# KRR: Activity 3 – Introduction to Modal Logics

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## Introduction

*Modal Logics* is a topic of study that became popular towards the end of the 19th century. Modal logic grew out of several endeavours to regiment reasoning about possible situations: utopias and ideals, hypothetical scenarios, the unknown future, responsibilities and *what might have happened*, that which is *possible* given our limited knowledge of the facts, and so on.

In artificial intelligence, modal logics can be a tool to say things about what an intelligent agent knows or believes, what the agent intends to do, what it is allowed to do, what it can do, what it has done and will eventually do.

In this activity, we will study one of the foundational types of *logical modality*: that of *what is possible* and *what is necessary*.

## Reasoning about possibility and necessity - alethic logic

In alethic logic we use the $\diamond$ sign to denote possibility and $\Box$ to denote necessity.

With this notation we can express ideas such as:

- *"With necessity, if I find my apartment door open, it must be because I forgot to lock it, or because i've been burglared"* – $\Box(DoorOpen \rightarrow (ForgotClose \vee Burgler)$.
- *"It is not possible for me to have left the door open and for there to have been a football incident"* – $\neg \diamond (ForgotClose \wedge FootbalIncident)$.

A semantic model for modal logics is constructed based on the notion of *Kripke Frames*.

**A Kripke Frame** is a tuple $\langle W, R \rangle$, such that:

- $W$ is a non-empty set of possible worlds
- $R \subseteq (W \times W)$ is a binary relation on $W$, if $wRv$ we say that $v$ is *accessible* from $w$

**A Kripke Model** is a tuple $M = \langle W, R, V \rangle$ where

- $\langle W, R \rangle$ is a Kripke frame
- $V : W \rightarrow Pow(VAR)$ is a valuation function for the set of atomic propositions $VAR$. Proposition $p$ is true in world $w$ if $p \in V(w)$ and false in $w$ if $p \notin V(w)$

Evaluation of the modal operators $\Box$ and $\diamond$ is then subject to the following conditions

- $\diamond p$ is *true* in a world $w$, **iff** $\exists v$ such that $wRv$ and $V_v(p) = true$ (i.e. $p$ is true in at least one world $v$, which is connected to $w$).

- $\Box p$ is *true* in a world $w$, **iff** $\forall v$ for which $wRv$, $V_v(p) = true$ (i.e. for all worlds $v$ which are connected to $w$, $p$ is true in those worlds).

See the *Tasks* section for a visual example.

# Semantic Tableaux Method for alethic propositional logic

The *Semantic Tableaux Method* can be extended with 3 rules to enable inference on modal logics.

- **Type 1 rules**: contains the rules we already discussed for propositional logic, **applied within a world**. It is augmented with the following additions for eliminating negation:
  - $\neg\Box A = \diamond(\neg A)$
  - $\neg \diamond A = \Box(\neg A)$
- **Type 2 rules**: are *world creating rules.* For a set of formulas, on an *open branch* which are *modalized* with the $\diamond$ symbol, introduce simultaneously a set of possible worlds where the formulas are copied over *without their modality symbol.*
- **Type 3 rules**: are *interworld copy rules.* If there is an open branch, from which *possible* worlds have been created (by applying rule 2), and the branch contains formulas modalized by the $\Box$ symbol (e.g. $\Box B1, \Box B2$, ...), copy over the formulas without the $\Box$ symbol (e.g. $B1, B2$,...) *for each* of the possible worlds.

### Order of rule application

- First apply all Type 1 rules in a given world, leaving that branch either closed or open.
- Then apply Type 2 rules for every $\diamond$-modalized formula in the current world.
- Lastly, apply Type 3 rules to copy over in each formulas that are modalized as *necessary* into each possible world created from rule 2.

**Semantic Tableaux Evaluation** For type 1 rules, the evaluation of an open or closed branch in a world follows the semantics explained for propositional logic.

For a possible world, if all branches of the tree within that possible world close, then the possible world closes too.

For a branch that leads into possible worlds, **if any one** of the possible worlds branching from a node *closes*, than that node is considered *closed*, regardless of what is possible in the *other* possible worlds.

# Tasks

Consider the Kripke frames in figure :

**Task 1** Write on paper the truth value for the following modal logic expressions (which are all evaluated in world $W_0$:

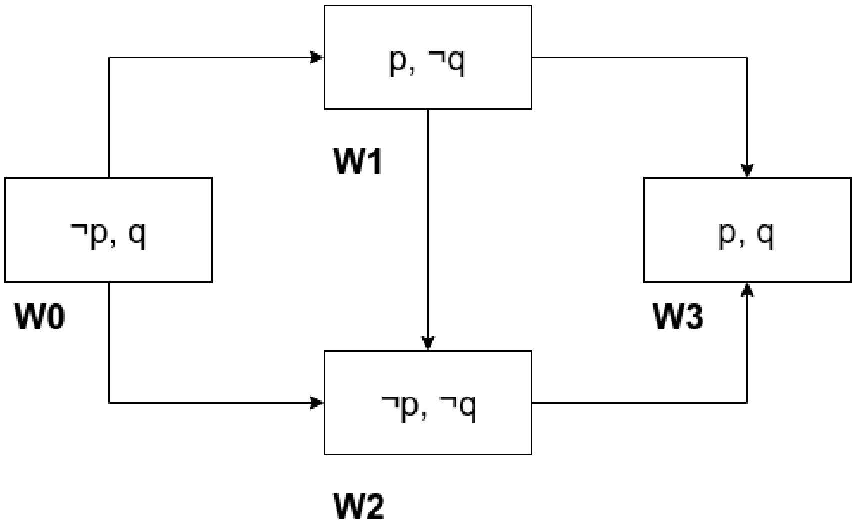


Figure 1: Kripke frame for exercise 1

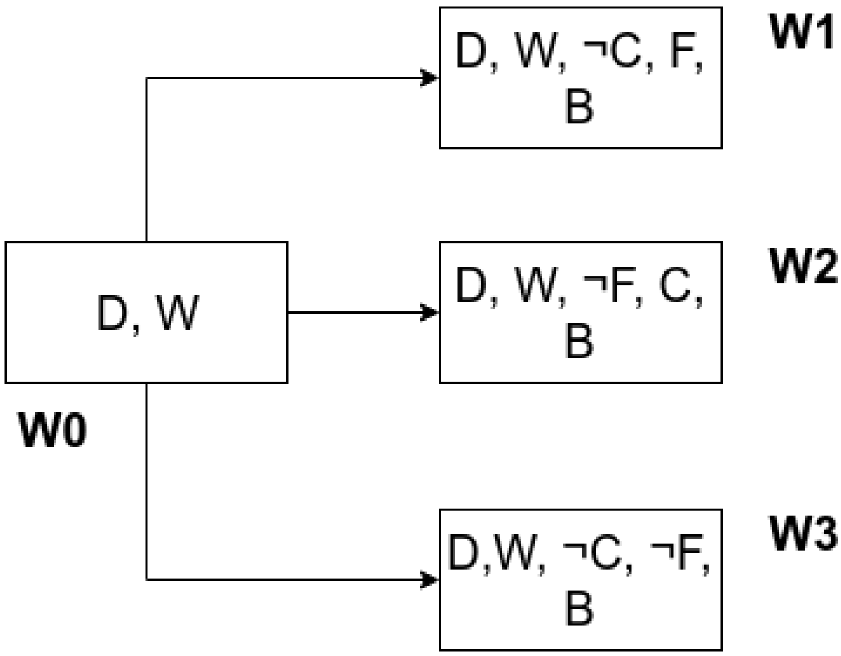


Figure 2: Kripke frame for exercise 2

- $\Diamond p$
- $\Box \neg q$
- $\Box \Box p$
- $\Diamond \Diamond p$
- $\Diamond \Diamond q$

**Task 2** Augment your propositional logic representation with operators for *possibility* and *necessity*.

Implement an evaluation algorithm, that takes as input a Kripke Model and a set of modal logic expressions and outputs the truth value of each of expression. Exemplify your algorithm on the Kripke model given in Figure and the following expressions:

- $\Box(D \to (C \vee B))$
- $\Box(W \to (F \vee B))$

- $\neg \diamond (C \wedge F)$
- $\Box B$

**Task 3 - Extension of Semantic Tableaux Algorithm (Bonus 2p)** Implement an extension of the Semantic Tableaux Algorithm to the case of modal logics in K-system logics (i.e. the simplest kind or accessibility relations). Use as guideline the indications given in this lab notes, as well as the attached paper on the Tableau method for modal logics. Check out Section **Tableaux Methods for Propositional Modal Logics** in particular.

Evaluate your implementation by proving the validity of the following argument:

> I know that, if the door is open, then either I forgot to close it, or I was burglarized. (P1)
> Secondly, I also know that, if the window is broken, then either there was a football accident, or I was burglarized. (P2)
> Now, as a matter of fact, the window is broken and the door is open. (P3 + P4)
> But, it is inconceivable that I both forgot to close the door and there was a football accident. (P5)
> Therefore, I know that I was burglared. (C)

In logical terms of modal logic expressions, the argument is written as follows:

$$(P1)\Box(DoorOpen \to (ForgotClose \vee Burglar))$$
$$(P2)\Box(WindowBroken \to (FootballAccident \vee Burglar))$$
$$(P3)DoorOpen$$
$$(P4)WinndowBroken$$
$$(P5)\neg \diamond (ForgotClose \wedge FootballAccident)$$
$$(C)\Box Burglar$$