Test 2

1. Prove that the following formula is satisfiable:

$$\varphi = \exists x \exists y \exists z (x \neq y \wedge x \neq z \wedge y \neq z \wedge \forall u (u = x \vee u = y \vee u = z))$$

   (a) Define a model $M = \langle D, I \rangle$ such that

$$M \models \exists x \exists y \exists z (x \neq y \wedge x \neq z \wedge y \neq z \wedge \forall u (u = x \vee u = y \vee u = z))$$

   (Hint: think of an intuitive interpretation of φ.)

   (b) Prove that $M \models \exists x \exists y \exists z (x \neq y \wedge x \neq z \wedge y \neq z \wedge \forall u (u = x \vee u = y \vee u = z))$ by using the truth conditions of first order logic.

2. Give a counterexample for the following argument and prove that it is invalid.

$$\{\exists x (p(x))\} \models \exists x \exists y (x \neq y)$$

   (a) Give a counterexample for the argument. (Hint: think of an intuitive interpretation of the predicates.)

   (b) Prove that the counterexample invalidates the argument by using the truth conditions of first order logic.

3. We want to show that

$$\neg(\exists y \forall x (p(x,y)) \wedge \forall x \exists y (p(x,y) \rightarrow p(y,x)) \wedge \neg \exists x (p(x,x)))$$

   is valid. To this end, we want to use resolution to derive the empty clause from

$$\exists y \forall x (p(x,y)) \wedge \forall x \exists y (p(x,y) \rightarrow p(y,x)) \wedge \neg \exists x (p(x,x)).$$

   (a) Convert the formula

$$\exists y \forall x (p(x,y)) \wedge \forall x \exists y (p(x,y) \rightarrow p(y,x)) \wedge \neg \exists x (p(x,x))$$

   into clausal form.

   (b) Take the separate clauses (i.e. disjunctions of literals) you obtained in the previous step, and derive the empty clause by means of the resolution rule. Also provide the substitution(s) you used to do so in each application of the resolution rule.

Answers

1. (a) [2 pt] Any model with exactly three objects, e.g. $M = \langle D, I\rangle$ with $D = \{1,2,3\}$ and arbitrary I (there are no constants or predicates in this formula).

   (b) [4 pt] Proof:

$$\begin{gathered}
M \models \exists x \exists y \exists z(x \neq y \wedge x \neq z \wedge y \neq z \wedge \forall u(u = x \vee u = y \vee u = z)) \\
\text{iff} \\
\text{for some } d_1, d_2, d_3 \in D: \\
M, V[d_1/x, d_2/y, d_3/z] \models x \neq y \wedge x \neq z \wedge y \neq z \wedge \forall u(u = x \vee u = y \vee u = z) \\
\text{iff} \\
\text{for some } d_1, d_2, d_3 \in D: M, V[d_1/x, d_2/y, d_3/z] \models x \neq y \text{ and} \\
M, V[d_1/x, d_2/y, d_3/z] \models x \neq z \text{ and } M, V[d_1/x, d_2/y, d_3/z] \models y \neq z \text{ and} \\
M, V[d_1/x, d_2/y, d_3/z] \models \forall u(u = x \vee u = y \vee u = z) \\
\text{iff} \\
\text{for some } d_1, d_2, d_3 \in D: I_{V[d_1/x, d_2/y, d_3/z]}(x) \neq I_{V[d_1/x, d_2/y, d_3/z]}(y) \text{ and} \\
I_{V[d_1/x, d_2/y, d_3/z]}(x) \neq I_{V[d_1/x, d_2/y, d_3/z]}(z) \text{ and} \\
I_{V[d_1/x, d_2/y, d_3/z]}(y) \neq I_{V[d_1/x, d_2/y, d_3/z]}(z) \text{ and for all} \\
e \in D: M, V[d_1/x, d_2/y, d_3/z, e/u] \models u = x \vee u = y \vee u = z \\
\text{iff} \\
\text{for some } d_1, d_2, d_3 \in D: d_1 \neq d_2 \text{ and } d_1 \neq d_3 \text{ and } d_2 \neq d_3 \text{ and for all} \\
e \in D: (I_{V[d_1/x, d_2/y, d_3/z, e/u]}(u) = I_{V[d_1/x, d_2/y, d_3/z, e/u]}(x) \text{ or} \\
I_{V[d_1/x, d_2/y, d_3/z, e/u]}(u) = I_{V[d_1/x, d_2/y, d_3/z, e/u]}(y) \text{ or} \\
I_{V[d_1/x, d_2/y, d_3/z, e/u]}(u) = I_{V[d_1/x, d_2/y, d_3/z, e/u]}(z)) \\
\text{iff} \\
\text{for some } d_1, d_2, d_3 \in D: d_1 \neq d_2 \text{ and } d_1 \neq d_3 \text{ and } d_2 \neq d_3 \text{ and for all} \\
e \in D: e = d_1 \text{ or } e = d_2 \text{ or } e = d_3 \\
\text{iff} \\
1, 2, 3 \in \{1,2,3\} \text{ and } 1 \neq 2 \text{ and } 1 \neq 3 \text{ and } 2 \neq 3 \text{ and for all} \\
e \in \{1,2,3\}: e = 1 \text{ or } e = 2 \text{ or } e = 3
\end{gathered}$$

2. (a) [2 pt] Counterexample should make the premise true and the conclusion false. For the latter, we need a singleton model, for the former, the single element should be in the interpretation of p. So for example the domain could be Beatrix, and p could stand for being the Dutch queen. Then $M = \langle D, I\rangle$ with $D = \{\text{Beatrix}\}$ and $I(p) = \{\text{Beatrix}\} = D$.

   (b) Proof:

   - [2 pt] M satisfies the premise:

$$\begin{gathered}
M \models \exists x(p(x)) \\
\text{iff} \\
\text{for some } d \in D: M, V[d/x] \models p(x) \\
\text{iff} \\
\text{for some } d \in D: I_{V[d/x]} \in I(p) \\
\text{iff} \\
\text{for some } d \in D: d \in D
\end{gathered}$$

- [2 pt] M does not satisfy the conclusion:

$$M \not\models \exists x \exists y (x \neq y)$$
iff
$$\text{there exist no } d, e \in D : M, V[d/x, e/y] \models x \neq y$$
iff
$$\text{for all } d, e \in D : M, V[d/x, e/y] \not\models x \neq y$$
iff
$$\text{for all } d, e \in D : I_{V[d/x, e/y]}(x) = I_{V[d/x, e/y]}(y)$$
iff
$$\text{for all } d, e \in D : d = e$$
iff
$$\text{for all } d, e \in \{\text{Beatrix}\} : d = e$$
iff
$$\text{Beatrix} = \text{Beatrix}$$

3. (a) [4 pt] First, eliminate implications, which results in:

$$\exists y \forall x (p(x,y)) \wedge \forall x \exists y (\neg p(x,y) \vee p(y,x)) \wedge \neg \exists x (p(x,x)).$$

Second, move $\neg$ inwards, which results in:

$$\exists y \forall x (p(x,y)) \wedge \forall x \exists y (\neg p(x,y) \vee p(y,x)) \wedge \forall x (\neg p(x,x)).$$

Third, standardize variables apart, which results in:

$$\exists y \forall x (p(x,y)) \wedge \forall u \exists v (\neg p(u,v) \vee p(v,u)) \wedge \forall z (\neg p(z,z)).$$

Fourth, skolemize and eliminate existential quantifiers, which results in:

$$\forall x (p(x,c)) \wedge \forall u (\neg p(u, f(u)) \vee p(f(u), u)) \wedge \forall z (\neg p(z,z)).$$

Fifth, drop universal quantifiers:

$$p(x,c) \wedge (\neg p(u, f(u)) \vee p(f(u), u)) \wedge \neg p(z,z).$$

There is no need to distribute $\vee$ over $\wedge$, so we are done.

(b) [2 pt] We obtained three clauses in the previous step: $p(x,c)$, $\neg p(u, f(u)) \vee p(f(u), u)$, $\neg p(z,z)$. From the first, and third, by using substitution $\theta = \{x/z, z/c\}$ we immediately obtain the empty clause and we are done.

Grade = # points/2 + 1