MAT237 Multivariable Calculus
Lecture Notes

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1 Critical Points

Definition A symmetric $n \times n$ matrix A is

1. **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$
2. **nonnegative definite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $x \in \mathbb{R}^n$

In addition, we say that A is

1. **negative definite** if -A is positive definite
2. **nonpositive definite** if -A is nonnegative definite

A matrix A is **indefinite** if none of the above holds. Equivalently, A is indefinite if there exist $\mathbf{x},\ \mathbf{y} \in \mathbb{R}$ such that $\mathbf{x}^T A\mathbf{x} < 0 < \mathbf{y}^T A\mathbf{y}$

Theorem 1 Assume that A is a symmetric matrix. Then

1. A is positive definite $\iff$ all its eigenvalues are positive
$\iff$ $\exists \lambda_1 > 0$ such that $\mathbf{x}^T A\mathbf{x} \geq \lambda_1 |\mathbf{x}|^2$ for all $\mathbf{x} \in \mathbb{R}^n$
2. A is nonnegative definite $\iff$ all its eigenvalues are nonnegative
3. A is indefinite $\iff$ A has both positive and negative eigenvalues

Remark If A is a symmetric matrix then
The smallest eigenvalue of A $= \min_{\{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}|=1\}} \mathbf{u}^T A\mathbf{u}$

Theorem 2 For the matrix $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$,

1. if $detA < 0$, then A is indefinite
2. if $detA > 0$, then
   if $\alpha > 0$ then A is positive definite
   if $\alpha < 0$ then A is negative definite
3. if $detA = 0$ then at least one eigenvalue equals zero.

Definition A critical point $\mathbf{a}$ of C^2 function $\mathbf{f}$ is degenerate if $\det(D_{\mathbf{H}}(\mathbf{a})) = 0$

Theorem 3 - first derivative test If $\mathbf{f} : S \in \mathbb{R}^n \to \mathbb{R}$ is differentiable, then every local extremum is a critical point.

Theorem 4 - second derivative test

1. If $f : S \to \mathbb{R}$ is C^2 and $\mathbf{a}$ is a local minimum point for f, then $\mathbf{a}$ is a critical point of f and $H(\mathbf{a})$ is nonnegative definite.
2. If $\mathbf{a}$ is a critical point and $H(\mathbf{a})$ is positive definite, then $\mathbf{a}$ is a local minimum point.

Corollary Assume that f is C^2 and $\nabla f(\mathbf{a}) = \mathbf{0}$

1. If H(a) is positive definite, then a is a local min;
2. If H(a) is negative definite, then a is a local max;
3. If H(a) is indefinite, then a is a saddle point;
4. If none of the above hold, then we cannot determine the character of the critical point without further thought.

2 The Implicit Function Theorem

Assume that S is an open subset of $\mathbb{R}^{n+k}$ and that $F : S \to \mathbb{R}^k$ is a function of class C^1. Assume also that $(\mathbf{a}, \mathbf{b})$ is a point in S such that **F(a, b)=0** and $\det D_{\mathbf{y}}\mathbf{F}(\mathbf{a},\ \mathbf{b}) \neq 0$
1. Then there exists $r_0, r_1 > 0$ such that for every $\mathbf{x} \in \mathbb{R}^n$ such that $|\mathbf{x} - \mathbf{a}| < r_0$, there exists a unique $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{b}| < r_1$

$$\mathbf{F}(\mathbf{x},\ \mathbf{y}) = \mathbf{0}(1)$$

In other words, equation (1) implicitly defines a function $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for $x \in \mathbb{R}^n$ near $\mathbf{a}$, with $\mathbf{y} = \mathbf{f}(\mathbf{x})$ close to $\mathbf{b}$. Note in particular that $\mathbf{b} = \mathbf{f}(\mathbf{a})$.
2. Moreover, the function $\mathbf{f} : B(r_0, \mathbf{a}) \to B(r_1, \mathbf{b}) \subset \mathbb{R}^k$ from part (1) above is of class C^1, and its derivatives may be determined by differentiating the identity

$$\mathbf{F}(\mathbf{x},\mathbf{f}(\mathbf{x})) = \mathbf{0}$$

and solving to find the partial derivatives of $\mathbf{f}$.

Remark

$$D\mathbf{f}(\mathbf{a}) = -[D_{\mathbf{y}}\mathbf{F}(\mathbf{a},\ \mathbf{b})]^{-1} D_{\mathbf{x}}\mathbf{F}(\mathbf{a},\ \mathbf{b})$$

3 The Inverse Function Theorem

Let U and V be open sets in $\mathbb{R}^n$, and assume that $\mathbf{f} : U \to V$ is a mapping of class C^1.
Assume that $\mathbf{a} \in U$ is a point such that $D\mathbf{f}(\mathbf{a})$ is invertible.
and let $\mathbf{b} := \mathbf{f}(\mathbf{a})$. Then there exist open sets $M \subset U$ and $N \subset V$ such that

1. $\mathbf{a} \in M$ and $\mathbf{b} \in N$
2. $\mathbf{f}$ is one-to-one from M onto N (hence invertible), and
3. the inverse function $f^{-1} : N \to M$ is of class C^1

Moreover, if $x \in M$ and $y = \mathbf{f}(\mathbf{x}) \in N$, then

$$D(\mathbf{f}^{-1})(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1}$$

In particular,

$$D(\mathbf{f}^{-1})(\mathbf{b}) = [D\mathbf{f}(\mathbf{a})]^{-1}$$

4 Theorems of 1-D Integral Calculus

Lemma: Refined partitions give better approximations Let P be some partition over an interval and let P' be a refinement of P, then

$$LS_{P'}f \geq LS_P f \wedge US_{P'} \leq US_P f$$

Where LS and US stands for lower sum and upper sum respectively.

Lemma: Lower sum is always less then or equal to upper sum If P and Q are any partitions of $[a, b]$, then $LS_P f \leq US_Q f$. The essence of this proof is to consider the common refinement of these two partitions.

Lemma. $\epsilon - \delta$ definition of integrability If f is a bounded function on $[a, b]$, the following conditions are equivalent:

1. f is integrable on $[a, b]$
2. $\forall \epsilon > 0, \exists P$ of $[a, b]$ such that $US_P f - LS_P f < \epsilon$

Theorem: Integration is "Linear"

1. Suppose $a < b < c$. If f is integrable on $[a, b]$ and on $[b, c]$, then f is integrable on $[a, c]$, further more

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

2. If f and g are integrable on $[a, b]$, then so is $f + g$, further more

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Theorem. Suppose f is integrable on $[a, b]$.

1. If $c \in \mathbb{R}$, the cf is integrable on $[a, b]$, and $\int_a^b cf(x) = c \int_a^b f(x)dx$
2. Of $[c, d] \subset [a, b]$, then f is integrable on $[c, d]$.
3. If g is integrable on $[a, b]$ and $f(x) \leq g(x), \forall x \in [a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
4. $|f|$ is integrable on $[a, b]$, and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$

Theorem: Bounded + monotone $\implies$ integrable If f is bounded and monotone on $[a, b]$, then f is integrable on $[a, b]$. The proof of this uses the $\epsilon - \delta$ definition of integrability

Theorem: Continuous $\implies$ integrable If f is continuous on $[a, b]$, then f is integrable on $[a, b]$. Note that continuous is a sufficient but not necessary condition of integrability

Theorem: discontinuous at only finite pts $\implies$ integrable If f is bounded on $[a, b]$ and continuous at all except finitely many points in $[a, b]$, then f is integrable on $[a, b]$. A easy example of this would be any $\mathbb{R}$ function that has a hole in it.

Theorem: Discontinuous at only zero content $\implies$ integrable If f is bounded on $[a, b]$ and the set of points in $[a, b]$ at which f is discontinuous has zero content, then f is integrable on $[a, b]$.

Proposition. Suppose f and g are integrable on $[a,b]$ and $f(x) = g(x)$ for all except finitely many points $x \in [a,b]$. Then $\int_a^b f(x)dx = \int_a^b g(x)dx$.

The Fundamental Theorem Of Calculus

1. Let f be an integrable function on $[a,b]$. For $x \in [a,b]$, let $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a,b]$; more-over, $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous,

2. Let F be a continuous function on $[a,b]$ that is differentiable except perhaps at finitely many points in $[a,b]$, and let f be a function on $[a,b]$ that agrees with F' at all points where the latter is defined. If f is integrable on $[a,b]$, then $\int_a^b f(t)dt = F(b) - F(a)$

Proposition. Suppose f is integrable on $[a,b]$. Given $\epsilon > 0, \exists \delta > 0$ such that if $P = \{x_0, ..., x_J\}$ is any partition of $[a,b]$ satisfying

$$max\{x_j - x_{j-1} | 1 \leq j \leq J\} < \delta$$

the sums $LS_P f$ and $US_P f$ differ from $\int_a^b f(x)dx$ by at most ϵ.

5 Generalized Integral Calculus