1. Start from Maxwell's equations

$$\nabla \times H = J + \frac{\partial D}{\partial t} \tag{1a}$$
$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{1b}$$
$$\nabla \cdot D = \rho \tag{1c}$$
$$\nabla \cdot B = 0 \tag{1d}$$

and constitutive equations (without dielectric)

$$D = \epsilon E \tag{2a}$$
$$B = \mu H \tag{2b}$$

2. Assume time harmonicity ($\Xi(r,t) = \Xi(r)e^{j\omega t}$), where $\Xi \in B, D, E, H$, and simplify:

$$\nabla \times H = J + j\omega D \tag{3a}$$
$$\nabla \times E = -j\omega B \tag{3b}$$
$$\nabla \cdot D = \rho \tag{3c}$$
$$\nabla \cdot B = 0 \tag{3d}$$

3. Introduce magnetic vector potential A and electric potential ϕ defined by:

$$B = \nabla \times A \tag{4a}$$
$$E = -\nabla\phi - \frac{\partial A}{\partial t} \tag{4b}$$

and fix Lorenz gauge:

$$\nabla \cdot A + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

Note that time-harmonicity also applies to A, and therefore will also apply to ϕ, so the Lorenz gauge reduces to

$$\nabla \cdot A = -\frac{j\omega}{c^2}\phi \tag{5}$$

4. Apply the above with some vector identities to reduce Ampere-Maxwell equation (3a) to Helmholtz equation:

   (a) Substitute the constitutive conditions (2) and potentials (4) respectively to (3a):

$$\frac{1}{\mu}\nabla \times B = J + j\omega\epsilon E$$
$$\implies \frac{1}{\mu}\nabla \times \nabla \times A = J - j\omega\epsilon\left(\nabla\phi + \frac{\partial A}{\partial t}\right)$$
$$\implies \frac{1}{\mu}\nabla \times \nabla \times A = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A$$

(b) Use the vector calculus identity $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$, so

$$\frac{1}{\mu}\left(\nabla(\nabla \cdot A) - \nabla^2 A\right) = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A$$

(c) Apply the Lorenz gauge condition (5):

$$\begin{aligned} &-\frac{1}{\mu}(\frac{j\omega}{c^2}\nabla\phi + \nabla^2 A) = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A \\ \implies &-\frac{1}{\mu}\nabla^2 A = J - \omega^2\epsilon A \\ \implies &\nabla^2 A + k^2 A = -\mu J. \end{aligned}$$

5. Solve for vector potential using Green functions to get

$$A = \mu \int J(r)G(r, r')dr' \tag{6}$$

6. Plug into electric field formula (4b) in terms of potentials:

$$A = \frac{j}{\omega}(\nabla\phi + E)$$

Use the far field assumption on the scalar potential $\nabla\phi \to 0$ - this can be worked out to die off quickly asymptotically. Finally we arrive at

$$\begin{aligned} &E \approx -\omega\mu j \int J(r)G(r, r')dr' \\ \implies &E \approx -\eta k j \int J(r)G(r, r')dr' \end{aligned}$$