

Math 431: Analysis

Homework Assignment #1 — Cardinalities

September 7, 2021

Recall that a function $f : S \rightarrow T$ is **one-to-one** if

$$\forall x \in S \forall y \in S (f(x) = f(y) \implies x = y).$$

(Note, this is the contrapositive of the definition on Schramm, p. 32). Also, f is **onto** if

$$\forall z \in T \exists x \in S (z = f(x)).$$

If both of these hold, then we say that f is a **one-to-one correspondence** and that S and T have the same cardinality.

Exercises

1. Let S be any non-empty set. Show that there is a one-to-one-correspondence from S to S .
2. Suppose $f : S \rightarrow T$ is a one-to-one correspondence. Find a one-to-one correspondence from T to S .
3. Suppose $f : S \rightarrow T$ and $g : T \rightarrow U$ are both one-to-one correspondences. Show that $g \circ f$ is a one-to-one correspondence from S to U . Recall that $(g \circ f)(x) = g(f(x))$.

4. Suppose

$$f(x) = \frac{x}{1+x}.$$

Use the function f to show that $S = (0, \infty)$ is in one-to-one correspondence with $T = (0, 1)$.

5. Suppose

$$f(x) = \frac{x}{1-x}.$$

Use the function f to show that $S = (-\infty, 0)$ is in one-to-one correspondence with $T = (-1, 0)$.

6. Use the previous two results to show that

$$f(x) = \frac{x}{1+|x|}$$

is a one-to-one correspondence from \mathbf{R} to $(-1, 1)$.

7. Find a one-to-one correspondence:

- (a) From the even natural numbers to N .
 - (b) From the odd natural numbers to N .
 - (c) From \mathbf{N} to \mathbf{Z} .
8. Show that the set of terminating decimals in $(0,1)$ is denumerable (see Schramm 2.2.4).
 9. Modify Schramm 2.2.4 to give a proof that the set of repeating decimals in $(0,1)$ is denumerable.
 10. Let S be the set of all infinite sequences of 0's and 1's. An element of S looks like $(0, 1, 1, 0, 0, 1, 0, \dots)$. Show that the set S is uncountable. Use a proof by contradiction.

Assignment 1: Cardinalities

1. Let S be a non-empty set. show that there is a one-to-one correspondence from S to S .

consider the function $f: S \rightarrow S$
" $x \mapsto x$ "
" $f(x) = x$ "

* one-to-one Suppose $\exists x, y \in S$ s.t. $f(x) = f(y)$.

Since $f(x) = x$ and $f(y) = y$, then $x = y$.

Hence, f is one-to-one.

* onto Suppose $y \in S$. ^{← "co-domain"}

Since $f(y) = y$ (b/c element in f map to themselves).

Thus, f is onto.

Since f is one-to-one and onto, then there is a one-to-one correspondence from S to S .

QED

2. Suppose $f: S \rightarrow T$ is a one-to-one correspondence.

Find a one-to-one correspondence from T to S .

consider the function $h: T \rightarrow S$
"x \mapsto y"

$$h = f^{-1} \text{ b/c } \begin{matrix} h: T \rightarrow S \\ f: S \rightarrow T \end{matrix}$$

$$h(x) = f^{-1}(f(x)) = x \quad \text{,,}$$
$$\hookrightarrow x = f(y)$$

Proof

Consider the function $h: T \rightarrow S$ defined by $h(x) = y$ iff $x = f(y)$.

* one-to-one: Suppose $\exists x, y \in T$ s.t. $h(x) = h(y)$.

Since $h(x) \in S$, $\exists m \in S$ s.t. $h(x) = m$.

If $h(x) = m$, then we have $x = f(m)$. (By definition)

Also $h(y) = m$ implying $y = f(m)$. (Again)

Since $h(x) = h(y) \Rightarrow x = y$, the h is one-to-one.

* onto: Suppose $y \in S$.

We want to show $\exists x \in T$ s.t. $h(x) = y$.

Since $y \in S$, $f(y) \in T$, implying $f(y) = p$, for some $p \in T$.

$$\underline{f(y) = p \text{ iff } h(p) = y;}$$

Hence, h is onto.

Since h is one-to-one and onto, there is a one-to-one correspondence from T to S .

Q.E.D

3. Suppose $f: S \rightarrow T$ and $g: T \rightarrow U$ are both one-to-one correspondences.
Show that $g \circ f$ is a one-to-one correspondence from S to U .
Recall that $(g \circ f)(x) = g(f(x))$.

Proof \circ We want to show $g \circ f: S \rightarrow U$ is a one-to-one correspondence.

* one-to-one \circ Suppose $\exists x, y \in S$ s.t. $g(f(x)) = g(f(y))$.
Since g is one-to-one \circ if $g(f(x)) = g(f(y))$, then $f(x) = f(y)$.
Since f is one-to-one \circ if $f(x) = f(y)$, then $x = y$.
Hence, $g \circ f$ is one-to-one.

* onto \circ Suppose $z \in U$.

We want to show $\exists x \in S$ s.t. $g \circ f(x) = g(f(x)) = z$.

Since g is onto, $\exists y \in T$ s.t. $g(y) = z$.
 \downarrow domain of g

Since $y \in T$ and f is onto, $\exists x \in S$ s.t. $f(x) = y$.

Since $f(x) = y$, $g(f(x)) = z$.

$\therefore g \circ f$ is a one-to-one correspondence from S to U .

QED.

4. Suppose $f(x) = \frac{x}{1+x}$.

Use the function f to show that $S = (0, \infty)$ is in one-to-one correspondence with $T = (0, 1)$.

consider \circ $f: S \rightarrow T$
 $(0, \infty) \rightarrow (0, 1)$

$$x \mapsto \frac{x}{1+x}$$

$f = \text{domain}$

* One-to-one \circ Suppose $\exists x, y \in (0, \infty)$ s.t. $f(x) = f(y)$.

$$\text{so, } \frac{x}{1+x} = \frac{y}{1+y}$$

$$x(1+y) = y(1+x)$$

$$x + xy = y + xy$$

$$x = y$$

Hence, f is one-to-one.

$T = \text{codomain}$

* Onto \circ Suppose $y \in (0, 1)$.

We want to show that $\exists x \in (0, \infty)$ s.t. $f(x) = y$.

$$\text{so, } \frac{x}{1+x} = y$$

$$x = y(1+x)$$

$$x = y + xy$$

$$x - xy = y$$

$$x(1-y) = y$$

$$x = \frac{y}{1-y}$$

\checkmark : Isolate x

$$\Rightarrow \text{Now plug } \circ \frac{x}{1+x} \rightarrow \frac{\frac{y}{1-y}}{\frac{1-y}{1-y} + \frac{y}{1-y}} = \frac{y}{1-y} \cdot \frac{1-y}{1-y+y} = y \checkmark: \text{goal}$$

Since $\exists x \in (0, \infty)$ s.t. $f(x) = y$, f is onto.

Therefore, f is a one-to-one correspondence.

from $S = (0, \infty)$ to $T = (0, 1)$.

QED

5. Suppose $f(x) = \frac{x}{1-x}$.

Use the function f to show that $S = (-\infty, 0)$ is in one-to-one correspondence with $T = (-1, 0)$.

consider $\phi: f: S \rightarrow T$
 $(-\infty, 0) \rightarrow (-1, 0)$
 $"x \mapsto \frac{x}{1-x}"$

*one-to-one ϕ Suppose $\exists x, y \in \overbrace{(-\infty, 0)}^{\text{domain}}$ s.t. $f(x) = f(y)$.

$$\begin{aligned}\frac{x}{1-x} &= \frac{y}{1-y} \\ x(1-y) &= y(1-x) \\ x - xy &= y - yx \\ x &= y \quad \checkmark\end{aligned}$$

Hence, f is one-to-one.

*onto ϕ Suppose $y \in \overbrace{(-1, 0)}^{\text{codomain}}$.
 We want to show that $\exists x \in \overbrace{(-\infty, 0)}^{\text{domain}}$ s.t. $f(x) = y$.

$$\begin{aligned}\frac{x}{1-x} &= y \Rightarrow x = y - yx \quad \checkmark: x \text{ is isolated} \\ x + yx &= y \Rightarrow x(1+y) = y \\ x &= \frac{y}{1+y}\end{aligned}$$

\Rightarrow Now plug into check $\phi: f(x) = f\left(\frac{y}{1+y}\right)$

$$= \frac{\frac{y}{1+y}}{1 - \frac{y}{1+y}} = \frac{\frac{y}{1+y}}{\left(\frac{1+y}{1+y}\right) - \frac{y}{1+y}}$$

$$= \frac{\frac{y}{1+y}}{\frac{1}{1+y}} = \frac{y}{1+y} \cdot \frac{1+y}{1} = y \quad \checkmark: \text{goal}$$

Since $\exists x \in (-\infty, 0)$ s.t. $f(x) = y$, then f is onto.

Therefore, f is one-to-one correspondence from S to T .

QED

6. Use the previous two results to show that $f(x) = \frac{x}{1+|x|}$,
is a one-to-one correspondence from \mathbb{R} to $(-1, 1)$.

consider $\phi: f: \mathbb{R} \rightarrow (-1, 1)$

$$x \mapsto \frac{x}{1+|x|}$$

$$f(x) = \begin{cases} \frac{x}{1+x} & \text{if } x \geq 0 \quad (x \in (0, \infty)) \\ \frac{x}{1-x} & \text{if } x < 0 \quad (x \in (-\infty, 0)) \\ 0 & \text{if } x = 0 \end{cases}$$

All the sets $(-\infty, 0) \cup \{0\} \cup (0, \infty) = \mathbb{R}$

Proof by cases

* one-to-one ϕ Suppose $\exists x, y \in \mathbb{R}$ s.t. $f(x) = f(y)$.

case 1: $f(x), f(y) \in (0, 1)$

By exercise 4, $x = y$.

case 2: $f(x), f(y) \in (-1, 0)$.

By exercise 5, $x = y$.

case 3: If $f(x) = f(y) = 0$, then $x = 0 = y$.

Hence, f is one-to-one.

* onto

Suppose $y \in (-1, 1)$.
We want to show $\exists x \in \mathbb{R}$ s.t. $f(x) = y$.

case 1: If $y \in (0, 1)$, by exercise 4,

$$f\left(\frac{y}{1-y}\right) = y$$

x from #4

case 2: If $y \in (-1, 0)$, by exercise 5,

$$f\left(\frac{y}{1+y}\right) = y$$

Hence, f is onto.

case 3: If $y = 0$,
then $x = 0$, which
is in \mathbb{R} .
(already defined
in the exercise)

Therefore, there is a one-to-one correspondence from \mathbb{R} to $(-1, 1)$. QED

7. Find a one-to-one correspondence:

(a) From the even natural numbers to \mathbb{N} .

$$\begin{array}{ccccccc} E = \{ 2, 4, 6, 8, \dots \} \\ \downarrow \downarrow \downarrow \downarrow \\ \mathbb{N} = \{ 1, 2, 3, 4, \dots \} \end{array}$$

Define function $\phi: E \rightarrow \mathbb{N}$
" $x \mapsto \frac{x}{2}$ "

$$\underline{\underline{f(x) = \frac{x}{2}}}$$

(b) From the odd natural numbers to \mathbb{N} .

$$\begin{array}{ccccccc} O = \{ 1, 3, 5, 7, 9, \dots \} \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ \mathbb{N} = \{ 1, 2, 3, 4, 5, \dots \} \end{array}$$

Define function $\phi: O \rightarrow \mathbb{N}$

$$\underline{\underline{f(x) = \frac{x+1}{2}}}$$

$$\frac{5+1}{2} = 3$$

$$\frac{7+1}{2} = 4$$

(c) From \mathbb{N} to \mathbb{Z}

$$\begin{array}{ccccccc} \mathbb{N} = \{ 1, 2, 3, 4, 5, \dots \} \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ \mathbb{Z} = \{ 0, 1, -1, 2, -2, \dots \} \end{array}$$

.... "create piecewise function"

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -(\frac{x-1}{2}) & \text{if } x \text{ is odd} \end{cases}$$

$$f(1) = -(\frac{1-1}{2}) = 0$$

$$f(3) = -(\frac{3-1}{2}) = -1$$

8. Show that the set of terminating decimals in $(0,1)$ is denumerable.
(see schramm 2.2, 4)

Notes

* terminating decimal - $0.50000\dots$
 $0.62000\dots$
 $0.001000\dots$

"At some point the #'s are zero"

plan 8 denumerable

• One-to-one correspondence w/ \mathbb{N}
 $= 1-1$
 $= 0.10$

* repeating decimals - "some sequence of #'s repeat forever"
 $0.424242\dots$
 $0.333333\dots$

* denumerable - a set is denumerable or countable if there is a one-to-one correspondence b/w the set and \mathbb{N} (natural numbers)

$\exists \varphi$ iff A is a denumerable set, \exists a 1-1 and onto function from A to \mathbb{N} (or from \mathbb{N} to A).

In schramm, terminating decimal $\varphi 0.d_1d_2\dots d_n 0000$
 $d_i \in \underbrace{\{0,1,2,\dots,9\}}_{\text{digits}}$

Proof φ Let M be the set of terminating decimals in $(0,1)$

$f: \mathbb{N} \rightarrow M$ or can also be $f: M \rightarrow \mathbb{N}$

consider the function $\varphi f: M \rightarrow \mathbb{N}$

Defined by $\varphi f(0.d_1d_2\dots d_n) = d_nd_{n-1}\dots d_2d_1$

* one-to-one φ suppose $\exists x,y \in M$ s.t. $f(x) = f(y)$.

Let $x = 0.d_1d_2d_3\dots d_n$ and $y = 0.a_1a_2\dots a_m$

Then $f(x) = d_nd_{n-1}\dots d_2d_1$ and $f(y) = a_ma_{m-1}\dots a_2a_1$

Since $f(x) = f(y)$, then $n = m$ (same # of digits)

It follows, $d_1 = a_1, d_2 = a_2, \dots$ and $d_n = a_n$.

Hence, $x = y$ which implies f is one-to-one.

* onto: Let $y \in \mathbb{N}$ ← codomain

Since y is a natural number,

We can write $y = d_1 d_2 \dots d_n$ where d_i is a digit.

We want to show $\exists x \in M$ s.t. $f(x) = y$.

Since $f(0.d_n d_{n-1} \dots d_1) = .d_1 d_2 \dots d_n$,

there exists $x \in M$ s.t. $f(x) = y$.

Hence f is onto.

Therefore, M is denumerable.

QED

9. Modify screenshot 2.2.4 to give a proof that the set of repeating decimals in $(0,1)$ is denumerable.

* Recall: denumerable = 1-1 correspondence w/ \mathbb{N} .
(i.e. countable)

Proof by contradiction

Assume to the contrary, the set of repeating decimals in $(0,1)$ is uncountable.

We'll call this set S .

If S is uncountable, it has a 1-1 correspondence w/ \mathbb{R} .

$$\text{Then } |S| = |\mathbb{R}|.$$

Now, since every repeating decimal can be written as a rational number $S \subset \mathbb{Q}$.

"repeating dec.
can be written
as fractions, \mathbb{Q} ."
"Not all so use \leq "

Note: If $A \subseteq B$, then $|A| \leq |B|$.

$$\text{Then, } |S| \leq |\mathbb{Q}|.$$

** We know $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$ (since \mathbb{N} , \mathbb{Z} , and \mathbb{Q} are countable)
↳ "known proofs"

**
known
and $|\mathbb{N}| < |\mathbb{R}|$.

Since $|\mathbb{Q}| = |\mathbb{N}|$, then $|\mathbb{Q}| < |\mathbb{R}|$ and

Since $|S| = |\mathbb{R}|$, then $|\mathbb{Q}| < |S|$.

"can't both be true oo"

We reach a contradiction since we said that $|S| \leq |\mathbb{Q}|$.

Therefore, S is denumerable.

QED

10. Let S be the set of all infinite sequences of 0's and 1's.
 An element of S looks like $(0, 1, 1, 0, 0, 1, 0, \dots)$.
 Show that the set S is uncountable.
 Use a proof by contradiction.

Proof by contradiction

Assume to contrary S is countable.

If S is countable, \exists a function $f: \mathbb{N} \rightarrow S$
 s.t. f is a one-to-one correspondence.

$$f: \mathbb{N} \rightarrow S$$

$f(1)$	1	\mapsto	$(0, 1, 0, 0, 1, \dots)$	} examples based on given
$f(2)$	2	\mapsto	$(1, 0, 1, 0, 0, \dots)$	
	3	\mapsto	$(0, 0, 0, 0, 1, \dots)$	
	4	\mapsto		
	5	\mapsto		

* Note: "I want to come up with a rule that gives me some element of S ."

↑ let $m \in S$ be defined using the following rule:

* Uses Cantor's Argument *

- If m_n denotes the n th coordinate of m ,
 then $\underline{m_n} = \begin{cases} 0 & \text{if the } n\text{th coordinate of } f(n) \text{ is } 1 \\ 1 & \text{if the } n\text{th coordinate of } f(n) \text{ is } 0 \end{cases}$

$$\begin{cases} m_1 = 1 \\ m_2 = 1 \\ m_3 = 1 \end{cases} \Rightarrow m = (1, 1, 1, \dots)$$

Since the first coordinate of $f(1)$ is 0, but the first coordinate of m is 1, then $f(1) \neq m$. Similarly $m \neq f(2)$ b/c their second coordinates are not equal.

We have that $m \neq f(3)$, $m \neq f(4)$, $m \neq f(5)$, and so on. Since m is not equal to any element of S , $m \notin S$. We reach a contradiction since m is sequence of 0's and 1's, hence $m \in S$.

∴ S is uncountable