Balanis Quick Reference

Andrew Whelan

Contents

Notation

∂_t	$\dfrac{\partial}{\partial t}$	
E	Electric Field Intensity	[V/m]
H	Magnetic Field Intensity	[A/m]
D	Electric Flux Density	[C/m^2]
B	Magnetic Flux Density	[Wb/m^2]
$J_{c/d/i}$	Electric Current Density $_{\text{(conduction/displacement/impressed)}}$	[A/m^2]
$\mathfrak{M}_{d/i}$	Magnetic Current Density $_{\text{(displacement/impressed)}}$	[V/m^2]
ρ_e	Electric Charge Density	[C/m^2]
ρ_m	Magnetic Charge Density	[Wb/m^2]

1 Basics

1.1 Maxwell's Equations

Differential Form

$$
\begin{aligned}
\text{Maxwell-Faraday :} &\qquad \nabla \times E = -\partial_t B - \mathfrak{M}_i && \text{(1a)} \\
\text{Ampère-Maxwell :} &\qquad \nabla \times H = \partial_t D + J_c + J_i && \text{(1b)} \\
\text{Gauss :} &\qquad \nabla \cdot D = \rho_e && \text{(1c)} \\
\text{Gauss (Magnetism) :} &\qquad \nabla \cdot B = 0 \quad = \rho_m && \text{(1d)}
\end{aligned}
$$

Integral Form

$$
\oint_C E \cdot dl = -\partial_t \iint_S B \cdot ds - \iint_S \mathfrak{M}_i \cdot ds \tag{2a}
$$

$$
\oint_C H \cdot dl = \partial_t \iint_S D \cdot ds + \iint_S (J_c + J_i) \cdot ds \tag{2b}
$$

$$
\oiint_S D \cdot ds = \iiint_V \rho_e \cdot dv \tag{2c}
$$

$$
\oiint_S B \cdot ds = \iiint_V \rho_m \cdot dv \tag{2d}
$$

1.2 Constitutive Relations

$$
D = \epsilon E \tag{3a}
$$

$$
B = \mu H \tag{3b}
$$

$$
J_c = \sigma E \tag{3c}
$$

2 P.E.C Field Derivation

1. Starting from Maxwell's equations (1) and constitutive equations (3), transform to frequency domain to simplify curl equations:

$$\nabla \times H = J + j\omega D \tag{4a}$$
$$\nabla \times E = -j\omega B \tag{4b}$$

2. Introduce magnetic vector potential A and electric potential ϕ defined by:

$$B = \nabla \times A \tag{5a}$$
$$E = -\nabla\phi - j\omega A \tag{5b}$$

and fix Lorenz gauge:

$$\nabla \cdot A = -\frac{j\omega}{c^2}\phi \tag{6}$$

3. Apply the above with some vector identities to reduce Ampere-Maxwell equation (4a) to Helmholtz equation:

   (a) Substitute the constitutive conditions (3) and potentials (5) respectively to (4a):

$$\begin{aligned} &\frac{1}{\mu}\nabla \times B = J + j\omega\epsilon E \\ \implies &\frac{1}{\mu}\nabla \times \nabla \times A = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A \end{aligned}$$

   (b) Use the vector calculus identity $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$, so

$$\frac{1}{\mu}\left(\nabla(\nabla \cdot A) - \nabla^2 A\right) = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A$$

   (c) Apply the Lorenz gauge condition (6):

$$\begin{aligned} &-\frac{1}{\mu}(\frac{j\omega}{c^2}\nabla\phi + \nabla^2 A) = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A \\ \implies &-\frac{1}{\mu}\nabla^2 A = J - \omega^2\epsilon A \\ \implies &\nabla^2 A + k^2 A = -\mu J. \end{aligned}$$

4. Solve for vector potential using Green functions to get

$$A = \mu \int J(r)G(r, r')dr' \tag{7}$$

5. Plug into electric field formula (5b) in terms of potentials:

$$A = \frac{j}{\omega}(\nabla\phi + E)$$

Use the far field assumption on the scalar potential $\nabla\phi \to 0$ - this can be worked out to die off quickly asymptotically. Finally we arrive at

$$E \approx -\omega\mu j \int J(r)G(r,r')dr'$$
$$\implies E \approx -\eta k j \int J(r)G(r,r')dr'$$