the num. of pivots (recursive calls) in i-th stage. Then $Y_i \leq X_i(3/4)^i n$.

$$\implies E[Y_i] \leq E[X_i(3/4)^i n] = 2(3/4)^i n \implies \text{Expected Total Running Time} \leq E[\sum_i Y_i] \leq \sum_i 2(3/4)^i n = O(n)$$

Example: i-th Smallest Element in Two Sorted Arrays

Given two sorted arrays $A1$ and $A2$ of sizes m and n. Design an algorithm to find the k-th smallest element in the union of the elements in $A1$ and $A2$ ($k \leq m + n$).

Algorithm 17: i-th Smallest Element in Two Sorted Arrays

```
Search(array A1, array A2, start1 1, end1 k, start2 1, end2 k, Order k)
Main Idea: Compare elements A1[k/2] and A2[k/2]
if A1[k/2] < A2[k/2] then
  | Eliminate first half of A1
  | return Search(A1, A2, k/2 + 1, end1, start2, end2, k/2)
end
else
  | Eliminate first half of A2
  | return Search(A1, A2, start1, end1, k/2 + 1, end2, k/2)
end
```

Time Complexity: $\Theta(\log k)$

4.3 Heapsort

4.3.1 Priority Queues

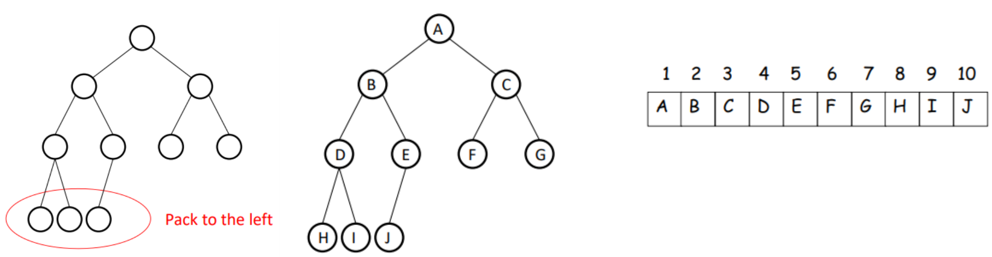
Main Idea: Processing the shortest job first - Extracting the smallest element from the queue.
A Priority Queue is an abstract data structure that supports two operations: Insert & Extract-Min.
Implementations:
1. Unsorted list + a pointer to the smallest element: $O(1)$ Insert & $O(n)$ Extract-Min
2. Sorted doubly linked list + a pointer to first element: $O(n)$ Insert & $O(1)$ Extract-Min

4.3.2 Binary Heap Implementation

- All levels are full except possibly the lowest level
- If the lowest level is not full, then nodes must be packed to the left
- The value of a node is at least the value of its parent —Min-heap
- Both Insert & Extract-Min can be done in $O(\log n)$ time

Notice
The binary tree here is DIFFERENT from the Binary Search Tree, which requires ALL left nodes $<$ parent, while ALL right nodes $>$ parent.

Array Implementation of Heap
- The root is in array position 1
- For any element in array position i, the left child is in position $2i$, the right child is in position $2i+1$, the parent is in position $\lfloor i/2 \rfloor$



4.3.3 Heapsort

Insert
- Add the new element to the next available position at the lowest level.
- Restore the min-heap property if violated.

Algorithm 18: Add item x to heap $A[1 \dots i-1]$

Insert(x, i)
$A[i] \leftarrow x$
$j = i$
while $j > 1$ *and* $A[j] < A[\lfloor j/2 \rfloor]$ **do** // $A[j]$ is less than its parent
  Swap $A[j]$ and $A[\lfloor j/2 \rfloor]$
end
$j = \lfloor j/2 \rfloor$

Time Complexity: $O(\log n)$

Extract-Min: Should preserve both min-heap property & completeness

• Copy the last element to the root (overwrite).

• Restore the min-heap property by percolating (or bubbling down): if the element is larger than either of its children, then interchange it with the smaller of its children.

Algorithm 19: Remove the smallest item $A[1]$ in the heap $A[1 \dots i]$

```
Extract-Min(i)
Output(A[1])
Swap A[1] and A[i]
A[i] = ∞, j = 1, l = A[2j], r = A[2j+1] // Left & Right Children
while A[j] > min(l, r) do  // if A[j] is larger than a child, swap with the smaller child
    if l < r then
        Swap A[j] with A[2j], j = 2j
    end
    else
        Swap A[j] with A[2j+1], j = 2j+1
    end
    l = A[2j], r = A[2j+1]
end
```

Time Complexity: $O(\log n)$

Total Time Complexity: Build a binary heap of n elements & Perform n Extract-Min operations: $O(n \log n)$

Example: Merging k Sorted Arrays

Suppose that you have k sorted arrays, each with n elements, and you want to combine them into a single sorted array of kn elements.

Solution 1: Merge the first two arrays, then merge it with the third, and so on. Time Complexity $= \sum_{i=2}^{k} in = O(k^2 n)$

Solution 2: Divide recursively k sorted arrays into two parts, conduct the merging for the subproblems. $T(k) = 2T(k/2) + kn \implies$ Time Complexity $= T(k) = O(kn \log k)$

Solution 3 (Heapsort): Insert the first element of each array into an empty min-heap. Extract-min every time and insert the next item of in the same array as the one being extracted. Time Complexity $= O(kn \log k)$

[Operation Implementation]

Decrease-Key: Decreases the value of one specified element (Used in Dijkstra's Algorithm)

Modification of the heaps to support it in $O(\log n)$ time: Change the heap tree to a binary search tree.

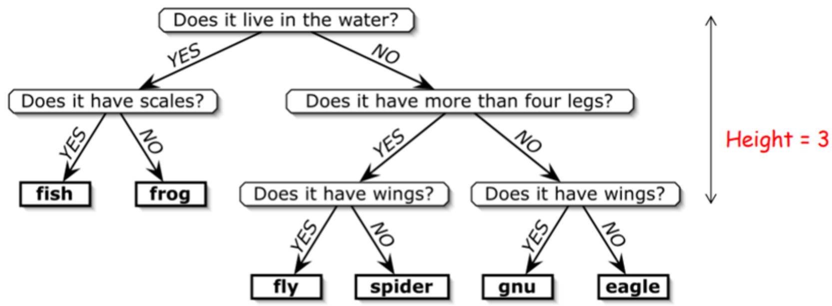
For more information, see: Binary Heap (Wikipedia)

Some websites markdown: Zhihu Web 1 Web 2

4.4 Linear-Time Sorting

4.4.1 Decision Trees and Lower Bounds

Decision Tree Model



A decision tree to choose one of six animals.

Fact: A binary tree with n leaves must have height $\Omega(\log n)$.

Theorem: Any algorithm for finding location of given element in a sorted array of size n must have running time $\Omega(\log n)$ in the decision-tree model.

Theorem: Any **comparison-based sorting algorithm** (only by using comparisons without using thetr accurate values) requires $\Omega(n \log n)$ time.

Given n numbers, there are $n!$ possible permutations, resulting in the tree height being $\Omega(\log(n!))$. Thus, the time complexity is bounded as $\Omega(\log(n!)) = \Omega(n \log n)$.

4.4.2 Linear-time Sorting

Counting-Sort

- Assumes that the elements are integers from 1 to k

Algorithm 20: Counting-Sort Algorithm

Input: $A[1 \cdots n]$ where $A[j] \in 1, 2, \cdots, k$
Output: $B[1 \cdots n]$, sorted
`Counting-Sort`(A, B, k)
Let $C[1 \cdots k]$ be a new array
for $i \leftarrow 1$ *to* k **do** // Initialize Counters
| $C[i] \leftarrow 0$
end
for $j \leftarrow 1$ *to* n **do** // Count the number of each element
| $C[A[j]] \leftarrow C[A[j]] + 1$
end
for $i \leftarrow 2$ *to* k **do** // Count the accumulative number of elements
| $C[i] \leftarrow C[i] + C[i-1]$
end
for $j \leftarrow n$ *to 1* **do** // Move the items into proper location
| $B[C[A[j]]] \leftarrow A[j]$
| $C[A[j]] \leftarrow C[A[j]] - 1$
end

Time Complexity: $\Theta(n + k)$
Space Complexity: $\Theta(n + k)$

Radix-Sort

Algorithm 21: Radix-Sort Algorithm

Input: An array of n numbers, each has at most d digits
Output: A sorted array

```
Radix-Sort(A, d)
for i ← 1 to d do
 | Use Counting-Sort to sort array A on digit i
end
```

Time Complexity: $\Theta(d(n+k))$

4.5 Sorting Reprise & Comparison

	Insertion Sort	Merge Sort	Quick Sort	Heap Sort	Radix Sort
Running Time	$\Theta(n^2)$	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(d(n+k))$
Randomized	No	No	Yes	No	No
Working Space	$\Theta(1)$	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$	$\Theta(n+k)$
Comparison-Based	Yes	Yes	Yes	Yes	No
Stable	Yes	Yes	No	No	Yes
Cache Performance	Good	Good	Good	Bad	Bad
Parallelization	No	Excellent	Good	No	No

5 Greedy Algorithms

A greedy algorithm always makes the choice that looks best at the moment and adds it to the current partial solution.

Greedy algorithms don't always yield optimal solutions, but when they do, they're usually the simplest and most efficient algorithms available.

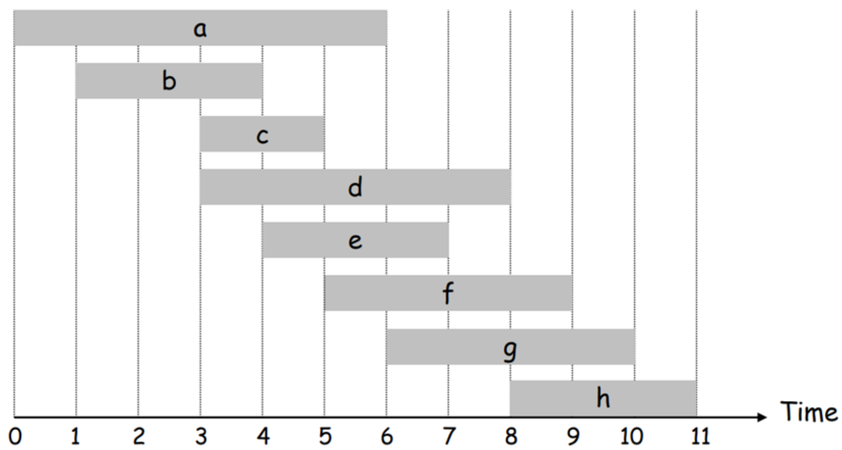
Usually, greedy algorithms involve a sorting step that dominates the total cost.

Example: Interval Scheduling

Job j starts at s_j and finishes at f_j.

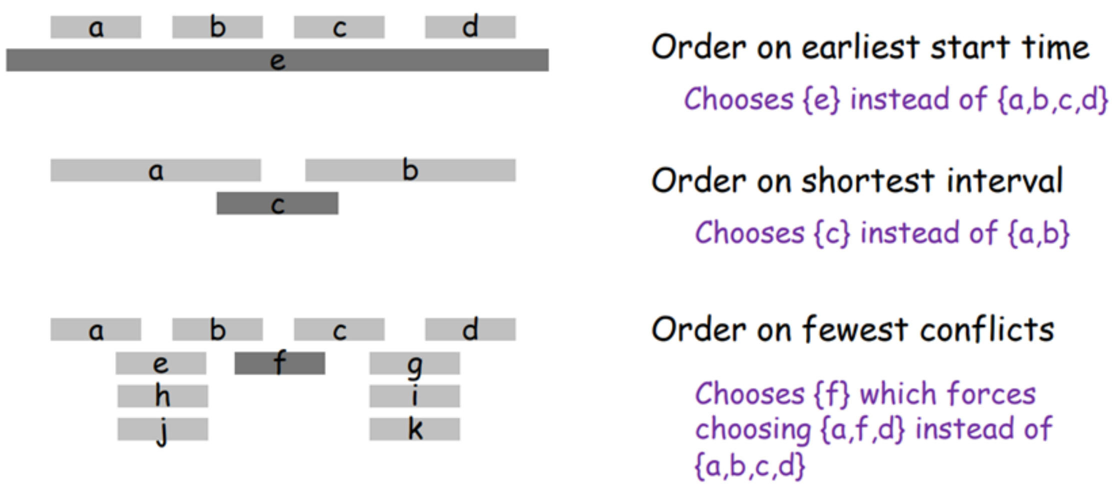
Two jobs are compatible if they don't overlap.

Goal: Find maximum size subset **(Note: Not the longest duration)** of mutually compatible jobs.



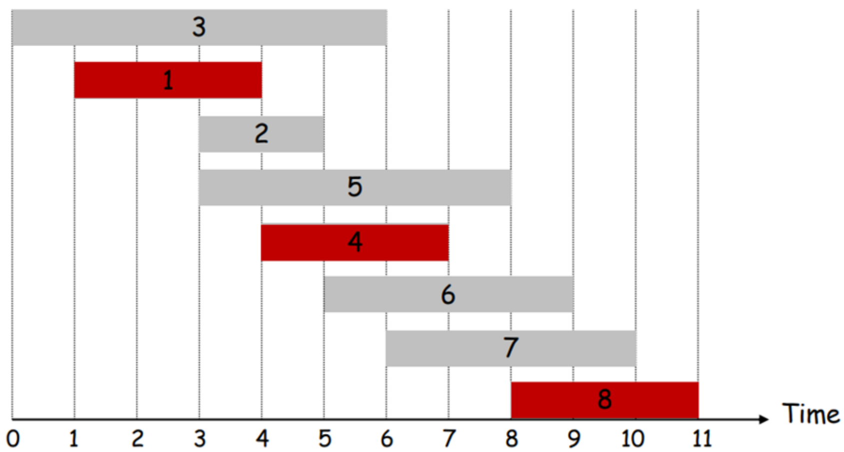
Three Possible Rules:

- [Earliest Start Time] Consider jobs in increasing order of start time s_j.
- [Shortest Interval] Consider jobs in increasing order of duration $f_j - s_j$.
- [Fewest Conflicts] Consider jobs in increasing order of number of conflicts c_j with other jobs.



All three rules may not yield optimal solutions.

Correct Greedy Algorithm:
Consider jobs in increasing order of finish time f_j. Take each job if compatible with all previously taken jobs.



Algorithm 22: Greedy Algorithm

Sort jobs by finish time f_j, i.e. $f_1 \le f_2 \le \cdots \le f_n$
$A \leftarrow \emptyset$, $last \leftarrow 0$
for $j \leftarrow 1$ *to* n **do**
  if $s_j \ge last$ **then**
    $A \leftarrow A \cup \{j\}$
    $last \leftarrow f_j$
  end
end
return A

See Page 13 of 11_Greedy_all.pdf for the proof.

Example: The Fractional Knapsack Problem
Input: Set of n items: item i has weight w_i and value v_i, and a knapsack with capacity W.
Goal: Find $0 \le x_1, \ldots, x_n \le 1$ to maximize $\sum_{i=1}^{n} v_i x_i$ subject to $\sum_{i=1}^{n} w_i x_i \le W$. (Note: x_i is the fraction of item i to be taken)

- The x_i must be 0 or 1: The 0/1 knapsack problem.
- The x_i can be any value in [0,1]: The fractional knapsack problem.

Algorithm 23: Fractional-Knapsack Algorithm

`Fractional-Knapsack`$(w, v, W, x[])$
Sort $\frac{v_i}{w_1}$ so that $\frac{v_1}{w_1} \ge \frac{v_2}{w_2} \ge \cdots \ge \frac{v_n}{w_n}$
for $i \leftarrow 1$ *to* n **do**
  if $w_i < W$ **then**
    $x_i \leftarrow 1$
    $W \leftarrow W - w_i$
  end
  else
    $x_i \leftarrow W / w_i$
    return x
  end
end
return x

Example: Hiking Problem

Suppose you are going on a hiking trip over multiple days. For safety reasons you can only hike during daytime. You can travel at most d kilometers per day, and there are n camping sites along the hiking trail where you can make stops at night. Assuming the starting point of the trail is at position $x_0 = 0$, the camping sites are at locations $x_1, \cdots, x_n$, and the end of the trail is at position x_n. Design an $O(n)$-time algorithm to find a plan that uses the minimum number of days to finish the trip. You can assume that $x_{i+1} - x_i \leq d$ for all i (otherwise there is no solution).

Idea: For each day i, stop at the furthest camping site, i.e. stop at the largest x_j such that x_j minus the start location of day i is at most d.

Example: Interval Partitioning

Lecture j starts at s_j and finishes at f_j. Two lectures are compatible if they don't overlap. Goal: Find minimum number of classrooms to schedule all lectures.

Idea: Sort the class by start time. Insert in order, opening new classroom when needed.

6 Graph Algorithms

6.1 Breadth & Depth First Search

6.2 Shortest Paths

6.3 Shortest Paths

6.4 Maximum Flow and Bipartite Matchings

7 AVL Trees

8 Basic String Matching

9 Hashing

Homework 1

Name: LIN, Xuanyu, SID: 20838295, Email Address: xlinbf@connect.ust.hk

Problem 1

For each pair of expressions (A, B) below, indicate whether A is $O, \Omega,$ or Θ of B. List all applicable relations. No explanation is needed.
(a) $A = n^3 - 100n$, $B = n^2 + 50n$
(b) $A = \log_2(n^2)$, $B = \log_{2.7}(n^4)$
(c) $A = 10^{10000}$, $B = \frac{n}{10^{10000}}$
(d) $A = 2^{n \log n}$, $B = n^{10} + 8n^2$
(e) $A = 2^n$, $B = 2^{n+\log n}$
(f) $A = 3^{3n}$, $B = 3^{2n}$
(g) $A = (\sqrt{2})^{\log n}$, $B = \sqrt{\log n}$

Solution.
(a) $A = \Omega(B)$
(b) $A = O(B), A = \Omega(B), A = \Theta(B)$
(c) $A = O(B)$
(d) $A = \Omega(B)$
(e) $A = O(B)$
(f) $A = \Omega(B)$
(g) $A = \Omega(B)$

Problem 2

Derive asymptotic upper bounds for T(n) in the following recurrences. Make your bounds as tight as possible. You may assume that n is a power of 2 for (a), n is a power of 4 for (b), and $\sqrt{n}$ is always an integer for (c).
(a) $T(1) = 1$; $T(n) = 4T(n/2) + n^2$ for $n > 1$.
(b) $T(1) = 1$; $T(n) = 16T(n/4) + n$ for $n > 1$.
(c) $T(2) = 1$; $T(n) = T(\sqrt{n}) + 1$ for $n > 1$.

Solution.
(a)

$$\begin{aligned}
T(n) &= 4T(n/2) + n^2 = 4[4T(n/4) + (n/2)^2] + n^2 = 4\{4[4T(n/8) + (n/4)^2] + (n/2)^2\} + n^2 \\
&= 4\{4\{4\{\cdots[4T(1) + 2^2] + 4^2\} + \cdots + (n/4)^2\} + (n/2)^2\} + n^2 \\
&= 4^{\log_2 n} + 4^{\log_2 n - 1} \times 2^2 + 4^{\log_2 n - 2} \times 4^2 + \cdots + 4^1 \times (n/2)^2 + n^2 \\
&= n^2 + \frac{n^2}{4} \times 2^2 + \frac{n^2}{4^2} \times 4^2 + \cdots + 4 \times (n/2)^2 + n^2 \\
&= n^2(\log_2 n + 1) = O(n^2 \log n)
\end{aligned}$$

(b)

$$\begin{aligned}
T(n) &= 16T(n/4) + n = 16[16T(n/4^2) + n/4] + n = 16\{16[16T(n/4^3) + n/4^2] + n/4\} + n \\
&= 16\{16\{16\{\cdots[16T(1) + 4] + 4^2\} + \cdots + n/4^2\} + n/4\} + n \\
&= 16^{\log_4 n} + 16^{\log_4 n - 1} \times 4 + 16^{\log_4 n - 2} \times 4^2 + \cdots + 16^1 \times n/4 + n \\
&= n^2 + \frac{n^2}{4} + \frac{n^2}{4^2} + \cdots + 4n + n \\
&= \frac{n - 4n^2}{1 - 4} = \frac{4}{3}n^2 - \frac{1}{3}n = O(n^2)
\end{aligned}$$

(c)

$$T(n) = T(\sqrt{n}) + 1 = T(n^{1/2^2}) + 2 = T(n^{1/2^3}) + 3 = \cdots = T(2) + \log_2(\log_2 n) = O(\log_2(\log_2 n))$$

Problem 3

(a) Describe a recursive algorithm that returns a list of all possible $n \times n$ binary arrays where n is a positive input integer. An array is binary if each of its entry is either 0 or 1. You can either describe your algorithm in text or in a documented pseudocode. Make sure that your algorithm is recursive. Make sure that your description is understandable.
(b) Write down the recurrence for the running time of your recursive algorithm in (a) with the boundary condition(s). Explain your notations. Solve your recurrence from scratch to obtain the the running time of your algorithm.

Solution.

(a) Main Idea: Recursively solve the problem by reducing the $k \times k$ array to $(k-1) \times (k-1)$ array. Then generate all the binary array of the remaining $1 \times (k-1)$, $(k-1) \times 1$ and 1×1 array. For each kind of array for the $(k-1) \times (k-1)$ array, we insert the remaining $(2k-1)$ elements into it, forming the $(k \times k)$ array.

Algorithm 24: All $n \times n$ Binary Arrays

`AllBinaryArrays`(n)
EmptyArray $\leftarrow$ array$[n][n]$
ArrayList<Type=array$[n][n]$> $\leftarrow$ [ ]
if n = *1* **then**
  ArrayList.append(array$[n][n] = 0$, array$[n][n] = 1$) // Set the last element to 0 & 1, others remaining 0
  return *ArrayList*
end
ArrayList_1, ArrayList_2, ArrayList_3 $\leftarrow$ `GenerateRemaining`(n)
PreviousArrayList[ ][2 : n][2 : n] $\leftarrow$ `AllBinaryArrays`$(n-1)$
// Insert the remaining $(2n-1)$ elements into the $n \times n$ array, forming all kinds of $(n \times n)$ array
foreach *Combination of ArrayList_1, ArrayList_2, ArrayList_3* **do**
  ArrayList.append(PreviousArrayList[ ][2 : n][2 : n].set(PreviousArrayList[ ][1][2 : n] $\leftarrow$ ArrayList_1,
   PreviousArrayList[ ][2 : n][1] $\leftarrow$ ArrayList_2, PreviousArrayList[ ][1][1] $\leftarrow$ ArrayList_3))
end
return *ArrayList*

/* Recursively generate all the kinds of $1 \times (n-1)$, $(n-1) \times 1$ and 1×1 array respectively */
`GenerateRemaining`(n)
ArrayList_1<Type=array[1]$[n-1]$> $\leftarrow$ [ ]
ArrayList_2<Type=array$[n-1]$[1]> $\leftarrow$ [ ]
ArrayList_3<Type=array[1][1]> $\leftarrow$ [ ]
if $n = 2$ **then**
  return $[[0],[1]],[[0],[1]],[[0],[1]]$ // To be assigned to $1 \times (n-1)$, $(n-1) \times 1$ and 1×1 array
end
ArrayList_1[ ][1][2 : $n-1$], ArrayList_2[ ][2 : $n-1$][1], ArrayList_3[ ][1][1] $\leftarrow$ `GenerateRemaining`$(n-1)$ //
 Assign all possible cases of the previous arrays into all the array in the corresponing list
// Set the new element inserted to be 0 and 1
ArrayList_1 = ArrayList_1[ ][1][2 : $n-1$].set[ ][1][1] $\leftarrow$ 0 + ArrayList_1[ ][1][2 : $n-1$].set[ ][1][1] $\leftarrow$ 1
ArrayList_2 = ArrayList_2[ ][2 : $n-1$][1].set[ ][1][1] $\leftarrow$ 0 + ArrayList_1[ ][2 : $n-1$][1].set[ ][1][1] $\leftarrow$ 1
return *ArrayList_1, ArrayList_2, ArrayList_3*
First Call: `AllBinaryArrays`(n)

(b) **Recurrence:** Suppose $T(n) = T(n-1) + O(f(n))$

$f(n)$ contains generating the remaining $1 \times (n-1)$, $(n-1) \times 1$ and 1×1 arrays as well as inserting such arrays into the original $n \times n$ array. The first part requires $O(n)$ time as a simple recursion while the second part requires $O(n \times 2^n)$ time, considering the insertion time.

Thus,

$$T(n) = T(n-1) + O(n \times 2^n) = \sum_{k=1}^{n} k \times 2^k = O(n2^n)$$

Problem 4

Let $A[1..n]$ be an array of n elements. One can compare in $O(1)$ time two elements of A to see if they are equal or not; however, the order relations $<$ and $>$ do not make sense. That is, one can check whether $A[i] = A[j]$ in $O(1)$ time, but the relations $A[i] < A[j]$ and $A[i] > A[j]$ are undefined and cannot be determined.
In the tutorial you developed an $O(n \log n)$-time divide-and-conquer algorithm for finding a majority element of A if one exists. In this assignment you need to generalize this problem.
Let $k \in [1..n]$ be a fixed integer. An element of $A[1..n]$ is a k-major element if its number of occurrences in A is greater than n/k. For example, if $n = 30$, then a 10-major element should occur greater than 3 times (i.e., at least 4 times). Note that it is possible that no k-major exists for a particular k; it is also possible that there are multiple k-major elements for a particular k.
This problem concerns with designing a divide-and-conquer algorithm for finding **all** 10-major elements in $A[1..n]$ in $O(n \log n)$ time; if there is no 10-major element, report so. Answer the following questions.

(a) What is the maximum number of 10-major elements in $A[1..n]$? Explain.

(b) Design a divide-and-conquer algorithm that finds all 10-major elements in $A[1..n]$ in $O(n \log n)$ time; if there is no 10-major element, your algorithm should report so. Recall that one can check whether $A[i] = A[j]$ in $O(1)$ time, but the relations $A[i] < A[j]$ and $A[i] > A[j]$ are undefined and cannot be computed.

Write your algorithm in documented pseudocode. Also, explain in text what your pseudocode does. Explain the correctness of your algorithm.

Since your algorithm uses the divide-and-conquer principle, it should be recursive in nature. That is, it should work on $A[1..n]$ in the first call to return all 10-major elements of $A[1..n]$, and in subsequent recursive calls, it may recurse on many subarrays $A[p..q]$ for some $p, q \in [1..n]$ to return all 10-major elements of $A[p..q]$
Given a particular subarray $A[p..q]$, a 10-major element of $A[p..q]$ is not necessarily a 10-major element of $A[1..n]$. Conversely, a 10-major element of $A[1..n]$ is not necessarily a 10-major element of $A[p..q]$.

(c) Derive a recurrence relation that describes the running time $T(n)$ of your algorithm. Explain your reasoning. State the boundary condition(s).

(d) Solve your recurrence **from scratch** to show that $T(n) = O(n \log n)$.

Solution.

(a) The maximum number of 10-major elements is 9. If there were 10 10-major elements in an array of n numbers, each 10-major elements should appear at least $\lfloor n/10 \rfloor + 1$ times, resulting in a total of at least $10\lfloor n/10 \rfloor + 10 > 10(n/10-1)+10 = n$ numbers to be presented. For 9 10-major elements, supposing there are 100 numbers in total, each such elements appear 11 times would satisfy the requirement.

(b) Since the 10-major elements require the elements to appear at least $\lfloor n/10 \rfloor + 1$ times, we can infer that, if we cut the array into 10 subarrays, such elements must also be a 10-major elements in at least one of the subarray. Reporting such elements in the subarray to the previous recurrsion and checking if they are still 10-major elements in the previous recurrsion can help us get all the 10-major elements.

Algorithm 25: Get All 10-Major Elements

```
GetMajorElements(A[n])
if len(A) = 0 or 1 then
    return A
end
// Return A itself as it stores nothing or the only 10-major element
PossibleMajorElements ← [ ]
TrueMajorElements ← [ ]
for i ← 0 to 9 do
    start ← ⌈ni/10⌉
    end ← ⌈n(i+1)/10⌉ − 1
    PossibleMajorElements.append(GetMajorElements(A[start : end]))
    // Check whether the elements in the PossibleMajorElements array are still 10-major elements in A
    foreach elements in PossibleMajorElements do
        Count its appearances in A, append it into TrueMajorElements if its appearances ≥ ⌊n/10⌋ + 1
    end
end
return TrueMajorElements
First Call: GetMajorElements(A[n])
```

Correctness: As any 10-major element must be a 10-major element in at least one of the subarray, we must get all the possible answers.

(c) **Recurrence:** Suppose $T(n) = 10T(n/10) + O(f(n))$ with $T(1) = 1$

$f(n)$ contains counting the appearance number of each element in PossibleMajorElements within the array A. We've derived that there are at most 9 10-major elements in any subarray, indicating that there are no more than $9 \times 10 = 90$ candidates stored in PossibleMajorElements. Comparing such 90 elements to the original array A costs $O(n)$ time complexity. Thus, $f(n) = n$

$$T(n) = 10T(n/10) + O(n) \text{ with } T(1) = 1$$

(d)

$$T(n) = 10T(n/10) + O(n) = 100T(n/100) + 11n = 1000T(n/1000) + 111n = \cdots = 10^{\log_{10} n}T(1) + 100n\log_{10} n = O(n \log n)$$

Homework 2

Name: LIN, Xuanyu, SID: 20838295, Email Address: xlinbf@connect.ust.hk

1. (20 points) Let $A[1..n]$ be an array that stores n possibly non-distinct numbers.

(a) (10 points) A *range query* specifies two integers x and y that form an interval $(x, y]$. The answer to a range query is the number of elements in A that are greater than x and less than or equal to y. There is NO requirement that x and y are elements of A.

As an example if $A = [4, 5, 2, 6, 8, 3, 4, 4]$, the range query for the interval $(3, 6]$ would return 5 because there are five elements in A that lie in $(3, 6]$, namely $A[1], A[2], A[4], A[7]$, and $A[8]$. Similarly, the range query for the interval $(7, 10]$ would return 1.

Describe an algorithm that generates a data structure C such that C uses $O(n)$ space and you can use C to answer any range query in $O(\log n)$ time. Explain the running time of your algorithm for constructing C. Describe in detail how you use C to answer a range query and explain why it takes $O(\log n)$ time.

(b) (10 points) Suppose that the numbers in $A[1..n]$ come from the range $[1..k]$ for some positive integer k given to you. A *range-sum query* specifies two integers x and y in the range $[1, k]$ that form an interval $(x, y]$. The answer to a range-sum query is the sum of elements in A that are greater than x and less than or equal to y.

As an example if $k = 9$ and $A = [4, 5, 2, 6, 8, 3, 4, 4]$, the range-sum query for the interval $(3, 6]$ would return 23 . The interval $(7, 10]$ does not define a range-sum query because 10 is outside the range $[1, 9]$.

Describe in detail how you would organize the data structure C and construct C so that each range-sum query can be answered in $O(1)$ time. Explain the running time of your algorithm for constructing C. Explain why a range-sum query can be answered in $O(1)$ time.

Solution.

(a) **Main Idea:** Construct a segment tree data structure such that each node stores the count of numbers within a specific range. The root represent the whole range of the sequence, with its left and right children represent two halves of its range. If a range doesn't contain any number, then the node ends.

In such tree structure, we can get the range query by starting at the root node of the segment tree and going through its nodes according to the rule:

- If the current node's range falls completely within the query range, return its count.
- If the current node's range does not overlap with the query range, return 0.
- Otherwise, recursively query the left and right child nodes and return the sum their results.

Space Occupied: $O(1 + 2 + 4 + \cdots + n) = O(2n - 1) = O(n)$

Time Complexity: $O(\log n)$ as it's a binary tree structure.

(b) **Main Idea:** Construct a array with length as large as the maximum number of the given array to store the sum of the numbers that are smaller than the index of the current position.

The detail are as follows:

- Suppose the largest element of the array is k, construct an array of k numbers, calling it $A[k]$.
- $A[0] = 0$. For $i = 0$ to k, $A[i] = A[i - 1] + i \times$ #Number of elements equal to i in the array
- To get the range-sum in $(i, j]$, simply substract $A[j]$ by $A[i]$ to get the answer in $O(1)$ time.

Space Occupied: $O(k)$ where k is the largest element in the array.

Time Complexity: $O(1)$

2. (20 points) Consider a switch with two inputs A and B and two outputs C and D. Each input is a single bit. Each output is also a single bit. The switch has two settings. In one setting, A is connected to C, and B is connected to D. That is, the outputs C and D are just the inputs A and B, respectively. In the other setting, A is connected to D, and B is connected to C. That is, the outputs C and D are the inputs B and A, respectively. Let n be a positive power of 2. Describe how to assemble such switches into one device such that the device has n inputs and n outputs, and by setting the switches appropriately in the device, the ouptuts can be any of the $n!$ permutations of the inputs. Your device should not use more than $O(n \log n)$ switches. Explain the correctness of your device.

Solution.

First Attempt: Make use of the Batcher's odd even merge sort algorithm, construct a sorting network. We can construct a sorting network which consists of $\log_2 n$ stages, with wach stage having $n/2$ switches. (No further idea about such method)

Reference: Wiki: Batcher's Odd Even Merge Sort

Second Attempt: Make use of the binary tree construction. First, construct a binary tree with n inputs being placed at n nodes in a tree with height $\log n$. At each node, a switch is placed to switch the position of, and only the position of its two children. The tree is constructed via pre-order tree transversal. After all the switches are set to a specific state, the output is extracted accordingly also via pre-order tree transversal.

Total switches used:

$$\sum_{i=1}^{\log n} \frac{n}{2^i} = O(n \log n)$$

Correctness: We are proving that different types of combination of switches yeilds distinct solutions, so that using $O(n \log n)$ switches, we are able to perform $2^{n \log n} \sim n!$ distinct types of output.

By modifying a switch at a node, we're only modifying the two children which is directly connected to the current node, i.e, whenever the states of the switches are not identical, the two trees are not identical, resulting in the different extracted result via pre-order tree transversal. Thus, $O(n \log n)$ switches are enough to generate all the $n!$ distinct permutations of the array.

Reference: Tree Traversal Techniques -Data Structure and Algorithm Tutorials

3. (20 points) You are given n numbers, where n is a positive power of 2. Describe an algorithm that finds the largest and second largest numbers in $n + \log_2 n - 2$ comparisons. Explain the correctness of your algorithm. Show that your number of comparisons is indeed $n + \log_2 n - 2$.

Solution.

Main Idea: Use divide-and-conquer to find the largest element, which requires $n - 1$ comparisons. During such comparison, the second largest element must be eliminated by the largest element. As a result, we are able to examine through the $\log_2 n$ to get the second larget element.

• To get the largest element, we've conducted the comparison like a "single elimination tournament" with guaranteed $n - 1$ comparisons.

• Note that the second largest element must be eliminated by the largest element. Thus, we construct a list for every of the n element to store all the elements eliminated by this number. Every time when we eliminate an element, we append the number eliminated by it (the "loser" of the comparison) to the list of the "winner".

• After we've got the largest element, we traverse through the list stored by the largest element to obtain the second largest element. The list contains $\log n$ elements, which requires $\log n - 1$ comparisons.

• Thus, we need $n + \log n - 2$ comparisons in total.

4. (20 points; from text book) Let RANDOM$(1,k)$ be a procedure that draws an integer uniformly at random from $[1,k]$ and returns it. We assume that a call of RANDOM takes $O(1)$ worst-case time. The following recursive algorithm RANDOM-SAMPLE generates a random subset of $[1,n]$ with $m \le n$ distinct elements. Prove that RANDOM-SAMPLE returns a subset of $[1,n]$ of size m drawn uniformly at random.

```
RANDOM-SAMPLE(m, n)
  if m = 0 then
    return ∅
  else
    S ← RANDOM-SAMPLE(m − 1, n − 1)
    i ← RANDOM(1, n)
    if i ∈ S then
      return S = S ∪ {n}
    else
      return S = S ∪ {i}
    end if
    return S
  end if
```

Proof.
The algorithm is identical as the procedure described as below:
• First, randomly choose a number in range $[1, n-m+1]$ to append to S.
• Next, randomly choose a number in range $[1, n-m+2]$ to append to S. If it's already in S, then append $n-m+2$ to S.
• Then, randomly choose a number in range $[1, n-m+3]$ to append to S. If it's already in S, then append $n-m+3$ to S.
• ···
• Finally, randomly choose a number in range $[1, n]$ to append to S. If it's already in S, then append n to S. (A total of m random pick is conducted)
We can check the consistency of the probability by calculating the probability of each number to be chosen.
Consider an arbitrary number k, evaluating the probability of k NOT to be chosen:
• If $k \le n-m+1$, then

$$\text{The number } k \text{ is NOT chosen} = \frac{n-m}{n-m+1} \cdot \frac{n-m+1}{n-m+2} \cdots \frac{n-1}{n} = \frac{n-m}{n}$$

• Suppose $n-m+1 < k \le n$, then we've already chosen $k-n+m-1$ numbers $\Longrightarrow$ At the random draw in range $[1,k]$, the probability of k NOT to be chosen is $[k-(k-n+m-1+1)]/k = (n-m)/k$. (Condition: At the draw in range $[1,k]$, neither the elements that have been drawn nor k itself are pulled out) In the remaining random pick, the calculation is the same as the above circumstance.

$$\text{The number } k \text{ is NOT chosen} = \frac{n-m}{k} \cdot \frac{k}{k+1} \cdots \frac{n-1}{n} = \frac{n-m}{n}$$

From the above discussion, we see that the probabilities of each elements to be drawn are identical, all equals to $1 - \frac{n-m}{n} = \frac{m}{n}$. This yields directly that the probability of each combination of RANDOM-SAMPLE to be drawn is also identical.

5. (20 points, from textbook) We explore a different analysis of the application of randomized quicksort to an array of size n.

(a) (2 points) For $i \in [1, n]$, let X_i be the indicator random variable for the event that the ith smallest number in the array is chosen as the pivot. That is, $X_i = 1$ if this event happens, and $X_i = 0$ otherwise. Derive $\mathrm{E}[X_i]$.

(b) (2 points) Let $T(n)$ be a random variable that denotes the running time of randomized quicksort on an array of size n. Prove that

$$\mathrm{E}\big[T(n)\big] = \mathrm{E}\left[\sum_{i=1}^{n} X_i \cdot (T(i-1) + T(n-i) + \Theta(n))\right].$$

(c) (2 points) Prove that $E\big[T(n)\big] = \frac{2}{n} \cdot \sum_{i=2}^{n-1} \mathrm{E}\big[T(i)\big] + \Theta(n)$.

(d) (7 points) Prove that $\sum_{k=1}^{n-1} k \log k \le \frac{1}{2} n^2 \log n - \frac{1}{8} n^2$. (Hint: Consider $k = 2, 3, \ldots, (n/2) - 1$ and $k = n/2, \ldots, n-1$ separately.)

(e) (7 points) Use (d) to show that the recurrence in (c) yields $\mathrm{E}\big[T(n)\big] = \Theta(n \log n)$. (Hint: Use substitution to show that $\mathrm{E}\big[T(n)\big] \le cn \log n$ for some positive constant c when n is sufficiently large.)

Note: The summation range should be from 2 to $n-1$ instead of 1 to $n-1$ for (d).

Solution.

(a) In the quicksort, the probabilities of each element to be chosen are identical, all equals to $1/n$, where n indicates the number of elements.

$$\implies E[X_i] = \frac{1}{n}$$

(b) Note that according to the quicksort, the array to be sorted is splitted into two subarrays according to the element that is randomly chosen. The two subarrays are of length $i-1$ and $n-i$ if the i-th smallest element is chosen randomly. According to the recursive algorithm, the ”Merge” step requires $\Theta(n)$ time. Thus,

$$E[T(n)] = E\left[\sum_{i=1}^{n} X_i \cdot (T(i-1) + T(n-i) + \Theta(n))\right]$$

(c) The deriviation is as follows:

$$E[T(n)] = E\left[\sum_{i=1}^{n} X_i \cdot (T(i-1) + T(n-i) + \Theta(n))\right] = E\left[\sum_{i=1}^{n} \frac{1}{n} \cdot (T(i-1) + T(n-i) + \Theta(n))\right]$$
$$= \Theta(n) + \frac{1}{n} \cdot E\left[\sum_{i=1}^{n}(T(i-1) + T(n-i))\right] = \Theta(n) + \frac{2}{n}\sum_{i=1}^{n} E[T(i-1)] = \Theta(n) + \frac{2}{n}\sum_{i=0}^{n-1} E[T(i)] = \Theta(n) + \frac{2}{n}\sum_{i=2}^{n-1} E[T(i)]$$

(d) According to the integral inequality, the left hand rule,

$$\sum_{k=2}^{n-1} k \log k \le \int_2^n k \log k = \frac{\frac{k^2}{2}\log k - \frac{k^2}{4}}{\log 2}\Bigg|_2^n = \frac{n^2}{2}\log n - \frac{n^2}{4\log 2} - 1$$

$$\implies \sum_{k=2}^{n-1} k \ln k \le \frac{n^2}{2}\log n - \frac{n^2}{4\log 2} - 1 \le \frac{n^2}{2}\log n - \frac{n^2}{8}$$

(e) Recurrence:

$$E[T(n)] = \Theta(n) + \frac{2}{n}\sum_{i=2}^{n-1} E[T(i)] \le \Theta(n) + \frac{2}{n}\sum_{i=2}^{n-1}[i \log i + \Theta(n)] \le \Theta(n) + \frac{2}{n}\sum_{i=2}^{n-1} i \log i$$
$$\le \frac{2}{n}\left[\frac{n^2 \log n}{2} - \frac{n^2}{8}\right] + \Theta(n) = n \log n - \frac{1}{4}n + \Theta(n) = n \log n + \Theta(n)$$

Problem 5

Solution.

Problem 6

Solution.

Problem 7

Solution.

Problem 8

Solution.