1. For the Kelly problem, let us assume that we only get to bet once.
(a) Write an expression for the expected value of the wealth after one bet. Plot this as a function of the betting fraction $f(0 \leq f \leq 1)$. Does this have a maxima?

Solution.
$\Rightarrow$ Let the initial wealth be $I_0 = \alpha + \beta$

Let's assume we bet $\boldsymbol{\beta}$ and get fraction $\boldsymbol{k_1}$ of our bet as reward if we win and lose a fraction $\boldsymbol{k_2}$ of our betting money if we lose the bet.

The probability of winning is $\boldsymbol{p}$ and that of losing is $\boldsymbol{q}$.
$\Rightarrow \beta = fI_0$, where f is the Kelly fraction.

Now, if he wins after one bet,

$$\begin{aligned} \Rightarrow I_1 &= \alpha + \beta + k_1\beta \\ &= I_0 + k_1\beta \\ &= I_0 + k_1 f I_0 \\ &= I_0(1 + k_1 f) \end{aligned}$$

$$\Rightarrow W = I_0(1 + k_1 f) \tag{1}$$

Now, if he loses after one bet,

$$\begin{aligned} \Rightarrow I_1 &= \alpha + \beta - k_2\beta \\ &= I_0 - k_2\beta \\ &= I_0 - k_2 f I_0 \\ &= I_0(1 - k_2 f) \end{aligned}$$

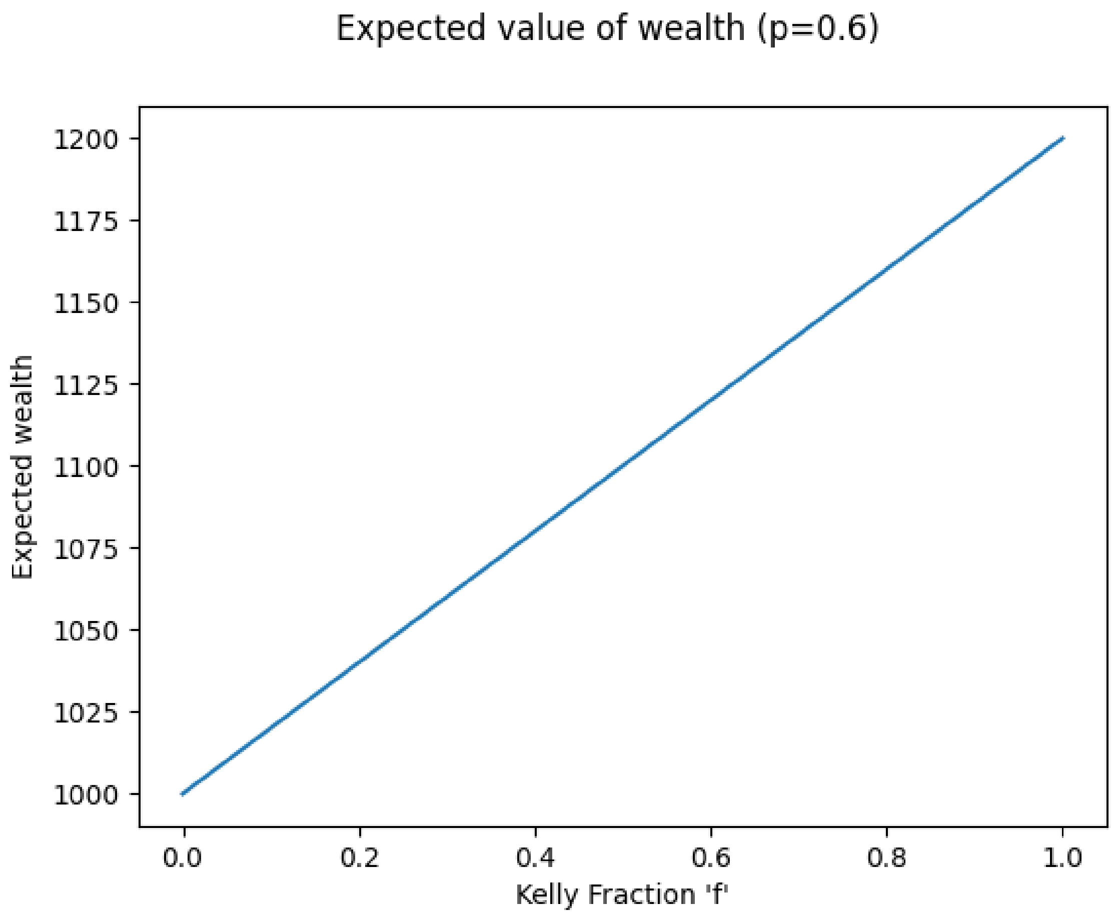
$$\Rightarrow L = I_0(1 - k_2 f) \tag{2}$$

Now the expectation of wealth after one bet is,

$$\begin{aligned} \Rightarrow E[I] &= pW + qL \\ &= pI_0(1 + k_1 f) + qI_0(1 - k_2 f) \\ &= pI_0(1 + k_1 f) + (1 - p)I_0(1 - k_2 f) \\ &= I_0(1 + pk_1 f + pk_2 f - k_2 f) \\ &= I_0(1 + f(p(k_1 + k_2) - k_2) \end{aligned}$$

if $k_1 = k_2 = k$,

$$\Rightarrow E[I] = I_0(1 + kf(2p - 1)) \tag{3}$$



$\therefore$ the function does not have a maxima ■

(b) Write an expression for the expected value of the log of wealth after one bet. Plot this as a function of the betting fraction $f(0 \leq f \leq 1)$. Does this have a maxima?

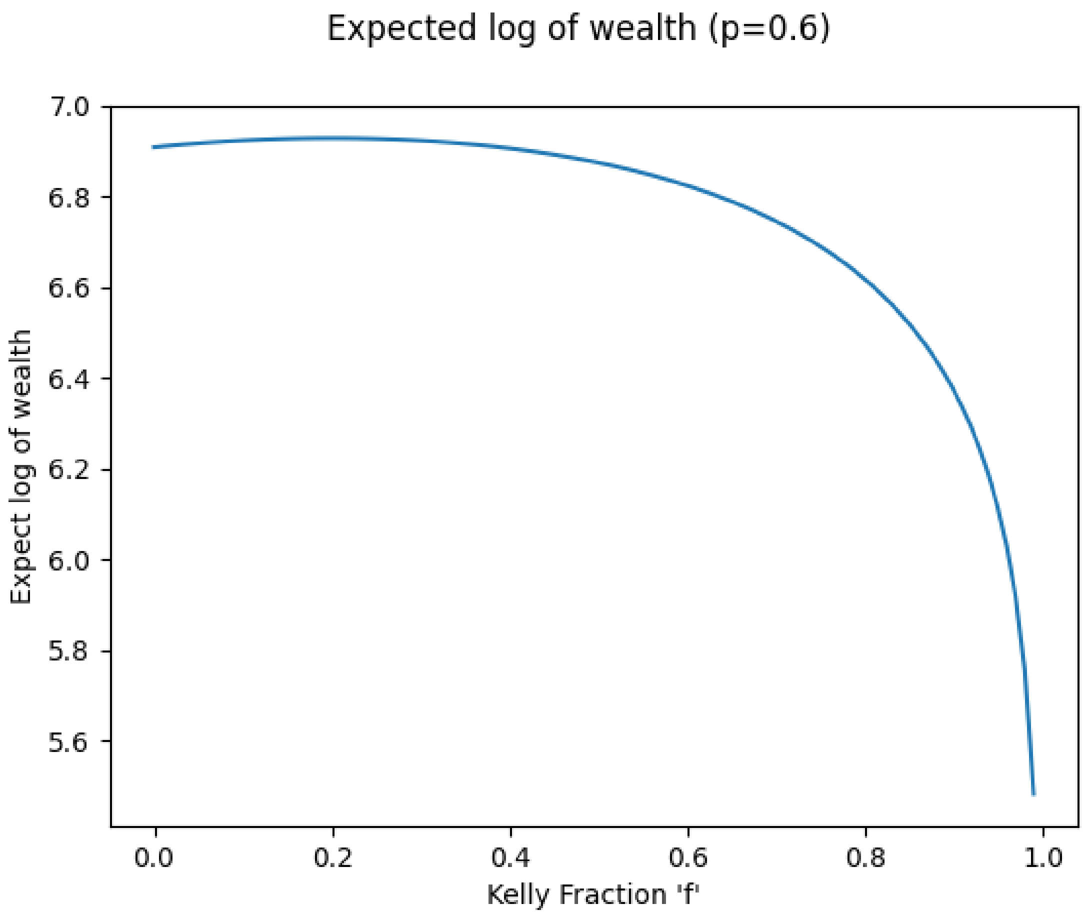
Solution.
From above equations,

$$\Rightarrow W = I_0(1 + k_1 f)$$
$$\Rightarrow L = I_0(1 - k_2 f)$$

$$\begin{aligned}
\Rightarrow E[\log I] &= p \log W + q \log L \\
&= p \log\left(I_0(1 + k_1 f)\right) + q \log\left(I_0(1 - k_2 f)\right) \\
&= p(\log I_0 + \log(1 + k_1 f)) + q(\log I_0 + \log(1 - k_2 f)) \\
&= (p + q) \log I_0 + p \log(1 + k_1 f) + q \log(1 - k_2 f) \\
&= \log I_0 + p \log(1 + k_1 f) + q \log(1 - k_2 f)
\end{aligned}$$

if $k_1 = k_2 = k$,

$$\Rightarrow E[\log(I)] = \log I_0 + p \log(1 + kf) + q \log(1 - kf) \tag{4}$$



$\therefore$ the function has a maxima at $f = 0.2$ for the given conditions $k_1 = k_2 = 1, p = 0.6, I_0 = 1000$ ■

2. Refer to Section 3 in the biological version of the Kelly-criterion in article [3] (also uploaded on LMS). Compute the steps leading from Eq. (5) to Eq. (8). Under what conditions does Eq. (8) reduce to the Kelly fraction, i.e., $G_{max} = 2P_Y - 1$? Derive an expression for the highest growth rate.

Solution.
Given equation,

$$\Rightarrow \lim_{N\to\infty} \frac{\log S_n}{N} = (1-P_Y)\log\left[(1-G)(1-D)\right] + P_Y \log\left[(1-G)(1-D)+GY\right] \quad (5)$$

Taking partial derivative wrt to G,

$$\Rightarrow \frac{\partial \lim_{N\to\infty} \frac{\log S_n}{N}}{\partial G} = -\frac{(1-P_Y)(1-D)}{(1-G)(1-D)} + P_Y \frac{Y-(1-D)}{(1-D)(1-G)+GY}$$
$$= -\frac{1-P_Y}{1-G} + P_Y \frac{Y+D-1}{(1-D)(1-G)+GY}$$

$$\Rightarrow \frac{\partial \lim_{N\to\infty} \frac{\log S_n}{N}}{\partial G} = -\frac{1-P_Y}{1-G} + P_Y \frac{Y+D-1}{(1-D)(1-G)+GY} \tag{6}$$

Setting equation above equation to zero,

$$\begin{aligned}
&\Rightarrow -\frac{1-P_Y}{1-G} + P_Y \frac{Y+D-1}{(1-D)(1-G)+GY} = 0 \\
&\Rightarrow P_Y \frac{Y+D-1}{(1-D)(1-G)+GY} = \frac{1-P_Y}{1-G} \\
&\Rightarrow P_Y \frac{Y+D-1}{1-P_Y} = (1-D) + \frac{GY}{1-G} \\
&\Rightarrow P_Y + DP_Y - P_Y + D - DP_Y - 1 + P_Y = GY\frac{1-P_Y}{1-G} \\
&\Rightarrow (YP_Y + D - 1)(1-G) = GY(1-P_Y) \\
&\Rightarrow YP_Y + D - 1 - GYP_Y - GD + G = GY - GYP_Y \\
&\Rightarrow YP_Y + D - 1 = G(Y+D-1) \\
&\Rightarrow G = \frac{YP_Y + D - 1}{Y+D-1}
\end{aligned}$$

$$\Rightarrow G = P_Y - (1-P_Y)\frac{1-D}{Y+D-1} \tag{7}$$

To reduce the equation to Kelly fraction,

$$\begin{aligned}
&\Rightarrow Y + D - 1 = 1 - D \\
&\Rightarrow Y + 2D = 2 \\
&\Rightarrow Y = 2(1-D)
\end{aligned}$$

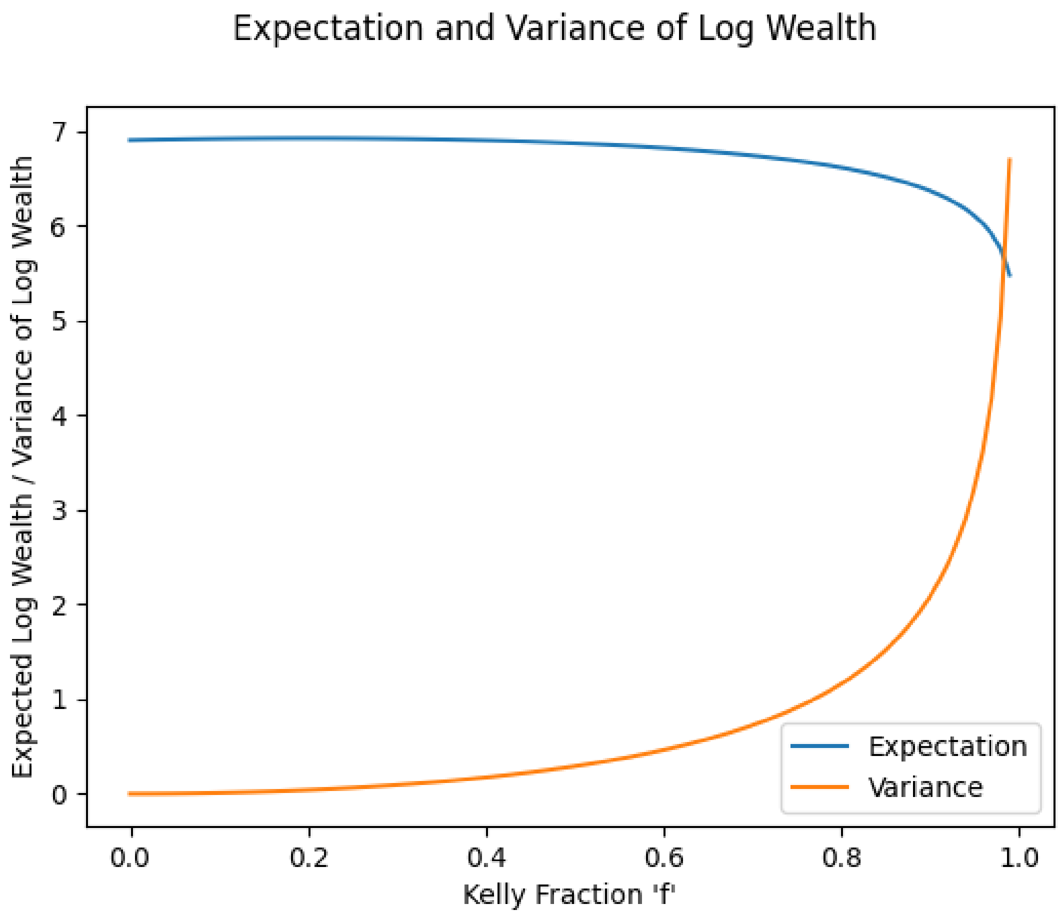
$$\Rightarrow G_{max} = 2P_Y - 1 \tag{8}$$

■

3. Write a numerical simulation to find the Kelly fraction in the way it was discussed in Lecture 2. You may use the template code provided or write your own version.

(a) Plot the expected log-wealth and the variance of log-wealth. Does the maxima of expected value simultaneously lead to a minima of the variance?

Solution.



Yes, the maxima of expected value simultaneously leads to a minima of the variance ■

(b) Generalize the code to find the optimal betting fraction when the risky prospect has three possible outcomes with probabilities $p_i\left(\sum p_i = 1\right)$ and returns r_i, where $i \in 1, 2, 3$.

Solution.

■