Problem #3

1) Let X, y be sets and $f: x \to y$ is a function.
To prove that
If $U \subseteq X$, then $f(X-U) \supseteq f(X) - f(U)$
Let $y \in f(X) - f(U)$
Then there exists $x \in X$ such that $f(x) = y$, but
$y \notin \{f(x) : x \in U\}$. Hence $x \in U$ or $x \in X - U$ which
implies $y = f(x) \in f(X-U)$.
Thus, $y \in f(X) - f(U) \Rightarrow y \in f(X-U)$.
So $f(X-V) \supseteq f(X) - f(U)$.
2) If $V \subset Y$, then to prove that $f^{-1}(Y-V) = f^{-1}(Y) - f^{-1}(V)$.
First to show that $f^{-1}(Y-V) \subseteq f^{-1}(Y) - f^{-1}(V)$.
Let, $x \in f^{-1}(Y-V) \Rightarrow f(x) \in Y - V$. So, $f(x) \in Y$ and $f(x) \notin V$,
$\Rightarrow x \in f^{-1}(Y)$ and $x \in f^{-1}(V)$, $\Rightarrow x \in f^{-1}(Y) - f^{-1}(V)$.
Thus $f^{-1}(Y-V) \subseteq f^{-1}(Y) - f^{-1}(V)$
To prove that $f^{-1}(Y-V) \supseteq f^{-1}(Y) - f^{-1}(V)$
Let $x \in f^{-1}(Y) - f^{-1}(V)$
$\Rightarrow x \in f^{-1}(Y)$ & $x \in f^{-1}(V)$
$\Rightarrow f(x) \in Y$ and $f(x) \notin V$,
$\Rightarrow f(x) \in Y - V$,
$\Rightarrow x \in f^{-1}(Y-V)$.
Hence $f^{-1}(Y-V) \supseteq f^{-1}(Y) - f^{-1}(V)$.
When combining, you get
$f^{-1}(Y-V) = f^{-1}(Y) - f^{-1}(V)$.