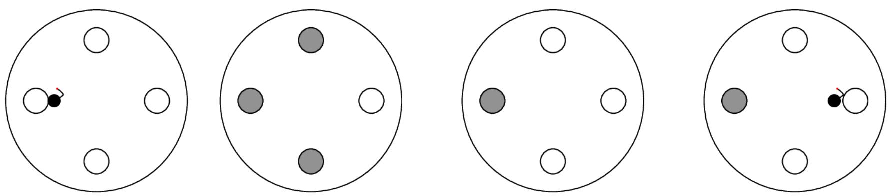
Boss Battle

You are stuck at a boss level of your favorite video game. The boss battle happens in a circular room with n indestructible pillars arranged evenly around the room. The boss hides behind an unknown pillar. Then the two of you proceed in turns.

- First, in your turn, you can throw a bomb past one of the pillars. The bomb will defeat the boss if it is behind that pillar or either of the adjacent pillars.
- Next, if the boss was not defeated, it may either stay where it is or use its turn to move to a pillar that is adjacent to its current position. With the smoke of the explosion, you cannot see this movement.

Given n where $1 \leq n \leq 100$. What is the maximum number of pillars you have to bomb, provided that you and the monster both played optimally?



first bomb after the blast after the boss moves second bomb

Observations

For $n \leq 3$, it DOES NOT matter which pillar you choose to bomb. The bomb will eliminate everything behind all the pillars. So, the answer is always 1.

For $n > 3$, the worst case would be if the monster is always behind one of the unaffected pillars, regardless of which one we choose to bomb at any stage of the game. An important observation we can make is that it is possible to narrow down the position of the monster.

If we choose a pillar to bomb, and we end up not defeating the monster, then the monster CANNOT move to the pillar that we just bombed, as it was not in one of the adjacent pillars to begin with. Therefore, we narrowed down the position of the monster by 1 pillar.

Based on this observation, how can we strategize on the choices of pillars? And based on this strategy, what would be the total number of pillars we have to bomb? **Hint:** Try it on small test cases, i.e., $n = 4, 5, 6$.

Solution

- If $n \leq 3$, it is obvious to see that $S(n \leq 3) = 1$. Now, we have to solve for $n > 3$.
- Based on the observation, we can see that if we bombed a pillar, the monster cannot move back to it on the immediate next turn, narrowing the total number of pillars by 1. However, what would be the next pillar we should pick?
- If pillar i is bombed, affecting pillars $i - 1$ and $i + 1$, then we choose pillar $i + 2$ to bomb, affecting pillars $i + 1$ and $i + 3$. We know that the monster cannot be on pillar $i + 2$. We also know that if the monster moved to pillar $i + 1$, it would be bombed on the next turn, so the monster is definitely not on pillar $i + 1$ as well. We managed to narrow the position of the monster by 2 pillars. It can be proven that you can continuing doing this until there are 3 pillars left, after which you can choose any final pillar to bomb, and that defeats the monster.
- Let the solution to the problem be $S(n)$, where n is the number of pillars. The answer to $S(n)$ can be solved recursively: $S(n) = 1 + S(n - 1)$.
- We get the following recursive formula:

$$S(n) = \begin{cases} 1, & n \leq 3 \\ 1 + S(n-1), & n > 3 \end{cases}$$

- The time complexity for this is $O(n)$. This is sufficient in passing all the test cases given for this problem. However, we can do better!
- We can expand the recurrence for $n > 3$:

$$\begin{aligned} S(n) &= 1 + S(n-1) \\ &= 1 + 1 + S(n-2) \\ &\ldots \\ &= n - 3 + S(3) = n - 3 + 1 \\ S(n) &= n - 2 \end{aligned}$$

- This improved our time complexity to O(1).