Effective Roughness Models

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TODO

The below derivations are not quite as nice as I'd like, however, I'm probably spending a little too much time on prettifying the assumptions so... here's a list of things to do after I've looked at the actual model comparisons and gotten some results (focusing on that this week):

1. Refine the notation,
2. Make a glossary,
3. Complete the assumptions,
4. Come up with a sensible taxonomy of assumptions, and format appropriately
5. Refine the derivations so that we only number significant results,

(N) Notation

The following generic notation is used throughout:

$$\mathcal{E} = \mathrm{Re}\{\mathbf{E}e^{j\omega t}\} \text{ is the electric field } (\mathbf{E} \text{ is the phasor}). \tag{N.1}$$

$$\mathcal{H} = \mathrm{Re}\{\mathbf{H}e^{j\omega t}\} \text{ is the magnetic field } (\mathbf{H} \text{ is the phasor}). \tag{N.2}$$

$$\mathbf{E} = \vec{E}e^{\phi_E} \tag{N.3}$$

$$\mathbf{H} = \vec{H}e^{\phi_H} \tag{N.4}$$

$$\vec{E} \cdot \vec{H} = |\vec{E}||\vec{H}| \cos\theta_{\mathcal{S}} \tag{N.5}$$

$$\mathcal{S} \overset{\text{def}}{=} \mathcal{E} \times \mathcal{H} \text{ is the Poynting vector.} \tag{N.6}$$

$$\phi_{\mathcal{S}} = \phi_E - \phi_H \tag{N.7}$$

$$Z \overset{\text{def}}{=} \frac{|\mathbf{E}|}{|\mathbf{H}|} \text{ is the EM-wave impedance} \tag{N.8}$$

$$|\langle \mathcal{S} \rangle| \overset{\text{def}}{=} \left| \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} S(t)dt \right| = \ldots = \frac{1}{2Z}|\mathbf{E}|^2 \sin(\theta_{\mathcal{S}}) \cos(\phi_{\mathcal{S}}) \tag{N.9}$$

(F) Field Assumptions

The Rx and Tx are not in the near-field (a reasonable assumption in an outdoor scenario), so we have

$$\theta_{\mathcal{S}} = \frac{\pi}{2} \tag{F.1}$$

$$\phi_{\mathcal{S}} = 0 \tag{F.2}$$

The wave is travelling in free space until it hits an obstruction, so

$$Z = \eta_0 \tag{F.3}$$

The contributions to E_s arriving at the Rx have random, uncorrelated phases

$$\implies |E_s|^2 = \int_X d(|E_s|^2), \text{ where} \tag{F.4}$$

- $d(|E_s|^2)$ is the squared-field contribution from dX over
- X, the wall surface.

(A) Core Assumptions

A.0 Generic Setup

A.0.1 Field Decomposition

The total electric field phasor at the boundary of a wall X ($E_{\partial X}$) can be decomposed into incident, transmitted, specularly-reflected, and diffusely-scattered component fields:

$$E_{\partial X} = E_i + E_t + E_r + E_s. \tag{A.0.1}$$

A.0.2 Field Continuity

$$E_i, E_t, E_r, E_s \in C^0(\mathbb{R}^n \times \mathbb{C}), \text{ where } n \leq 3. \tag{A.0.2}$$

A.1 Local Power Balance

S_i is the phasor Poynting vector for the incoming wave. So, we have

Letting P_z denote the total power associated with a given field component, dP_z denote the total power associated with the portion incident on dX, and

$$\begin{aligned} dP_i &= dP_t + dP_r + dP_s \\ S_i & \\ \implies \frac{|E_i|^2}{2\eta_0} dX_i &= P_t + \frac{R^2|\Gamma|^2|E_i|^2 d\Omega_i r_i^2}{} \end{aligned}$$

A constant fraction (S^2) of the total power incident on dX (dP_i) is converted into diffusely scattered power (dP_s):

$$dP_s = S^2 dP_i. \tag{A.1}$$

A.2 Scattering Pattern $\mathfrak{S}$

$\mathfrak{S}$ is The radiant intensity I is given by a constant times a scattering pattern $K \cdot \mathfrak{S}$. This relates to the above as follows:

$$\begin{aligned} dP_s &= \int_{\Omega_X} K\mathfrak{S}(\Omega) d\Omega_s \\ &= \int_{\Omega_X} \left(\frac{\mathfrak{S}(\Omega) dP_s}{\int_{\Omega_X} \mathfrak{S}(\Omega'_s) d\Omega'_s} \right) d\Omega_s, \quad \text{where} \end{aligned} \tag{A.2}$$

- $\mathfrak{S}$ is the scattering pattern,
- K is a normalization constant relating to the maximum intensity,
- Ω is the set of angular coordinates,
- Ω_X is the scattering domain. Formally, $\Omega_X = \{\Omega \mid E_s(r, \Omega) \neq 0\}$, and
- $d\Omega_s$ is the solid angle of the scattered ray-tube from dX.

1 2D Lambertian-Model

Let

- $d\theta_{Rx}$ be the Rx aperture,
- $dl_{Rx} = r_s d\theta_{Rx}$ be the Rx aperture length, and
- $dP_{Rx} = \frac{d(|E_s|^2)}{2\eta_0} dl_{Rx}$ be the power through the Rx aperture.

$$
\begin{aligned}
\implies d(|E_s|^2) &= \frac{2\eta_0}{dl_{Rx}} dP_{Rx} \\
&= \frac{2\eta_0}{dl_{Rx}} \int_{\theta_s - \frac{d\theta_{Rx}}{2}}^{\theta_s + \frac{d\theta_{Rx}}{2}} I(\theta_s) d\theta \\
&\overset{d\theta_{Rx}\to 0}{\approx} \frac{2\eta_0 I(\theta_s) d\theta_{Rx}}{dl_{Rx}} = \frac{2\eta_0 d\theta_{Rx}}{dl_{Rx}} I(\theta_s) \\
&= \frac{2\eta_0 d\theta_{Rx}}{dl_{Rx}} \left(\frac{\cos\theta_s dP_s}{\int_{\Omega_s} \cos\theta_s' d\Omega_s'} \right) \\
&= \frac{2\eta_0 \cos\theta_s d\theta_{Rx} dP_s}{dl_{Rx} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta_s' d\theta_s'} = 2\eta_0 \cos\theta_s \frac{d\theta_{Rx}}{2dl_{Rx}} dP_s = \frac{\cos\theta_s}{r_s} \eta_0 dP_s \\
&\overset{(A.1)}{=} \frac{S^2 \cos\theta_s}{r_s} \eta_0 dP_i = \frac{S^2 \cos\theta_s}{r_s} \eta_0 \frac{|E_i|^2}{2\eta_0} \cos\theta_i dx \\
&= \frac{S^2 \cos\theta_i \cos\theta_s}{2r_s} |E_i|^2 dx
\end{aligned}
\tag{1.1}
$$

A far-field cylindrical incoming wave allows us to assume

$$
|E_i|^2 = \frac{A}{r_i} \tag{1.2}
$$

Then, combining (1.1) and (1.2) with the expression for power, we get

$$
\begin{aligned}
P_i &= \int_0^{2\pi} \frac{|E_i|^2}{2\eta_0} r_i d\theta \\
&= \frac{A}{2\eta_0} \int_0^{2\pi} d\theta \\
&= \frac{A\pi}{\eta_0} \\
\implies A &= \frac{\eta_0 P_i}{\pi} \\
\implies |E_i|^2 &= \frac{\eta_o P_i}{\pi} \frac{1}{r_i} \\
\implies d(|E_s|^2) &= \frac{S^2 \eta_0 P_i \cos\theta_i \cos\theta_s}{2\pi r_s r_i} \ dx
\end{aligned} \tag{1.3}
$$

Coordinate Transformations

Our setup consists of a transmitter at (x_T, y_T) and a receiver at (x_R, y_R). We can express $\theta_i, \theta_s, r_i, r_s$ as follows:

$$
\begin{aligned}
r_s &= \sqrt{y_R^2 + (x_R - x)^2} \\
\cos\theta_s &= \frac{y_R}{r_s} \\
r_i &= \sqrt{y_T^2 + (x - x_T)^2} \\
\cos\theta_i &= \frac{y_T}{r_i}
\end{aligned}
$$

Plugging these into (1.3) we get

$$
d(|E_s|^2) = \frac{S^2 \eta_0 P_i \ y_T y_R}{2\pi (y_R^2 + (x_R - x)^2)(y_T^2 + (x - x_T)^2)} \ dx \tag{1.4}
$$

We can also express x in terms of t by inverting the equation

$$
(ct) = \sqrt{y_T^2 + (x - x_T)^2} + \sqrt{y_R^2 + (x_R - x)^2} \tag{1.5}
$$

In order to invert this, we need to square twice to get rid of any square roots. However, in order for the x^4 and x^3 terms to cancel, we need to make a further coordinate transformation so the origin is at the specular point of reflection:

$$
\begin{aligned}
x \rightarrow x' &= x - x_T - y_T \left(\frac{x_R - x_T}{y_R + y_T} \right) \\
\implies x' &= x - \left(\frac{x_T y_R + x_R y_T}{y_R + y_T} \right)
\end{aligned} \tag{1.6}
$$

For the Tx term $(x - x_T)$:

$$x - x_T = x' + \frac{y_T(x_R - x_T)}{y_R + y_T} \tag{1.7}$$

For the Rx term $(x_R - x)$:

$$\begin{aligned} x_R - x &= -x' + x_R - \left(\frac{x_T y_R + x_R y_T}{y_R + y_T} \right) \\ &= -x' + \frac{x_R(y_R + y_T) - (x_T y_R + x_R y_T)}{y_R + y_T} \\ \implies x_R - x &= -x' + \frac{y_R(x_R - x_T)}{y_R + y_T} \end{aligned} \tag{1.8}$$

Rewriting (1.5) with these new expressions:

$$ct = \sqrt{y_T^2 + \left(x' + \frac{y_T(x_R - x_T)}{y_R + y_T} \right)^2} + \sqrt{y_R^2 + \left(\frac{y_R(x_R - x_T)}{y_R + y_T} - x' \right)^2}$$

To simplify, define the distances from the transmitter and receiver to the specular point on the surface:

$$\begin{aligned} R_{T0} &:= \sqrt{y_T^2 + \left(\frac{y_T(x_R - x_T)}{y_R + y_T} \right)^2} \\ R_{R0} &:= \sqrt{y_R^2 + \left(\frac{y_R(x_R - x_T)}{y_R + y_T} \right)^2} \end{aligned}$$

We then get

$$ct = \sqrt{R_{T0}^2 + 2x' \frac{y_T(x_R - x_T)}{y_R + y_T} + (x')^2} + \sqrt{R_{R0}^2 - 2x' \frac{y_R(x_R - x_T)}{y_R + y_T} + (x')^2}$$

Isolate the second square root and square both sides:

$$\begin{aligned} &(ct)^2 - 2ct\sqrt{R_{T0}^2 + 2x' \frac{y_T(x_R - x_T)}{y_R + y_T} + (x')^2} + \left(R_{T0}^2 + 2x' \frac{y_T(x_R - x_T)}{y_R + y_T} + (x')^2 \right) \\ &= R_{R0}^2 - 2x' \frac{y_R(x_R - x_T)}{y_R + y_T} + (x')^2 \end{aligned}$$

Now collect terms and cancel:

$$2x'(x_R - x_T) + (ct)^2 + R_{T0}^2 - R_{R0}^2 = 2ct\sqrt{R_{T0}^2 + 2x' \frac{y_T(x_R - x_T)}{y_R + y_T} + (x')^2}$$

Squaring both sides:

$$
\begin{aligned}
&4\left((x_R - x_T)^2 + (ct)^2\right)^2 (x')^2 + \\
&4(x_R - x_T)\left(R_{T0}^2 - R_{R0}^2 + (ct)^2\left(\frac{y_R - y_T}{y_R + y_T}\right)\right) x' + \\
&\left((ct)^2 + R_{T0}^2 - R_{R0}^2\right)^2 - 4(ct)^2 R_{T0}^2
\end{aligned}
$$

Finally, we can solve for x :

$$
\begin{aligned}
x &= \frac{x_T y_R + x_R y_T}{y_R + y_T} - \frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ad}}{2a}, \text{ where} \\
a &= 4\left((x_R - x_T)^2 + (ct)^2\right)^2 \\
b &= 4(x_R - x_T)\left(R_{T0}^2 - R_{R0}^2 + (ct)^2\left(\frac{y_R - y_T}{y_R + y_T}\right)\right), \\
d &= \left((ct)^2 + R_{T0}^2 - R_{R0}^2\right)^2 - 4(ct)^2 R_{T0}^2 \\
R_{T0}^2 &= y_T^2 + \left(\frac{y_T(x_R - x_T)}{y_R + y_T}\right)^2, \text{ and} \\
R_{R0}^2 &= y_R^2 + \left(\frac{y_R(x_R - x_T)}{y_R + y_T}\right)^2
\end{aligned}
\tag{1.9}
$$

2 Plane Wave Incident on PEC

2.1 Questions for Conor for Fri 20 Jun 2025

1. Discussion of the above work, any thoughts on further refinements apart from what I've listed already in the TODO section (I've spent too long on this part but it will be nice to have for a writeup. I'm going to focus on the MATLAB code now)?

2. In the above I've done the derivation for the power assuming a cylindrical incoming wave. However, in the original MATLAB code setup you seem to have assumed the Tx is in the far-region (hence a plane wave approximation). Do you think it's sensible to adjust the above from (1.1) for $|E_i|$?

3. What measures specifically do you think would be good to compare first? I think last time you mentioned the power profiles as a first step. So, do you think calibrating the model correctly