1 ER Model

Assumptions

- **E-field decomposition**:

$$E = E_i + E_r + E_s + E_t, \text{ where} \tag{1}$$

  - E is the total phasor (spatial, or time-averaged) component of the electric field,
  - E_i is the incident field,
  - E_r is the specularly reflected field,
  - E_s is the diffusely scattered field, and
  - E_t is the transmitted field.

- **E_s-incoherence in Rx-region**: The various contributions to E_s arriving at the Rx have random, uncorrelated phases. Consequently, the total squared-magnitude $|E_s|^2$ of the scattered field satisfies:

$$|E_s|^2 = \int_W d(|E_s|^2), \text{ where} \tag{2}$$

  - $d(|E_s|^2)$ is the squared-field contribution from
  - dW, the infinitesimal scattering element on
  - W, the wall surface.

- **Constant Local Power Balance (diffuse scattering coefficient S)**: A constant fraction S^2 of the power incident on a patch dW is converted into diffusely scattered power:

$$S^2 := \frac{dP_s}{dP_i}, \text{ where} \tag{3}$$

  - dP_i is the power incident on dW, and
  - dP_s is the total power diffusely scattered from dW into the surrounding hemisphere.

- **Lambertian Scattering Pattern**:

$$\frac{d(dP_s)}{d\Omega_s} = K \cos\theta_s, \text{ where} \tag{4}$$

  - $d\Omega_s$ is the infinitesimal solid angle of the scattered ray-tube, and
  - K is a normalization constant, found by integrating over the entire forward scattering space (a hemisphere in 3D, a semicircle in 2D) and setting the result equal to the total scattered power, dP_s.

2D-Setup

dP_{Rx} - power through Rx aperture

From the scattering element's (dW) POV, we have:

$$dP_{Rx} \approx \left(\frac{d(dP_s)}{d\theta_s} \right) d\theta_{Rx} \tag{5}$$

Integrating (4) over $-\frac{\pi}{2} < \theta_s \leq \frac{\pi}{2}$, then plugging into (5) we get

$$dP_{Rx} = \frac{dP_s}{2} \cos\theta_s d\theta_{Rx} = \frac{dP_s}{2r_s} \cos\theta_s dl_{Rx} \tag{6}$$

From the Rx's POV, we have

$$dP_{Rx} = \frac{d(|E_s|^2)}{2\eta_0} dl_{Rx} \tag{7}$$

Combining (3), (6) and (7) we get

$$d(|E_s|^2) = \frac{\eta_0 S^2 \cos\theta_s}{r_s} dP_i \tag{8}$$

dP_i - power incident on dW

From the incoming waves POV, we have

$$dP_i = \frac{|E_i|^2}{2\eta_0} \cos\theta_i dx. \tag{9}$$

Combining (9) with (8), we get

$$d(|E_s|^2) = \frac{S^2 \cos\theta_i \cos\theta_s}{2r_s} |E_i|^2 dx \tag{10}$$

$|E_i|$ as a function of P_i and r_i

A far-field cylindrical incoming wave allows us to assume

$$|E_i|^2 = \frac{A}{r_i} \tag{11}$$

$$
\begin{aligned}
\implies P_i &= \int_0^{2\pi} \frac{|E_i|^2}{2\eta_0} r_i d\theta \\
&= \frac{A}{2\eta_0} \int_0^{2\pi} d\theta \\
&= \frac{A\pi}{\eta_0} \\
\implies A &= \frac{\eta_0 P_i}{\pi}
\end{aligned}
$$

$$
\implies |E_i|^2 = \frac{\eta_o P_i}{\pi} \frac{1}{r_i} \tag{12}
$$

Then, combining (12) with (10), we get

$$
d(|E_s|^2) = \frac{S^2 \eta_0 P_i \cos\theta_i \cos\theta_s}{2\pi r_s r_i} \; dx \tag{13}
$$

Coordinate Transformations

Our setup consists of a transmitter at (x_T, y_T) and a receiver at (x_R, y_R). We can express $\theta_i, \theta_s, r_i, r_s$ as follows:

$$
\begin{aligned}
r_s &= \sqrt{y_R^2 + (x_R - x)^2} \\
\cos\theta_s &= \frac{y_R}{r_s} \\
r_i &= \sqrt{y_T^2 + (x - x_T)^2} \\
\cos\theta_i &= \frac{y_T}{r_i}
\end{aligned}
$$

Plugging these into (13) we get

$$
d(|E_s|^2) = \frac{S^2 \eta_0 P_i \; y_T y_R}{2\pi (y_R^2 + (x_R - x)^2)(y_T^2 + (x - x_T)^2)} \; dx \tag{14}
$$

We can also express x in terms of t by inverting the equation

$$
(ct) = \sqrt{y_T^2 + (x - x_T)^2} + \sqrt{y_R^2 + (x_R - x)^2}.
$$

In order to invert this, we need to square twice to get rid of any square roots. However, in order for the x^4 and x^3 terms to cancel, we need to make a further coordinate transformation so the origin is at the specular point of reflection:

$$
x \to x' = x - x_T - y_T \left(\frac{x_R - x_T}{y_R + y_T} \right) \tag{15}
$$