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FA19-BCS-015
Assignment No: 2
Linear Algebra.

Q1:- what is determinant of a Matrix?

The determinant is a special number that can be calculated from a matrix.
The matrix has to be square. The determinant helps us to find the inverse of a matrix, tells us things about the matrix that are useful in systems of linear equations, calculus and more.

For $n \geq 2$ the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{ij} \det A_{ij}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In Symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

QNo:2: What are the properties of a determinant? Explain each property with the help of an example?

Properties of Determinants

- Row Operations:

Let A be a square matrix:

1: If a multiple of one row of A is added to another row to produce a matrix B, then
$\det B = \det A$.

Example:-

let, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Add 2 times row 1 to row 2. This will give matrix

$$B = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$$

Now,

$$\det A = \det B$$
$$(4-6) = 8-10$$
$$-2 = -2 \quad \text{(Proved).}$$

2: If two rows of A are interchanged to produce B, then $\det B = -\det A$.

Example:-

let,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

after interchanging row 1 and row 2.

$$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$-\det A = +\det B$$
$$(4-6) = (6-4)$$
$$-2 = 2 \quad \text{(Proved)}$$

3. If one row of A is multiplied by K to produce B, then det B = K · det A.

Example:-

let, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

multiply row 1 by $K = 3$ to get B.

$B = \begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix}$

$$\det B = K \cdot \det(A)$$
$$((12) - 18) = 4 - 6$$
$$-6 = -2$$
$$3(-2)$$
$$3 \cdot \det A$$

(Proved)

- Column Operations:-

If A is an $n \times n$ matrix than $\det A^T = \det A$

Example:-

let, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$\det A^T = \det A$$
$$(1 \times 4) - (3 \times 2) = (1 \times 4) - (3 \times 2)$$
$$4 - 6 = 4 - 6$$
$$-2 = -2$$

(Proved)

- Multiplicative Property:-

If A and B are $n \times n$ matrix then $\det AB = (\det A)(\det(B))$.

Example:-

$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

$$\det AB = (\det A) \det(B)$$
$$(25 \times 13) - (20 \times 14) = (12 - 3)(8 - 3)$$
$$325 - 280 = 9 \times 5$$
$$45 = 45 \quad \text{(Proved)}$$

- A linearity property of a determinant:-

Suppose $A = [a_1 \dots a_j \dots a_n]$ is an $n \times n$ matrix with column a_1 a_j a_n. Suppose one column say a_j is replaced by a variable column x from $\mathbb{R}^n$. let, $T(x) = \det[a_1\ x\ a_n]$. $T(x)$ is a number So, $x \rightarrow T(x)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^1$.

T is a linear transformation, this means:-

(i) $T(cx) = cT(x)$ for all x in $\mathbb{R}^n$ and all scalars c.

(ii) $T(x+y) = T(x) + T(y)$ for all x, y in $\mathbb{R}^n$

(ii) is true because of a property we already know about how factoring a number out of a column effects the determinant.

$$T(cx) = \det[a_1 .. cx ... a_n]$$
$$= c \det[a_1 \quad x \quad a_n]$$
$$= c\,T(x)$$

<u>Example:-</u>

let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ replace column 2 by ar variable vector x from $\mathbb{R}^3$.

$$T(x) = \det \begin{bmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 6 \\ 7 & x_3 & 9 \end{bmatrix}$$

multiplying a scalar with 2nd column.

$$T(x) = \det \begin{bmatrix} 1 & 2x_1 & 3 \\ 4 & 2x_2 & 6 \\ 7 & 2x_3 & 9 \end{bmatrix}$$
$$= 2\det \begin{bmatrix} 1 & x_1 & 3 \\ 4 & x_2 & 6 \\ 7 & x_3 & 9 \end{bmatrix}$$
$$= 2\,(T(x)) \quad \text{(Proved)}$$
$$= c\,(T(x))$$

(ii) is checked by expanding the determinants for $T(x+y)$, $T(x)$ and $T(y)$ down the j^{th} column.

Example:-

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$ replace column 2 by a ~~var~~ variable vector x from $\mathbb{R}^3$.

$$T(x) = \det \begin{bmatrix} 1 & x_1 & 7 \\ 2 & x_2 & 8 \\ 3 & x_3 & 9 \end{bmatrix}$$

Since, T is linear.

$$T(x+y) = \det \begin{bmatrix} 1 & x_1+y_1 & 7 \\ 2 & x_2+y_2 & 8 \\ 3 & x_3+y_3 & 9 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & x_1 & 7 \\ 2 & x_2 & 8 \\ 3 & x_3 & 9 \end{bmatrix} + \det \begin{bmatrix} 1 & y_1 & 7 \\ 2 & y_2 & 8 \\ 3 & y_3 & 9 \end{bmatrix}$$

$$= T(x) + T(y)$$ (Proved)