MAT237 Multivariable Calculus
Lecture Notes

Yuchen Wang, Tingfeng Xia

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1 Critical Points

1.1 symmetric matrices

Definition A symmetric $n \times n$ matrix A is

1. **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$
2. **nonnegative definite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $x \in \mathbb{R}^n$

In addition, we say that A is

1. **negative definite** if -A is positive definite
2. **nonpositive definite** if -A is nonnegative definite

A matrix A is **indefinite** if none of the above holds. Equivalently, A is indefinite if there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ such that $\mathbf{x}^T A\mathbf{x} < 0 < \mathbf{y}^T A\mathbf{y}$

Theorem 1 Assume that A is a symmetric matrix. Then

1. A is positive definite $\iff$ all its eigenvalues are positive
   $\iff \exists \lambda_1 > 0$ such that $\mathbf{x}^T A\mathbf{x} \geq \lambda_1 |\mathbf{x}|^2$ for all $\mathbf{x} \in \mathbb{R}^n$
2. A is nonnegative definite $\iff$ all its eigenvalues are nonnegative
3. A is indefinite $\iff$ A has both positive and negative eigenvalues

Remark If A is a symmetric matrix then
The smallest eigenvalue of A $= \min_{\{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}|=1\}} \mathbf{u}^T A\mathbf{u}$

Theorem 2 For the matrix $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$,

1. if $detA < 0$, then A is indefinite
2. if $detA > 0$, then
   if $\alpha > 0$ then A is positive definite
   if $\alpha < 0$ then A is negative definite
3. if $detA = 0$ then at least one eigenvalue equals zero.

Definition A critical point $\mathbf{a}$ of C^2 function $\mathbf{f}$ is degenerate if $det(D_{\mathbf{H}}(\mathbf{a})) = 0$

Theorem 3 - first derivative test If $\mathbf{f} : S \in \mathbb{R}^n \to \mathbb{R}$ is differentiable, then every local extremum is a critical point.

Theorem 4 - second derivative test

1. If $f : S \to \mathbb{R}$ is C^2 and $\mathbf{a}$ is a local minimum point for f, then $\mathbf{a}$ is a critical point of f and $H(\mathbf{a})$ is nonnegative definite.
2. If $\mathbf{a}$ is a critical point and $H(\mathbf{a})$ is positive definite, then $\mathbf{a}$ is a local minimum point.

Corollary Assume that f is C^2 and $\nabla f(\mathbf{a}) = \mathbf{0}$

1. If H(a) is positive definite, then a is a local min;
2. If H(a) is negative definite, then a is a local max;
3. If H(a) is indefinite, then a is a saddle point;
4. If non of the above hold, then we cannot determine the character of the critical point without further though.