1. Functions

Functions, as we defined, are a special case of relations. Recall that a binary relation $R \subseteq S_1 \times S_2$ is a (partial) function if for each $x \in S_1$, there exists (at most) one $y \in S_2$ such that $(x, y) \in R$.

Predomain, codomain, domain and range.
For a functional relation $R_f \subseteq S_1 \times S_2$, the set S_1 is called the _pre-domain_ and S_2 is called the _co-domain_ of the function. We usually write $f : S_1 \to S_2$ instead of $R_f \subseteq S_1 \times S_2$ and write $y = f(x)$ when $(x, y) \in R_f$. We write $predom(f) = S_1$ and $codom(f) = S_2$; the relation R_f is often referred to as the "_extension_" or _graph_ of the function f.

A partial function need not have a value $y = f(x)$ for each $x \in S_1$.
We say that function f is defined at $x \in S_1$ if there exists a $y \in S_2$ such that $(x, y) \in R_f$. The _domain_ of a partial function $f : S_1 \to S_2$ is defined as $dom(f) = \{x \in S_1 \ \exists y \in S_2 : y = f(x)\}$, i.e., the elements $x \in S_1$ for which $f(x) \in S_2$ is defined. Note that $dom(f) \subseteq S_1 = predom(f)$.
The _range_ of a partial function $f : S_1 \to S_2$ is defined as $range(f) = \{y \in S_2 \ \exists x \in S_1 : y = f(x)\}$, i.e., the elements $y \in S_2$ such that $y = f(x)$ for some $x \in S_1$. Note that $range(f) \subseteq S_2 = codom(f)$..

Total function. If the functional relation R_f is _total_, i.e., for each $x \in S_1$ there exists a $y \in S_2$ such that $y = f(x)$, we call f a total function and $dom(f) = S_1 = predom(f)$.

Mappings and notation. We will use the notation of mappings "$x \mapsto y$" to indicate that the element $x \in S_1$ is mapped by a function $f : S_1 \to S_2$ to the element $y \in S_2$. Note that the "mapping" arrow "$\mapsto$" tells us how _elements_ are mapped by a function, and _notice the little vertical bar on the left of the arrow_. In contrast, $S_1 \to S_2$ is a _set-level construction_ that depicts a functional relation between S_1, S_2.

Function Restriction. Suppose $f : S_1 \to S_2$ is a function. Let $A \subseteq S_1$. Then the restriction of function f to (pre)domain A, written as f_A is defined as the mapping that associates $f(x)$ to each element $x \in A$. ($f_A(x) = f(x)$ for all $x \in A$). That is f_A behaves just as function f would but takes as its predomain only the set A.

Onto. If a function f (namely, its functional relation R_f) is _onto_, i.e., for each $y \in S_2$ there exists an $x \in S_1$ such that $y = f(x)$, then we call f an _onto (surjective, epi)_ function and $range(f) = S_2 = codom(f)$.

(1-1 or injection). A partial function $f : S_1 \to S_2$ is called _1-1 (injective, mono)_ if whenever $f(x_1) = f(x_2) \in S_2$, then $x_1 = x_2 \in S_1$.

(Cardinality) The _cardinality_ of set S_2 (written $|S_2|$) is at least as great as that of set S_1 (i.e., $|S_1| \leq |S_2|$) if there exists a <u>1-1 total function</u> $f : S_1 \to S_2$.

Exercise: If $S_1 \subseteq S_2$ then $\;S_1 \;\leq\; S_2\;$.

Exercise. $\;\; \mathbf{0} \;\leq\; S\;$ for any set S.

(Bijection) A function $f : S_1 \rightarrow S_2$ is called a _bijection_ if it is both _1-1 and onto_. Note that a bijection has a natural inverse _function_, written f^- or more commonly f^{-1}, whose graph is defined relationally as the symmetric inverse R_f^- of relation R_f.

Exercise: Show that the _identity_ function on any set $id_S : S \rightarrow S$ defined by the mapping "$x \mapsto x$" is (i) total; (ii) 1-1; (iii) onto and (iv) a bijection.

Two sets S_1, S_2 are said to be in bijection with each other (written $S_1 \cong S_2$) if there exists a bijective function $f : S_1 \rightarrow S_2$. Note that if $S_1 \cong S_2$, then $\;S_1 \;=\; S_2$

Exercise. Recall the set $\mathbf{1}$, with its canonical element written as ().
[Note: this is the empty tuple, and resembles writing a 0]

Show that $S \times \mathbf{1} \cong S \cong \mathbf{1} \times S$

Natural numbers and **Peano's Axioms:**
The set of Natural _numbers_ $\mathbb{N}$ is an infinite set defined inductively in terms of two constructors: (zero) 0 and (successor) $s(\_)$. Natural numbers can be defined by the following Peano's Axioms.

1. (Base constant Zero) $0 \in \mathbb{N}$.
2. (Equality-Reflexive) For all $x \in \mathbb{N}, x = x$.
3. (Equality-Symmetric) For all $x, y \in \mathbb{N}$, if $x = y$ then $y = x$
4. (Equality-Transitive) For all $x, y, z \in \mathbb{N}$, if $x = y$ and $y = z$ then $x = z$
5. (Closure-Equality) For all x, y, if $x \in \mathbb{N}$, and $x = y$ then $y \in \mathbb{N}$.
6. (Closure-Successor) For all $x \in \mathbb{N}, s(x) \in \mathbb{N}$.
7. (Successor-Injective) For all $x, y \in \mathbb{N}$, if $s(x) = s(y)$ then $x = y$.
8. (Zero-not-Successor) For all $x \in \mathbb{N}, s(x) \neq 0$.
9. (Induction-Set) Let $A \subseteq \mathbb{N}$. **If** (i) $0 \in A$; and (ii) for all $x \in \mathbb{N}$: if $x \in A$ then $s(x) \in A$; **then** $A = \mathbb{N}$.

(9') (Induction-Pred) For _any_ unary predicate P: **If** (i) $P(0)$; and (ii) for all $x \in \mathbb{N}$: if $P(x)$ implies $P(s(x))$; **then** for all $x \in \mathbb{N}$: $P(x)$.

Axioms 1 and 6 define the constructors, Axioms 2-5 characterise equality and closure of the set with respect to it, and Axiom 7 and 8 define (non)equality properties of the constructors. Axiom 9 is an induction scheme. (Axiom 9') is an alternative way to state Axiom 9 on predicates.

Finite and Denumerable Sets.

A prefix of cardinality n of the natural numbers $\mathbb{N}$ is defined as the set
$\mathbb{N}_n = \{0, 1, \ldots (n-1)\}$.

Let $N' \subseteq \mathbb{N}$. A set A is called _denumerable_ or _countable_ if there exists a 1-1 total function $f : A \rightarrow N'$, i.e., there is an injective mapping _from all of A to_ a _subset_ of the _natural numbers._

A _finite_ set is a set whose cardinality is equal to some $n \in \mathbb{N}$. Clearly every finite set of size $n \in \mathbb{N}$ has a 1-1 total function mapping to $\mathbb{N}_n$.

> Note that _every_ natural number is finite, but the _set of natural numbers_ $\mathbb{N}$ is denumerably infinite (countably infinite). $\mathbb{N}$ is not in bijection with any $\mathbb{N}_n$ for $n \in \mathbb{N}$.

Every finite set is trivially denumerable.

Finite-domain functions. A function $f : S_1 \rightarrow S_2$ is called a _finite-domain function_ if its domain is _finite_. That is $dom(f)$ is a finite subset of S_1. We write $f : S_1 \rightarrow_{fin} S_2$ to highlight $dom(f)$ is finite.

Function composition is a special case of relational composition where the relations involved are (partial) functions: Suppose $f : S_1 \rightarrow S_2$, and $g : S_2 \rightarrow S_3$ are (partial) functions. Then their composition, which we will write as $f;g : S_1 \rightarrow S_3$ is defined as the mapping $x \mapsto g(f(x))$ provided element $g(f(x)) \in S_3$ is defined (that is, there must exist $y = f(x) \in S_2$, and $z = g(y) \in S_3$).

Exercise: Show that function composition is associative, i.e., if $f : S_1 \rightarrow S_2$, $g : S_2 \rightarrow S_3$ and $h : S_3 \rightarrow S_4$, then $f;(g;h) = (f;g);h : S_1 \rightarrow S_4$.

Exercise: Show that $dom(f;g) = \{x \in dom(f) \ f(x) \in dom(g)\}$ and $range(f;g) = \{g(y) \in range(g) \ y \in range(f) \cap dom(g)\}$.

Exercise: Show that total functions are closed under composition. That is, if $f : S_1 \rightarrow S_2$ and $g : S_2 \rightarrow S_3$ are total functions, then so is $f;g : S_1 \rightarrow S_3$.

Exercise: Show that 1-1 functions are closed under composition. That is, if $f : S_1 \rightarrow S_2$ and $g : S_2 \rightarrow S_3$ are 1-1 functions, then so is $f;g : S_1 \rightarrow S_3$.

Exercise: Show that onto functions are closed under composition. That is, if $f : S_1 \rightarrow S_2$ and $g : S_2 \rightarrow S_3$ are onto functions, then so is $f;g : S_1 \rightarrow S_3$.

Exercise: Show that bijections are closed under composition. That is, if $f : S_1 \rightarrow S_2$ and $g : S_2 \rightarrow S_3$ are bijections, then so is $f;g : S_1 \rightarrow S_3$.

Exercise: Show that finite-domain functions are closed under composition. That is, if $f : S_1 \rightarrow_{fin} S_2$ and $g : S_2 \rightarrow_{fin} S_3$ are finite-domain functions, then so is $f;g : S_1 \rightarrow_{fin} S_3$.

Some denumerable sets

Proposition: (i) The set of even natural numbers is denumerable but not finite.
(ii) The set of odd natural numbers is denumerable but not finite.
(iii) The cardinalities of the even naturals (respectively, odd naturals) is the same as that of $\mathbb{N}$.

Proof hint: (Trivial) Define a total 1-1 function from the even (respectively odd) natural numbers to a subset of the naturals. (More interesting) Also show that the cardinality of these sets is at least that of the natural numbers by showing a total 1-1 function from the naturals to each of them.

Proposition: The set of integers $\mathbb{Z}$ is denumerable (countably infinite).
Proof hint: Define a total 1-1 function from $\mathbb{Z}$ to (a subset of) $\mathbb{N}$.

Proposition: (First diagonal argument — Cauchy) The set of pairs of natural numbers $\mathbb{N} \times \mathbb{N}$ is denumerable (countably infinite).
Proof hint: Define a total 1-1 function from $\mathbb{N} \times \mathbb{N}$ to (a subset of) $\mathbb{N}$.

Proposition: (Corollary) The set of pairs of integers $\mathbb{Z} \times \mathbb{Z}$ is denumerable (countably infinite).

Proposition: (Corollary) The set of rational numbers $\mathbb{Q}$ is denumerable (countably infinite).
Proof: $\mathbb{Q} = \{m/n \ \ m, n \in \mathbb{Z} \land n > 0 \land m, n \text{ co-prime}\}$.
Consider $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ defined by the mapping (for $q \in \mathbb{Q} = m/n$), $q \mapsto (m, n)$.

Proposition: (Corollary) Let A be a denumerable set. Let us define
$$A^0 \cong \mathbf{1}$$
$$A^{s(n)} \cong A \times A^n$$
For each $n \in N$, the set A^n of lists over A of length n is a denumerable set.
Proof Hint: Proof by induction on $n \in N$.

Theorem (Cauchy): The denumerable union of denumerable sets is denumerable.

Proposition: (Corollary) Let A be a denumerable set. Let us define
$$A* = \bigcup_{n \geq 0} A^n$$
Then the set $A*$ of all lists (sequences) built over elements from A is a denumerable set.

Proposition: The cardinality $\mathbb{R}$ of the set of real numbers $\mathbb{R}$ is the same as that of any open interval (a, b), in particular $(0,1)$ or $(-\pi/2, \ \pi/2)$.

Theorem (Cantor: Second Diagonal Argument): The set of real numbers $\mathbb{R}$ is *not* denumerable.
Proof by Cantordiction.
We rely on the fact that if any subset of a $\mathbb{R}$ is *not* denumerable, then $\mathbb{R}$ is *not* denumerable.

(1) Assume $(0,1)$ is denumerable.

(2) Consider the set D of *all* non-terminating decimals in $(0,1)$. D is a subset of $(0,1)$, and so must be denumerable (a subset of a denumerable set is also denumerable). That is, there is an total 1-1 function $e : D \to \mathbb{N}$.

Therefore we can write down *all* the non-terminating decimal elements of $(0,1)$, i.e., the elements of D, in *some* enumerated order as a series of "rows":

$$r_0 = d_{00}\ d_{01}\ldots d_{0j}\ldots$$
$$r_1 = d_{10}\ d_{11}\ldots d_{1j}\ldots$$
$$\vdots\ \vdots\ \vdots\quad\ \vdots\ \ldots\ \vdots\ \ldots$$
$$r_i = d_{i0}\ d_{i1}\ldots\ d_{ij}\ldots$$
$$\vdots\ \vdots\ \vdots\quad\ \vdots\ \ldots\ \vdots\ \ldots$$

where each of the rows (r_i) represents a non-terminating decimal of the form $0.d_{i0}d_{i1}\ldots\ d_{ij}\ldots$ [Note $r_i \in (0,1)$, and if $e(x) = i$, then x is represented as the ith row r_i.]

(Note that all finite decimals, which have finitely many digits followed by an infinite number of 0's at the end can instead be represented by an indistinguishable form "in the limit"— with one less in the last non-zero digit and then an infinite series of 9's. For example, instead of 0.5 let us write $0.49999\ldots$)

(Note: we don't know — *and don't care* — what the order of enumeration is. It is not at all likely to be the usual $<$ ordering).

Now consider a number represented as $r' = r'_0 r'_1 \ldots r'_i \ldots$ such that for *each* $j \in \mathbb{N}$: $r'_j \neq d_{jj} \wedge r'_j \neq 0$. (The second conjunct is to ensure we do not get a terminating decimal). That is, the r'_j is different from the "diagonal digit" d_{jj}. [Note: There are many many such choices for r'.]

Claim: $r' \in D$. By construction, $r' = r'_0 r'_1 \ldots r'_i \ldots$ is a non-terminating decimal in $(0,1)$.

Therefore by (2), $r' = r_k$ for some $k \in \mathbb{N}$, since D is assumed denumerable, and so $e(r') = k$ for some $k \in \mathbb{N}$.

But $r' \neq r_k$ because its kth digit $r'_k \neq d_{kk}$, which is k^{th} digit of the r_k. (Note that by construction, for each $j \in \mathbb{N}$, the jth digit $r'_j \neq d_{jj}$).

Therefore $r' \in D$ but $e(r') \notin \mathbb{N}$. So e is either not total, or not 1-1.

So D is not denumerable. This contradicts (2), and thus contradicts Assumption (1). So $\mathbb{R}$ is not denumerable.