Balanis Quick Reference

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Notation

Mathematical

∂_t	$\dfrac{\partial}{\partial t}$	
j	Imaginary unit ($j^2 = -1$)	

Microscopic Fields

E	Electric Field Intensity	[V m^{-1}]
H	Magnetic Field Intensity	[A m^{-1}]

Macroscopic Fields

D	Electric Flux Density	[C m^{-2}]
B	Magnetic Flux Density	[Wb m^{-2}]

Field Sources

$J_{c/d/i}$	Electric Current Density (conduction/displacement/impressed)	[A m^{-2}]
$\mathfrak{M}_{d/i}$	Magnetic Current Density (displacement/impressed)	[V m^{-2}]
ρ_e	Electric Charge Density	[C m^{-2}]
ρ_m	Magnetic Charge Density	[Wb m^{-2}]

Constitutive Parameters

ϵ	Permittivity	[F m^{-1}]
μ	Permeability	[H m^{-1}]
σ	Conductivity	[S m^{-1}]

Waves

f	Frequency	[s^{-1}]
$k^{(11)}$	Wavenumber	[m^{-1}]
$\alpha^{(10)}$	Wave Attenuation	[m^{-1}]
$\lambda := \frac{2\pi}{k}$	Wavelength	[m]
$\omega := 2\pi f$	Angular Frequency	[s^{-1}]
$\eta := \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$	Wave Impedance	[S^{-1}]
$v_g := \partial_k \omega$	Group velocity of wave (envelope velocity)	[ms^{-1}]
$v_p := \frac{\omega}{k}$	Phase velocity of wave (peak/trough velocity)	[ms^{-1}]

1 Basics

1.1 Maxwell's Equations

Differential Form

$$
\begin{aligned}
\text{Maxwell-Faraday :} \quad && \nabla \times E = -\partial_t B - \mathfrak{M}_i && \text{(1a)}\\
\text{Ampère-Maxwell :} \quad && \nabla \times H = \partial_t D + J_c + J_i && \text{(1b)}\\
\text{Gauss :} \quad && \nabla \cdot D = \rho_e && \text{(1c)}\\
\text{Gauss (Magnetism) :} \quad && \nabla \cdot B = 0 = \rho_m && \text{(1d)}
\end{aligned}
$$

Integral Form

$$\oint_C E \cdot dl = -\partial_t \iint_S B \cdot ds - \iint_S \mathfrak{M}_i \cdot ds \tag{2a}$$

$$\oint_C H \cdot dl = \partial_t \iint_S D \cdot ds + \iint_S (J_c + J_i) \cdot ds \tag{2b}$$

$$\oiint_S D \cdot ds = \iiint_V \rho_e \cdot dv \tag{2c}$$

$$\oiint_S B \cdot ds = 0 = \iiint_V \rho_m \cdot dv \tag{2d}$$

1.2 Constitutive Relations

$$D = \epsilon E \tag{3a}$$

$$B = \mu H \tag{3b}$$

$$J_c = \sigma E \tag{3c}$$

1.3 Boundary Conditions

	General	**Finite σ, no source/charge**	**Medium 1 PEC**	**Medium 1 PMC**
$E_\parallel := n \times E$	$E_{\parallel 2} - E_{\parallel 1} = -\mathfrak{M}_s$	$E_{\parallel 2} - E_{\parallel 1} = 0$	$E_{\parallel 2} = 0$	$E_{\parallel 2} = -\mathfrak{M}_s$
$H_\parallel := n \times H$	$H_{\parallel 2} - H_{\parallel 1} = J_s$	$H_{\parallel 2} - H_{\parallel 1} = 0$	$H_{\parallel 2} = J_s$	$H_{\parallel 2} = 0$
$D_\perp := n \cdot D$	$D_{\perp 2} - D_{\perp 1} = \phi_{es}$	$D_{\perp 2} - D_{\perp 1} = 0$	$D_{\perp 2} = \phi_{es}$	$D_{\perp 2} = 0$
$B_\perp := n \cdot B$	$B_{\perp 2} - B_{\perp 1} = \phi_{ms}$	$B_{\perp 2} - B_{\perp 1} = 0$	$B_{\perp 2} = 0$	$B_{\perp 2} = \phi_{ms}$

1.4 Material Considerations

All the constitutive parameters of (3) are typically time/space-varying tensors. Furthermore, they are complex-valued in order to model dissipation for time-varying fields.

Generally, we can classify materials into categories described below.

1.4.1 Magnets

The magnetization is the net effect of the microscopic magnetic dipoles created by orbiting electrons. A large value of μ indicates a stronger magnetization.

1.4.2 Dielectrics/Insulators

Here, the dominant charges are on the boundary of the material creating an overall electric dipole. A large value of ϵ indicates a stronger ability to store charge, but must be weighed vs. σ and ω also. The condition for a good dielectric is

$$\frac{\sigma}{\omega\epsilon} \ll 1 \tag{4}$$

1.4.3 Conductors

Here, there are free charges creating currents throughout the material, due to valence electrons that aren't tightly bound. The condition here is the opposite of the above:

$$\frac{\sigma}{\omega\epsilon} \gg 1 \tag{5}$$

1.4.4 Semiconductors

These are roughly in between an insulator and a conductor, with the condition

$$\frac{\sigma}{\omega\epsilon} = O(1) \tag{6}$$

1.5 Wave Equation

E and H obey equations:

$$(\mu\epsilon\partial_t^2 + \mu\sigma\partial_t - \nabla^2)E + \nabla \times \mathfrak{M}_i + \mu\partial_t J_i + \frac{1}{\epsilon}\nabla\rho_e = 0 \tag{7a}$$

$$(\mu\epsilon\partial_t^2 + \mu\sigma\partial_t - \nabla^2)H - \nabla \times J_i + \epsilon\partial_t\mathfrak{M}_i + \frac{1}{\mu}\nabla\rho_m + \sigma\mathfrak{M}_i = 0 \tag{7b}$$

To obtain solutions we usually look at time-harmonic solutions, and can then use Fourier series to express other forms in terms of these. The time harmonic forms of (7) are obtained by replacements $\partial_t \to \omega j$, $\partial_t^2 \to -\omega^2$

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)E + \nabla \times \mathfrak{M}_i + \mu\omega j J_i + \frac{1}{\epsilon}\nabla\rho_e = 0 \tag{8a}$$

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)H - \nabla \times J_i + \epsilon\omega j\mathfrak{M}_i + \frac{1}{\mu}\nabla\rho_m + \sigma\mathfrak{M}_i = 0 \tag{8b}$$

1.5.1 Source-Free Solutions

The source-free ($\rho_e = \rho_m = J_i = \mathfrak{M}_i = 0$) versions of (8) are

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)E = 0 \tag{9a}$$

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)H = 0 \tag{9b}$$

Solutions to (9) can be obtained by:

1. Expressing the field in terms of coordinate functions, and
2. Using separation of variables.

The solutions obtained in this way are expressible in terms of complex exponentials and Bessel functions. Note that the quantity $-\mu\epsilon\omega^2 + \mu\sigma\omega j$ can be expressed as the square of a single complex number $\gamma = \alpha + kj$. Solving for α and k, we get

$$\alpha = \omega\sqrt{\mu\epsilon}\left(\frac{1}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right)\right)^{\frac{1}{2}} \tag{10}$$

and

$$k = \omega\sqrt{\mu\epsilon}\left(\frac{1}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right)\right)^{\frac{1}{2}} \tag{11}$$

which, for lossless materials ($\sigma = 0$) reduce to

$$\alpha = 0 \tag{12}$$

and

$$k = \omega\sqrt{\mu\epsilon} \tag{13}$$

1.5.2 Transverse Modes

Transverse modes are solutions of (8) whose E and/or H fields have no component for a given set of coordinates (it is said to be "transverse to" this set) over time for a given spatial point, e.g.

- TE^y means that the electric field has no y component,
- TM^z means that the magnetic field has no z component,
- TEM means that the electric and magnetic field are both contained in a plane,
- If equiphase planes are parallel, then it's a plane wave.

2 Reflection and Transmission

2.1 Normal Incidence

Assuming the wave vector in the z direction, the electric field is polarized in the x direction, defining E_0, Γ, T respectively by

$$
\begin{aligned}
E^i &= E_0 e^{-(\alpha_1 + jk_1)z} \mathbf{e}_x \\
E^r &= \Gamma E_0 e^{(\alpha_1 + jk_1)z} \mathbf{e}_x \\
E^t &= T E_0 e^{-(\alpha_2 + jk_2)z} \mathbf{e}_x
\end{aligned}
$$

applying right-hand-rule and enforcing continuity of tangential components ($\Theta^i + \Theta^r = \Theta^t$ at $z = 0$, where $\Theta \in E, H$) leads to

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \tag{15a}$$

$$T = 1 + \Gamma \tag{15b}$$

2.2 Oblique Incidence

The formulae for oblique angles are simple to obtain in a similar fashion. For the electric field perpendicular to the plane of incidence we replace (15a) by

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \tag{16}$$

and when it is polarized parallel, it instead becomes

$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{17}$$

which can both be derived from the formulae for plane-wave impedances for transverse modes (the first is TM^z corresponding to $\eta_p \to \frac{\eta_p}{\cos\theta_p}$, and the second is TE^z corresponding to $\eta_p \to \eta_p \cos\theta_p$)

3 Vector Potentials

The magnetic vector potential A is defined for source-free regions (guaranteed experimentally, since there are no magnetic monopoles: $\rho_m = 0$) by

$$B_A = \nabla \times A \tag{18}$$

The electric scalar potential ϕ_e is then defined by

$$E_A = -\nabla\phi_e - j\omega A \tag{19}$$

Similarly, if there are no electric charges ($\rho_e = 0$), then we can define the electric vector potential F by

$$D_F = -\nabla \times F \tag{20}$$

and the magnetic scalar potential ϕ_m by

$$H_F = -\nabla\phi_m - j\omega F \tag{21}$$

We can specify ϕ_e, ϕ_m arbitrarily (doing so is called ”fixing a gauge”). The Lorenz gauge is defined by

$$\nabla \cdot \Theta + \mu\epsilon\partial_t\phi_\theta = 0 \tag{22}$$

Applying this to both potentials leads to the Helmholtz equations:

$$\nabla^2 A + k^2 A = -\mu J \tag{23a}$$

$$\nabla^2 F + k^2 F = -\epsilon\mathfrak{M} \tag{23b}$$