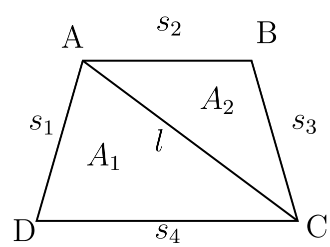
Janitor Troubles

Given 4 side lengths s_1, s_2, s_3, s_4 such that $2s_i < \sum_{j=1}^{4} s_j$ for all i, and $1 \leq s_i \leq 1000$. Compute the maximum area of any quadrilateral that can be formed using these side lengths.

Hint

- The condition $2s_i < \sum_{j=1}^{4} s_j$ for all i is to ensure that the side lengths can be used to construct the quadrilateral for all permutations (not arrangements) of s_1, s_2, s_3, s_4.
- Before we can compute the answer to this problem, we must find a way to get an area of a quadrilateral using the four side lengths s_1, s_2, s_3, s_4.
- The most intuitive way (for those who have little no geometry background) is to divide the quadrilateral into two triangles. Here is the following visualization for a *specific* arrangement of s_1, s_2, s_3, s_4:



- The area of the quadrilateral is now the sum $A = A_1 + A_2$ where A_1 and A_2 are the areas of ΔADC and ΔABC respectively (See the above diagram). Computing the areas of the triangles is (or should be) trivial.

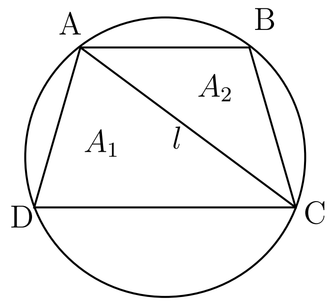
$$\begin{aligned} A &= A_1 + A_2 \\ &= \frac{1}{2} s_1 s_4 \sin \angle ADC + \frac{1}{2} s_2 s_3 \sin \angle ABC \\ &= \frac{1}{2} \Big(s_1 s_4 \sin \angle ADC + s_2 s_3 \sin \angle ABC \Big) \end{aligned}$$

- The question is, how can we maximize this value? There is something we need to consider: The angles $\angle ABC$ and $\angle ADC$ are unknown. However, there is a special case in which we can relate the two angles. **Hint:** Try with the case where $s_1 = s_2 = s_3 = s_4$.

Solution

- It turns out, in the most optimal case, $\angle ADC = 180^{\text{o}} - \angle ABC$. This happens when the quadrilateral is inscribed in a circle. Let $\angle ADC = \theta$. Then, $\sin\angle ADC = \sin\angle ABC = \sin\theta$. We also have to be careful because $\angle ADC$ and $\angle ABC$ are complementary, so $\cos\angle ABC = -\cos\theta$.

- At this point, it is possible to solve for θ, under the condition: $0 \leq \theta \leq \pi$. However, there is another way that shortens this step, and it is by computing the diagonal l using the Law of Cosines:



$$\begin{aligned}
l^2 &= s_1^2 + s_4^2 - 2s_1s_4\cos\angle ADC \\
&= s_2^3 + s_3^2 - 2s_2s_3\cos\angle ABC \\
s_1^2 + s_4^2 - 2s_1s_4\cos\theta &= s_2^3 + s_3^2 + 2s_2s_3\cos\theta \\
s_1^2 + s_4^2 - s_3^2 - s_2^3 &= 2(s_2s_3 + s_1s_4)\cos\theta
\end{aligned}$$

- Plugging $\sin\theta$ into A gives:

$$\begin{aligned}
A_{\text{max}} &= \frac{1}{2}\sin\theta(s_1s_4 + s_2s_3) \\
A_{\text{max}}^2 &= \frac{1}{4}\sin^2\theta(s_1s_4 + s_2s_3)^2 \\
4A_{\text{max}}^2 &= \sin^2\theta(s_1s_4 + s_2s_3)^2 \\
4A_{\text{max}}^2 &= (1 - \cos^2\theta)(s_1s_4 + s_2s_3)^2 \\
4A_{\text{max}}^2 &= (s_1s_4 + s_2s_3)^2 - \cos^2\theta(s_1s_4 + s_2s_3)^2 \\
16A_{\text{max}}^2 &= 4(s_1s_4 + s_2s_3)^2 - 4\cos^2\theta(s_1s_4 + s_2s_3)^2
\end{aligned}$$

- We can see that $4\cos^2\theta(s_1s_4 + s_2s_3)^2 = (s_1^2 + s_4^2 - s_3^2 - s_2^3)^2$. Plugging $(s_1^2 + s_4^2 - s_3^2 - s_2^3)^2$ into A gives the maximum area of the cyclic quadrilateral $ABCD$ in terms of the given side lengths:

$$16A_{\text{max}}^2 = 4(s_1s_4 + s_2s_3)^2 - (s_1^2 + s_4^2 - s_3^2 - s_2^3)^2$$

- We can probably compute $A_{\max}$ based on the given equation without overflow errors since the side lengths $s_i \leq 10^3$. However, simplifying the expression further gives the following elegant formula:

$$A_{\max} = \sqrt{(S - s_1)(S - s_2)(S - s_3)(S - s_4)}$$

  where $S = \frac{s_1+s_2+s_3+s_4}{2}$. This is also known as *Brahmagupta's Formula*!!

- Time Complexity: $O(1)$