Balanis Quick Reference

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Notation

Mathematical

∂_t	$\frac{\partial}{\partial t}$	
j	Imaginary unit ($j^2 = -1$)	

Microscopic Fields

E	Electric Field Intensity	[V m^{-1}]
H	Magnetic Field Intensity	[A m^{-1}]

Macroscopic Fields

D	Electric Flux Density	[C m^{-2}]
B	Magnetic Flux Density	[Wb m^{-2}]

Field Sources

$J_{c/d/i}$	Electric Current Density (conduction/displacement/impressed)	[A m^{-2}]
$\mathfrak{M}_{d/i}$	Magnetic Current Density (displacement/impressed)	[V m^{-2}]
ρ_e	Electric Charge Density	[C m^{-2}]
ρ_m	Magnetic Charge Density	[Wb m^{-2}]

Constitutive Parameters

ϵ	Permittivity	[F m^{-1}]
μ	Permeability	[H m^{-1}]
σ	Conductivity	[S m^{-1}]

Waves

f	Frequency	[s^{-1}]
$k^{(11)}$	Wavenumber	[m^{-1}]
$\alpha^{(10)}$	Wave Attenuation	[m^{-1}]
$\lambda := \frac{2\pi}{k}$	Wavelength	[m]
$\omega := 2\pi f$	Angular Frequency	[s^{-1}]
$\eta := \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$	Wave Impedance	[S^{-1}]
$v_g := \partial_k \omega$	Group velocity of wave (envelope velocity)	[ms^{-1}]
$v_p := \frac{\omega}{k}$	Phase velocity of wave (peak/trough velocity)	[ms^{-1}]

1 Basics

1.1 Maxwell's Equations

Differential Form

$$
\begin{aligned}
&\text{Maxwell-Faraday}: && \nabla \times E = -\partial_t B - \mathfrak{M}_i && (1a)\\
&\text{Ampère-Maxwell}: && \nabla \times H = \partial_t D + J_c + J_i && (1b)\\
&\text{Gauss}: && \nabla \cdot D = \rho_e && (1c)\\
&\text{Gauss (Magnetism)}: && \nabla \cdot B = 0 = \rho_m && (1d)
\end{aligned}
$$

Integral Form

$$\oint_C E \cdot dl = -\partial_t \iint_S B \cdot ds - \iint_S \mathfrak{M}_i \cdot ds \tag{2a}$$

$$\oint_C H \cdot dl = \partial_t \iint_S D \cdot ds + \iint_S (J_c + J_i) \cdot ds \tag{2b}$$

$$\oiint_S D \cdot ds = \iiint_V \rho_e \cdot dv \tag{2c}$$

$$\oiint_S B \cdot ds = 0 = \iiint_V \rho_m \cdot dv \tag{2d}$$

1.2 Constitutive Relations

$$D = \epsilon E \tag{3a}$$

$$B = \mu H \tag{3b}$$

$$J_c = \sigma E \tag{3c}$$

1.3 Boundary Conditions

	General	**Finite σ, no source/charge**	**Medium 1 PEC**	**Medium 1 PMC**
$E_\parallel := n \times E$	$E_{\parallel 2} - E_{\parallel 1} = -\mathfrak{M}_s$	$E_{\parallel 2} - E_{\parallel 1} = 0$	$E_{\parallel 2} = 0$	$E_{\parallel 2} = -\mathfrak{M}_s$
$H_\parallel := n \times H$	$H_{\parallel 2} - H_{\parallel 1} = J_s$	$H_{\parallel 2} - H_{\parallel 1} = 0$	$H_{\parallel 2} = J_s$	$H_{\parallel 2} = 0$
$D_\perp := n \cdot D$	$D_{\perp 2} - D_{\perp 1} = \phi_{es}$	$D_{\perp 2} - D_{\perp 1} = 0$	$D_{\perp 2} = \phi_{es}$	$D_{\perp 2} = 0$
$B_\perp := n \cdot B$	$B_{\perp 2} - B_{\perp 1} = \phi_{ms}$	$B_{\perp 2} - B_{\perp 1} = 0$	$B_{\perp 2} = 0$	$B_{\perp 2} = \phi_{ms}$

1.4 Material Considerations

All the constitutive parameters of (3) are typically time/space-varying tensors. Furthermore, they are complex-valued in order to model dissipation for time-varying fields.

Generally, we can classify materials into categories described below.

1.4.1 Magnets

The magnetization is the net effect of the microscopic magnetic dipoles created by orbiting electrons. A large value of μ indicates a stronger magnetization.

1.4.2 Dielectrics/Insulators

Here, the dominant charges are on the boundary of the material creating an overall electric dipole. A large value of ϵ indicates a stronger ability to store charge, but must be weighed vs. σ and ω also. The condition for a good dielectric is

$$\frac{\sigma}{\omega\epsilon} \ll 1 \tag{4}$$

1.4.3 Conductors

Here, there are free charges creating currents throughout the material, due to valence electrons that aren't tightly bound. The condition here is the opposite of the above:

$$\frac{\sigma}{\omega\epsilon} \gg 1 \tag{5}$$

1.4.4 Semiconductors

These are roughly in between an insulator and a conductor, with the condition

$$\frac{\sigma}{\omega\epsilon} = O(1) \tag{6}$$

1.5 Wave Equation

E and H obey equations:

$$(\mu\epsilon\partial_t^2 + \mu\sigma\partial_t - \nabla^2)E + \nabla \times \mathfrak{M}_i + \mu\partial_t J_i + \frac{1}{\epsilon}\nabla\rho_e = 0 \tag{7a}$$

$$(\mu\epsilon\partial_t^2 + \mu\sigma\partial_t - \nabla^2)H - \nabla \times J_i + \epsilon\partial_t\mathfrak{M}_i + \frac{1}{\mu}\nabla\rho_m + \sigma\mathfrak{M}_i = 0 \tag{7b}$$

To obtain solutions we usually look at time-harmonic solutions, and can then use Fourier series to express other forms in terms of these. The time harmonic forms of (7) are obtained by replacements $\partial_t \to \omega j$, $\partial_t^2 \to -\omega^2$

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)E + \nabla \times \mathfrak{M}_i + \mu\omega j J_i + \frac{1}{\epsilon}\nabla\rho_e = 0 \tag{8a}$$

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)H - \nabla \times J_i + \epsilon\omega j\mathfrak{M}_i + \frac{1}{\mu}\nabla\rho_m + \sigma\mathfrak{M}_i = 0 \tag{8b}$$

1.5.1 Source-Free Solutions

The source-free ($\rho_e = \rho_m = J_i = \mathfrak{M}_i = 0$) versions of (8) are

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)E = 0 \tag{9a}$$

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)H = 0 \tag{9b}$$

Solutions to (9) can be obtained by:

1. Expressing the field in terms of coordinate functions, and
2. Using separation of variables.

The solutions obtained in this way are expressible in terms of complex exponentials and Bessel functions. Note that the quantity $-\mu\epsilon\omega^2 + \mu\sigma\omega j$ can be expressed as the square of a single complex number $\gamma = \alpha + kj$. Solving for α and k, we get

$$\alpha = \omega\sqrt{\mu\epsilon}\left(\frac{1}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right)\right)^{\frac{1}{2}} \tag{10}$$

and

$$k = \omega\sqrt{\mu\epsilon}\left(\frac{1}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right)\right)^{\frac{1}{2}} \tag{11}$$

which, for lossless materials ($\sigma = 0$) reduce to

$$\alpha = 0 \tag{12}$$

and

$$k = \omega\sqrt{\mu\epsilon} \tag{13}$$

1.5.2 TEM (Transverse ElectroMagnetic) Modes)

TEM modes are solutions of (8) whose E and H fields form a fixed plane over time for a given spatial point. If equiphase planes are parallel, then it's a plane wave.

TE^y means that the electric field has no y component.

2 P.E.C Field Derivation

1. Starting from Maxwell's equations (1) and constitutive equations (3), transform to frequency domain to simplify curl equations:

$$\nabla \times H = J + j\omega D \tag{14a}$$

$$\nabla \times E = -j\omega B \tag{14b}$$

2. Introduce magnetic vector potential A and electric potential ϕ defined by:

$$B = \nabla \times A \tag{15a}$$

$$E = -\nabla\phi - j\omega A \tag{15b}$$

and fix Lorenz gauge:

$$\nabla \cdot A = -\frac{j\omega}{c^2}\phi \tag{16}$$

3. Apply the above with some vector identities to reduce Ampere-Maxwell equation (14a) to Helmholtz equation:

   (a) Substitute the constitutive conditions (3) and potentials (15) respectively to (14a):

$$\begin{aligned} &\frac{1}{\mu}\nabla \times B = J + j\omega\epsilon E \\ \Longrightarrow\ &\frac{1}{\mu}\nabla \times \nabla \times A = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A \end{aligned}$$

   (b) Use the vector calculus identity $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$, so

$$\frac{1}{\mu}\left(\nabla(\nabla \cdot A) - \nabla^2 A\right) = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A$$

   (c) Apply the Lorenz gauge condition (16):

$$\begin{aligned} &-\frac{1}{\mu}(\frac{j\omega}{c^2}\nabla\phi + \nabla^2 A) = J - j\omega\epsilon\nabla\phi - \omega^2\epsilon A \\ \Longrightarrow\ &-\frac{1}{\mu}\nabla^2 A = J - \omega^2\epsilon A \\ \Longrightarrow\ &\nabla^2 A + k^2 A = -\mu J. \end{aligned}$$

4. Solve for vector potential using Green functions to get

$$A = \mu \int J(r)G(r, r')dr' \tag{17}$$

5. Plug into electric field formula (15b) in terms of potentials:

$$A = \frac{j}{\omega}(\nabla\phi + E)$$

Use the far field assumption on the scalar potential $\nabla\phi \to 0$ - this can be worked out to die off quickly asymptotically. Finally we arrive at

$$E \approx -\omega\mu j \int J(r)G(r, r')dr'$$
$$\implies E \approx -\eta k j \int J(r)G(r, r')dr'$$