Assignment 3: Linear Algebra Review

CS321 Spring 2018

Due Date: 2018.3.22 12:00

Follow the instructions carefully. If you encounter any problems in the setup, please do not hesitate to reach out to TA.

Introduction

This assignment reviews basic mathematical tools you will use throughout our course. In the end of this lab, you will setup a useful mathematic library in OpenGL, which can help you finish the upcoming problems in an elegant way.

Vector Operations

Exercise 2. *Letting* $\mathbf{u} := (9,8,5)$ *and* $\mathbf{v} := (6,6,3)$, *calculate the following quantities:*

(a) $\mathbf{u} - \mathbf{v}$

(b) $\mathbf{u} + 5\mathbf{v}$

Exercise 3. *So far we have been working with vectors in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$, *but it is important to remember that other objects, like functions, also behave like vectors in the sense that we can add them, subtract them, multiply them by scalars,* etc. *Calculate the following quantities for the two polynomials* $p(x) := 9x^2 + 8x + 5$ *and* $q(x) := 6x^2 + 6x + 3$, *and evaluate the result at the point* $x = 7$:

(a) $p(x) - q(x)$

(b) $p(x) + 5q(x)$

Inner Products

Exercise 5. *Any inner product* $\langle \cdot, \cdot \rangle$ *determines a* norm, *given by* $|\mathbf{u}| := \sqrt{\langle \mathbf{u}, \mathbf{u}}$. *Using the standard Euclidean inner product, compute the norm* $|\mathbf{u}|$ *of the vector* $\mathbf{u} := (7,3,8)$.

Exercise 6. *Suppose we define an alternative operation on 2-vectors, given by*

$$\langle \mathbf{u}, \mathbf{v} \rangle := 6u_1v_1 + u_1v_2 + u_2v_1 + 3u_2v_2.$$

Compute the following quantities, thinking about how they help verify that $\langle \cdot, \cdot \rangle$ *is a valid inner product:*

(a) $\langle \mathbf{x}, \mathbf{x} \rangle$, *for* $\mathbf{x} := (1,0)$.

(b) $\langle \mathbf{y}, \mathbf{y} \rangle$, *for* $\mathbf{y} := (0,1)$.

(c) $\langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle$, *for* $\mathbf{u} := (6,8)$ *and* $\mathbf{v} := (3,2)$.

(d) $\langle 8\mathbf{u} + \mathbf{v}, \mathbf{w} \rangle - (8\langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle)$, *for* $\mathbf{u} = (4,4)$, $\mathbf{v} = (4,9)$, *and* $\mathbf{w} = (6,3)$.

Exercise 7. *Just as we can take inner products of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, we can also take inner products of functions. In particular, if $f(x)$ and $g(x)$ are two real-valued functions over the unit interval $[0,1]$, we can define the so-called L_2 inner product*

$$\langle\langle f, g \rangle\rangle := \int_0^1 f(x)g(x)\,dx.$$

Intuitively, this inner product measures how much the two functions "line up" (just like the dot product in $\mathbb{R}^n$). Check that $\langle\langle \cdot, \cdot \rangle\rangle$ behaves like an inner product by evaluating the following expressions:

(a) $\langle\langle f, f \rangle\rangle$, *for* $f(x) := ax^2 + b$, *and evaluate the result for* $a = 2$, $b = 6$.

(b) $\langle\langle f, f \rangle\rangle$, *for* $f(x) := 9e^{7x}$.

(c) $\langle\langle f, g \rangle\rangle - \langle\langle g, f \rangle\rangle$, *for* $f(x) := 6x + 3$ *and* $g(x) := 6x^2$.

Linear Maps

Exercise 11. *Determine whether the maps $F(f)(x) := 4 + \frac{d}{dx} f(x)$, $G(f)(x) := \int_0^1 f(x)\,dx$, and $H(f)(x) := f(0)$ are linear* [a] *by computing the quantities below for functions $f(x) := \sin(x)$ and $g(x) := e^x$, evaluating the final result at the point $x = 5$. [**Hint:** For some of these calculations you can save yourself a lot of time and trouble by first computing the difference in terms of generic functions $f(x)$ and $g(x)$ rather than immediately plugging in the functions $\sin(x)$ and e^x, or the point $x = 5$.]*

(a) $F(f+g) - (F(f) + F(g))$

(b) $F(8f) - 8F(f)$

(c) $G(f+g) - (G(f) + G(g))$

(d) $G(4f) - 4G(f)$

(e) $H(f+g) - (H(f) + H(g))$

(f) $H(2f) - 2H(f)$

Basis and Span

Exercise 12. *Consider the basis vectors $\mathbf{e}_1 := (1/\sqrt{2}, 1/\sqrt{2})$ and $\mathbf{e}_2 := (-1/\sqrt{2}, 1/\sqrt{2})$, and the vector $\mathbf{u} := (9,4)$ (using the standard Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle := u_1 v_1 + u_2 v_2$ throughout).*

(a) *Compute the projection $a := \langle \mathbf{e}_1, \mathbf{u} \rangle$ of $\mathbf{u}$ onto $\mathbf{e}_1$.*

(b) *Compute the projection b of $\mathbf{u}$ onto $\mathbf{e}_2$.*

(c) *Compute the difference $\mathbf{u} - (a\mathbf{e}_1 + b\mathbf{e}_2)$.*

Exercise 15. *The* determinant *of a collection of vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k \in \mathbb{R}^n$ is the (signed) volume of the parallelpiped with these edge vectors. For instance, for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, the signed volume of the corresponding parallelogram is* $\det(u, v) := u_1 v_2 - u_2 v_1$.

(a) *Compute the determinant of the vectors $\mathbf{e}_1 := (7,5)$ and $\mathbf{e}_2 := (14,10)$.*

(b) *Compute the value $a := \langle \mathbf{e}_1, \mathbf{w} \rangle$ for $\mathbf{w} = (2,8)$.*

(c) *Compute the value $b := \langle \mathbf{e}_2, \mathbf{w} \rangle$ for the same vector $\mathbf{w}$.*

(d) *Compute the vector $a\mathbf{e}_1 + b\mathbf{e}_2$. (Think about a couple different reasons why this procedure does or does not reconstruct the original vector $\mathbf{w}$.)*

Linear Equation

Exercise 16. *Solve the system of linear equations*

$$\begin{array}{rcl} 9x + 8y & = & 4 \\ -8x + 9y & = & 3 \end{array}$$

for the unknown pair $(x, y) \in \mathbb{R}^2$.

Quadratic forms

Exercise 20. *An extremely common energy in computer graphics (used in image processing, geometry, physically-based animation, ...) measures the failure of the derivative of a function* f *to match some fixed function* u. *A simple version of this energy is given by the expression*

$$E(f) := \left|\left|\tfrac{df}{dx} - u\right|\right|^2,$$

where $||\cdot||$ *denotes the* L_2 *norm on functions over the unit interval,* i.e., $||f||^2 := \int_0^1 f(x)^2\, dx$.

If you expand the norm, you will get three terms: one that is quadratic in f (i.e., *involving a square of* f*), one that is linear in* f (i.e., *where* f *appears only once), and one that does not depend on* f *at all. Derive an expression for the bilinear form* B *corresponding to the quadratic part, which maps a pair of functions* $f, g : [0, 1] \to \mathbb{R}$ *to a real scalar value* $B(f, g)$. *Once you've determined* B, *evaluate it on the functions* $f(x) := 6x$ *and* $g(x) = e^{2x}$.

*[**Hint:** For most of this exercise, it will greatly simplify your calculations to write the* L_2 *norm and inner product as* $||\cdot||$ *and* $\langle\langle \cdot, \cdot \rangle\rangle$, *respectively, rather than expanding everything out in terms of integrals over the real line. Since we've already checked that these operations obey the same rules as for vectors in* $\mathbb{R}^n$, *you can manipulate functions just like you would manipulate ordinary vectors—this is part of the power of interpreting functions as vectors. However, if at any point you find these manipulations confusing, it may help to go back to the definition of the inner product in terms of an integral.]*

Matrices and Vectors

Exercise 22. *Suppose linear maps* $f : \mathbb{R}^p \to \mathbb{R}^q$ *and* $g : \mathbb{R}^q \to \mathbb{R}^r$ *are represented by matrices* $\mathbf{A} \in \mathbb{R}^{q\times p}$ *and* $B \in \mathbb{R}^{r\times q}$, *respectively. Then the product of these matrices encodes composition of the linear maps. In other words, for any vector* $\mathbf{x} \in \mathbb{R}^p$

$$\mathbf{BAx} := g(f(\mathbf{x})).$$

(a) In row-major order, give the matrix $\mathbf{A} \in \mathbb{R}^{3\times 2}$ *representing the linear map*

$$f : \mathbb{R}^2 \to \mathbb{R}^3; \quad (x_1, x_2) \mapsto (5x_1, 6x_2, x_1 + x_2).$$

(b) In row-major order, give the matrix $\mathbf{B} \in \mathbb{R}^{3\times 3}$ *representing the linear map*

$$g : \mathbb{R}^3 \to \mathbb{R}^3; \quad (x_1, x_2, x_3) \mapsto 6(x_2, x_3, x_1).$$

(c) Compute the matrix-matrix product $\mathbf{BA} \in \mathbb{R}^{3\times 2}$, *which represents the map* $g \circ f$.

Exercise 25. *Consider a linear map f represented as a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$, and a quadratic form Q represented as a matrix $\mathbf{B} \in \mathbb{R}^{2\times 2}$. Suppose we originally expressed these matrices in terms of an orthonormal basis $(\mathbf{u}, \mathbf{v})$, but now we need to apply them to vectors expressed in a different orthonormal basis, $(\mathbf{x}, \mathbf{y})$. This situation is quite common in computer graphics, since (for instance) objects in a scene might experience some kind of transformation before we apply a given operation.*

(a) *Compute the matrix $\mathbf{E} \in \mathbb{R}^{2\times 2}$ expressing the change of basis from $\mathbf{u} = (7/\sqrt{53}, 2/\sqrt{53})$, $\mathbf{v} = (-2/\sqrt{53}, 7/\sqrt{53})$ to $\mathbf{x} = (8/\sqrt{68}, 2/\sqrt{68})$, $\mathbf{y} = (-2/\sqrt{68}, 8/\sqrt{68})$. In other words, $\mathbf{E}$ should satisfy the relationships $\mathbf{Eu} = \mathbf{x}$ and $\mathbf{Ev} = \mathbf{x}$ (i.e., the change of basis should be achieved via **left**-multiplication). The result should be given in row-major form. [**Hint:** Take an arbitrary vector $\mathbf{a} = a_1\mathbf{u} + a_2\mathbf{v}$ and project it onto the two new directions $\mathbf{x}, \mathbf{y}$. How can you express this operation as a matrix? You may want to double-check your answer by applying your final matrix to your vector u. It may also be helpful to work out the solution symbolically first, plugging in the numerical values only at the very end.]*

(b) *Suppose we now want to apply the linear map $f(\mathbf{a}) := (7a_1 + 5a_2, 8a_1 + 6a_2)$ to a vector $\mathbf{a}$ that is expressed in the basis $(\mathbf{x}, \mathbf{y})$, but express the result in the basis $(\mathbf{u}, \mathbf{v}$. Write (in row-major form) the matrix that represents this operation. [**Hint:** Be very careful about which direction you change bases! It may be helpful to consider the previous exercise (on the transpose).]*

(c) *Similarly, suppose we now want to apply the quadratic form $Q(\mathbf{a}) := 6a_1^2 + 8a_1a_2 + 5a_2^2$ to a vector $\mathbf{a}$ expressed in the basis $(\mathbf{x}, \mathbf{y})$. What matrix should we use (in row-major form)? [**Hint:** The matrix representation of a quadratic form generally looks like $\mathbf{w}^\mathsf{T}\mathbf{Bw}$ for some symmetric matrix $\mathbf{B}$, where $\mathbf{w}$ is the argument. Think very carefully about when and how the argument has to be transformed from one basis into another.]*

In practice

Now that we've explained all the theory behind transformations, it's time to see how we can use this knowledge to our programs. OpenGL does not have any built-in data structure of matrix or vector knowledge, so we have to define our own mathematics classes and functions. Luckily, there is an easy-to-use and tailored-for-OpenGL mathematics library called GLM.

GLM

GLM stands for Open**GL** **M**athematics and is a header-only library, which means that we only need to include the proper header files and we're done; no linking and compiling necessary. GLM can be downloaded from their website. Copy the root directory of the header files into your *includes* folder and done.

**There may be some initialize problem when you use some code from the Internet, because they use older version.

There are 3 headers files that we commonly use:

```
#include <glm/glm.hpp>
#include <glm/gtc/matrix_transform.hpp>
#include <glm/gtc/type_ptr.hpp>
```

Let's see an example of translating a vector of (1,0,0) by (1,1,0):

```
glm::vec4 vec(1.0f, 0.0f, 0.0f, 1.0f);
glm::mat4 trans;
trans = glm::translate(trans, glm::vec3(1.0f, 1.0f, 0.0f));
vec = trans * vec;
```

```
std::cout << vec.x << vec.y << vec.z << std::endl;
```

And the other example of scale and rotation:

```
glm::mat4 trans;
trans = glm::rotate(trans, glm::radians(90.0f), glm::vec3(0.0, 0.0, 1.0));
trans = glm::scale(trans, glm::vec3(0.5, 0.5, 0.5));
```

Exercise:

1. Finish the exercises with any resources that can help you understand the concepts. You can use calculator, etc. to perform numerical computation, ONLY numerical computation.
2. Setup the glm library, try the examples by yourself.