Portfolio Optimization

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Outline

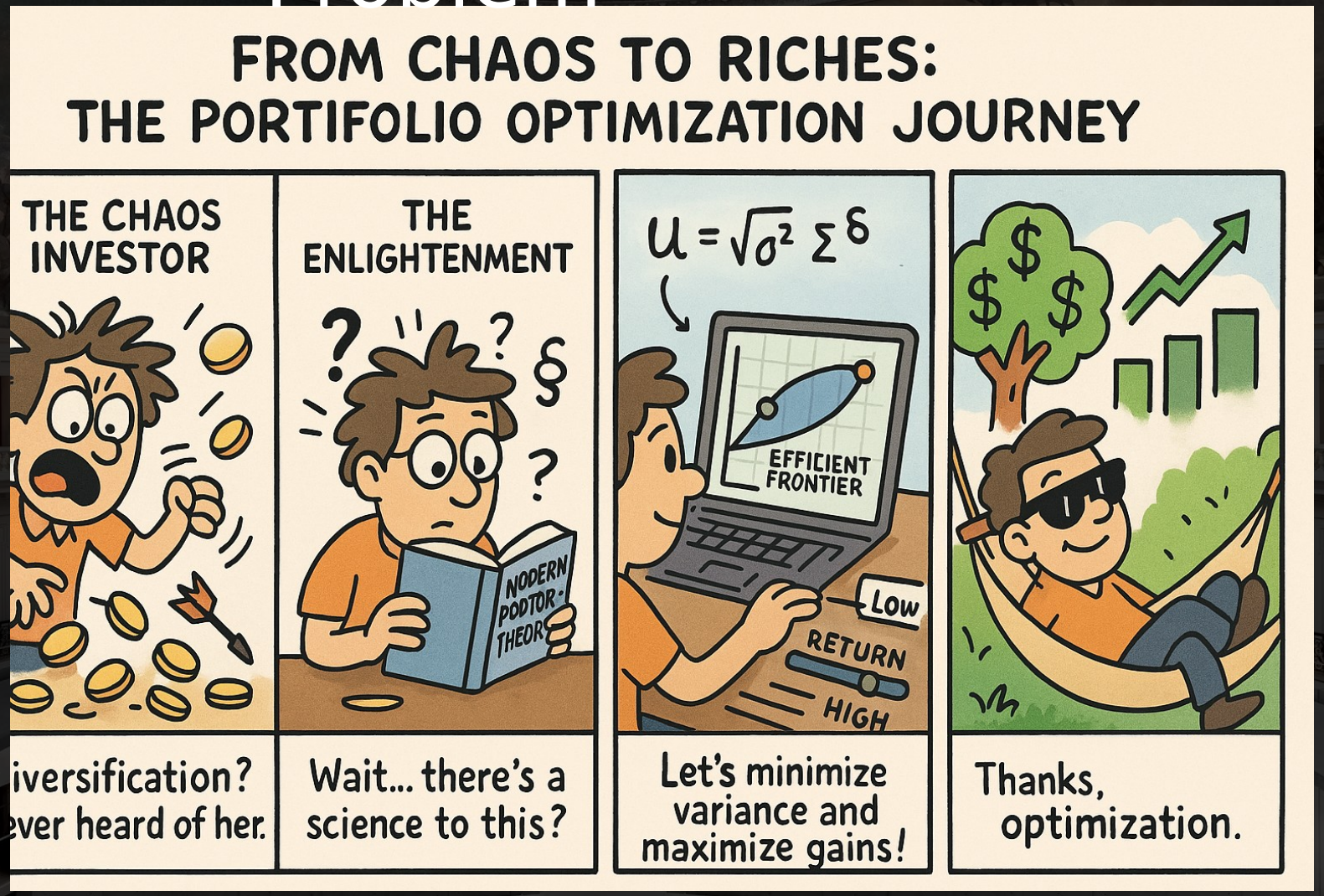
The problem of portfolio optimization

Markowitz / CVar

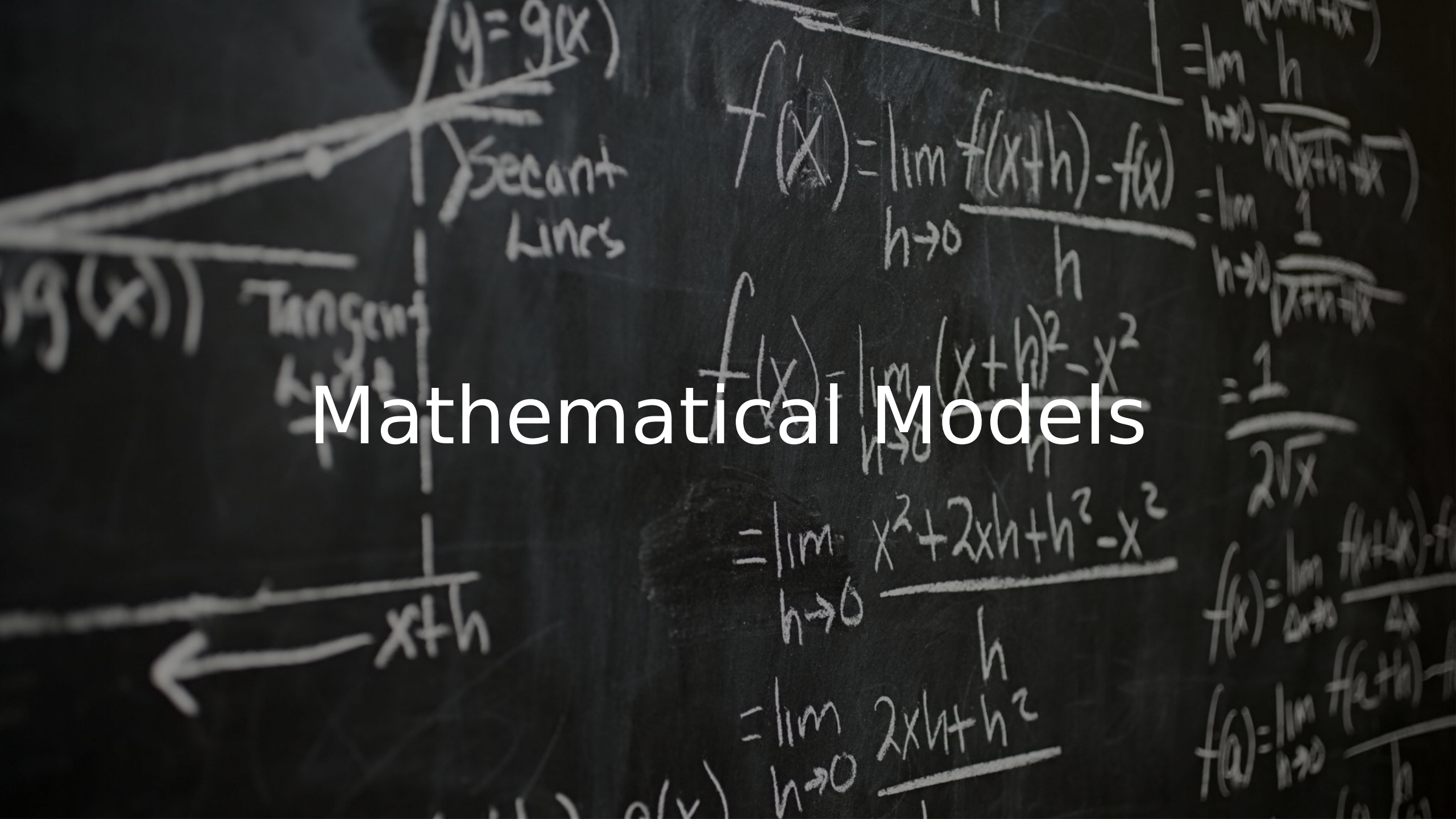
Implementations

- QP - SLSQP
- GA
- CP-SAT
- MINLP

The Portfolio Optimization Problem



Mathematical Models



Markowitz Model
Sharpe Ratio Maximization

Sharpe Ratio

$$S(w) = \frac{w^T r - R_f}{\sqrt{w^T \Sigma w}}$$

$$\frac{\text{Expected Return}_{\text{Portfolio}} - \text{Risk-Free Rate}}{\text{Standard Deviation}_{\text{Portfolio}}}$$

Objective Function

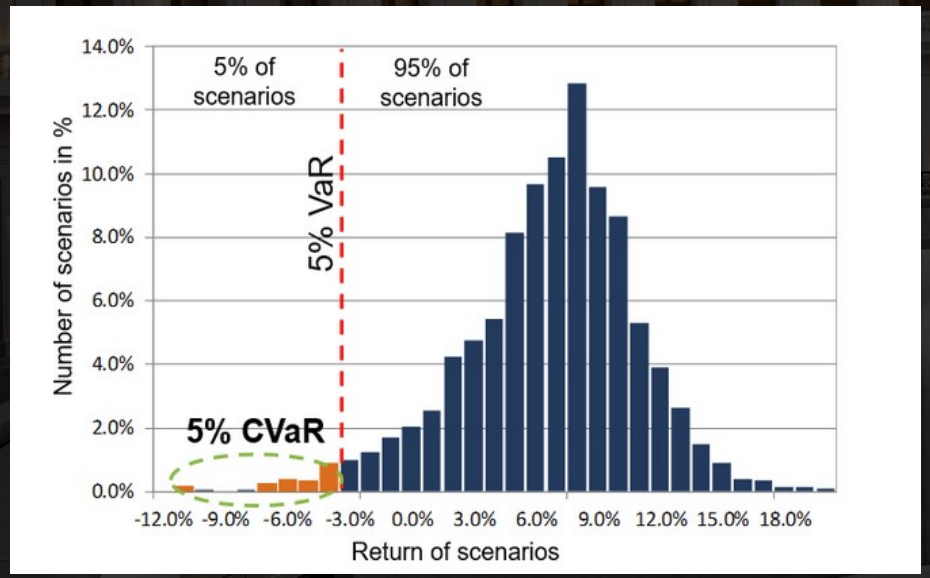
$$\max_{w} \quad \frac{w^T r - R_f}{\sqrt{w^T \Sigma w}}$$

$$\sum_{i=1}^{n} w_i = 1$$

CVaR Model
CVaR Minimization

CVaR Formula

$$CVaR = \frac{1}{1-c}\int_{-1}^{VaR} xp(x)\,dx$$

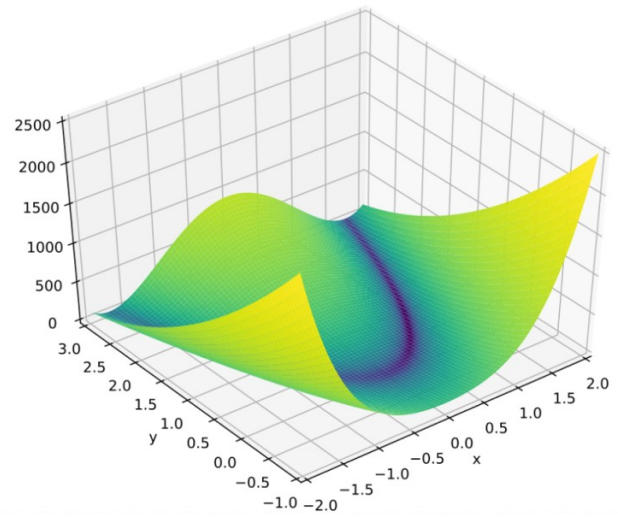


Objective Function: Rockafellar-Uryasev

$$\mathrm{CVaR}_\alpha(X) = \min_{\eta\in\mathbb{R}}\left(\eta + \frac{1}{1-\alpha}\mathbb{E}[(X-\eta)_+]\right)$$

Implementations

Generic QP Problem



$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}\mathbf{x}^{\mathrm{T}} Q \mathbf{x} + \mathbf{c}^{\mathrm{T}} \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \preceq \mathbf{b}, \end{aligned}$$

QP Markowitz Reformulation

$$\max_x \frac{\mu^T x}{\sqrt{x^T Q x}}$$

there is a common trick (section 5.2) used to put the problem in convex form. You add a variable κ such that $x = y/\kappa$ choose κ s.t. $\mu^T y = 1$. Changing the problem to the simple convex problem

$$\min_{y,\kappa} y^T Q y \text{ where } \mu^T y = 1, \kappa > 0$$

- Interior Point Method
- Active Set
- Conjugate Gradient (some reformulation tointegrate inequalities)
- Augmented Lagrangian

$$min_{w,k}\ w^T \Sigma w$$

$$w_i \geq 0$$

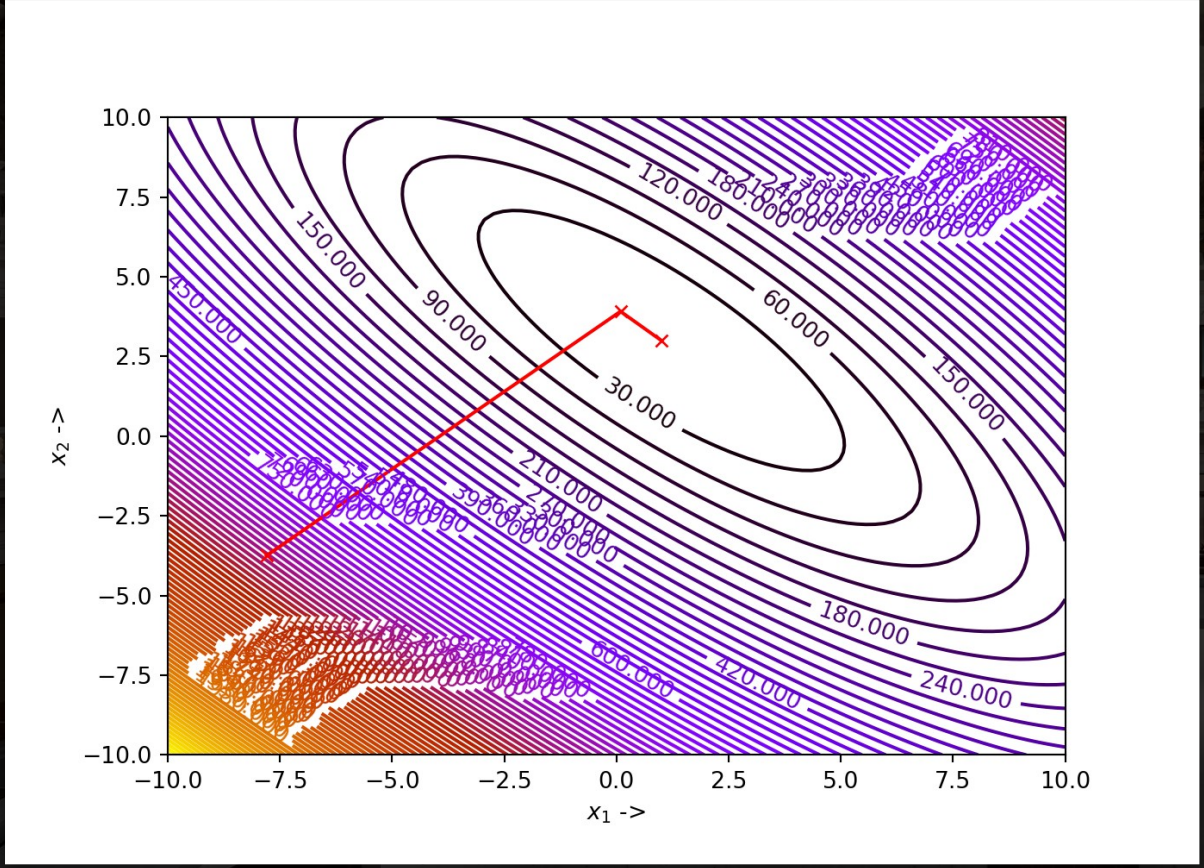
$$\mathbf{1}^T w = 1$$

$$r^T w = 1$$

$$\mathbf{1}^T k \geq 1$$

<u>Problems</u>: No cardinality, no risks

Non-Linear QP - SLSQP



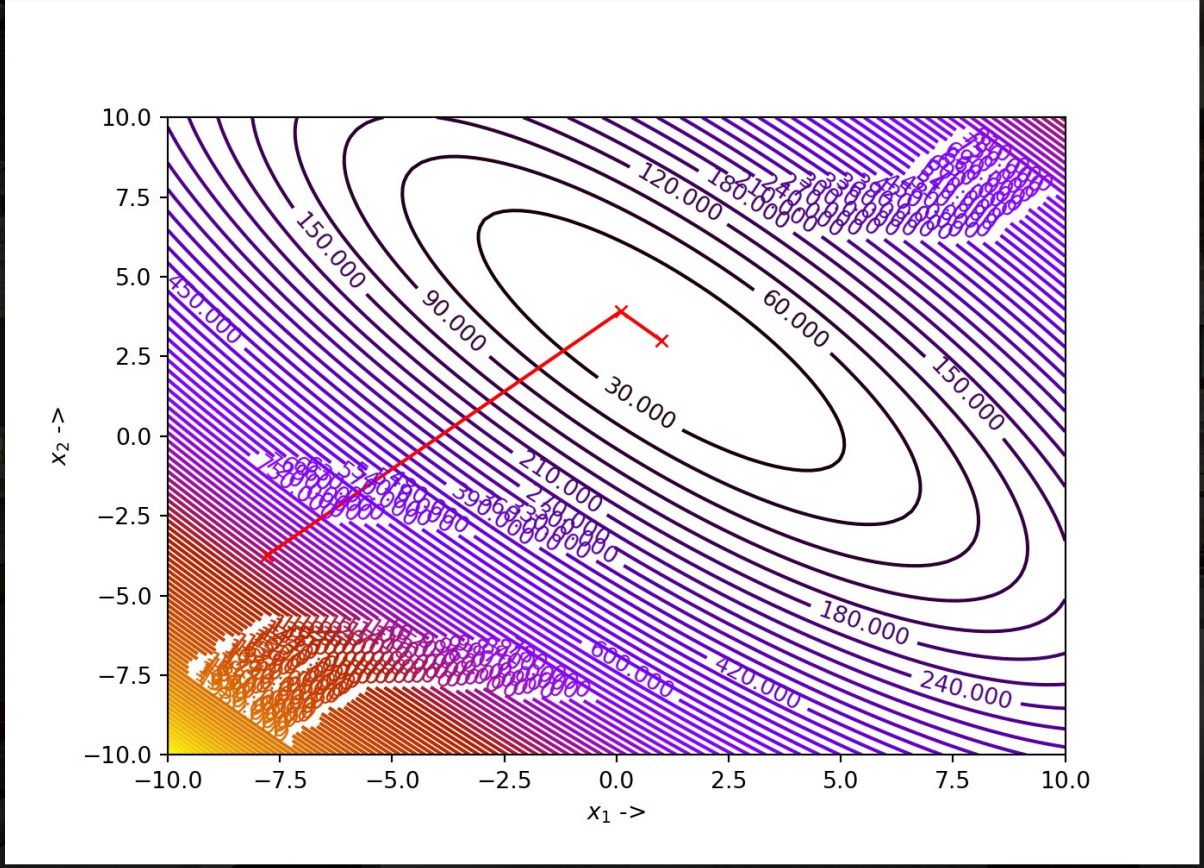
Newton's Method

$$f(x_k + t) \approx f(x_k) + f'(x_k)t + \frac{1}{2}f''(x_k)t^2.$$

$$x_{k+1} = x_k + t = x_k - \frac{f'(x_k)}{f''(x_k)}.$$

Problems: No cardinality

Non-Linear QP - SLSQP



BFGS Method (Quasi Newton)

The Hessian approximation B is chosen to satisfy

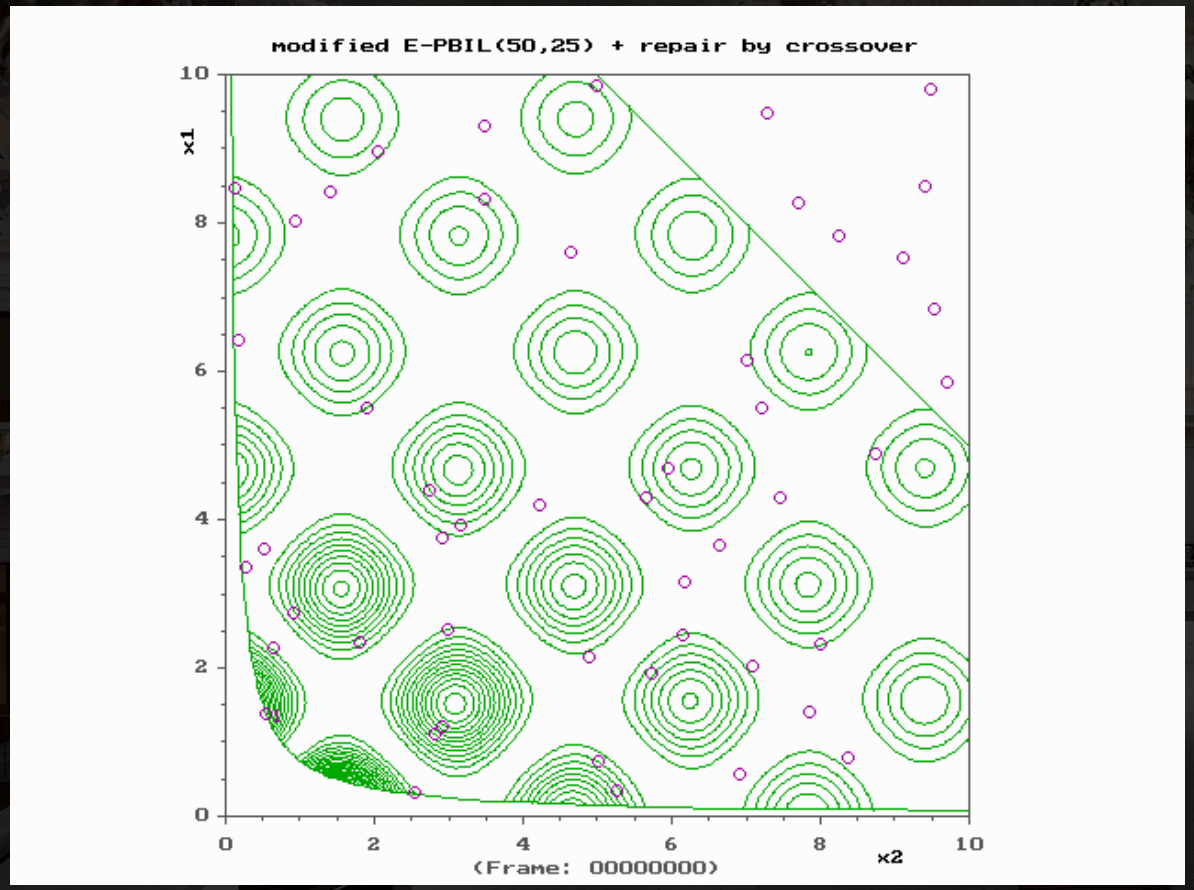
$$\nabla f(x_k + \Delta x) = \nabla f(x_k) + B\,\Delta x,$$

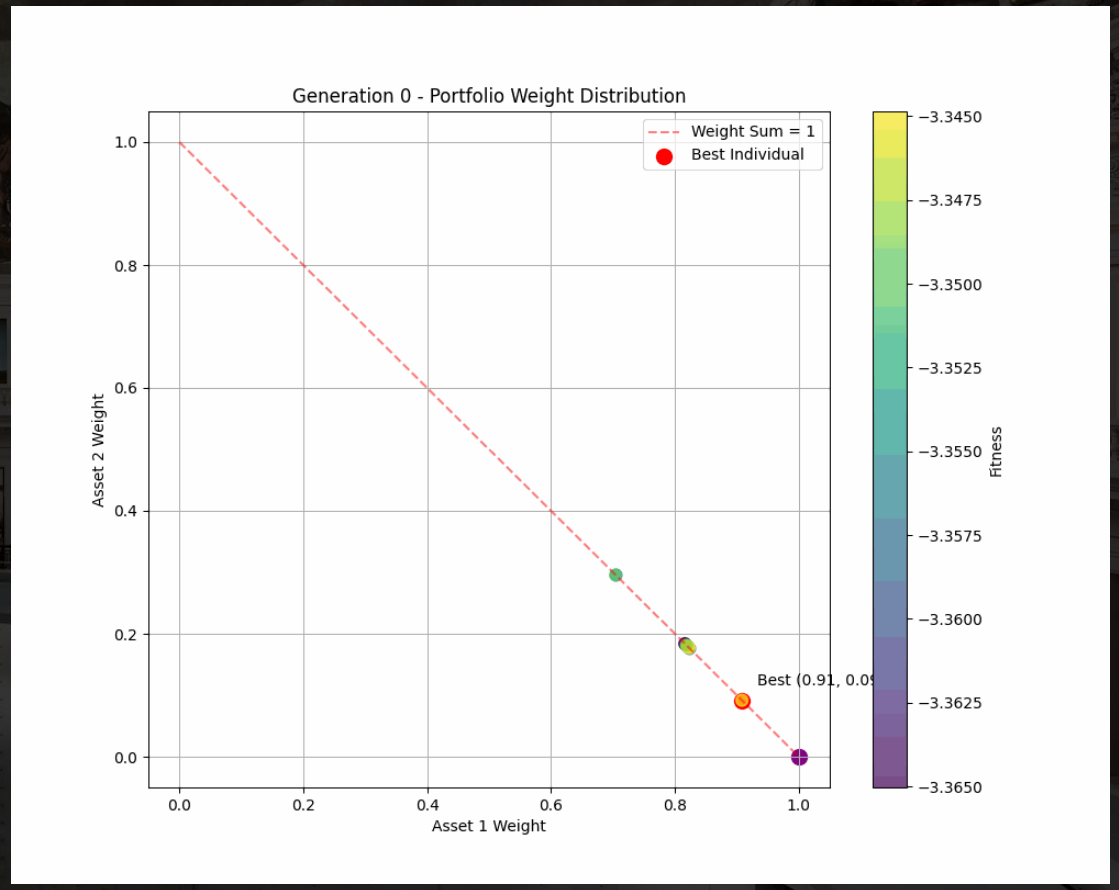
Algorithm 1 Deterministic BFGS Method

initial parameter w_0, inverse Hessian approximation $H_0, \varepsilon > 0$;
$k = 0$;
while $\|\nabla F(w_k)\| > \varepsilon$ **do**;
  Compute search direction $p_k \leftarrow -H_k \nabla F(w_k)$;
  Set $w_{k+1} = w_k + \alpha_k p_k$ where α_k is computed from a line search procedure;
  Define $s_k = w_{k+1} - w_k$ and $y_k = \nabla F(w_{k+1}) - \nabla F(w_k)$;
  $H_{k+1} = (\boldsymbol{I} - \rho_k s_k y_k^\top) H_k (\boldsymbol{I} - \rho_k y_k s_k^\top) + \rho_k s_k s_k^\top$
  $k = k + 1$;
end while

SLSQP: solve QP until convergence using BFGS updates

Evolutionary Algorithm - Genetic Algorithm





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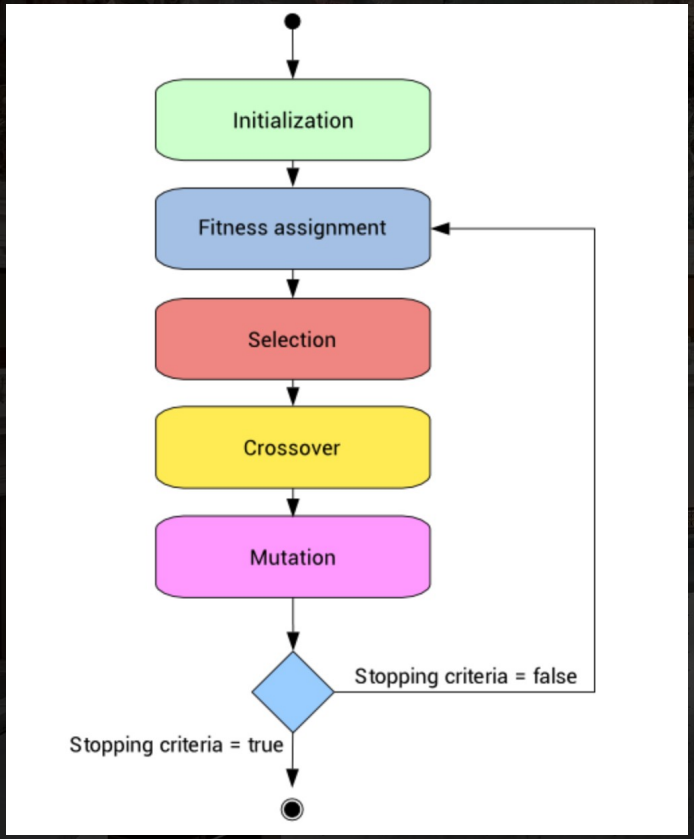
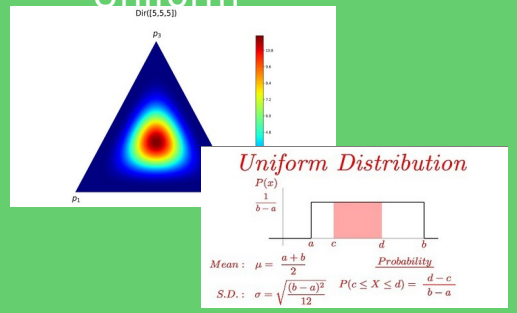
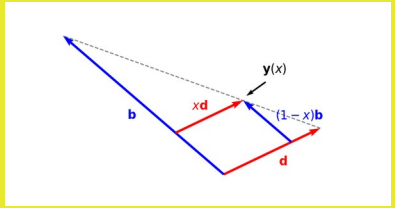


Dirichlet / Uniform



Convex / Arithmetic

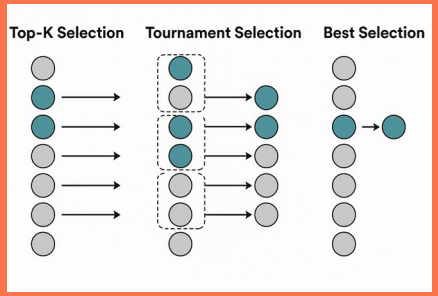
$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$



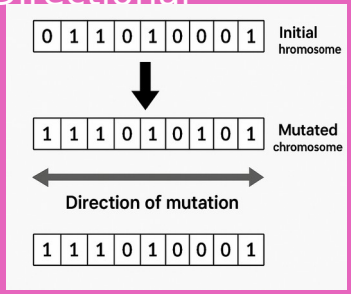
Sharpe Ratio

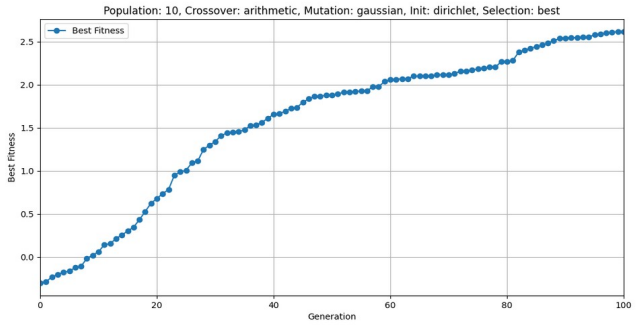
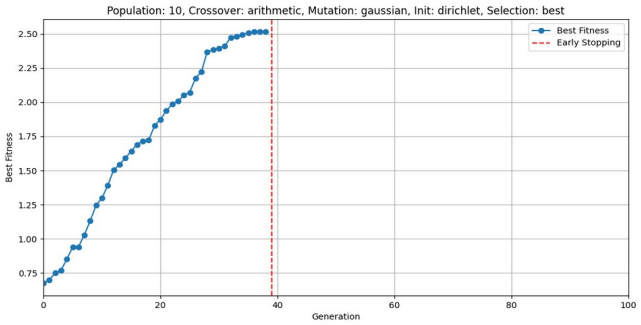
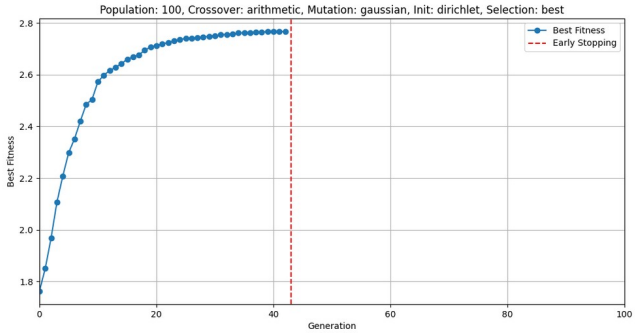
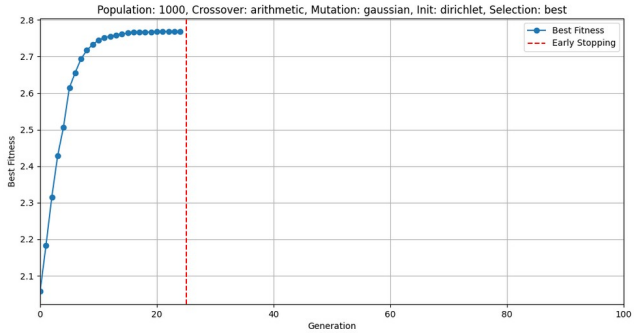
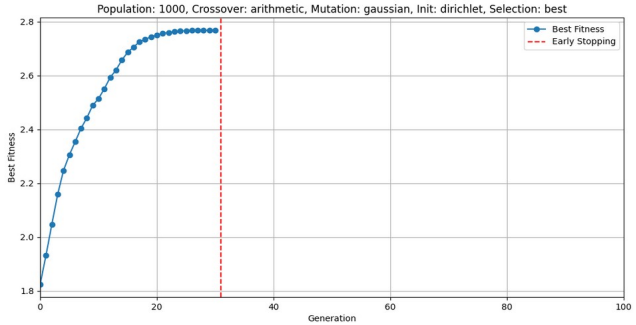
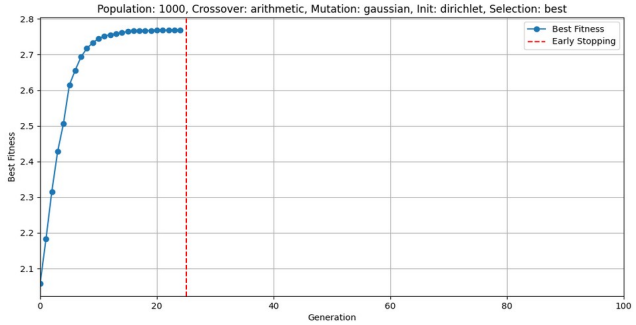
$$S(w) = \frac{w^T r - R_f}{\sqrt{w^T \Sigma w}}$$

Top K / Tournament / Best

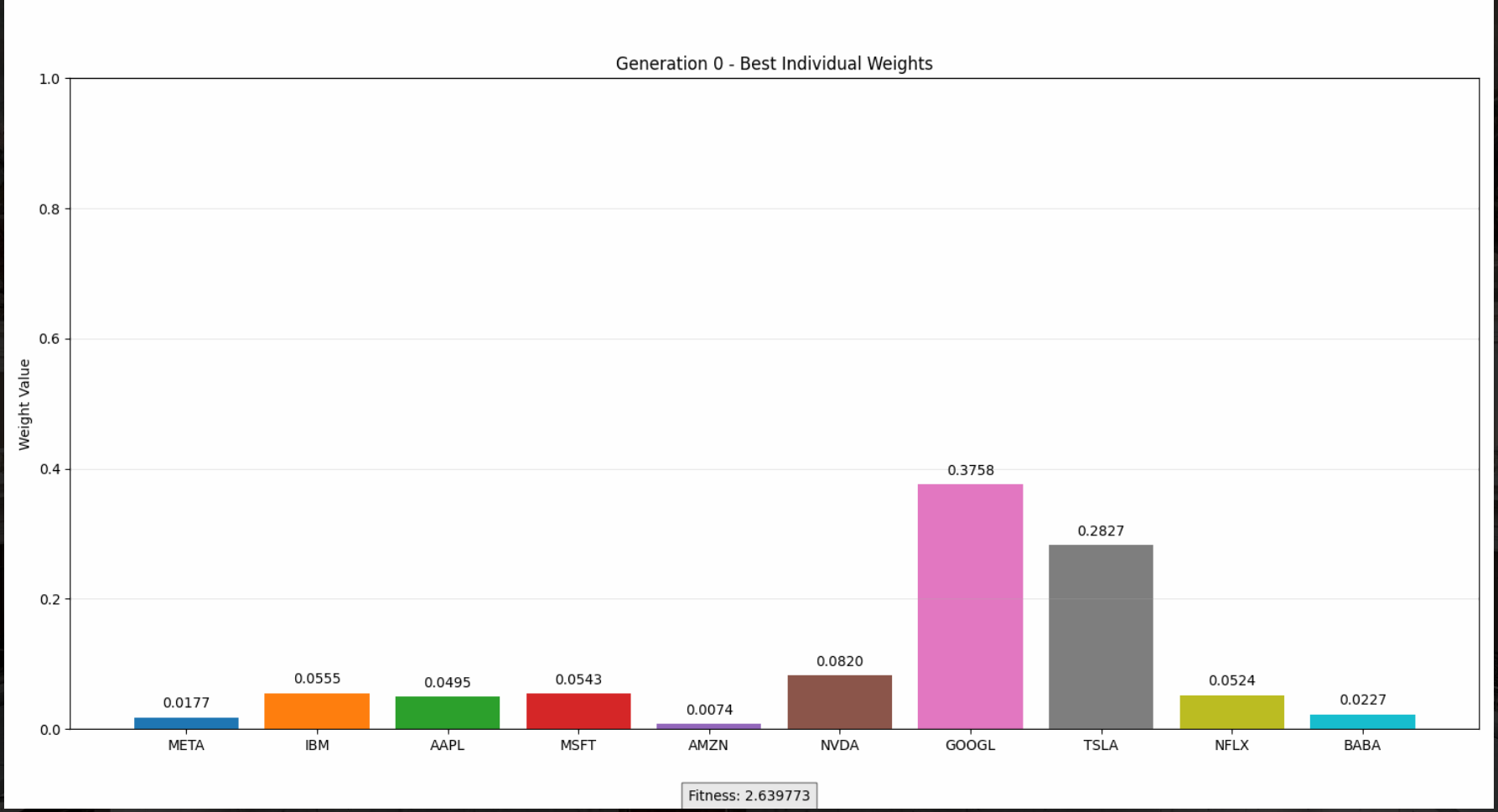


Gaussian / Directional



	Convergence	Convergence with more mutations / crossovers	Convergence	Best Fitness	Time (s)
GA – 10 genomes			100/40	1.67	0.6 - 2
GA – 100 genomes			45/25	2.76	7 - 13
GA – 1000 genomes			35/25	2.8	23 -50

Evolutionary Algorithm - Genetic Algorithm



Problems: Local Optima in High dimensions, slow

CP-SAT

Optimize a portfolio by maximizing a surrogate Sharpe ratio

Expected Return: Weighted sum of asset returns
Risk: Weighted sum of asset volatilities (standard deviations)

Tradeoff between excess return (after risk-free rate) and risk penalty using integer weights.

*risk rate * (portfolio excess return) − non risk rate * (portfolio risk)*

Why CP-SAT ?

- Portfolio solver, concurrently executes diverse algorithms and strategies, each with unique strengths and weaknesses.
- Branch and Cut techniques, including linear relaxations and cutting planes
- Lazy Clause Generation

Preprocessing Steps

Model Loading and Verification (constraints, variables, objective function)

Pre-processing

- Presolve (domain reduction)
- Expansion of higher-level constraints
- Detection of equivalent variables and affine relations
- Substitution with canonical representations
- Variable probing

Solution Search Steps

- **Parallel Subsolvers**
  Multiple solvers run in parallel, each with a unique strategy (e.g., aggressive restarts, focus on bounds, linearized models)

- **Feasibility Searchers**
  Additional threads aim to quickly find any valid solution

- **Improvement Phase (LNS)**
  Once feasible, local search heuristics (LNS) take over, rotating strategies in a Round Robin fashion to refine the solution

- **Optimality**
  Any subsolver can find the optimal solution, some just get there faster

MILP/MINLP : Mixed Integers (Non-)Linear Programming

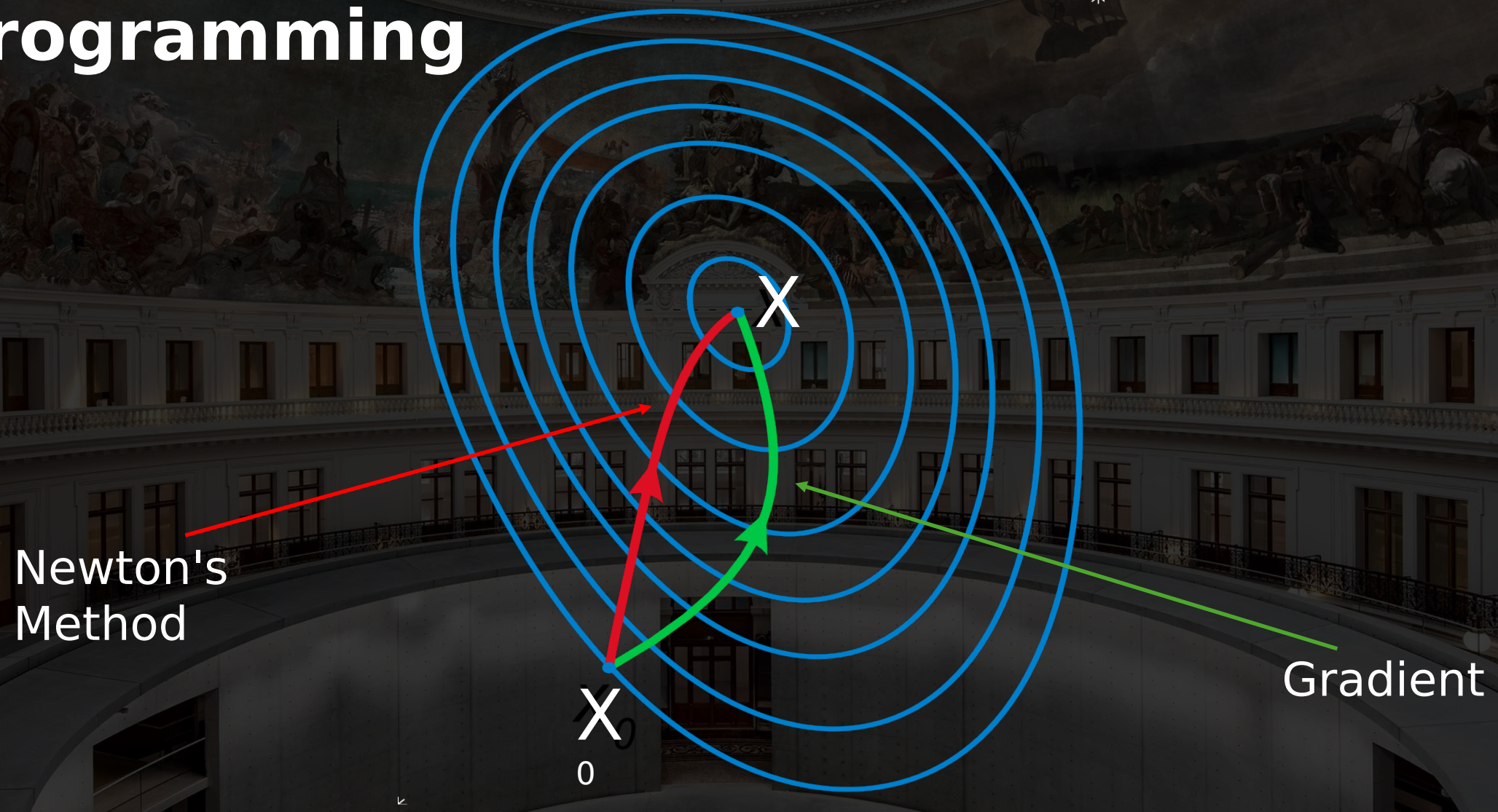
- **Integer Constraints**
  Binary Constraints (Do I select the asset ? Yes or No = 1 or 0)

- **Karush-Kuhn-Tucker (KKT) conditions**
  Define the feasibility space, describe the optimality

- **Newton's Method**
  Iterative method to solve the KKT system matrix and find an approximation of the roots

- **Search direction**
  Solving Δx = Tells the optimizer where and how far to move

MILP/MINLP : Mixed Integers (Non-)Linear Programming



User Interface

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Documentation

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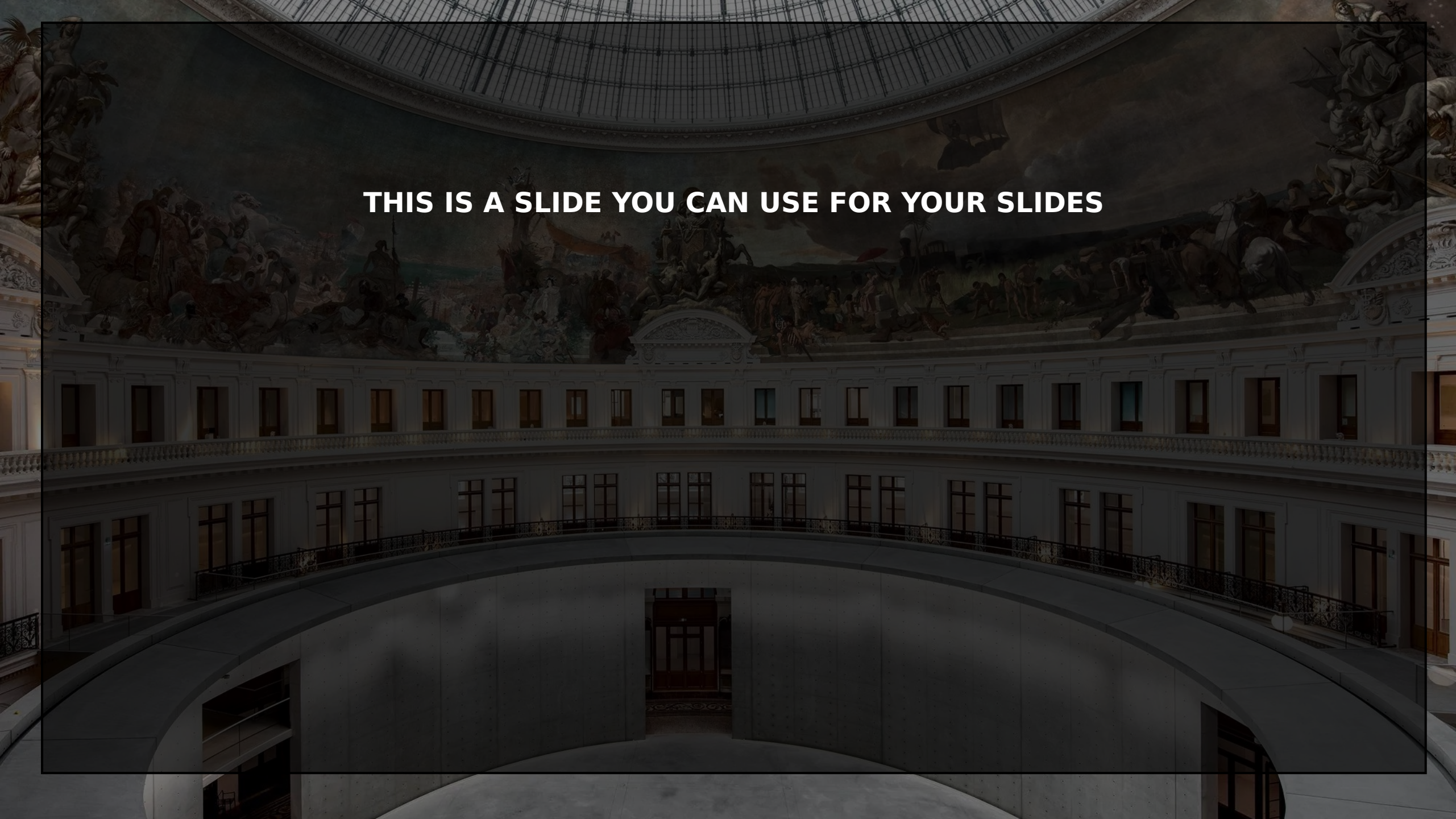
SciPy: https://docs.scipy.org/doc/scipy/
PyGad: https://pygad.readthedocs.io/en/latest/
MEALPY: https://mealpy.readthedocs.io/en/latest/
Streamlit: https://docs.streamlit.io/
Quandl: https://pypi.org/project/Quandl/
NLopt: https://nlopt.readthedocs.io/en/latest/
Gekko: https://gekko.readthedocs.io/en/latest/
Ortools: https://developers.google.com/optimization/reference/python/sat/python/cp_model
CVXPY: https://www.cvxpy.org/index.html



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