MAT237 Multivariable Calculus
Lecture Notes

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January 26, 2019

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1 Critical Points

Definition A symmetric $n \times n$ matrix A is

1. **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$
2. **nonnegative definite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $x \in \mathbb{R}^n$

In addition, we say that A is

1. **negative definite** if -A is positive definite
2. **nonpositive definite** if -A is nonnegative definite

A matrix A is **indefinite** if none of the above holds. Equivalently, A is indefinite if there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ such that $\mathbf{x}^T A\mathbf{x} < 0 < \mathbf{y}^T A\mathbf{y}$

Theorem 1 Assume that A is a symmetric matrix. Then

1. A is positive definite $\iff$ all its eigenvalues are positive
$\iff \exists \lambda_1 > 0$ such that $\mathbf{x}^T A\mathbf{x} \geq \lambda_1 |\mathbf{x}|^2$ for all $\mathbf{x} \in \mathbb{R}^n$
2. A is nonnegative definite $\iff$ all its eigenvalues are nonnegative
3. A is indefinite $\iff$ A has both positive and negative eigenvalues

Remark If A is a symmetric matrix then
The smallest eigenvalue of A $= \min_{\{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}|=1\}} \mathbf{u}^T A\mathbf{u}$

Theorem 2 For the matrix $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$,

1. if $detA < 0$, then A is indefinite
2. if $detA > 0$, then
   if $\alpha > 0$ then A is positive definite
   if $\alpha < 0$ then A is negative definite
3. if $detA = 0$ then at least one eigenvalue equals zero.

Definition A critical point $\mathbf{a}$ of C^2 function $\mathbf{f}$ is degenerate if $\det(D_{\mathbf{H}}(\mathbf{a})) = 0$

Theorem 3 - first derivative test If $\mathbf{f} : S \in \mathbb{R}^n \to \mathbb{R}$ is differentiable, then every local extremum is a critical point.

Theorem 4 - second derivative test

1. If $f : S \to \mathbb{R}$ is C^2 and $\mathbf{a}$ is a local minimum point for f, then $\mathbf{a}$ is a critical point of f and $H(\mathbf{a})$ is nonnegative definite.
2. If $\mathbf{a}$ is a critical point and $H(\mathbf{a})$ is positive definite, then $\mathbf{a}$ is a local minimum point.

Corollary Assume that f is C^2 and $\nabla f(\mathbf{a}) = \mathbf{0}$

1. If H(a) is positive definite, then a is a local min;
2. If H(a) is negative definite, then a is a local max;
3. If H(a) is indefinite, then a is a saddle point;
4. If none of the above hold, then we cannot determine the character of the critical point without further thought.

2 The Implicit Function Theorem

Assume that S is an open subset of $\mathbb{R}^{n+k}$ and that $F : S \to \mathbb{R}^k$ is a function of class C^1. Assume also that $(\mathbf{a}, \mathbf{b})$ is a point in S such that $\mathbf{F(a, b)=0}$ and $\det D_{\mathbf{y}}\mathbf{F(a, b)} \neq 0$
1. Then there exists $r_0, r_1 > 0$ such that for every $\mathbf{x} \in \mathbb{R}^n$ such that $|\mathbf{x} - \mathbf{a}| < r_0$, there exists a unique $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{b}| < r_1$

$$\mathbf{F(x, y) = 0}(1)$$

In other words, equation (1) implicitly defines a function $\mathbf{y = f(x)}$ for $x \in \mathbb{R}^n$ near $\mathbf{a}$, with $\mathbf{y = f(x)}$ close to $\mathbf{b}$. Note in particular that $\mathbf{b = f(a)}$.
2. Moreover, the function $\mathbf{f} : B(r_0, \mathbf{a}) \to B(r_1, \mathbf{b}) \subset \mathbb{R}^k$ from part (1) above is of class C^1, and its derivatives may be determined by differentiating the identity

$$\mathbf{F(x,f(x)) = 0}$$

and solving to find the partial derivatives of $\mathbf{f}$.

Remark

$$D\mathbf{f(a)} = -[D_{\mathbf{y}}\mathbf{F(a, b)}]^{-1} D_{\mathbf{x}}\mathbf{F(a, b)}$$

3 The Inverse Function Theorem

Let U and V be open sets in $\mathbb{R}^n$, and assume that $\mathbf{f} : U \to V$ is a mapping of class C^1.
Assume that $\mathbf{a} \in U$ is a point such that $D\mathbf{f(a)}$ is invertible.
and let $\mathbf{b} := \mathbf{f(a)}$. Then there exist open sets $M \subset U$ and $N \subset V$ such that

1. $\mathbf{a} \in M$ and $\mathbf{b} \in N$
2. $\mathbf{f}$ is one-to-one from M onto N (hence invertible), and
3. the inverse function $f^{-1} : N \to M$ is of class C^1

Moreover, if $x \in M$ and $y = \mathbf{f(x)} \in N$, then

$$D(\mathbf{f}^{-1})(\mathbf{y}) = [D\mathbf{f(x)}]^{-1}$$

In particular,

$$D(\mathbf{f}^{-1})(\mathbf{b}) = [D\mathbf{f(a)}]^{-1}$$