Balanis Quick Reference

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Notation

∂_t	$\frac{\partial}{\partial t}$	
E	Electric Field Intensity	[V m^{-1}]
H	Magnetic Field Intensity	[A m^{-1}]
D	Electric Flux Density	[C m^{-2}]
B	Magnetic Flux Density	[Wb m^{-2}]
$J_{c/d/i}$	Electric Current Density $_{\text{(conduction/displacement/impressed)}}$	[A m^{-2}]
$\mathfrak{M}_{d/i}$	Magnetic Current Density $_{\text{(displacement/impressed)}}$	[V m^{-2}]
ρ_e	Electric Charge Density	[C m^{-2}]
ρ_m	Magnetic Charge Density	[Wb m^{-2}]
ϵ	Permittivity	[F m^{-1}]
μ	Permeability	[H m^{-1}]
σ	Conductivity	[S m^{-1}]
$\eta := \sqrt{\frac{\mu}{\epsilon}}$	Wave Impedance	[S^{-1}]
f	Frequency	[s^{-1}]
λ	Wavelength	[m]
$\omega := 2\pi f$	Angular Frequency	[s^{-1}]
$k := \frac{2\pi}{\lambda}$	Wavenumber	[m^{-1}]
v_f	Velocity of front part of wave	[ms^{-1}]
v_g	Group velocity of wave (envelope velocity)	[ms^{-1}]
v_p	Phase velocity of wave (peak/trough velocity)	[ms^{-1}]

1 Basics

1.1 Maxwell's Equations

Differential Form

$$
\begin{aligned}
\text{Maxwell-Faraday :} \quad & \nabla \times E = -\partial_t B - \mathfrak{M}_i && \text{(1a)} \\
\text{Ampère-Maxwell :} \quad & \nabla \times H = \partial_t D + J_c + J_i && \text{(1b)} \\
\text{Gauss :} \quad & \nabla \cdot D = \rho_e && \text{(1c)} \\
\text{Gauss (Magnetism) :} \quad & \nabla \cdot B = 0 = \rho_m && \text{(1d)}
\end{aligned}
$$

Integral Form

$$\oint_C E \cdot dl = -\partial_t \iint_S B \cdot ds - \iint_S \mathfrak{M}_i \cdot ds \tag{2a}$$

$$\oint_C H \cdot dl = \partial_t \iint_S D \cdot ds + \iint_S (J_c + J_i) \cdot ds \tag{2b}$$

$$\oint\!\!\!\!\!\iint_S D \cdot ds = \iiint_V \rho_e \cdot dv \tag{2c}$$

$$\oint\!\!\!\!\!\iint_S B \cdot ds = 0 = \iiint_V \rho_m \cdot dv \tag{2d}$$

1.2 Constitutive Relations

$$D = \epsilon E \tag{3a}$$

$$B = \mu H \tag{3b}$$

$$J_c = \sigma E \tag{3c}$$

1.3 Boundary Conditions

	General	**Finite σ, no source/charge**	**Medium 1 PEC**	**Medium 1 PMC**
$E_\parallel := n \times E$	$E_{\parallel 2} - E_{\parallel 1} = -M_s$	$E_{\parallel 2} - E_{\parallel 1} = 0$	$E_{\parallel 2} = 0$	$E_{\parallel 2} = -M_s$
$H_\parallel := n \times H$	$H_{\parallel 2} - H_{\parallel 1} = J_s$	$H_{\parallel 2} - H_{\parallel 1} = 0$	$H_{\parallel 2} = J_s$	$H_{\parallel 2} = 0$
$D_\perp := n \cdot D$	$D_{\perp 2} - D_{\perp 1} = \phi_{es}$	$D_{\perp 2} - D_{\perp 1} = 0$	$D_{\perp 2} = \phi_{es}$	$D_{\perp 2} = 0$
$B_\perp := n \cdot B$	$B_{\perp 2} - B_{\perp 1} = \phi_{ms}$	$B_{\perp 2} - B_{\perp 1} = 0$	$B_{\perp 2} = 0$	$B_{\perp 2} = \phi_{ms}$

1.4 Material Considerations

All the constitutive parameters of (3) are typically time/space-varying tensors. Furthermore, they are complex-valued in order to model dissipation for time-varying fields.

Generally, we can classify materials into categories described below.

1.4.1 Magnets

The magnetization is the net effect of the microscopic magnetic dipoles created by orbiting electrons. A large value of μ indicates a stronger magnetization.

1.4.2 Dielectrics/Insulators

Here, the dominant charges are on the boundary of the material creating an overall electric dipole. A large value of ϵ indicates a stronger ability to store charge, but must be weighed vs. σ and ω also. The condition for a good dielectric is

$$\frac{\sigma}{\omega\epsilon} \ll 1 \tag{4}$$

1.4.3 Conductors

Here, there are free charges creating currents throughout the material, due to valence electrons that aren't tightly bound. The condition here is the opposite of the above:

$$\frac{\sigma}{\omega\epsilon} \gg 1 \tag{5}$$

1.4.4 Semiconductors

These are roughly in between an insulator and a conductor, with the condition

$$\frac{\sigma}{\omega\epsilon} = O(1) \tag{6}$$

2 P.E.C Field Derivation

1. Starting from Maxwell's equations (1) and constitutive equations (3), transform to frequency domain to simplify curl equations:

$$\nabla \times H = J + j\omega D \tag{7a}$$
$$\nabla \times E = -j\omega B \tag{7b}$$

2. Introduce magnetic vector potential A and electric potential ϕ defined by:

$$B = \nabla \times A \tag{8a}$$
$$E = -\nabla\phi - j\omega A \tag{8b}$$

   and fix Lorenz gauge:

$$\nabla \cdot A = -\frac{j\omega}{c^2}\phi \tag{9}$$

3. Apply the above with some vector identities to reduce Ampere-Maxwell equation (7a) to Helmholtz equation:

   (a) Substitute the constitutive conditions (3) and potentials (8) respectively to (7a):

$$\begin{aligned} &\frac{1}{\mu}\nabla \times B = J + j\omega\epsilon E \\ \implies &\frac{1}{\mu}\nabla \times \nabla \times A = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A \end{aligned}$$

   (b) Use the vector calculus identity $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$, so

$$\frac{1}{\mu}\left(\nabla(\nabla \cdot A) - \nabla^2 A\right) = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A$$

   (c) Apply the Lorenz gauge condition (9):

$$\begin{aligned} &-\frac{1}{\mu}(\frac{j\omega}{c^2}\nabla\phi + \nabla^2 A) = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A \\ \implies &-\frac{1}{\mu}\nabla^2 A = J - \omega^2\epsilon A \\ \implies &\nabla^2 A + k^2 A = -\mu J. \end{aligned}$$

4. Solve for vector potential using Green functions to get

$$A = \mu\int J(r)G(r,r')dr' \tag{10}$$

5. Plug into electric field formula (8b) in terms of potentials:

$$A = \frac{j}{\omega}(\nabla\phi + E)$$

Use the far field assumption on the scalar potential $\nabla\phi \to 0$ - this can be worked out to die off quickly asymptotically. Finally we arrive at

$$E \approx -\omega\mu j \int J(r)G(r,r')dr'$$
$$\implies E \approx -\eta k j \int J(r)G(r,r')dr'$$