Effective Roughness Models

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Contents

Glossary

Mathematical Symbols

Spatial Coordinates	$\vec{r} \in \mathbb{R}^3$.
Basis Vectors	$\mathbf{e}_i \in \mathbb{R}^3$ is the i-th spatial coordinate basis vector: $\vec{r} = r^i \mathbf{e}_i$, using Einstein summation convention. When using specific well known coordinate systems, e.g. angular coordinates, shorthands like $\mathbf{e}_\phi$ will also be used.
Space-Time Coordinates	$(\vec{r}, t) \in \mathbb{R}^{3+1}$.
Space-Frequency Coordinates	$(\vec{r}, \omega) \in \mathbb{R}^{3+1}$.
Temporal Fourier Transform	$\mathcal{F}_t(f(\vec{r}, t)) = \int_{-\infty}^{+\infty} f(\vec{r}, t) e^{-j\omega t} dt$.
Spatial Fourier Transform	$\mathcal{F}_{\vec{r}}(f(\vec{r}, t)) = \int_{\mathbb{R}^3} f(\vec{r}, t) e^{-j\vec{k}\cdot\vec{r}} d\vec{r}$ is also known as the angular spectrum (or momentum representation) of f.
Surface (generic)	$\Sigma : D \to \mathbb{R}^3$, where $D \subseteq \mathbb{R}^2$.
Surface Normal	$\hat{n}_\Sigma(\vec{r})$ is the unit surface normal along Σ at $\vec{r}$, as determined by the right hand rule. When Σ itself is specified with a subscript (e.g. Σ_k), then the notation is shortened to use the surface subscript (e.g. $\hat{n}_k \stackrel{\text{def}}{=} \hat{n}_{\Sigma_k}$).
Vector Surface Element	$d\vec{A}_\Sigma = dA_\Sigma \ \hat{n}_\Sigma$, where dA_Σ is the standard surface element determined by coordinate choice and surface. The same shorthands as those mentioned for surface normal above also apply here.

Geometric Setup

Media Boundary	Σ_i is the boundary between two different media.
Incident Point	$\vec{r}_i$ is a point along Σ_i.
Transmitter	$\vec{r}_t$ is the position of a point transmitter.
Receiver	$\vec{r}_r$ is the position of a point receiver.
Reflected Solid Angle	Ω_Σ is the solid angle of the reflected field with respect to Σ.

Fields

Electric Field (time-domain (t))	$\vec{\mathcal{E}} : \mathbb{R}^{3+1} \to \mathbb{R}^3$.
Magnetic Field (time-domain (t))	$\vec{\mathcal{H}} : \mathbb{R}^{3+1} \to \mathbb{R}^3$.
Electric Field (frequency-domain (ω))	$\mathbf{E} : \mathbb{R}^{3+1} \to \mathbb{C}^3$. $\mathbf{E} \stackrel{\text{def}}{=} \mathcal{F}_t(\vec{\mathcal{E}})$.
Magnetic Field (frequency-domain (ω))	$\mathbf{H} : \mathbb{R}^{3+1} \to \mathbb{C}^3$. $\mathbf{H} \stackrel{\text{def}}{=} \mathcal{F}_t(\vec{\mathcal{H}})$.
Poynting Vector	$\vec{\mathcal{S}} \stackrel{\text{def}}{=} \vec{\mathcal{E}} \times \vec{\mathcal{H}}$.
Time-averaged Poynting Vector	$\langle \vec{\mathcal{S}} \rangle \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \vec{\mathcal{S}}(\vec{r}, t) dt$.
Incremental Power	$dP_\Sigma \stackrel{\text{def}}{=} \langle \vec{\mathcal{S}} \rangle \cdot d\vec{A}$.
Energy Propagation Angle	$\theta_\Sigma \stackrel{\text{def}}{=} \cos^{-1}\left(\frac{\langle \vec{\mathcal{S}} \rangle \cdot \hat{n}_\Sigma}{\lvert\langle \vec{\mathcal{S}} \rangle\rvert}\right)$.

Material Properties

Electrical Permittivity	$\epsilon : \mathbb{C}^3 \times \mathbb{R} \to \mathbb{C}^3$.
Magnetic Permeability	$\mu : \mathbb{C}^3 \times \mathbb{R} \to \mathbb{C}^3$.
Electrical Conductivity	$\sigma : \mathbb{C}^3 \times \mathbb{R} \to \mathbb{C}^3$.
Wave Impedance	η is the intrinsic wave impedance for a medium.
Fresnel Reflection Coefficient	Γ relates the incident and reflected polarized components of $\mathbf{E}$.
Reflectance	$\mathcal{R}$ is the portion of dP reflected to Ω.
Transmittance	$\mathcal{T}$ is the portion of dP transmitted through Σ_i.
Specular Reflectance	$\mathcal{R}_r \stackrel{\text{def}}{=} R^2\lvert\Gamma\rvert^2$ is the specular part of $\mathcal{R}$.
Diffuse Reflectance	$\mathcal{R}_s \stackrel{\text{def}}{=} S^2\lvert\Gamma\rvert^2$ is the diffuse part of $\mathcal{R}$.

Phase and Polarization

Jones Vector	$\hat{\mathbf{e}}_E$ is the electric-field Jones vector. The relative phase between the two components determines the type of polarization.
Global Phase	$\mathbf{E} = e^{j\chi_E}\lvert\mathbf{E}\rvert\hat{\mathbf{e}}_E$. Then χ_H is determined by Maxwell's equations, and the global phase difference is an invariant describing balance between reactive and active power.
Polarization Coupling	$\phi \stackrel{\text{def}}{=} \arg(\hat{\mathbf{e}}_E^\dagger \hat{\mathbf{e}}_H)$ gives the geometric relationship between the $\mathbf{E}$ and $\mathbf{H}$ polarization ellipses..

A Model Assumptions

A.1 Homogeneous Scattering Medium

The scattering surface is modelled as the boundary of a homogeneous medium. The roughness and subsurface scattering effects are therefore statistically uniform, and as a consequence, the local diffusely scattered power is related to radiant intensity I by

$$S^2|\Gamma|^2 dP_i = \iint_\Omega I(\theta, \phi)\ d\Omega \tag{A.1}$$

A.2 Incoherent Scattering

Scattered wave phases are random & uncorrelated, so local scattered power at the receiver is an incoherent sum from contributions along Σ_i :

$$\begin{aligned} dP_r &= dP_{r,\text{diff}} + dP_{r,\text{spec}},\ \text{where} && \text{(A.2a)} \\ dP_{r,\text{diff}} &= \int_{\Sigma_i} \left(\int_{\theta_R - \frac{d\theta_R}{2}}^{\theta_R + \frac{d\theta_R}{2}} \int_{\phi_R - \frac{d\phi_R}{2}}^{\phi_R + \frac{d\phi_R}{2}} I(\theta, \phi)\ d\phi\ d\theta \right) \\ &= \int_{\Sigma_i} I(\theta_R, \phi_R)\ d\phi_R\ d\theta_R,\ \text{and} && \text{(A.2b)} \\ dP_{r,\text{spec}} &= && \text{(A.2c)} \end{aligned}$$

A.3 Effective Roughness Parameter S

$$\begin{aligned} &S \in [0,1] \text{ is constant across } \Sigma_i, \\ &\qquad \text{and represents 'effective roughness' of } \Sigma_i \\ &\qquad \text{i.e. surface, and subsurface sources of diffuse scattering.} \end{aligned} \tag{A.3}$$

A.4 Decoupling of Effective Roughness and Transmittance

Transmittance is only weakly dependent on effective roughness :

$$\partial_S \mathcal{T} \approx 0 \tag{A.4a}$$

$$\implies R \approx \sqrt{1 - S^2}, \tag{A.4b}$$

where (A.4b) can be derived from a combination of

1. Local power balance on dP_S,
2. Equation (A.4a), and

3. Normalizing against the known behaviour of a smooth surface.

It is only approximately true, and becomes less plausible under extremely rough scenarios.

F Field Assumptions

F.1 Remote Antennas

Antennas are not in the near-field (an outdoor scenario), so we have

$$\phi = \frac{\pi}{2}, \quad \chi_E = \chi_H \tag{F.1}$$

F.2 Medium 1

The wave is travelling in free space until it hits an obstruction:

$$(\epsilon_1, \mu_1, \sigma_1) = (\epsilon_0, \mu_0, \sigma_0). \tag{F.2}$$

F.3 Medium 2

The obstruction is a PEC (this can be adjusted later):

$$(\epsilon_2, \mu_2, \sigma_2) = (\epsilon_0, \mu_0, \infty). \tag{F.3}$$

D Direct Consequences

D.1 Local Incremental Power

The formula for local incremental power can be simplified into two useful forms:

$$\begin{aligned} dP_\Sigma(\vec{r}) &= \langle \vec{\mathcal{S}}(\vec{r}) \rangle \cdot d\vec{A}_\Sigma(\vec{r}) \\ &= \mathrm{Re}(\mathbf{E}(\vec{r}) \times \mathbf{H}(\vec{r})^*) \cdot \hat{\mathbf{n}}_\Sigma(\vec{r}) && \text{(D.1a)} \\ &= |\langle \vec{\mathcal{S}} \rangle| \cos(\theta_\Sigma) dA_\Sigma && \text{(D.1b)} \end{aligned}$$

D.2 Transmittance Approximation

(**??**) and (**??**) mean that we can adjust S and R without adversely affecting T. So, considering the perfectly specular case $S = 0, R = 1$, we get

$$\begin{aligned} &\mathcal{T} \approx 1 - |\Gamma|^2 \\ \implies &R^2 + S^2 \approx 1 \end{aligned} \tag{D.2}$$

1 2D Lambertian-Model

Here, we have a restricted 2d setup, and assume a cylindrical wave source:

$$|\langle \vec{\mathcal{S}} \rangle| = \frac{K}{|\vec{r} - \vec{r}_T|} \tag{1.1}$$

We can calculate K by integrating over a wavefront that doesn't intersect Σ_i:

$$\begin{aligned} P_0 &= \int_0^{2\pi} |\langle \vec{\mathcal{S}} \rangle| dl = \int_0^{2\pi} \frac{K}{|\vec{r} - \vec{r}_T|} dl \\ &= \int_0^{2\pi} \frac{K}{|\vec{r} - \vec{r}_T|} |\vec{r} - \vec{r}_T| d\theta = 2\pi K \\ \overset{(1.1),(D.1b)}{\implies} dP_S(\vec{r}) &= \frac{P_0 \cos\theta_S}{2\pi |\vec{r} - \vec{r}_T|} dl_S \end{aligned} \tag{1.2}$$

The Lambertian scattering assumption in 2D is (see (A.1)):

$$I(\theta) = D\cos\theta \tag{1.3}$$

$$\begin{aligned} \implies S^2|\Gamma|^2 dP_S &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D\cos\theta d\theta = 2D \\ \overset{(1.3),(1.2)}{\implies} I(\theta) &= \frac{S^2|\Gamma|^2 P_0 \cos\theta_S dl_S}{4\pi|\vec{r} - \vec{r}_T|} \cos\theta \\ \overset{(A.2b)}{\implies} dP_{\Sigma_{R,S}} &= \frac{1}{S^2} \int_{\Sigma_\partial} \left(\int_{\theta_{\Sigma_R} - \frac{d\theta_{\Sigma_R}}{2}}^{\theta_{\Sigma_R} + \frac{d\theta_{\Sigma_R}}{2}} \frac{P_0 S^2 |\Gamma|^2}{4\pi|\vec{r}_\partial - \vec{r}_T|} \cos\theta d\theta \right) \cos\theta_S dl_S \\ &= \int_{\Sigma_\partial} \left(\frac{P_0|\Gamma|^2 \cos\theta_{\Sigma_R}}{4\pi|\vec{r}_\partial - \vec{r}_T|} d\theta_{\Sigma_R} \right) \cos\theta_S dl_S \\ &= \frac{P_0}{4\pi} \left(\int_{\Sigma_\partial} \frac{|\Gamma|^2 \cos\theta_{\Sigma_R} \cos\theta_S}{|\vec{r}_\partial - \vec{r}_T||\vec{r}_R - \vec{r}_\partial|} dl_S \right) dl_{\Sigma_R} \end{aligned} \tag{1.4}$$

$dP_{\Sigma,GO}$, the portion as calculated via Geometrical Optics, is given by:

$$dP_{\Sigma,GO} = \mathbb{1}_{\Sigma_\partial}(\vec{r}_{\text{spec}}) \frac{P_0}{2\pi} \left(\frac{|\Gamma|^2 \cos\theta_{\text{spec}}}{|\vec{r}_{\text{spec}} - \vec{r}_T| + |\vec{r}_R - \vec{r}_{\text{spec}}|} \right) dl_{\Sigma_R} \tag{1.5}$$

1.1 Setup Simplifications

Assume that the second medium is a PEC strip starting at $(w_{x0}, 0)$ and ending at $(w_{xe}, 0)$:

$$\Sigma_i = \{(w_x, 0) \in \mathbb{R}^2 \mid w_x \in [w_{x0}, w_{xe}]\} \tag{1.6}$$

So that

$$\vec{r}_{\text{spec}} = (w_{\text{spec}}, 0) \tag{1.7}$$

The line of receivers is parallel to the strip:

$$\Sigma_R = \{(r_x, r_y) \in \mathbb{R}^2 \mid r_x \in [r_{x0}, r_{xe}]\} \tag{1.8}$$

The transmitter is fixed:

$$\vec{r}_T = (t_x, t_y) \tag{1.9}$$

We also assume that the source is linearly polarized with the electric field normal to the plane of incidence, so that

$$|\Gamma|^2 = 1 \tag{1.10}$$

We then get:

$$\begin{aligned}
dx &= dl_S = dl_{\Sigma_R} \\
\frac{\cos\theta_{\Sigma_R}}{|\vec{r}_\partial - \vec{r}_T|} &= \frac{t_y}{t_y^2 + (w_x - t_x)^2} \\
\frac{\cos\theta_S}{|\vec{r}_R - \vec{r}_\partial|} &= \frac{r_y}{r_y^2 + (r_x - w_x)^2} \\
w_{\text{spec}} &= t_x + t_y\left(\frac{r_x - t_x}{r_y + t_y}\right) \\
&= r_x - r_y\left(\frac{r_x - t_x}{r_y + t_y}\right) \\
\implies \cos\theta_{\text{spec}} &= \frac{r_y + t_y}{\sqrt{(r_x - t_x)^2 + (r_y + t_y)^2}}, \quad \text{and} \\
|\vec{r}_{\text{spec}} - \vec{r}_T| + |\vec{r}_R - \vec{r}_{\text{spec}}| &= \sqrt{(r_x - t_x)^2 + (r_y + t_y)^2}
\end{aligned}$$

which lead to the explicit spatial power distributions along the line of receivers:

$$dP_{\Sigma_{R,S}} = \frac{P_0}{4\pi}\left(\int_{w_{x0}}^{w_{xe}} \frac{t_y r_y}{\left(t_y^2 + (x' - t_x)^2\right)\left(r_y^2 + (x - x')^2\right)} dx'\right) dx \tag{1.11}$$

$$dP_{\Sigma_{R,GO}} = \frac{P_0}{4\pi}\begin{cases} 2 \cdot \dfrac{t_y + r_y}{(x - t_x)^2 + (r_y + t_y)^2} dx & \text{if } w_{x0} \le w_{\text{spec}} \le w_{xe} \\ 0 & \text{otherwise} \end{cases} \tag{1.12}$$

1.2 Example Setup

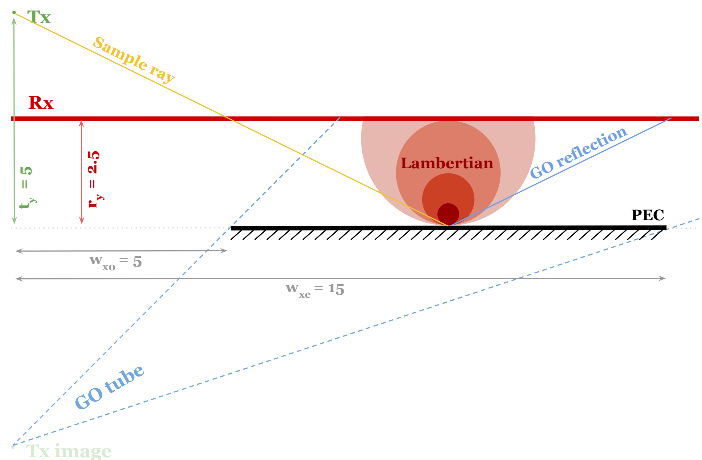

Figure 1: Line of receivers and fixed point source for a PEC strip of fixed width

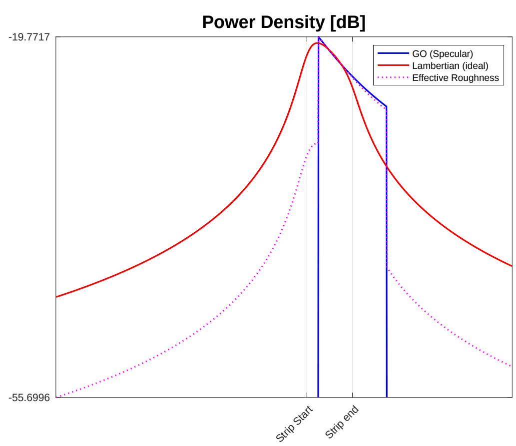

Figure 2: Power Density Profile for setup in Figure 1.

1.3 Lambertian Asymptotic

A virtue of implementing the ER model is the convergence of (1.11) under small values of the numerical parameter N_{strip}, the number of points along the strip for the integration, which is likely a consequence of the simple surface profile.

An even further simplification can be made, if we consider that the integrand in (1.11) is the product of two Cauchy distributions, and the whole integral, in the limit of an infinite wall, is the convolution of two Cauchy distributions. Under this limit, we can simplify (1.11) to

$$dP_{\Sigma_{R,S}} \sim \frac{P_0}{4} \cdot \frac{t_y + r_y}{(x - t_x)^2 + (t_y + r_y)^2}. \tag{1.13}$$

A comparison of (1.11) and (1.13) is shown in Figure 3 (limiting case).

1.4 Validation vs. Physical Optics Approximation

It's instructive to compare (1.11) to the physical optics case. Here, the incident electric field is given by a Hankel function:

$$E^i(\vec{r}) = E_0 H_0^{(2)}(k|\vec{r} - \vec{r}_T|)\hat{\mathbf{e}}_z \tag{1.14}$$

The incident magnetic field can be calculated from Maxwell's equations:

$$\begin{aligned}
H^i(\vec{r}) &= \frac{j}{\omega\mu_0} \nabla \times E^i(\vec{r}) \\
&= \frac{j}{\omega\mu_0} \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ 0 & 0 & E_0 H_0^{(2)}(k|\vec{r} - \vec{r}_T|) \\ \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \end{vmatrix} \\
&= \frac{jE_0}{\omega\mu_0} \left(\left(\partial_y (H_0^{(2)}(k|\vec{r} - \vec{r}_T|)) \right) \hat{\mathbf{e}}_x - \left(\partial_x (H_0^{(2)}(k|\vec{r} - \vec{r}_T|)) \right) \hat{\mathbf{e}}_y \right) \\
&= -\frac{jkE_0 H_1^{(2)}(k|\vec{r} - \vec{r}_T|)}{\omega\mu_0 |\vec{r} - \vec{r}_T|} \left((y - t_y)\hat{\mathbf{e}}_x - (x - t_x)\hat{\mathbf{e}}_y \right)
\end{aligned} \tag{1.15}$$

The Physical Optics surface current is then given by

$$\begin{aligned}
J^{PO}(\vec{r}) &= 2\hat{n} \times H^i(\vec{r}) = 2\hat{\mathbf{e}}_y \times H^i(\vec{r}) \\
&= \frac{2jkE_0(y - t_y) H_1^{(2)}(k|\vec{r} - \vec{r}_T|)}{\omega\mu_0 |\vec{r} - \vec{r}_T|} \hat{\mathbf{e}}_z
\end{aligned} \tag{1.16}$$

In our case $\vec{r} \in \{\ (x, 0) \mid x \in (w_{x0}, w_{xe})\ \}$, so $y = 0$. Using this, we can find the scattered electric field:

$$\begin{aligned}
E_S^{PO}(\vec{r}) &= \frac{k\eta_0}{4} \int_{\Sigma_i} J^{PO}(\vec{r}\,') H_0^{(2)}(k|\vec{r} - \vec{r}\,'|) dl' \\
&= -\frac{jkE_0 t_y}{2} \int_{\Sigma_i} \frac{H_1^{(2)}(k|\vec{r}\,' - \vec{r}_T|)\ H_0^{(2)}(k|\vec{r} - \vec{r}\,'|)\hat{\mathbf{e}}_z}{|\vec{r}\,' - \vec{r}_T|} dx'
\end{aligned} \tag{1.17}$$

Now, compute the scattered magnetic field (complex-conjugate), again from Maxwell:

$$\begin{aligned} H_S^{PO*} &= \frac{-j}{\omega\mu_0}\left(\nabla \times E_S^{PO}\right)^* \\ &= -\frac{E_0 t_y}{2\eta_0}\left(\int_{\Sigma_i} \frac{H_1^{(2)}(k|\vec{r}' - \vec{r}_T|)\ \nabla \times (H_0^{(2)}(k|\vec{r} - \vec{r}'|)\hat{\mathbf{e}}_z)}{|\vec{r}' - \vec{r}_T|} dx'\right)^* \\ &= -\frac{kE_0 t_y}{2\eta_0}\left(\int_{\Sigma_i} \frac{H_1^{(2)}(k|\vec{r}' - \vec{r}_T|) H_1^{(2)}(k|\vec{r} - \vec{r}'|)(r_y\hat{\mathbf{e}}_x + ((x' - x)\hat{\mathbf{e}}_y))}{|\vec{r}' - \vec{r}_T||\vec{r} - \vec{r}'|} dx'\right)^* \\ &= -\frac{kE_0 t_y}{2\eta_0}\int_{\Sigma_i} \frac{H_1^{(1)}(k|\vec{r}' - \vec{r}_T|) H_1^{(1)}(k|\vec{r} - \vec{r}'|)(r_y\hat{\mathbf{e}}_x + ((x' - x)\hat{\mathbf{e}}_y))}{|\vec{r}' - \vec{r}_T||\vec{r} - \vec{r}'|} dx' \end{aligned}$$

From this, we get

$$\begin{aligned} dP_S^{PO} &= \frac{1}{2}\hat{\mathbf{n}} \cdot \mathrm{Re}(E_S^{PO} \times H_S^{PO*}) \\ &= -\frac{k^2 P_0 t_y^2 r_y}{4} \\ &\mathrm{Im}\left(\int_{\Sigma_i} \frac{H_1^{(1)}(k|\vec{r}_{T,\partial}|) H_1^{(1)}(k|\vec{r}_{\partial,R}|)}{|\vec{r}_{T,\partial}||\vec{r}_{\partial,R}|} dx' \int_{\Sigma_i} \frac{H_1^{(2)}(k|\vec{r}_{T,\partial}|)\ H_0^{(2)}(k|\vec{r}_{\partial,R}|)\hat{\mathbf{e}}_z}{|\vec{r}_{T,\partial}|} dx'\right) \end{aligned} \quad (1.18)$$

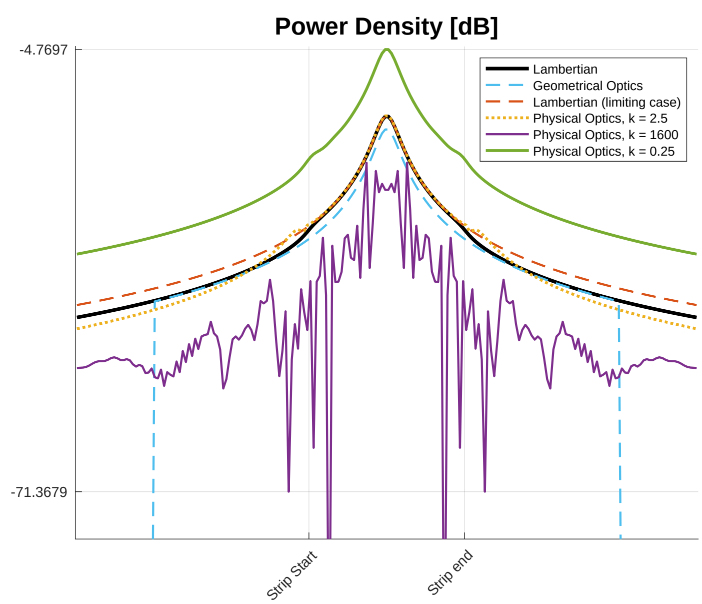

Figure 3: Model Validation vs. Physical Optics Approximation

Figure 3 is a revealing comparison and validation of the models seen so far for the simple setup. Among other things, it shows that:

- There is an inherent overestimation in the Lambertian models, independent of the scattering parameter S,
- This overestimation would likely carry over to directional models,
- So, new parametric models could certainly improve the accuracy and computational efficiency of the legacy models.

1.5 Power Delay Profile

We can also derive temporal power distributions (taking care with t-labels) from (1.11) and (1.12) by converting from spatial x, x' coordinates to temporal t, t' coordinates, via

$$ct = \sqrt{t_y^2 + (x - t_x)^2} + \sqrt{r_y^2 + (r_x - x)^2}, \quad \text{and} \tag{1.19}$$

$$ct' = \sqrt{t_y^2 + (x' - t_x)^2} \tag{1.20}$$

Inverting (1.20) for x' is a simple algebraic manipulation, whereas (1.19) inversion for x requires a change of coordinates so that the origin is at the specular point of reflection. Upon squaring twice, the x^4 and x^3 terms then cancel.

1.6 MATLAB Code

```matlab
% Assume a cylindrical wave incident field
% The models used to compare are:
% 1. Effective Roughness (ER) Lambertian model
%    (overall ER contribution is an integral along the points of the wall), and
% 2. Geometrical Optics  (GO).
%
% Compare the time-averaged power density for the setup:
%
%     (r_x0, r_y)               (r_x, r_y)         (r_xe, r_y)
%                                                 /
%                (t_x, t_y)                    /
%                     \                     /
%                      \                 /
%                       \             /
%                        \         /
%                         \     /
%                          \/
% (w_x0,0)       (w_x, 0)           (w_xe, 0)
%
function dP = GO_Power_Density( x, ...              % Position along receiver line
                                P0, ...             % Source power
                                r_y, ...            % Rx antenna height
                                t_x, t_y, ...       % Tx antenna positions
                                w_x0, w_xe )        % Start and end of strip
% Formula 1.12 in derivations
   w_spec = t_x + t_y * ( (x - t_x)/(r_y + t_y) );
   if( w_x0 > w_spec || w_spec > w_xe )
```

```
      dP = 0;
   else
      dP = P0 * ( r_y + t_y ) / ...
               ( 2 * pi * ((x - t_x)^2 + (r_y + t_y)^2) );
   end
end

function dP = Lambertian_Power_Density( x, ...
                                        P0, ...
                                        r_y, ...
                                        t_x, t_y, ...
                                        w_x0, w_xe, ...
                                        N_strip )   % # Points along strip
% Formula 1.11 in derivations
   w_len = w_xe - w_x0;
   dx_w = w_len / N_strip;
   dP = 0;
   for( N = 1:N_strip )
      w_x = w_x0 + (N-0.5)*dx_w;
      numerator = t_y * r_y * dx_w;
      denominator = ((t_y)^2 + (w_x - t_x)^2) * ((r_y)^2 + (x - w_x)^2);
      ddP = ( numerator / denominator );
      dP = dP + ddP;
   end
   dP = dP * P0 / (4 * pi);
end

function dP = PO_Power_Density( x, ...
                                P0, ...
                                k, ...
                                r_y, ...
                                t_x, t_y, ...
                                w_x0, w_xe, ...
                                N_strip )   % # Points along strip
% Formula 1.20 in derivations
   w_len = w_xe - w_x0;
   dx_w = w_len / N_strip;
   int1 = 0;
   int2 = 0;
   for( N=1:N_strip )
      w_x = w_x0 + (N-0.5)*dx_w;
      r_TB = sqrt( (w_x - t_x)^2 + (t_y)^2 );
      r_BR = sqrt( (x - w_x)^2 + (r_y)^2 );
      num1 = besselh(1, 1, k*r_TB) * besselh(1, 1, k*r_BR );
      denom1 = r_TB * r_BR;
      num2 = besselh(1, 2, k*r_TB) * besselh( 0, 2, k*r_BR );
      denom2 = r_TB;
      int1 = int1 + dx_w*(num1/denom1);
      int2 = int2 + dx_w*(num2/denom2);
   end
   dP = imag( int1 * int2 );
   dP = - dP * (k*t_y)^2 * P0 * (r_y) / 4.0;
end

function dP = Lambertian_Power_Density_PO( x, ...
                                           P0, ...
                                           r_y, ...
                                           t_x, t_y, ...
                                           w_x0, w_xe )
% Formula 1.13 in derivations
    dP = (t_y + r_y)/((x - t_x)^2 + (t_y + r_y)^2);
    dP = P0 * dP / 4;%(4*pi);
end

%---Special Parameter Markings-------------------------------------------------------
%   [!] Interesting
%   [n] Numerical
%------------------------------------------------------------------------------------
```

```
clc
clear all
close all
% Rx antennas------------------------------------------------------------------------
r_x0 = -200;                               % [!] First Rx
r_xe = 200;                                % [!] Last Rx
r_y = 5;                                   % [!] Rx height wrt strip
N_rx  = 200;                               % [n] Number of receivers
r_spread = r_xe - r_x0;                    %     Rx spread
dx_r = r_spread / N_rx;                    %     dx along Rx-line
r_x = zeros(1, N_rx);                      %     Rx positions
for( N = 1:N_rx )
   r_x( N ) = r_x0 + (N-0.5) * dx_r;
end
% Tx antenna-------------------------------------------------------------------------
t_x = 0;                                   %     Tx position
t_y = 2.5;                                 % [!] Tx height wrt strip
% Strip------------------------------------------------------------------------------
w_x0 = -50;                                % [!] Strip start
w_xe = 50;                                 % [!] Strip end
N_strip = 600;                             % [n] # Calc points along strip
w_len = w_xe - w_x0;                       %     Strip length
% Source waves-----------------------------------------------------------------------
P0 = 1;                                    %     Source power
k_sweet = 2.5e0;                           % [!] Wavenumber, sweetspot
k_low = 0.1 * k_sweet;                     % [>] Wavenumber, low
k_high = 1600;                             % [!] Wavenumber, high (EHF)
% Models-----------------------------------------------------------------------------
S = sqrt(0.1);                             % [!] Effective Roughness
R = sqrt( 1.0 - (S)^2);                    %     Specular reflectance reduction

Lambertian_Density = zeros(1, N_rx);
Lambertian_Density_PO = zeros(1,N_rx);
PO_Density_Sweet = zeros(1,N_rx);
PO_Density_High = zeros(1,N_rx);
PO_Density_Low = zeros(1,N_rx);
GO_Density = zeros(1, N_rx);
ER_Density = zeros(1, N_rx);
r_spec_start = NaN; %prealloc
r_spec_end = NaN; %prealloc
for(N_r = 1:N_rx)
   x = r_x( N_r );
   this_w_spec = t_x + t_y * ( (x - t_x)/(r_y + t_y) );
   if( w_x0  <= this_w_spec && this_w_spec <= w_xe && isnan(r_spec_start) )
       r_spec_start = x;
   elseif( w_xe < this_w_spec && isnan(r_spec_end) )
       r_spec_end = x;
   end
   PO_Density_Sweet( N_r ) = PO_Power_Density( x, ...
                                               P0, k_sweet, ...
                                               r_y, ...
                                               t_x, t_y, ...
                                               w_x0, w_xe, ...
                                               N_strip );   % # Points along strip
   PO_Density_High( N_r ) = PO_Power_Density( x, ...
                                              P0, k_high, ...
                                              r_y, ...
                                              t_x, t_y, ...
                                              w_x0, w_xe, ...
                                              N_strip );   % # Points along strip
   PO_Density_Low( N_r ) = PO_Power_Density( x, ...
                                             P0, k_low, ...
                                             r_y, ...
                                             t_x, t_y, ...
                                             w_x0, w_xe, ...
                                             N_strip );   % # Points along strip
   GO_Density( N_r ) = GO_Power_Density( x, ...
                                         P0, r_y, t_x, t_y, ...
```

```
                                          w_x0, w_xe );
    Lambertian_Density( N_r ) = Lambertian_Power_Density( x, ...
                                                          P0, r_y, t_x, t_y, ...
                                                          w_x0, w_xe, ...
                                                          N_strip );
    ER_Density( N_r ) = (S^2) * Lambertian_Density( N_r ) + (R^2) * GO_Density( N_r );
    Lambertian_Density_PO( N_r ) = Lambertian_Power_Density_PO( x, ...
                                                                P0, r_y, t_x, t_y, ...
                                                                w_x0, w_xe );
    % Use decibels:
    GO_Density( N_r ) = 10 * log10( GO_Density( N_r ));
    Lambertian_Density( N_r ) = 10 * log10( Lambertian_Density( N_r ));
    ER_Density( N_r ) = 10 * log10( ER_Density( N_r ));
    Lambertian_Density_PO( N_r ) = 10 * log10( Lambertian_Density_PO( N_r ));
    PO_Density_Sweet( N_r ) = 10 * log10( PO_Density_Sweet( N_r ));
    PO_Density_High( N_r ) = 10 * log10( PO_Density_High( N_r ));
    PO_Density_Low( N_r ) = 10 * log10( PO_Density_Low( N_r ));
end
% handle -inf for dB graphs
for(N_r = 1:N_rx)
    % replace complex values with -inf
    if( imag(PO_Density_Sweet(N_r)) > 1e-12 )
        PO_Density_Sweet(N_r) = -inf;
    else
        PO_Density_Sweet(N_r) = real(PO_Density_Sweet(N_r));
    end
    if( imag(PO_Density_High(N_r)) > 1e-12 )
        PO_Density_High(N_r) = -inf;
    else
        PO_Density_High(N_r) = real(PO_Density_High(N_r));
    end
    if( imag(PO_Density_Low(N_r)) > 1e-12 )
        PO_Density_Low(N_r) = -inf;
    else
        PO_Density_Low(N_r) = real(PO_Density_Low(N_r));
    end
end
all_data = [ GO_Density(:), Lambertian_Density(:), ER_Density(:), ...
             PO_Density_Sweet(:), PO_Density_High(:), PO_Density_Low(:) ];
finite_values = all_data(isfinite(all_data));
min_finite_val = min( finite_values );
max_finite_val = max( finite_values );
inf_display_val = min_finite_val - 0.5 * ( max_finite_val - min_finite_val );
for(N_r = 1:N_rx)
    if( GO_Density(N_r) == -inf )
        GO_Density(N_r) = inf_display_val;
    end
    if( Lambertian_Density(N_r) == -inf )
        Lambertian_Density(N_r) = inf_display_val;
    end
    if( ER_Density(N_r) == -inf )
        ER_Density( N_r ) = inf_display_val;
    end
    if( PO_Density_Sweet(N_r) == -inf )
        PO_Density_Sweet( N_r ) = inf_display_val;
    end
    if( PO_Density_High(N_r) == -inf )
        PO_Density_High( N_r ) = inf_display_val;
    end
    if( PO_Density_Low(N_r) == -inf )
        PO_Density_Low( N_r ) = inf_display_val;
    end
end

% default color ordering
new_colors = colororder;
figure;
%plot( r_x, GO_Density, ...
```

```
%      'b', ...
%      'LineWidth', 1.5, ...
%      'DisplayName', 'GO (Specular)' );
hold on;
plot( r_x, Lambertian_Density, ...
      'Color', 'k', ...
      'LineWidth', 2.5, ...
      'DisplayName', 'Lambertian' );
plot( r_x, GO_Density, '--', ...
      'Color', new_colors(6,:), ...
      'LineWidth', 1.5, ...
      'DisplayName', "Geometrical Optics" );
plot( r_x, Lambertian_Density_PO, '--', ...
      'Color', new_colors(2,:), ...
      'LineWidth', 1.5, ...
      'DisplayName', 'Lambertian␣(limiting␣case)' );
plot( r_x, PO_Density_Sweet, ':', ...
      'Color', new_colors(3,:), ...
      'LineWidth',2.0, ...
      'DisplayName', "Physical Optics, k = " + k_sweet );
plot( r_x, PO_Density_High, ...
      'Color', new_colors(4,:), ...
      'LineWidth', 1.5, ...
      'DisplayName', "Physical Optics, k = " + k_high );
plot( r_x, PO_Density_Low, ...
      'Color', new_colors(5,:), ...
      'LineWidth', 2.0, ...
      'DisplayName', "Physical Optics, k = " + k_low );
grid on;
ylim([min_finite_val*1.1, max_finite_val*0.9]);
title_string = "Power Density [dB]"
title( title_string, ...
       'FontSize', 17, ...
       'FontWeight', 'bold' )
ax = gca;
special_locations = [ w_x0, w_xe ];
special_labels = { 'Strip␣Start',  'Strip␣end' };
[ sorted_locations, sort_order] = sort(special_locations);
sorted_labels = special_labels(sort_order);
ax.XTick = sorted_locations;
ax.XTickLabel = sorted_labels;
ax.XTickLabelRotation = 45;
ax.YTick = [ min_finite_val, max_finite_val ];
hold off;
legend;
```