Exercises

1.1. For the given matrix/vector pairs, compute the following quantities: a_{ii}, $a_{ij}a_{ij}$, $a_{ij}a_{jk}$, $a_{ij}b_j$, $a_{ij}b_ib_j$, b_jb_j, b_ib_i. For each cases, point out whether the result is a scalar, vector or metrix.

a) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$a_{ii} = a_{11} + a_{22} + a_{33} = 1+4+1 = 6$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32}$

$+ a_{33}a_{33}$

$= 1+1+1+0+16+4+0+1+1 = 25$ (scalar)

$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix}$ (matrix)

$a_{ij}b_j = a_{11}b_1 + a_{12}b_2 + a_{13}b_3$

$a_{21}b_1 + a_{22}b_2 + a_{23}b_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ (vector.

$a_{31}b_1 + a_{32}b_2 + a_{33}b_3$

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3$

$+ a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 1+0+2+0+0+0+0+0+4 = 7$ (scalar)

$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_3 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ Matrix.

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3$

$= 1+0+4 = 5$ (Scalar)

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b) $a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$ $b_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$a_{ii} = a_{11} + a_{22} + a_{33} = 1+2+2 = 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1+4+0+0+4+1+0+16+4 = 30$ (scalar)

$a_{ij}a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix}$ (Matrix)

$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$ (Vector)

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3$

$a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 4+4+0+0+2+1+0+4+2 = 17$ (scalar)

$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_3 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ (matrix)

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 4+1+1 = 6$ (scalar

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c) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ $\quad b_i = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$a_{ii} = a_{11} + a_{22} + a_{33} = 1+0+4 = 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1+1+1+1+0+4+0+1+16 = 25$ (Scalar)

$b_i b_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (Matrix)

$b_i b_i = b_1b_1 + b_2b_2 + b_3b_3$

$= 1 + 1 + 0 = 2$ (scalar)

1.2 Use the decomposition result (1.2.10) to express a_{ij} from Exercise 1.1 in terms of the sum of symmetric and antisymmetric matrixes. Verify that $a_{(ij)}$ and $a_{[ij]}$ satisfy the conditions given in the last paragraph of section 1.2

a) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$.

$= \frac{1}{2}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix}\right)$

$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate condition.

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b) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions.

c) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$.

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate condition.

1.3 If a_{ij} is symmetric and b_{ij} is antisymmetric, prov in general that the product $a_{ij} b_{ij}$ is zero. Verify the result for the specific case by using the symmetric and antisymmetric terms from Exercise 1-2.

$$a_{ij} a_{ij} = -a_{ji} b_{ji} = -a_{ij} b_{ij} \Rightarrow 2a_{ij} b_{ij} = 0 \Rightarrow a_{ij} b_{ij} = 0$$

a) $$a_{(ij)} a_{[ij]} = \frac{1}{4} tr\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$$

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b) $a_{(ij)} a_{[ij]} = \frac{1}{4} tr\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T \right) = 0$

c) $a_{(ij)} a_{[ij]} = \frac{1}{4} tr\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T \right) = 0.$

1-4. Explicitly verify the following properties of the Kronecker delta.

$\delta_{ij} a_j = a_i$

$\delta_{ij} a_{jk} = a_{ik}.$

$$\delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3 = \begin{bmatrix} \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 \\ \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 \\ \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

$$\delta_{ij} a_{jk} = \begin{bmatrix} \delta_{11} a_{11} + \delta_{12} a_{21} + \delta_{13} a_{31} & \delta_{11} a_{12} + \delta_{12} a_{22} + \delta_{13} a_{32} & \delta_{11} a_{13} + \delta_{12} a_{23} + \delta_{13} a_{33} \\ \delta_{21} a_{11} + \delta_{22} a_{21} + \delta_{23} a_{31} & \delta_{21} a_{12} + \delta_{22} a_{22} + \delta_{23} a_{32} & \delta_{21} a_{13} + \delta_{22} a_{23} + \delta_{23} a_{33} \\ \delta_{31} a_{11} + \delta_{32} a_{21} + \delta_{33} a_{31} & \delta_{31} a_{12} + \delta_{32} a_{22} + \delta_{33} a_{23} & \delta_{31} a_{13} + \delta_{32} a_{23} + \delta_{33} a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{jk}.$$

1-5 Formally expand the ~~expansion~~ expression (1.3.4) for the determinant and justify that either index notation form yield a result that matches the traditional form for det $[a_{ij}]$.

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$$\det(a_{ij}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{123} a_{11} a_{22} a_{33} + \varepsilon_{231} a_{12} a_{23} a_{31} + \varepsilon_{312} a_{13} a_{21} a_{32}$$
$$+ \varepsilon_{321} a_{13} a_{22} a_{31} + \varepsilon_{132} a_{11} a_{23} a_{32} + \varepsilon_{213} a_{12} a_{21} a_{33}$$
$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}$$
$$= a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{12}(a_{21} a_{33} - a_{23} a_{31}) + a_{13}(a_{21} a_{32} - a_{22} a_{31})$$
$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

1-6 Determine the component of vector b_i and matrix a_{ij} given in Exercise 1-1 in a new coordinate system found through a rotation of 45 ($\pi/4$ radians) about the x_1-axis. The rotation direction follows the positive sense presented in Example 1-2.

45° rotation about x_1-axis $\Rightarrow Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

From Exercise 1-1 (a) $b'_i = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$
$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

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From exercise 1-1 (b) $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

From exercise 1-1 (c) : $b'_i = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 3.5 & 2.5 \\ -\sqrt{2}/2 & 1.5 & 0.5 \end{bmatrix}$$

1-8 Show that the second-order tensor $a\delta_{ij}$, where a is an arbitrary constant, retains its form under any transformation Q_{ij}. This form is then an isotropic second order tensor.

$$a'\delta'_{ij} = Q_{ip} Q_{jp} a \delta_{pq} = a Q_{ip} Q_{jp} \delta = a \delta_{ij}.$$

1-7 Consider the two dimensional coordinate transformation show in fig 1-7. Through the counter clockwise rotation θ, a new polar coordinate system is created. Show that the transformation matrix for this case is given by.

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$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

If $b_i = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $a_{ij} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ are the components of first and second order tensor in the x_1, x_2 system, Calculate their components in the rotated polar coordinate System.

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90-\theta) \\ \cos(90+\theta) & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_1\sin\theta + b_2\cos\theta \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta & a_{12}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta \\ a_{21}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{12}\sin^2\theta & a_{11}\sin^2\theta - (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta \end{bmatrix}$$

1-9 The most general form of 4th order isotropic tensor can be expressed by

$$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

where $\alpha, \beta, \& \gamma$ are arbitrary constants. Verify that this form remains the same under the general transformation given by (1.5.1)$_5$.

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$$\alpha' \delta'_{ij} \delta'_{kl} + \beta' \delta'_{ik} \delta'_{jl} + \gamma' \delta'_{il} \delta'_{jk}$$

$$= Q_{im} Q_{jn} Q_{kp} Q_{lq} \left(\alpha \delta_{mn} \delta_{pq} + \beta \delta_{mp} \delta_{nq} + \gamma \delta_{mq} \delta_{np} \right)$$

$$= \alpha Q_{im} Q_{jm} Q_{kp} Q_{lp} + \beta Q_{in} Q_{jn} Q_{km} Q_{lm} + \gamma Q_{in} Q_{jn} Q_{kn} Q_{ln}$$

$$= \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}.$$

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Question no 10:

For the fourth-order isotropic tensor given in ex 1-9, show that if $\beta = \gamma$, then the tensor will have the following symmetry $C_{ijkl} = C_{klij}$

$$C_{ijkl} = \alpha \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk}$$

$$= \alpha \delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$= \alpha \delta_{kl}\delta_{ij} + \beta(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li})$$

$$= C_{klij}$$

Question no 11

Show that the fundamental invariants can be expressed in terms of principal values as given by relation (16.5)

If $a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3$$

Question no 12:

Determine the invariants and principal values and directions of the following matrices. Use the determine principal directions to establish a

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principal coordinate system, and following the procedures in example 1.3 .....

(a) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I_a = -1, \; II_a = -2, \; III_a = 0$$

Characteristic eq'n is: $-\lambda^3 - \lambda^2 + 2\lambda = 0 \Rightarrow \lambda(\lambda^2 + \lambda - 2) = 0.$

$$\lambda(\lambda+2)(\lambda-1) = 0.$$

Roots are; ~~$\lambda_1 = 0$~~, $\lambda_3 = 1$, $\lambda_2 = 0$, $\lambda_1 = -2$.

For $\lambda_1 = -2$,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{array}{l} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1 \end{array}$$

$$\Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2, \; n^{(1)} = \pm(\sqrt{2}/2)(-1, 1, 0)$$

For $\lambda_2 = 0$ Case:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0, \\ n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} = 1 \end{array}$$

$$\Rightarrow n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow n^{(2)} = \pm(\sqrt{2}/2)(1, 1, 0)$$

For $\lambda_3 = 1$;

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \\ n_1^{(3)2} + n_2^{(3)2} + n_3^{(3)2} = 1 \end{array} \Rightarrow n_1 = n_2 = 0, \; n_3^{(3)} = 1$$

$$\Rightarrow n^{(3)} = \pm(0, 0, 1)$$

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The rotation matrix is given by $Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$ and

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -4,\ II_a = 3,\ III_a = 0.$

$$a_{ij} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Characteristic Eqn is $-\lambda^3 - 4\lambda^2 - 3\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 + 4\lambda + 3) = 0 \Rightarrow \lambda(\lambda+3)(\lambda+1) = 0.$$

Roots are: $\lambda_1 = -3,\ \lambda_2 = -1,\ \lambda_3 = 0.$

$\rightarrow$ For $\lambda_1 = -3$; Case;

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{array}{l} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1 \end{array}$$

$$\Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2,\ n^{(1)} = \pm(\sqrt{2}/2)(-1, 1, 0)$$

$\rightarrow \lambda_2 = -1$, Case:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0 \\ n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} = 1. \end{array}$$

$$\Rightarrow n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow n^{(2)} = \pm(\sqrt{2}/2)(1, 1, 0)$$

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$\longrightarrow \lambda_3 = 0$ Case:

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \\ n_1^{(3)^2} + n_2^{(3)^2} + n_3^{(3)^2} = 1 \end{array}$$

$$\Rightarrow n_1 = n_2 = 0,\ n_3^{(3)} = 1 \Rightarrow n^{(3)} = \pm(0,0,1)$$

The rotation matrix is given by $Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$

and $a'_{ij} = Q_{ip}Q_{jp}a_{pq} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$I_a = -2,\ II_a = 0,\ III_a = 0;$

$\therefore$ Characteristic Eqn is $-\lambda^3 - 2\lambda^2 = 0$ or $\lambda^2(\lambda+2) = 0$

Roots are: $\lambda_1 = -2,\ \lambda_2 = \lambda_3 = 0.$

For $\lambda_1 = -2$; Case:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{array}{l} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)^2} + n_2^{(1)^2} + n_3^{(1)^2} = 1 \end{array}$$

$$\Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2;\ n^{(1)} = \pm\sqrt{2}/2\,(-1,1,0)$$

For $\lambda_2 = \lambda_3 = 0$; Case:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -n_1 + n_2 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1 \end{array} \Rightarrow n_1 = n_2,\ n_3^2 = 1 - 2n_1^2$$

$$\Rightarrow n = \pm(k, k, \sqrt{1-2k^2})$$

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for arbitrary k, and thus directions are not uniquely determined. For convenience we may choose $k = \sqrt{2}/2$ and 0 to get $n^{(2)} = \pm\sqrt{2}/2\,(1,1,0)$ and

$$n^{(3)} = \pm(0,0,1)$$

The rotation matrix is given by $Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$

and $$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 14:

Calculate the quantities $\nabla \cdot u$, $\nabla \times u$, $\nabla^2 u$, ∇u, $tr(\nabla u)$ for the following cartesian vector fields:

(a) $u = x_1 e_1 + x_1 x_2 e_2 + 2x_1 x_2 x_3 e_3$

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3} = 1 + x_1 + 2x_1 x_2$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x & x_1 x_2 & 2x_1 x_2 x_3 \end{vmatrix} = 2x_1 x_3 e_1 - 2x_2 x_3 e_2 + x_2 e_3$$

$$\nabla^2 u = 0e_1 + 0e_2 + 0e_3 = 0.$$

$$\nabla u = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2 x_3 & 2x_1 x_3 & 2x_1 x_2 \end{bmatrix}, \quad tr(\nabla u) = 1 + x_1 + 2x_1 x_2.$$

(b) $u = x_1^2 e_1 + 2x_1 x_2 e_2 + x_3^3 e_3$.

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3} = 2x_1 + 2x_1 + 3x_3^2$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1 x_2 & x_3^3 \end{vmatrix} = 0e_1 - 0e_2 + 2x_2 e_3$$

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$$\nabla^2 u = 2e_1 + 0e_2 + 6x_3 e_3 = 0.$$

$$\nabla u = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix}, \quad tr(\nabla u) = 4x_1 + 3x_3^2.$$

(c) $u = x_2^2 e_1 + 2x_2 x_3 e_2 + 4x_1^2 e_3.$

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3} = 0 + 2x_3 + 0$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2 x_3 & 4x_1^2 \end{vmatrix} = -2x_2 e_1 - 8x_1 e_2 - 2x_2 e_3.$$

$$\nabla^2 u = 2e_1 + 0e_2 + 8e_3 = 0.$$

$$\nabla u = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix}, \quad tr(\nabla u) = 3x_3$$

Q-15:

The dual vector a_i of an antisymmetric second-order tensor a_{ij} is defined by $a_i = -1/2\,\varepsilon_{ijk} a_{jk}$. Show that this expression can be inverted to get $a_{jk} = -\varepsilon_{ijk} a_i$.

$$a_i = \frac{-1}{2} \varepsilon_{ijk} a_{jk}.$$

$$\varepsilon_{imn} a_i = \frac{-1}{2} \varepsilon_{ijk} \cdot \varepsilon_{imn} a_{jk}$$

$$= \frac{-1}{2} \begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} a_{jk} = -\frac{1}{2}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_{jk}$$

$$= \frac{-1}{2}(a_{mn} - a_{nm}) = \frac{-1}{2}(a_{mn} + a_{mn}) = -a_{mn}.$$

$$\therefore a_{jk} = -\varepsilon_{ijk} a_i$$

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Ques 16:

Using index notation, explicitly verify the vector identities:

(a) $(1.8.5)_{1,2,3}$

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k} = \phi\psi_{,kk} + \phi_{,k}\psi_{,k}$$

$$+ \phi_{,k}\psi_{,k} + \phi_{,kk}\psi$$

$$= \phi_{,kk}\psi + \phi\psi_{,kk}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot(\phi u) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k}u_k = \nabla\phi\cdot u + \phi(\nabla\cdot u)$$

(b) $(1.8.5)_{4,5,6,7}$

$$\nabla\times(\phi u) = \varepsilon_{ijk}(\phi u_k)_{,j} = \varepsilon_{ijk}(\phi u_{k,j} + \phi_{,j}u_k) = \varepsilon_{ijk}\phi_{,j}u_k + \phi\,\varepsilon_{ijk}u_{k}$$

$$= \nabla\phi\times u + \phi(\nabla\times u)$$

$$\nabla\cdot(u\times v) = (\varepsilon_{ijk}u_j v_k)_{,i} = \varepsilon_{ijk}(u_j v_{k,i} + u_{j,i}v_k) =$$

$$v_k\varepsilon_{ijk}u_{j,i} + u_j\varepsilon_{ijk}v_{k,i} = \nabla\cdot(\nabla\times u) - u\cdot(\nabla\times v)$$

$$\nabla\times\nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0 \text{ because of symmetry}$$

& antisymmetry in jk.

$$\nabla\cdot\nabla\phi = (\phi_{,k})_k = \phi_{,kk} = \nabla^2\phi.$$

(c) $(1.8.5)_{8,9,10}$

$$\nabla\cdot(\nabla\times u) = (\varepsilon_{ijk}u_{k,j})_{,i} = \varepsilon_{ijk}u_{k,ji} = 0, \text{ because of symmetry}$$

& antisymmetry in ij

$$\nabla\times(\nabla\times u) = \varepsilon_{mni}(\varepsilon_{ijk}u_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk}u_{k,jn} =$$

$$(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})u_{k,jn} = u_{n,nm} - u_{m,nn}$$

$$= \nabla(\nabla\cdot u) - \nabla^2 u.$$

$$u\times(\nabla\times u) = \varepsilon_{ijk}u_j(\varepsilon_{kmn}u_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn}u_j u_{n,m} =$$

$$(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_j u_{n,m} = u_n u_{n,i} - u_m u_{i,m}$$

$$= \frac{1}{2}\nabla(u\cdot u) - u\cdot\nabla u$$

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Question 17:

Extend the results found in examp 1-5, and determine the forms of ∇f, $\nabla \cdot u$, $\nabla^2 f$, and $\nabla \times u$ for a three dimensional cylinderical coordinate system:

Cylinderical coordinates; $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = z$.

$$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2 \Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = 1.$$

$$\hat{e}_r = \cos\theta\, e_1 + \sin\theta\, e_2,\quad \hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2,\quad \hat{e}_z = e_3.$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta,\quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r,\quad \frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \frac{\partial \hat{e}_z}{\partial \theta} = \frac{\partial \hat{e}_z}{\partial z} = 0.$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}.$$

$$\nabla f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_z \frac{\partial f}{\partial z}.$$

$$\nabla \cdot u = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

$$\nabla \times u = \left[\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right]\hat{e}_r + \left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right]\hat{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right]\hat{e}_z$$

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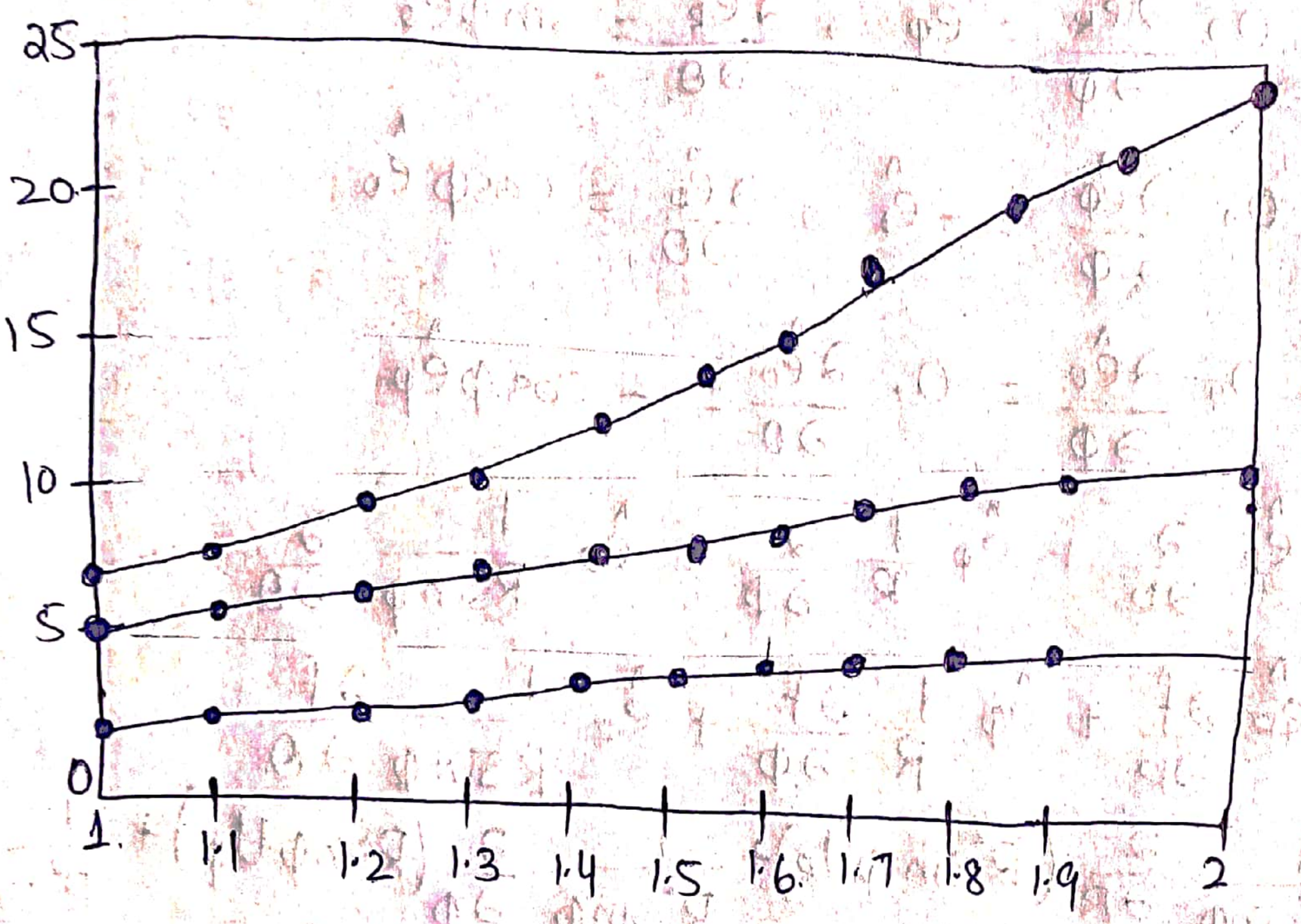
Ques:

A second-order symmetric field is given by.

$$a_{ij} = \begin{bmatrix} 2x_1 & x_1 & 0 \\ x_1 & -6x_1^2 & 0 \\ 0 & 0 & 5x_1 \end{bmatrix}$$

Using Matlab (or similar software), investigate the nature of variation of principal values and directions over the interval $1 \le x_1 \le 2$. Formally Plot the variation of absolute value of each principal value over the range $1 \le x_1 \le 2$.



Question no 18:

For the spherical coordinate system (R, ϕ, θ) in Fig 1-6, show that

$$h_1 = 1, \; h_2 = R, \; h_3 = R\sin\phi$$

Spherical coordinates: $\xi^1 = R, \; \xi^2 = \phi, \; \xi^3 = \theta$

$$x^1 = \xi^1 \sin\xi^2 \cos\xi^3, \; x^2 = \xi^1 \sin\xi^2 \sin\xi^3, \; x^3 = \xi^1 \cos\xi^2.$$

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Scale factors;

$$(h_1)^2 = \frac{\partial x^k \partial x^k}{\partial \xi^1 \partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1 \Rightarrow h_1 = 1$$

$$(h_2)^2 = \frac{\partial x^k \partial x^k}{\partial \xi^2 \partial \xi^2} = R^2 \Rightarrow h_2 = R.$$

$$(h_3)^2 = \frac{\partial x^k \partial x^k}{\partial \xi^3 \partial \xi^3} = R^2\sin^2\phi \Rightarrow h_3 = R\sin\phi$$

Unit vectors;

$$\hat{e}_R = \cos\theta\sin\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\phi\, e_3.$$

$$\hat{e}_\phi = \cos\theta\cos\phi\, e_1 + \sin\theta\cos\phi\, e_2 - \sin\phi\, e_3.$$

$$\hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2.$$

$$\frac{\partial \hat{e}_R}{\partial R} = 0, \quad \frac{\partial \hat{e}_R}{\partial \phi} = \hat{e}_\phi, \quad \frac{\partial \hat{e}_R}{\partial \theta} = \sin\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\phi}{\partial R} = 0, \quad \frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_1, \quad \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\theta}{\partial R} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \phi} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\phi\, \hat{e}_\phi$$

$$\Rightarrow \nabla = \hat{e}_R \frac{\partial}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}.$$

$$\nabla f = \hat{e}_R \frac{\partial f}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial f}{\partial \phi} + \hat{e}_z \frac{1}{R\sin\phi}\frac{\partial f}{\partial \theta}$$

$$\nabla \cdot u = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}(R^2\sin\phi\, u_R) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}(R\sin\phi\, u_\phi) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}(R u_\theta)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\, u_\phi) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}(u_\theta)$$

$$\nabla^2 f = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left[R^2\sin\phi\frac{\partial f}{\partial R}\right] + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left[R^2\frac{\partial f}{\partial R}\right] + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}.$$

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$$\nabla \times u = \left[\frac{1}{R^2 \sin\phi}\left[\frac{\partial}{\partial\phi}(R\sin\phi u_\theta) - \frac{\partial}{\partial\theta}(R u_\phi)\right]\right]\hat{e}_R + \left[\frac{1}{R\sin\phi}\left[\frac{\partial}{\partial\theta}(u_R) - \frac{\partial}{\partial R}(R\sin\phi u_\theta)\right]\right]\hat{e}_\phi + \left[\frac{1}{R}\frac{\partial}{\partial R}\left[(R u_\phi) - \frac{\partial}{\partial\phi}(u_R)\right]\right]\hat{e}_\theta$$

$$= \left[\frac{1}{R\sin\phi}\left(\frac{\partial}{\partial\phi}(\sin\phi\, u_\theta) - \frac{\partial u_\phi}{\partial\theta}\right)\right]\hat{e}_R + \left[\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial\theta} - \frac{1}{R}\frac{\partial}{\partial R}(R u_\theta)\right]\hat{e}_\phi + \left[\frac{1}{R}\left(\frac{\partial}{\partial R}(R u_\phi) - \frac{\partial u_R}{\partial\phi}\right)\right]\hat{e}_\theta.$$

Question:

Transform strain-displacement from cartesian to cylinderical coordinates:

Cylinderical coordinates

$u_x = u_r\cos\theta - u_\theta\sin\theta$

$u_y = u_r\sin\theta + u_\theta\cos\theta$

$u_z = u_z$

$\because$ Strain - displacement relation; $e = \frac{1}{2}(\nabla u + (\nabla u)^T)$; e = strain; ∇u = displacement.

$\rightarrow$ derivatives of $x = r\cos\theta$; $y = r\sin\theta$; $z = z$ where.

$r = \sqrt{x^2+y^2}$; $\theta = \arctan(y/x)$ is given by

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}.$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

It follows that

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right).$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \left(\cos\theta\frac{\partial}{\partial r}\right)\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r}\right)$$

$$= \cos^2\theta\frac{\partial}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \cos\theta\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)$$

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$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \cos\theta\sin\theta\left[\frac{-1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right]$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r}.$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta} -$$

$$\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \sin^2\theta\left[\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] + 2\sin\theta\cos\theta\left[\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right]$$

Likewise,

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial r^2} + \cos^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) - 2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)$$

Finally we can determine the expression;

$$e_{xx} = \frac{\partial u_x}{\partial x} = \cos\theta\frac{\partial}{\partial r}(u_r\cos\theta - u_\theta\sin\theta) - \frac{\sin\theta}{r}\left(\frac{\partial}{\partial\theta}(u_r\cos\theta - u_\theta\sin\theta)\right)$$

$$= \frac{\partial u_r}{\partial r}\cos^2\theta - \frac{\partial u_\theta}{\partial r}\sin\theta\cos\theta - \frac{\partial u_r}{\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{u_r}{r}\sin^2\theta + \frac{\partial u_\theta}{\partial\theta}\frac{\sin^2\theta}{r}$$

$$+ \frac{u_\theta}{r}\sin\theta\cos\theta.$$

$$= \frac{\partial u_r}{\partial r}\cos^2\theta + \left(\frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\sin\theta\cos\theta + \left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\sin^2\theta.$$

$$e_{yy} = \frac{\partial u_y}{\partial y} = \sin\theta\frac{\partial}{\partial r}(u_r\sin\theta + u_\theta\cos\theta) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}(u_r\sin\theta + u_\theta\cos\theta).$$

Using the same method to obtain expressions for e_{xy}-e_{xy}

$$e_{xy} = 2\left(\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x}\right)$$

Hence we obtain.

$$e_{rr} = \frac{\partial u_r}{\partial r}\ ;\ e_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right)\ ;\ e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

and $e_{zz} = \dfrac{\partial u_z}{\partial z}$.

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Question:

Verify that the alternator e_{ijk} has the property that

$$e_{ijk} = Q_{ip} Q_{jq} Q_{kr} \, e_{pqr} \text{ for all proper}$$

orthogonal matrices $[Q]$,

but that more generally

$$e_{ijk} \neq Q_{ip} Q_{jq} Q_{kr} \, e_{pqr} \text{ for all orthogonal matrices } [Q].$$

Note from this that the alternator is not an isotropic 3-tensor.

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