

MAT237 Multivariable Calculus

Lecture Notes

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January 27, 2019

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1 Critical Points

Definition A symmetric $n \times n$ matrix A is

1. **positive definite** if $\mathbf{x}^T A \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$
2. **nonnegative definite** if $\mathbf{x}^T A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$

In addition, we say that A is

1. **negative definite** if $-A$ is positive definite
2. **nonpositive definite** if $-A$ is nonnegative definite

A matrix A is **indefinite** if none of the above holds. Equivalently, A is indefinite if there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x}^T A \mathbf{x} < 0 < \mathbf{y}^T A \mathbf{y}$

Theorem 1 Assume that A is a symmetric matrix. Then

1. A is positive definite \iff all its eigenvalues are positive
 $\iff \exists \lambda_1 > 0$ such that $\mathbf{x}^T A \mathbf{x} \geq \lambda_1 |\mathbf{x}|^2$ for all $\mathbf{x} \in \mathbb{R}^n$
2. A is nonnegative definite \iff all its eigenvalues are nonnegative
3. A is indefinite \iff A has both positive and negative eigenvalues

Remark If A is a symmetric matrix then

The smallest eigenvalue of $A = \min_{\{\mathbf{u} \in \mathbb{R}^n: |\mathbf{u}|=1\}} \mathbf{u}^T A \mathbf{u}$

Theorem 2 For the matrix $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$,

1. if $\det A < 0$, then A is indefinite
2. if $\det A > 0$, then
 - if $\alpha > 0$ then A is positive definite
 - if $\alpha < 0$ then A is negative definite
3. if $\det A = 0$ then at least one eigenvalue equals zero.

Definition A critical point \mathbf{a} of C^2 function \mathbf{f} is degenerate if $\det(D_{\mathbf{H}}(\mathbf{a})) = 0$

Theorem 3 - first derivative test If $f : S \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable, then every local extremum is a critical point.

Theorem 4 - second derivative test

1. If $f : S \rightarrow \mathbb{R}$ is C^2 and \mathbf{a} is a local minimum point for f , then \mathbf{a} is a critical point of f and $H(\mathbf{a})$ is nonnegative definite.
2. If \mathbf{a} is a critical point and $H(\mathbf{a})$ is positive definite, then \mathbf{a} is a local minimum point.

Corollary Assume that f is C^2 and $\nabla f(\mathbf{a}) = \mathbf{0}$

1. If $H(\mathbf{a})$ is positive definite, then \mathbf{a} is a local min;
2. If $H(\mathbf{a})$ is negative definite, then \mathbf{a} is a local max;
3. If $H(\mathbf{a})$ is indefinite, then \mathbf{a} is a saddle point;
4. If none of the above hold, then we cannot determine the character of the critical point without further thought.

E.Knight's approach to critical points. In solving a question of $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ we could use the following "quick check" approach:

1. Calculate the gradient of F , equating it to zero to find the critical points
2. Calculate the Hessian of F , find the corresponding matrices for each critical points, where the Hessian is defined as

$$H(f) = \begin{bmatrix} \partial_{xx}f & \partial_{xy}f = \partial_{yx}f \\ \partial_{xy}f = \partial_{yx}f & \partial_{yy}f \end{bmatrix}$$

3. Calculate the determinant of the hessian, and there are the following cases to consider
 - (a) $\det H < 0$, then $\text{sig}(H) = (1, 1)$ and the point is a saddle point
 - (b) $\det H > 0$, then
 - i. $\text{tr}(H) < 0 \implies \text{sig}(H) = (2, 0)$ and the point is a local minimum
 - ii. $\text{tr}(H) > 0 \implies \text{sig}(H) = (0, 2)$ and the point is a local maximum
 - (c) $\det H = 0$, then the test is inconclusive. We have to do this case by staring at it.

2 The Implicit Function Theorem

Assume that S is an open subset of \mathbb{R}^{n+k} and that $F : S \rightarrow \mathbb{R}^k$ is a function of class C^1 . Assume also that (\mathbf{a}, \mathbf{b}) is a point in S such that $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ and $\det D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b}) \neq 0$

1. Then there exists $r_0, r_1 > 0$ such that for every $\mathbf{x} \in \mathbb{R}^n$ such that $|\mathbf{x} - \mathbf{a}| < r_0$, there exists a unique $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{b}| < r_1$

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \quad (1)$$

In other words, equation (1) implicitly defines a function $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for $x \in \mathbb{R}^n$ near \mathbf{a} , with $\mathbf{y} = \mathbf{f}(\mathbf{x})$ close to \mathbf{b} . Note in particular that $\mathbf{b} = \mathbf{f}(\mathbf{a})$.

2. Moreover, the function $\mathbf{f} : B(r_0, \mathbf{a}) \rightarrow B(r_1, \mathbf{b}) \subset \mathbb{R}^k$ from part (1) above is of class C^1 , and its derivatives may be determined by differentiating the identity

$$\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0}$$

and solving to find the partial derivatives of \mathbf{f} .

Remark

$$D\mathbf{f}(\mathbf{a}) = -[D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})]^{-1} D_{\mathbf{x}}\mathbf{F}(\mathbf{a}, \mathbf{b})$$

3 The Inverse Function Theorem

Let U and V be open sets in \mathbb{R}^n , and assume that $\mathbf{f} : U \rightarrow V$ is a mapping of class C^1 .

Assume that $\mathbf{a} \in U$ is a point such that $D\mathbf{f}(\mathbf{a})$ is invertible.

and let $\mathbf{b} := \mathbf{f}(\mathbf{a})$. Then there exist open sets $M \subset U$ and $N \subset V$ such that

1. $\mathbf{a} \in M$ and $\mathbf{b} \in N$
2. \mathbf{f} is one-to-one from M onto N (hence invertible), and
3. the inverse function $f^{-1} : N \rightarrow M$ is of class C^1

Moreover, if $x \in M$ and $y = \mathbf{f}(\mathbf{x}) \in N$, then

$$D(\mathbf{f}^{-1})(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1}$$

In particular,

$$D(\mathbf{f}^{-1})(\mathbf{b}) = [D\mathbf{f}(\mathbf{a})]^{-1}$$

4 Some important coordinate systems

4.1 polar coordinates in \mathbb{R}^2

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \mathbf{f}(r, \theta)$$

For \mathbf{f} to be a bijection between open sets, we have to restrict its domain and range. A common choice is to specify that \mathbf{f} is a function $U \rightarrow V$ where

$$U := \{(r, \theta) : r > 0, |\theta| < \pi\}, V := \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\}$$

(Note that there is a half of the x-axis missing)

4.2 spherical coordinates in \mathbb{R}^3

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \theta \sin \varphi \\ r \sin \theta \sin \varphi \\ r \cos \varphi \end{pmatrix} = \mathbf{f}(r, \theta, \varphi)$$

5 Zero content

Zero content in 1-D A set $S \subset \mathbb{R}$ is said to have zero content if

$$\forall \epsilon > 0, \exists \text{ intervals } I_1, \dots, I_n \text{ s.t. } S \subseteq \bigcup_{i=1}^n I_i \wedge \sum_{i=1}^n \text{Len}(I_i) < \epsilon$$

Multidimensional zero content. A set $S \subset \mathbb{R}^n$ is said to have zero content if

$$\forall \epsilon > 0, \exists \text{ boxes } B_1, \dots, B_n \text{ s.t. } S \subseteq \bigcup_{i=1}^n B_i \wedge \sum_{i=1}^n \text{Area}(B_i) < \epsilon$$

6 Theorems of 1-D Integral Calculus

Lemma: Refined partitions give better approximations Let P be some partition over an interval and let P' be a refinement of P , then

$$LS_{P'} f \geq LS_P f \wedge US_{P'} \leq US_P f$$

Where LS and US stands for lower sum and upper sum respectively.

Lemma: Lower sum is always less than or equal to upper sum If P and Q are any partitions of $[a, b]$, then $LS_P f \leq US_Q f$. The essence of this proof is to consider the common refinement of these two partitions.

Lemma. $\epsilon - \delta$ definition of integrability If f is a bounded function on $[a, b]$, the following conditions are equivalent:

1. f is integrable on $[a, b]$
2. $\forall \epsilon > 0, \exists P$ of $[a, b]$ such that $US_P f - LS_P f < \epsilon$

Theorem: Integration is “Linear”

1. Suppose $a < b < c$. If f is integrable on $[a, b]$ and on $[b, c]$, then f is integrable on $[a, c]$, further more

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

2. If f and g are integrable on $[a, b]$, then so is $f + g$, further more

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Theorem. Suppose f is integrable on $[a, b]$.

1. If $c \in \mathbb{R}$, the cf is integrable on $[a, b]$, and $\int_a^b cf(x) = c \int_a^b f(x)dx$
2. Of $[c, d] \subset [a, b]$, then f is integrable on $[c, d]$.
3. If g is integrable on $[a, b]$ and $f(x) \leq g(x), \forall x \in [a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
4. $|f|$ is integrable on $[a, b]$, and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$

Theorem: Bounded + monotone \implies integrable If f is bounded and monotone on $[a, b]$, then f is integrable on $[a, b]$. The proof of this uses the $\epsilon - \delta$ definition of integrability

Theorem: Continuous \implies integrable If f is continuous on $[a, b]$, then f is integrable on $[a, b]$. Note that continuous is a sufficient but not necessary condition of integrability

Theorem: discontinuous at only finite pts \implies integrable If f is bounded on $[a, b]$ and continuous at all except finitely many points in $[a, b]$, then f is integrable on $[a, b]$. A easy example of this would be any \mathbb{R} function that has a hole in it.

Theorem: Discontinuous at only zero content \implies integrable If f is bounded on $[a, b]$ and the set of points in $[a, b]$ at which f is discontinuous has zero content, then f is integrable on $[a, b]$.

Proposition. Suppose f and g are integrable on $[a, b]$ and $f(x) = g(x)$ for all except finitely many points $x \in [a, b]$. Then $\int_a^b f(x)dx = \int_a^b g(x)dx$.

The Fundamental Theorem Of Calculus

1. Let f be an integrable function on $[a, b]$. For $x \in [a, b]$, let $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a, b]$; more-over, $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous,
2. Let F be a continuous function on $[a, b]$ that is differentiable except perhaps at finitely many points in $[a, b]$, and let f be a function on $[a, b]$ that agrees with F' at all points where the latter is defined. If f is integrable on $[a, b]$, then $\int_a^b f(t)dt = F(b) - F(a)$

Proposition. Suppose f is integrable on $[a, b]$. Given $\epsilon > 0, \exists \delta > 0$ such that if $P = \{x_0, \dots, x_J\}$ is any partition of $[a, b]$ satisfying

$$\max\{x_j - x_{j-1} | 1 \leq j \leq J\} < \delta$$

the sums $LS_P f$ and $US_P f$ differ from $\int_a^b f(x)dx$ by at most ϵ .

7 Generalized Integral Calculus

Theorems of double integrals

1. If f_1 and f_2 are integrable on the bounded set S and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2$ is integrable on S , and

$$\int \int_S [c_1 f_1 + c_2 f_2] dA = c_1 \int \int_S f_1 dA + c_2 \int \int_S f_2 dA$$

2. Let S_1 and S_2 be bounded sets with no points in common (intersection $= \emptyset$), and let f be a bounded function. If f is integrable on S_1 and on S_2 , then f is integrable on $S_1 \cup S_2$, in which case

$$\int \int_{S_1 \cup S_2} f dA = \int \int_{S_1} f dA + \int \int_{S_2} f dA$$

3. If f and g are integrable on S and $f(\mathbf{x}) \leq g(\mathbf{x})$ for $\mathbf{x} \in S$, then $\int \int_S f dA \leq \int \int_S g dA$
4. If f is integrable on S , then so is $|f|$, and

$$\left| \int \int_S f dA \right| \leq \int \int_S |f| dA$$

Theorem. Suppose f is a bounded function on the rectangle R . If the set of points in R at which f is discontinuous has zero content, then f is integrable on R .

Proposition: on zero content

1. If $Z \subset \mathbb{R}^2$ has zero content and $U \subset Z$, then U has zero content.
2. If Z_1, \dots, Z_k have zero content, then so does $\bigcup_1^k Z_j$
3. $\mathbf{f} : (a_0, b_0) \rightarrow \mathbb{R}^2$ is of class C_1 , then $\mathbf{f}([a, b])$ has zero content whenever $a_0 < a < b < b_0$

Discontinuity of characteristic function The function χ_S is discontinuous at \mathbf{x} if and only if \mathbf{x} is in the boundary of S .

Theorem. Let S be a measurable subset of \mathbb{R}^2 . Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is bounded and the set of points in S at which f is discontinuous has zero content. Then f is integrable on S .

Remark on this theorem: The only points where f_{χ_S} can be discontinuous are those points in the closure of S where either f or χ_S is discontinuous. Both of these cases are discontinuity on a set of zero content. And we can definitely fix S inside of a rectangle, then by the previously stated theorem (The theorem directly above), such function is integrable.

Proposition: Integration on a set of zero content evaluates to zero.

Suppose $Z \subset \mathbb{R}^2$ has zero content. If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is bounded, then f is integrable on Z and $\int_Z f dA = 0$

Corollary

1. Suppose that f is integrable on the set $S \subset \mathbb{R}^2$. If $g(\mathbf{x}) = f(\mathbf{x})$ except for \mathbf{x} in a set of zero content, then g is integrable on S and $\int_S g dA = \int_S f dA$
2. Suppose that f is integrable on S and on T , and $S \cap T$ has zero content. Then f is integrable on $S \cup T$, and $\int_{S \cup T} f dA = \int_S f dA + \int_T f dA$