Assignment 1 : Cardinalities

1. Let S be a non-empty set. Show that there is a one-to-one correspondence from S to S.

Consider the function $f: S \to S$
"$x \mapsto x$"
"$f(x) = x$"

* **one-to-one :** Suppose $\exists_{x,y} \in S$ s.t. $f(x) = f(y)$.

Since $f(x) = x$ and $f(y) = y$, then $x = y$.

Hence, f is one-to-one.

* **onto :** Suppose $y \in S$. ← "co-domain"

Since $f(y) = y$ (b/c element in f map to themselve).

Thus, f is onto.

Since f is one-to-one and onto, then there is a one-to-one correspondence from S to S.

QED

2. Suppose $f: S \to T$ is a one-to-one correspondence.

Find a one-to-one correspondence from T to S.

"$h = f^{-1}$ b/c $h: T \to S$, $f: S \to T$

$h(x) \overset{f}{=} (f^{-1}(x)) = f(y)$

$\hookrightarrow x = f(y)$"

consider the function : $h: T \to S$
"$x \mapsto y$"

Proof

Consider the function $h: T \to S$ defined by $h(x) = y$ iff $x = f(y)$.

* one-to-one : Suppose $\exists\, x, y \in T$ s.t. $h(x) = h(y)$.

Since $h(x) \in S$, $\exists\, m \in S$ s.t. $h(x) = m$.

If $h(x) = m$, then we have $x = f(m)$. (By definition)

Also $h(y) = m$ implying $y = f(m)$. (Again)

Since $h(x) = h(y) \Rightarrow x = y$, the h is one-to-one.

* onto : Suppose $y \in S$.

We want to show $\exists\, x \in T$ s.t $h(x) = y$.

Since $y \in S$, $f(y) \in T$, implying $f(y) = p$, for some $p \in T$.

$f(y) = p$ iff $h(p) = y$;

Hence, h is onto.

Since h is one-to-one and onto, there is a one-to-one correspondence from T to S.

QED

3. Suppose $f : S \to T$ and $g : T \to U$ are both one-to-one correspondeces.

Show that $g \circ f$ is a one-to-one correspondence from S to U.

Recall that $(g \circ f)(x) = g(f(x))$.

Proof : We want to show $g \circ f : S \to U$ is a one-to-one correspondence.

* one-to-one : Suppose $\exists_{x,y} \in S$ s.t. $g(f(x)) = g(f(y))$.

Since g is one-to-one : if $g(f(x)) = g(f(y))$, then $f(x) = f(y)$.

Since f is one-to-one : if $f(x) = f(y)$, then $x = y$.

Hence, $g \circ f$ is one-to-one.

* onto : Suppose $z \in U$.

We want to show $\exists_x \in S$ s.t. $g \circ f(x) = g(f(x)) = z$:

Since g is onto, $\exists_y \in T$ s.t. $\underbrace{g(y) = z}$. T domain of g

Since $y \in T$ and f is onto, $\exists_x \in S$ s.t. $\underbrace{f(x) = y}$.

Since $\underbrace{f(x) = y, \; g(f(x)) = z}$.

$\therefore$ $g \circ f$ is a one-to-one correspondence from S to U.

QED.

4. Suppose $f(x) = \frac{x}{1+x}$.

Use the function f to show that $S = (0, \infty)$ is in one-to-one correspondence with $T = (0,1)$.

consider: $f : S \to T$

$$(0,\infty) \to (0,1)$$

$$\text{"} x \mapsto \frac{x}{1+x} \text{"}$$

* One-to-one: Suppose $\exists\ x, y \in \overbrace{(0,\infty)}^{S = \text{domain}}$ s.t. $f(x) = f(y)$.

So, $\frac{x}{1+x} = \frac{y}{1+y}$

$$x(1+y) = y(1+x)$$

$$x + \cancel{xy} = y + \cancel{xy}$$

$$x = y$$

Hence, f is one-to-one.

* Onto: Suppose $y \in \overbrace{(0,1)}^{T = \text{codomain}}$.

We want to show that $\exists\ x \in (0,\infty)$ s.t. $f(x) = y$.

So, $\frac{x}{1+x} = y$

$$x = y(1+x)$$

$$x = y + xy$$

$$x - xy = y$$

$$x(1-y) = y$$

$$x = \frac{y}{1-y}$$ ✓: Isolate x

$\Rightarrow$ Now plug: $\frac{x}{1+x} \to \frac{\frac{y}{1-y}}{\frac{1-y}{1-y} \cdot 1 + \frac{y}{1-y}} = \frac{y}{\cancel{1-y}} \cdot \frac{\cancel{1-y}}{1} = y$ ✓: goal

Since $\exists\ x \in (0,\infty)$ s.t. $f(x) = y$, f is onto.

Therefore, f is a one-to-one correspondence from $S = (0,\infty)$ to $T = (0,1)$.

QED

5. Suppose $f(x) = \dfrac{x}{1-x}$.

Use the function f to show that $S = (-\infty, 0)$ is in one-to-one correspondence with $T = (-1, 0)$.

Consider: $f : S \to T$

$$(-\infty, 0) \to (-1, 0)$$

$$'' \quad x \mapsto \frac{x}{1-x} \quad ''$$

* one-to-one: Suppose $\exists x, y \in \overbrace{(-\infty, 0)}^{\text{domain}}$ s.t. $f(x) = f(y)$.

$$\frac{x}{1-x} = \frac{y}{1-y}$$

$$x(1-y) = y(1-x)$$

$$x - \cancel{xy} = y - \cancel{yx}$$

$$x = y \quad ✓$$

Hence, f is one-to-one.

* onto: Suppose $y \in \overbrace{(-1, 0)}^{\text{codomain}}$.

We want to show that $\exists x \in \overset{\text{domain}}{(-\infty, 0)}$ s.t. $f(x) = y$.

$$\frac{x}{1-x} = y \quad \Rightarrow \quad x = y - yx \quad ✓: x \text{ is isolated}$$

$$x = y(1-x) \qquad x + yx = y \rightarrow x(1+y) = y$$

$$x = \frac{y}{1+y}$$

$\Rightarrow$ Now plug into check: $f(x) = f\left(\frac{y}{1+y}\right)$

$$= \frac{\frac{y}{1+y}}{1 - \frac{y}{1+y}} = \frac{\frac{y}{1+y}}{\left(\frac{1+y}{1+y}\right)1 - \frac{y}{1+y}}$$

$$= \frac{\frac{y}{1+y}}{\frac{1}{1+y}} = \frac{y}{\cancel{1+y}} \cdot \frac{\cancel{1+y}}{1} = y \quad ✓: \text{goal}$$

Since $\exists x \in (-\infty, 0)$ s.t. $f(x) = y$, then f is onto.

Therefore, f is one-to-one correspondence from S to T.

QED

6. Use the previous two results to show that $f(x) = \frac{x}{1+|x|}$,
is a one-to-one correspondence from R to $(-1,1)$.

consider: $f : \mathbb{R} \to (-1,1)$
$x \mapsto \frac{x}{1+|x|}$

#4

$$f(x) = \begin{cases} \frac{x}{1+x} & \text{if } x \geq 0 \quad (x \in (0,\infty)) \\ \frac{x}{1-x} & \text{if } x < 0 \quad (x \in (-\infty, 0)) \\ 0 & \text{if } x = 0 \end{cases}$$

#5

All the sets: $(-\infty,0) \cup \{0\} \cup (0,\infty) = \mathbb{R}$

Proof by cases:

* one-to-one: suppose $\exists x,y \in \mathbb{R}$ s.t. $f(x) = f(y)$.

case 1: $f(x), f(y) \in \overbrace{(0,1)}^{\text{codain of 4}}$

By exercise 4, $x = y$.

case 2: $f(x), f(y) \in \overbrace{(-1,0)}^{\text{codam of 5}}$.

By exercise 5, $x = y$.

case 3: If $f(x) = f(y) = 0$, then $x = 0 = y$.

Hence, f is one-to-one.

* onto: suppose $y \in (-1,1)$. suppose $f(x) = y$.
We want to show $\exists x \in \mathbb{R}$ s.t. $f(x) = y$.

case 1: If $y \in (0,1)$, by exercise 4,

$$f\left(\frac{y}{1-y}\right) = y$$

x from #4

case 2: If $y \in (-1,0)$, by exercise 5,

x from #5

$$f\left(\frac{y}{1+y}\right) = y$$

case 3: If $y = 0$, then $x = 0$, which is in $\mathbb{R}$. (already defined in the piecewise)

Hence, f is onto.

Therefore, there is a one-to-one correspondence from $\mathbb{R}$ to $(-1,1)$. QED

7. Find a one-to-one correspondence:

(a) From the even natural numbers to $\mathbb{N}$.

$$E = \{2, 4, 6, 8, \ldots\}$$
$$\downarrow \ \downarrow \ \downarrow \ \downarrow$$
$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Define function: $f: E \to \mathbb{N}$

"$x \mapsto \frac{x}{2}$"

$$f(x) = \frac{x}{2}$$

$\frac{5+1}{2} = 3$

$\frac{7+1}{2} = 4$

(b) From the odd natural numbers to $\mathbb{N}$.

$$O = \{1, 3, 5, 7, 9, \ldots\}$$
$$\downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow$$
$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

Define function: $f: O \to \mathbb{N}$

$$f(x) = \frac{x+1}{2}$$

(c) From $\mathbb{N}$ to $\mathbb{Z}$

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$
$$\downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow$$
$$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$$

.... "create piecewise function"

$f(1) = -\left(\frac{1-1}{2}\right) = 0$

$f(3) = -\left(\frac{3-1}{2}\right) = -1$

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\left(\frac{x-1}{2}\right) & \text{if } x \text{ is odd} \end{cases}$$

8. Show that the set of terminating decimals in (0,1) is denumerable.
(see schramm 2.2.4)

Plan: denumerable
- One-one correspondence w/ $\mathbb{N}$
  - 1-1
  - onto

Notes

* terminating decimal – 0.5 0 0 0 0 ...
  0.62 0 0 0 ...
  0.001 0 0 0 ...

  "At some point the #'s are zero"

* repeating decimals – "some sequence of #'s repeat forever"
  0.424242...
  0.333333...

* denumerable – a set is denumerable or countable if there is a One-to-one correspondence b/w the set and $\mathbb{N}$ (natural numbers)

Ex: If A is a denumerable set, $\exists$ a 1-1 and onto function from A to $\mathbb{N}$ (or from $\mathbb{N}$ to A).

In schramm, terminating decimal: $0.d_1d_2\ldots d_n0000$

$d_i \in \underbrace{\{0,1,2,\ldots,9\}}_{\text{digits}}$

Proof: Let M be the set of terminating decimals in (0,1)

$f: \mathbb{N} \to M$ or can also be $f: M \to \mathbb{N}$

Consider the function: $f: M \to \mathbb{N}$

Defined by: $f(0.d_1d_2\ldots d_n) = d_nd_{n-1}\ldots d_2d_1$

* One-to-one: Suppose $\exists\, x,y \in M$ s.t. $f(x) = f(y)$.

Let $x = 0.d_1d_2d_3\ldots d_n$ and $y = 0.a_1a_2\ldots a_m$

Then $f(x) = d_nd_{n-1}\ldots d_2d_1$ and $f(y) = a_ma_{m-1}\ldots a_2a_1$

Since $f(x) = f(y)$, then $n = m$ (same # of digits)

It follows, $d_1 = a_1$, $d_2 = a_2$, ... and $d_n = a_n$.

Hence, $x = y$ which implies f is one-to-one.

* **Onto:** Let $y \in \mathbb{N}$ ← codaim

Since y is a natural number,

We can write $y = d_1 d_2 \dots d_n$ where d_i is a digit.

We want to show $\exists x \in M$ s.t. $f(x) = y$.

Since $f(0.d_n d_{n-1} \dots d_1) = d_1 \dots d_n$,

there exists $x \in M$ s.t. $f(x) = y$.

Hence f is onto.

Therefore, M is denumerable.

QED

9. Modify Schroemm 2.2.4 to give a proof that the set of repeating decimals in $(0,1)$ is denumerable.

* Recall: denumerable = 1-1 correspondence w/ $\mathbb{N}$.
(ie countable)

Proof by contradiction

Assume to the contrary, the set of repeating decimals in $(0,1)$ is uncountable.

We'll call this set S.

If S is uncountable, it has a 1-1 correspondence w/ $\mathbb{R}$.

Then $\underbrace{|S| = |\mathbb{R}|}$.

Now, since every repeating decimal can be written as a rational number $S \subset \mathbb{Q}$.

" repeating dec. can be written as fractions, $\mathbb{Q}$." (Not all so use $\subseteq$ ")

Note: If $A \subseteq B$, then $|A| \leq |B|$.

Then, $|S| \leq |\mathbb{Q}|$.

(since $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countable)
↳ " known proofs"

** We know $\underbrace{|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|}$

known

and $\underbrace{|\mathbb{N}| < |\mathbb{R}|}$. **

Since $\underbrace{|\mathbb{Q}| = |\mathbb{N}|}$, then $\underbrace{|\mathbb{Q}| < |\mathbb{R}|}$ and

Since $\underbrace{|S| = |\mathbb{R}|}$, then $\underbrace{|\mathbb{Q}| < |S|}$. " can't both be true"

We reach a contradiction since we said that $\underbrace{|S| \leq |\mathbb{Q}|}$.

Therefore, S is denumerable.

QED

10. Let S be the set of all infinite sequences of 0's and 1's.
An element of S looks like $(0,1,1,0,0,1,0,\dots)$.
Show that the set S is uncountable.
Use a proof by contradiction.

Proof by contradiction

Assume to contrary S is countable.

If S is countable, $\exists$ a function $f: \mathbb{N} \to S$

s.t. f is a one-to-one correspondence.

$$f: \mathbb{N} \to S$$

$f(1)$ $1 \mapsto (0,1,0,0,1,\dots)$

$f(2)$ $2 \mapsto (1,0,1,0,0, \quad)$

$3 \mapsto (0,0,0,0,1,\dots)$

$4 \mapsto$

$5 \mapsto$

example's based on given

* Note: "I want to come up with a rule that gives me some element of S."

Uses Kantor's Argument *

Let $m \in S$ be defined using the following rule:

- If m_n denotes the nth coordinate of m,

then $m_n = \begin{cases} 0 & \text{if the } n\text{th coordinate of } f(n) \text{ is } 1 \\ 1 & \text{if the } n\text{th coordinate of } f(n) \text{ is } 0 \end{cases}$

$$\begin{cases} m_1 = 1 \\ m_2 = 1 \\ m_3 = 1 \end{cases} \Rightarrow m = (1,1,1,\dots)$$

Since the first coordinate of $f(1)$ is 0, but the first coordinate of m is 1, then $f(1) \neq m$. Similarly $m \neq f(2)$ b/c their second coordinates are not equal. We have that $m \neq f(3)$, $m \neq f(4)$, $m \neq f(5)$, and so on. Since m is not equal to any element of S, $m \notin S$. We reach a contradiction since m is sequence of 0's and 1's, hence $m \in S$. QED

Last sentence: $\therefore S$ is uncountable