# MAT237 Multivariable Calculus
# Lecture Notes

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# 1 Critical Points

**Definition** A symmetric $n \times n$ matrix A is

1. **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$
2. **nonnegative definite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $x \in \mathbb{R}^n$

In addition, we say that A is

1. **negative definite** if -A is positive definite
2. **nonpositive definite** if -A is nonnegative definite

A matrix A is **indefinite** if none of the above holds. Equivalently, A is indefinite if there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ such that $\mathbf{x}^T A\mathbf{x} < 0 < \mathbf{y}^T A\mathbf{y}$

**Theorem 1** Assume that A is a symmetric matrix. Then

1. A is positive definite $\iff$ all its eigenvalues are positive $\iff \exists\lambda_1 > 0$ such that $\mathbf{x}^T A\mathbf{x} \geq \lambda_1 |\mathbf{x}|^2$ for all $\mathbf{x} \in \mathbb{R}^n$
2. A is nonnegative definite $\iff$ all its eigenvalues are nonnegative
3. A is indefinite $\iff$ A has both positive and negative eigenvalues

**Remark** If A is a symmetric matrix then
The smallest eigenvalue of A $= \min_{\{\mathbf{u}\in\mathbb{R}^n : |\mathbf{u}|=1\}} \mathbf{u}^T A\mathbf{u}$

**Theorem 2** For the matrix $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$,

1. if $detA < 0$, then A is indefinite
2. if $detA > 0$, then
   if $\alpha > 0$ then A is positive definite
   if $\alpha < 0$ then A is negative definite
3. if $detA = 0$ then at least one eigenvalue equals zero.

**Definition** A critical point $\mathbf{a}$ of $C^2$ function $\mathbf{f}$ is <u>degenerate</u> if $\det(D_{\mathbf{H}}(\mathbf{a})) = 0$

**Theorem 3 - first derivative test** If $\mathbf{f} : S \in \mathbb{R}^n \to \mathbb{R}$ is differentiable, then every local extremum is a critical point.

**Theorem 4 - second derivative test**

1. If $f : S \to \mathbb{R}$ is $C^2$ and $\mathbf{a}$ is a local minimum point for $f$, then $\mathbf{a}$ is a critical point of $f$ and $H(\mathbf{a})$ is nonnegative definite.

2. If $\mathbf{a}$ is a critical point and $H(\mathbf{a})$ is positive definite, then $\mathbf{a}$ is a local minimum point.

**Corollary** Assume that $f$ is $C^2$ and $\nabla f(\mathbf{a}) = \mathbf{0}$

1. If H(a) is positive definite, then a is a local min;

2. If H(a) is negative definite, then a is a local max;

3. If H(a) is indefinite, then a is a saddle point;

4. If none of the above hold, then we cannot determine the character of the critical point without further thought.

## 2 The Implicit Function Theorem

Assume that S is an open subset of $\mathbb{R}^{n+k}$ and that $F : S \to \mathbb{R}^k$ is a function of class $C^1$. Assume also that $(\mathbf{a}, \mathbf{b})$ is a point in S such that **F(a, b)=0** and $\det D_{\mathbf{y}}\mathbf{F}(\mathbf{a},\ \mathbf{b}) \neq 0$
1. Then there exists $r_0, r_1 > 0$ such that for every $\mathbf{x} \in \mathbb{R}^n$ such that $|\mathbf{x} - \mathbf{a}| < r_0$, there exists a unique $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{b}| < r_1$

$$\mathbf{F(x,\ y)} = \mathbf{0}(1)$$

In other words, equation (1) implicitly defines a function $\mathbf{y} = \mathbf{f(x)}$ for $x \in \mathbb{R}^n$ near $\mathbf{a}$, with $\mathbf{y} = \mathbf{f(x)}$ close to $\mathbf{b}$. Note in particular that $\mathbf{b} = \mathbf{f(a)}$.
2. Moreover, the function $\mathbf{f} : B(r_0, \mathbf{a}) \to B(r_1, \mathbf{b}) \subset \mathbb{R}^k$ from part (1) above is of class $C^1$, and its derivatives may be determined by differentiating the identity

$$\mathbf{F(x,f(x))} = \mathbf{0}$$

and solving to find the partial derivatives of $\mathbf{f}$.

**Remark**

$$D\mathbf{f(a)} = -[D_{\mathbf{y}}\mathbf{F(a,\ b)}]^{-1} D_{\mathbf{x}}\mathbf{F(a,\ b)}$$

## 3 The Inverse Function Theorem

Let U and V be open sets in $\mathbb{R}^n$, and assume that $\mathbf{f} : U \to V$ is a mapping of class $C^1$.
Assume that $\mathbf{a} \in U$ is a point such that $D\mathbf{f(a)}$ is invertible.
and let $\mathbf{b} := \mathbf{f(a)}$. Then there exist open sets $M \subset U$ and $N \subset V$ such that

1. $\mathbf{a} \in M$ and $\mathbf{b} \in N$
2. $\mathbf{f}$ is one-to-one from M onto N (hence invertible), and
3. the inverse function $f^{-1} : N \to M$ is of class $C^1$

Moreover, if $x \in M$ and $y = \mathbf{f(x)} \in N$, then

$$D(\mathbf{f}^{-1})(\mathbf{y}) = [D\mathbf{f(x)}]^{-1}$$

In particular,

$$D(\mathbf{f}^{-1})(\mathbf{b}) = [D\mathbf{f(a)}]^{-1}$$

## 4 Theorems of 1-D Integral Calculus

**Lemma: Refined partitions give better approximations** Let $P$ be some partition over an interval and let $P'$ be a refinement of $P$, then

$$LS_{P'}f \geq LS_P f \wedge US_{P'} \leq US_P f$$

Where LS and US stands for lower sum and upper sum respectively.

**Lemma: Lower sum is always less then or equal to upper sum** If $P$ and $Q$ are any partitions of $[a, b]$, then $LS_P f \leq US_Q f$. The essence of this proof is to consider the common refinement of these two partitions.

**Lemma.** $\epsilon - \delta$ **definition of integrability** If $f$ is a bounded function on $[a, b]$, the following conditions are equivalent:

1. $f$ is integrable on $[a, b]$
2. $\forall \epsilon > 0, \exists P$ of $[a, b]$ such that $US_P f - LS_P f < \epsilon$

**Theorem: Integration is "Linear"**

1. Suppose $a < b < c$. If $f$ is integrable on $[a, b]$ and on $[b, c]$, then $f$ is integrable on $[a, c]$, further more

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

2. If $f$ and $g$ are integrable on $[a, b]$, then so is $f + g$, further more

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

**Theorem.** Suppose $f$ is integrable on $[a, b]$.

1. If $c \in \mathbb{R}$, the $cf$ is integrable on $[a, b]$, and $\int_a^b cf(x) = c\int_a^b f(x)dx$
2. Of $[c, d] \subset [a, b]$, then $f$ is integrable on $[c, d]$.
3. If $g$ is integrable on $[a, b]$ and $f(x) \leq g(x), \forall x \in [a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
4. $|f|$ is integrable on $[a, b]$, and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$

**Theorem: Bounded + monotone $\implies$ integrable** If $f$ is bounded and monotone on $[a, b]$, then $f$ is integrable on $[a, b]$. The proof of this uses the $\epsilon - \delta$ definition of integrability

**Theorem: Continuous $\implies$ integrable** If $f$ is continuous on $[a, b]$, then $f$ is integrable on $[a, b]$. Note that continuous is a sufficient but not necessary condition of integrability

**Theorem: discontinuous at only finite pts $\implies$ integrable** If $f$ is bounded on $[a, b]$ and continuous at all except finitely many points in $[a, b]$, then $f$ is integrable on $[a, b]$. A easy example of this would be any $\mathbb{R}$ function that has a hole in it.

**Theorem: Discontinuous at only zero content $\implies$ integrable** If $f$ is bounded on $[a, b]$ and the set of points in $[a, b]$ at which $f$ is discontinuous has zero content, then $f$ is integrable on $[a, b]$.

**Proposition.** Suppose $f$ and $g$ are integrable on $[a, b]$ and $f(x) = g(x)$ for all except finitely many points $x \in [a, b]$. Then $\int_a^b f(x)dx = \int_a^b g(x)dx$.

**The Fundamental Theorem Of Calculus**

1. Let $f$ be an integrable function on $[a, b]$. For $x \in [a, b]$, let $F(x) = \int_a^x f(t)dt$. Then $F$ is continuous on $[a, b]$; more-over, $F'(x)$ exists and equals $f(x)$ at every $x$ at which $f$ is continuous,

2. Let $F$ be a continuous function on $[a, b]$ that is differentiable except perhaps at finitely many points in $[a, b]$, and let $f$ be a function on $[a, b]$ that agrees with $F'$ at all points where the latter is defined. If $f$ is integrable on $[a, b]$, then $\int_a^b f(t)dt = F(b) - F(a)$

**Proposition.** Suppose $f$ is integrable on $[a, b]$. Given $\epsilon > 0, \exists \delta > 0$ such that if $P = \{x_0, ..., x_J\}$ is any partition of $[a, b]$ satisfying

$$max\{x_j - x_{j-1} | 1 \leq j \leq J\} < \delta$$

the sums $LS_P f$ and $US_P f$ differ from $\int_a^b f(x)dx$ by at most $\epsilon$.

# 5 Generalized Integral Calculus