Writing Exercise

1 Exercise 1

`Pf:`

$$\begin{aligned}
\mathrm{Var}(X+Y) &= \mathrm{E}(X+Y-\mathrm{E}(X)-\mathrm{E}(Y))^2 \\
&= \mathrm{E}((X-\mathrm{E}(X))+(Y-\mathrm{E}(Y)))^2 \\
&= \mathrm{E}(X-\mathrm{E}(X))^2+\mathrm{E}(Y-\mathrm{E}(Y))^2+2\mathrm{E}(X-\mathrm{E}(X))\mathrm{E}(Y-\mathrm{E}(Y)) \\
&= \mathrm{Var}(X)+\mathrm{Var}(Y)+2\mathrm{Cov}(X,Y)
\end{aligned}$$

`Q.E.D`

2 Exercise 2

`Solve:`

$$\begin{aligned}
A &= \text{Have the disease} \\
B &= \text{Test positive} \\
\Pr(B \mid A) &= \Pr(\text{not } B \mid \text{not } A) = 0.99 \\
\Pr(A) &= 0.0001 \\
\Pr(B) &= \Pr(B \mid A)\Pr(A) + \Pr(B \mid \text{not } A)\Pr(\text{not } A) \\
&= \Pr(B \mid A)\Pr(A) + (1-\Pr(\text{not } B \mid \text{not } A))(1-\Pr(A)) \\
&= 0.99 \cdot 0.0001 + (1-0.99)(1-0.0001) \\
&= 0.010098 \\
\Pr(A \mid B) &= \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)} \\
&= \frac{0.99 \cdot 0.0001}{0.010098} \\
&\approx 0.0098 \\
&< 1.0\%
\end{aligned}$$

3 Exercise 3

1. (a) `Pf:`

$$\begin{aligned}
\frac{\mathrm{d}\sigma(a)}{\mathrm{d}(a)} &= \frac{0+-e^{-a}}{(1+e^{-a})^2} \\
&= \frac{-e^{-a}}{(1+e^{-a})^2} \\
&= \frac{1}{1+e^{-a}}(1-\frac{1}{1+e^{-a}}) \\
&= \sigma(a)(1-\sigma(a))
\end{aligned}$$

Q.E.D

2. (b) Pf:

$$\begin{aligned}
l(\beta) &= \sum_{i=1}^{N}\{y_i\beta^T x_i - log(1+e^{\beta^T x_i})\} \\
\frac{\partial l(\beta)}{\partial \beta} &= \sum_{i=1}^{N}\{x_i y_i - \frac{x_i e^{\beta^T x_i}}{1+e^{\beta^T x_i}}\} \\
&= \sum_{i=1}^{N} x_i(y_i - \frac{e^{\beta^T x_i}}{1+e^{\beta^T x_i}}) \\
&= \sum_{i=1}^{N} x_i(y_i - p(x_i;\beta))
\end{aligned}$$

Q.E.D

3. (c) Pf:

$$\begin{aligned}
\text{Let } z &= \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_p \end{bmatrix} \\
X &= \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N1} & \cdots & x_{Np} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \\
W &= \begin{bmatrix} w_{11} & 0 & \cdots & 0 \\ 0 & w_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{NN} \end{bmatrix}
\end{aligned}$$

As for any non-zero vector z has:

$$\begin{aligned}
z^T(X^TWX)z &= (Xz)^TW(Xz) \\
&= \begin{bmatrix} z\cdot X_1 \\ z\cdot X_2 \\ \vdots \\ z\cdot X_N \end{bmatrix}^T \times \begin{bmatrix} w_{11} & 0 & \cdots & 0 \\ 0 & w_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{NN} \end{bmatrix} \times \begin{bmatrix} z\cdot X_1 \\ z\cdot X_2 \\ \vdots \\ z\cdot X_N \end{bmatrix} \\
&= \sum_{i=1}^{N} w_{ii}(z\cdot X_i)^2 \\
\because w_{ii} &= p(x_i;\beta)(1-p(x_i;\beta)) \\
\therefore \text{All } w_{ii} &> 0 \\
\therefore z^T(X^TWX)z &> 0
\end{aligned}$$

$\therefore X^TWX$ is positive definite.

Q.E.D