Modelling of Diffuse Scattering Effects for Outdoor Ray Tracing

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Contents

Notation

Mathematical

∂_t	$\dfrac{\partial}{\partial t}$
j	Imaginary unit ($j^2 = -1$)
$R_{x,\theta}, R_{y,\theta}, R_{z,\theta}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$
P_{xz}	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Microscopic Fields

E	Electric Field Intensity	[V m^{-1}]
H	Magnetic Field Intensity	[A m^{-1}]

Macroscopic Fields

D	Electric Flux Density	[C m^{-2}]
B	Magnetic Flux Density	[Wb m^{-2}]

Field Sources

$J_{c/d/i}$	Electric Current Density $_{\text{(conduction/displacement/impressed)}}$	[A m^{-2}]
$\mathfrak{M}_{d/i}$	Magnetic Current Density $_{\text{(displacement/impressed)}}$	[V m^{-2}]
ρ_e	Electric Charge Density	[C m^{-2}]
ρ_m	Magnetic Charge Density	[Wb m^{-2}]

Constitutive Parameters

ϵ	Permittivity	[F m^{-1}]
μ	Permeability	[H m^{-1}]
σ	Conductivity	[S m^{-1}]

Waves

f	Frequency	[s^{-1}]
$k^{(B.6b)}$	Wavenumber	[m^{-1}]
$\alpha^{(B.6a)}$	Wave Attenuation	[m^{-1}]
$\lambda := \frac{2\pi}{k}$	Wavelength	[m]
$\omega := 2\pi f$	Angular Frequency	[s^{-1}]
$\eta := \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$	Intrinsic Impedance	[S^{-1}]
$v_g := \partial_k \omega$	Group velocity of wave (envelope velocity)	[ms^{-1}]
$v_p := \frac{\omega}{k}$	Phase velocity of wave (peak/trough velocity)	[ms^{-1}]

1 Effective Roughness Model

This model is a modification of Geometrical Optics in which, in addition to the specular component, each point impinging on the middle of a surface element dW gives a diffuse contribution dE_d to the scattered field E_s whose amplitude $|dE_d|$ is given by

$$|dE_d| \propto \sqrt{\frac{dW \cos\theta_i \cos\theta_d}{\pi}} \frac{1}{r_i r_d} \tag{1.1}$$

with the constant of proportionality depending on the incident amplitude and a scattering parameter S. Specifically, we have

$$|dE_d| = S\Upsilon\sqrt{dW}, \quad \text{where} \tag{1.2a}$$

$$\Upsilon = \sqrt{\frac{60 G_t P_t \cos\theta_i \cos\theta_d}{\pi}} \frac{1}{r_i r_d} \tag{1.2b}$$

1.1 Uniform Plane Wave Incident on PEC

We start with a setup as per Figure 1.

$$k_i = \sin(\theta_i)\mathrm{e}_x - \cos(\theta_i)\mathrm{e}_y \tag{1.3a}$$

$$k_r = \sin(\theta_i)\mathrm{e}_x + \cos(\theta_i)\mathrm{e}_y \tag{1.3b}$$

$$k_d = \sin(\theta_d)\mathrm{e}_x + \cos(\theta_d)\mathrm{e}_y \tag{1.3c}$$

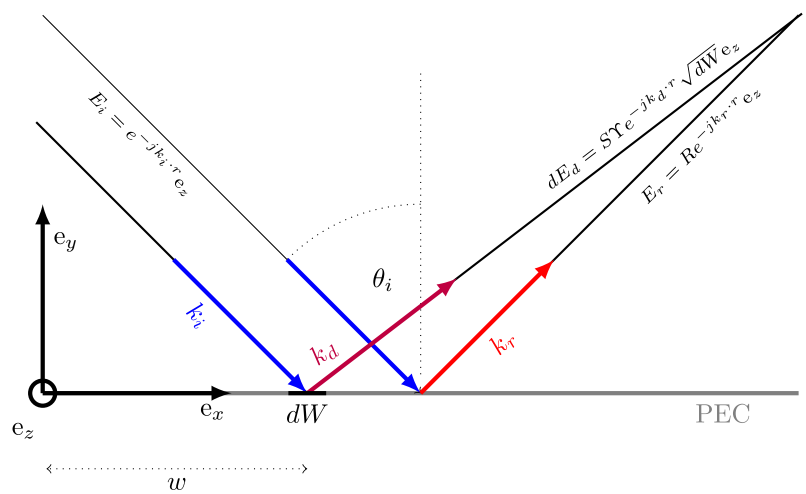

Figure 1: A uniform plane wave strikes a PEC. The overall scattered wave is $E_s = E_r + \int_W dE_d$. E_r is just the usual specular component of geometrical optics, multiplied by R, a roughness parameter. E_d is the diffuse component - a sum of non-coherent contributions along the wall, one of which is shown here, for a drawn surface element dW.

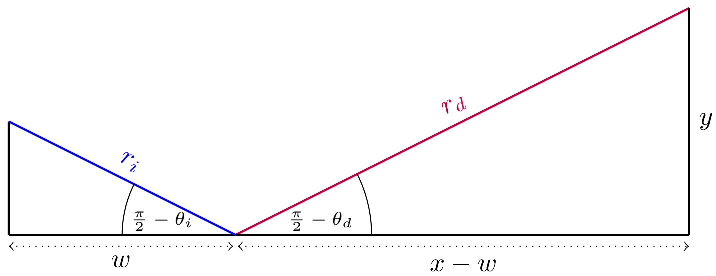

Figure 2: Geometry setup implies that $r_i = \frac{w}{\sin\theta_i}$, and $r_d = \sqrt{y^2 + (x-w)^2}$, so that $\frac{1}{r_i r_d} = \frac{\sin\theta_i}{w\sqrt{y^2+(x-w)^2}}$

Then, since $G_t = 1$ (0 gain), $P_t = 1$ (uniformity of wave), $dW = hdx$, $|\Gamma| = 1$, and referring to the geometry of Figure 2, we get

$$E_i = e^{-j(x\sin\theta_i - y\cos\theta_i)}\mathrm{e}_z \tag{1.4a}$$

$$E_r = \sqrt{1-S^2}\; e^{-j(x\sin\theta_i + y\cos\theta_i)}\mathrm{e}_z \tag{1.4b}$$

$$E_d = S\;\sin\theta_i\sqrt{\frac{60h\cos\theta_i}{\pi}}\int_W \frac{1}{w}\sqrt{\frac{\cos\theta_d\; dw}{y^2+(x-w)^2}}e^{-j(x\sin\theta_d + y\cos\theta_d)}\mathrm{e}_z \tag{1.4c}$$

Appendices

A Mathematical Identities and Theorems

$$\oint_C A \cdot dl = \iint_S (\nabla \times A) \cdot ds \tag{A.1a}$$

$$\oiint_S A \cdot ds = \iiint_V (\nabla \cdot A) dv \tag{A.1b}$$

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B) \tag{A.2a}$$

B Maxwell's Equations

B.1 Differential Form

$$\nabla \times E = -\partial_t B - \mathfrak{M}_i \tag{B.1a}$$
$$\nabla \times H = \partial_t D + J_c + J_i \tag{B.1b}$$
$$\nabla \cdot D = \rho_e \tag{B.1c}$$
$$\nabla \cdot B = 0 \quad = \rho_m \tag{B.1d}$$

B.2 Constitutive Relations

The material properties of a medium can be modelled without specifying microscopic structure:

$$D = \epsilon E \tag{B.2a}$$
$$B = \mu H \tag{B.2b}$$
$$J_c = \sigma E \tag{B.2c}$$

B.3 Integral Form

The below can be derived from taking the surface integral of the curl equations, the volume integral of the divergence equations, and applying (A.1):

$$\oint_C E \cdot dl = -\partial_t \iint_S B \cdot ds - \iint_S \mathfrak{M}_i \cdot ds \tag{B.3a}$$
$$\oint_C H \cdot dl = \partial_t \iint_S D \cdot ds + \iint_S (J_c + J_i) \cdot ds \tag{B.3b}$$
$$\oiint_S D \cdot ds = \iiint_V \rho_e \cdot dv \tag{B.3c}$$
$$\oiint_S B \cdot ds = 0 = \iiint_V \rho_m \cdot dv \tag{B.3d}$$

B.4 Uncoupled Form

$$(\mu\epsilon\partial_t^2 + \mu\sigma\partial_t - \nabla^2)E + \nabla \times \mathfrak{M}_i + \mu\partial_t J_i + \frac{1}{\epsilon}\nabla\rho_e = 0 \tag{B.4a}$$
$$(\mu\epsilon\partial_t^2 + \mu\sigma\partial_t - \nabla^2)H - \nabla \times J_i + \epsilon\partial_t\mathfrak{M}_i + \frac{1}{\mu}\nabla\rho_m + \sigma\mathfrak{M}_i = 0 \tag{B.4b}$$

B.5 Time-Harmonic Form

To obtain solutions we usually look at time-harmonic solutions, and can then use Fourier series to express other forms in terms of these. The time harmonic forms of (B.4) are obtained by replacements $\partial_t \to \omega j$, $\partial_t^2 \to -\omega^2$:

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)E + \nabla\times\mathfrak{M}_i + \mu\omega j J_i + \frac{1}{\epsilon}\nabla\rho_e = 0 \tag{B.5a}$$

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)H - \nabla\times J_i + \epsilon\omega j\mathfrak{M}_i + \frac{1}{\mu}\nabla\rho_m + \sigma\mathfrak{M}_i = 0 \tag{B.5b}$$

Note that the quantity $-\mu\epsilon\omega^2 + \mu\sigma\omega j$ can be expressed as the square of a single complex number $\gamma = \alpha + kj$. Solving for α and k, we get

$$\alpha = \omega\sqrt{\mu\epsilon}\left(\frac{1}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right)\right)^{\frac{1}{2}} \tag{B.6a}$$

$$k = \omega\sqrt{\mu\epsilon}\left(\frac{1}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right)\right)^{\frac{1}{2}} \tag{B.6b}$$

B.6 Source-Free Time Harmonic Equations

The source-free ($\rho_e = \rho_m = J_i = \mathfrak{M}_i = 0$) versions of (B.5) are

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)E = 0 \tag{B.7a}$$

$$(-\mu\epsilon\omega^2 + \mu\sigma\omega j - \nabla^2)H = 0 \tag{B.7b}$$

B.7 Boundary Conditions

	General	**Finite σ, no source/charge**	**Medium 1 PEC**	**Medium 1 PMC**
$E_\parallel := n\times E$	$E_{\parallel 2} - E_{\parallel 1} = -\mathfrak{M}_s$	$E_{\parallel 2} - E_{\parallel 1} = 0$	$E_{\parallel 2} = 0$	$E_{\parallel 2} = -\mathfrak{M}_s$
$H_\parallel := n\times H$	$H_{\parallel 2} - H_{\parallel 1} = J_s$	$H_{\parallel 2} - H_{\parallel 1} = 0$	$H_{\parallel 2} = J_s$	$H_{\parallel 2} = 0$
$D_\perp := n\cdot D$	$D_{\perp 2} - D_{\perp 1} = \phi_{es}$	$D_{\perp 2} - D_{\perp 1} = 0$	$D_{\perp 2} = \phi_{es}$	$D_{\perp 2} = 0$
$B_\perp := n\cdot B$	$B_{\perp 2} - B_{\perp 1} = \phi_{ms}$	$B_{\perp 2} - B_{\perp 1} = 0$	$B_{\perp 2} = 0$	$B_{\perp 2} = \phi_{ms}$

B.8 Material Considerations

All the constitutive parameters of (B.2) are typically time/space-varying tensors. Furthermore, they are complex-valued in order to model dissipation for time-varying fields.

Generally, we can classify materials into categories described below.

Magnets The magnetization is the net effect of the microscopic magnetic dipoles created by orbiting electrons. A large value of μ indicates a stronger magnetization.

Dielectrics/Insulators Here, the dominant charges are on the boundary of the material creating an overall electric dipole. A large value of ϵ indicates a stronger ability to store charge, but must be weighed vs. σ and ω also. The condition for a good dielectric is

$$\frac{\sigma}{\omega\epsilon} \ll 1 \tag{B.8}$$

Conductors Here, there are free charges creating currents throughout the material, due to valence electrons that aren't tightly bound. The condition here is the opposite of the above:

$$\frac{\sigma}{\omega\epsilon} \gg 1 \tag{B.9}$$

Semiconductors These are roughly in between an insulator and a conductor, with the condition

$$\frac{\sigma}{\omega\epsilon} = O(1) \tag{B.10}$$

C Electromagnetic Theorems

C.1 Conservation of Energy

Scalar multiply (B.1a) by H, and (B.1b) by E, subtract the two equations and apply (A.2a) to derive

$$\nabla \cdot (E \times H) + H \cdot (M_i + M_d) + E \cdot (J_i + J_c + J_d) = 0 \tag{C.1}$$

If we look at the units of H [A m^{-1}] and reduce the units of E to SI base units [V m^{-1}] $\rightarrow$ [kg m s^{-3} A^{-1}] we can see that $E \times H$ (labelled the *Poynting Vector*) has units [kg s^{-3}] = [J s^{-1} m^{-2}]. If we integrate (C.1) over a volume and apply (A.1b), we end up with

$$\oiint_S (E \times H) \cdot ds + \iiint_V [H \cdot (M_i + M_d) + E \cdot (J_i + J_c + J_d)] dv = 0 \tag{C.2}$$

which states that the supplied (M_i and J_i terms) power is equal to the power exiting ($E \times H$ term) plus the dissipated (J_c term) power plus the rate of change of the electromagnatic field energy (M_d and J_d terms).

C.2 Duality

Essentially a theoretical symmetrical relationship allowing quicker transformation of one problem to another. For example, a problem with electric sources and no magnetic ones can easily be transformed into one with magnetic sources and no electric ones.

C.3 Uniqueness

Essentially, a lossy ($\sigma \neq 0$) region with sources $J_i, \mathfrak{M}_i$ has unique solutions whenever the tangential components of E and/or H are specified over the boundary.

C.4 Lorentz Reciprocity

$$\nabla \cdot (E_2 \times H_1 - E_1 \times H_2) = (E_2 \cdot J1 - E_1 \cdot J2) - (H_2 \cdot M_1 - H_1 \cdot M_2) \tag{C.3}$$

C.5 Image Theory

Assuming a wall of infinite extent, and a PEC ($\sigma = \infty$) below, reflection can be considered for infinitesimal vertical and horizontal dipoles respectively. Reflected components can be considered by constructing virtual sources below the wall. Boundary conditions will enforce polarization of the virtual sources.

C.6 Equivalence Theorems

Volume Equivalence

$$\nabla \times E^s = -\mathfrak{M}_{\text{eq}} - j\omega\mu_0 H^s := -j\omega(\mu - \mu_0)H - j\omega\mu_0 H^s \tag{C.4a}$$

$$\nabla \times H^s = J_{\text{eq}} + j\omega\epsilon_0 E^s := j\omega(\epsilon - \epsilon_0)E + j\omega\epsilon_0 E^s \tag{C.4b}$$

Surface Equivalent

Start with a problem for unbounded setup:

$$(E_u, H_u, \epsilon_1, \mu_1, \sigma_1, \mathfrak{M}_1, J_1) \tag{C.5}$$

Now introduce a volume V bounded by surface S and construct the *surface equivalent* problem:

$$\begin{align} &\to ((E_u, E_b), (H_u, H_b), \epsilon_1, \mu_1, \sigma_1, ?, ?) \tag{C.6a}\\ J_S &= n \times (H_u - H_b) \tag{C.6b}\\ \mathfrak{M}_S &= -n \times (E_u - E_b) \tag{C.6c}\end{align}$$

Since we don't care about the contents of the fields E_b, H_b inside the medium, we can change them as we like. For instance we can make them be 0.

Induction Equivalent

Start off with an unbounded problem (C.5) Introduce a a volume V enclosed by a surface S, this will change the problem to:

$$\begin{align} &\to ((E_u + E_s, E_b), (H_u + H_s, H_b), (\epsilon_1, \epsilon_2), (\mu_1, \mu_2), (\sigma_1, \sigma_2), \mathfrak{M}_1, J_1.) \tag{C.7a}\\ 0 &= n \times (E_u + E_s - E_b) \tag{C.7b}\\ 0 &= n \times (H_u + H_s - H_b) \tag{C.7c}\end{align}$$

Transform this problem to an *induction equivalent* which is just the surface equivalent of (C.6) applied to the outside instead of the inside:

$$\begin{align} &\to ((E_s, E_b), (H_s, H_b), (\epsilon_1, \epsilon_2), (\mu_1, \mu_2), (\sigma_1, \sigma_2), ?, ?) \tag{C.8a}\\ J_S &= n \times (H_s - H_b) = -n \times H_u \tag{C.8b}\\ \mathfrak{M}_S &= -n \times (E_s - E_b) = n \times E_u \tag{C.8c}\end{align}$$

Physical Equivalent

Start off with a problem for PEC:

$$((E_i + E_s, 0), (H_i + H_s, 0), \epsilon_1, \mu_1, \sigma_1 = \infty, \mathfrak{M}_1, J_1)\,, \tag{C.9}$$

and transform to the *physical equivalent*:

$$\begin{align} &\to ((E_s, -E_i), (H_s, -H_i), \epsilon_1, \mu_1, \sigma_1, ?, ?) \tag{C.10a}\\ J_S &= n \times (H_s + H_i) \tag{C.10b}\\ \mathfrak{M}_S &= -n \times (E_s + E_b) = 0 \tag{C.10c}\end{align}$$

D Solutions to Maxwell's equations

D.1 Field Configurations

A field configuration is a restriction of E or H to a given geometry. A *mode* is a solution for a given field configuration. *Transverse modes* are solutions of (B.5) whose E and/or H fields have no component for a given set of coordinates (it is said to be ”transverse to” this set) over time for a given spatial point, e.g.

- TE^y means that the electric field has no y component,
- TM^z means that the magnetic field has no z component,
- TEM means that the electric and magnetic field are both contained in a plane,
- If equiphase planes are parallel, then it's a plane wave.

D.2 Separation of Variables

Under certain ideal scenarios, solutions to (B.7) can be obtained by:

1. Expressing the field in terms of coordinate functions,
2. Equating the vector components of expanded equation, and
3. Using separation of variables on uncoupled equations.

The solutions obtained in this way are expressible in terms of complex exponentials and Bessel functions.

D.3 Unbounded Uniform Plane Waves

TE^y

For purposes of simplifying coordinate transformations, we can simplify the formula given in Balanis for the E field. Essentially, it requires multiplication by P_{xz}:

$$E = \left(E_1^+ e^{-\hat{\gamma}^+ \cdot r} + E_0^- e^{-\hat{\gamma}^- \cdot r}\right) R_{y,\theta+\frac{\pi}{2}} P_{xz} \; \mathrm{e}_x \tag{D.1a}$$

$$H = \frac{j}{\omega\mu} \nabla \times E \tag{D.1b}$$

$$\hat{\gamma}^{\pm} = \pm(\alpha + jk) R_{y,\theta} P_{xz} \; \mathrm{e}_x \tag{D.1c}$$

Expanding out (D.1a) and plugging into (D.1b):

$$
\begin{aligned}
E &= \left(E_0^+ e^{-(\alpha+jk)(x\sin\theta+z\cos\theta)} + E_0^- e^{(\alpha+jk)(x\sin\theta+z\cos\theta)}\right)(\cos\theta\ \mathrm{e}_x - \sin\theta\ \mathrm{e}_z) \\
&= (U_1+U_2)(\cos\theta\ \mathrm{e}_x - \sin\theta\ \mathrm{e}_z) \\
H &= -\frac{1}{j\omega\mu}\begin{vmatrix} \mathrm{e}_x & \mathrm{e}_y & \mathrm{e}_z \\ \partial_x & \partial_y & \partial_z \\ (U_1+U_2)\cos\theta & 0 & -(U_1+U_2)\sin\theta \end{vmatrix} \\
&= -\frac{1}{j\omega\mu}\left(\sin\theta\ \partial_x(U_1+U_2) + \cos\theta\ \partial_z(U_1+U_2)\right)\mathrm{e}_y \\
&= -\frac{\gamma}{j\omega\mu}\left(-U_1\sin^2\theta\ + U_2\sin^2\theta - U_1\cos^2\theta\ + U_2\cos^2\theta\right)\mathrm{e}_y \\
&= \frac{\gamma}{j\omega\mu}(U_1-U_2)\mathrm{e}_y
\end{aligned}
$$

So,

$$E = \left(E_0^+ e^{-(\alpha+jk)(x\sin\theta+z\cos\theta)} + E_0^- e^{(\alpha+jk)(x\sin\theta+z\cos\theta)}\right)(\cos\theta\ \mathrm{e}_x - \sin\theta\ \mathrm{e}_z) \tag{D.2a}$$

$$M = \frac{1}{\eta}\left(E_0^+ e^{-(\alpha+jk)(x\sin\theta+z\cos\theta)} - E_0^- e^{(\alpha+jk)(x\sin\theta+z\cos\theta)}\right)\mathrm{e}_y \tag{D.2b}$$

$$\eta := \frac{j\omega\mu}{\gamma} = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\epsilon}} \tag{D.2c}$$

D.4 Vector Potentials

The magnetic vector potential A is defined for source-free regions (guaranteed experimentally, since there are no magnetic monopoles: $\rho_m = 0$) by

$$B_A = \nabla \times A \tag{D.3}$$

The electric scalar potential ϕ_e is then defined by

$$E_A = -\nabla\phi_e - j\omega A \tag{D.4}$$

Similarly, if there are no electric charges ($\rho_e = 0$), then we can define the electric vector potential F by

$$D_F = -\nabla \times F \tag{D.5}$$

and the magnetic scalar potential ϕ_m by

$$H_F = -\nabla\phi_m - j\omega F \tag{D.6}$$

We can specify ϕ_e, ϕ_m arbitrarily (doing so is called ”fixing a gauge”). The Lorenz gauge is defined by

$$\nabla \cdot \Theta + \mu\epsilon\partial_t\phi_\theta = 0 \tag{D.7}$$

Applying this to both potentials leads to the Helmholtz equations:

$$\nabla^2 A + k^2 A = -\mu J \tag{D.8a}$$
$$\nabla^2 F + k^2 F = -\epsilon \mathfrak{M} \tag{D.8b}$$

D.5 Reflection and Transmission

Uniform Plane Waves

Normal Incidence: Assuming the wave vector in the z direction, the electric field is polarized in the x direction, defining E_0, Γ, T respectively by

$$\begin{aligned} E^i &= E_0 e^{-(\alpha_1 + jk_1)z} \mathbf{e}_x \\ E^r &= \Gamma E_0 e^{(\alpha_1 + jk_1)z} \mathbf{e}_x \\ E^t &= T E_0 e^{-(\alpha_2 + jk_2)z} \mathbf{e}_x \end{aligned}$$

applying right-hand-rule and enforcing continuity of tangential components ($\Theta^i + \Theta^r = \Theta^t$ at $z = 0$, where $\Theta \in E, H$) leads to

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \tag{D.10a}$$
$$T = 1 + \Gamma \tag{D.10b}$$

Oblique Incidence: The formulae for oblique angles are simple to obtain in a similar fashion. For the electric field perpendicular to the plane of incidence we replace (D.10a) by

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \tag{D.11}$$

and when it is polarized parallel, it instead becomes

$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{D.12}$$

which can both be derived from the formulae for plane-wave impedances for transverse modes (the first is TM^z corresponding to $\eta_p \to \frac{\eta_p}{\cos\theta_p}$, and the second is TE^z corresponding to $\eta_p \to \eta_p \cos\theta_p$)

D.6 Geometrical Optics

Eikonal Surfaces are those normal to the ray.

Conservation of Energy Flux in addition to far-field assumption shows that

$$|E|\sqrt{dA} = \text{const} \tag{D.13}$$

TODO

GO Chapter 13.2, [763e], [744b] specifically the formulae and LK Series

Formal Statement of Problem

Papers of Degli-Esposti

Integral Equation Method Chapter 12.2-12.3, [690e], [671b]

Green's functions Chapter 14, [870e], [851b]

Construction of Solutions using Potentials Chapter 6.5+, [280e], [261b]

Polarization Characteristics on Reflection Chapter 5.6, [255e], [236b]

Infinite Line-Source Cylindrical Wave Radiation Chapter 11.2, [590e], [571b]

Eigenfunctions Arising from Boundary Conditions for Waveguides Chapter 8.2.1, [374e], [355b]

Plane Wave Scattering from Flat Surfaces Chapter 11.3, [596e], [577b]

Wave Theorems and Transformations Chapters 11.4, 11.7 [614e, 664e], [595b, 645b]

GTD Chapter 13.3, [784e], [765b]