DS 288 (AUG) 3:0 Numerical Methods
Homework–6 [1]

`Due date: November 28, 2024 (Thursday); 10:00 A.M.`

1. Solve the initial-value problem
$$\frac{dy}{dt} = \frac{1}{t^2} - \frac{y}{t} - y^2 \tag{1}$$
on the interval $1 \leq t \leq 2$ subject to initial conditions $y(1)$ = -1 using Euler's method with step-sizes $\triangle t = 0.1(2^{-n})$ for n = 0, 1, 2, 3, and 4 and compare with the exact solution $y(t) = -1/t$. Specifically, for each step-size compare your computed value of $y(2)$ with its analytic counterpart. Repeat the same series of calculations for Modified Euler's method. Compare errors. Plot the absolute value of the error for each method versus $1/\triangle t$ on a log-log scale. Do the trends that you observe agree with our knowledge of the order of accuracy (as $\triangle t$ decreases) of these methods?. [`5 points`]

2. Consider a simple ecosystem of two species competing for the same food supply. The population of the two species can be modeled by the coupled pair of non-linear first-order differential equations:
$$\frac{dN_1}{dt} = N_1(A_1 - B_1 N_1 - C_1 N_2); \ \frac{dN_2}{dt} = N_2(A_2 - B_2 N_2 - C_2 N_1) \tag{2}$$
where t is time, $N_\alpha = N_\alpha(t)$ is the number of species α ($\alpha = 1$ or 2). In these equations $A_\alpha N_\alpha$ is the birth rate, $B_\alpha N_\alpha^2$ is the death rate due to disease, and $C_\alpha N_1 N_2$ is the death rate due to competition for the food supply. Assume that $N_1(0) = N_2(0) = 1.0\text{x}10^5$, $A_1 = A_2 = 0.1$, $B_1 = B_2 = 8.0\text{x}10^{-7}$, $C_1 = 1.0\text{x}10^{-6}$, $C_2 = 1.0\text{x}10^{-7}$, and calculate $N_1(t)$ and $N_2(t)$ for $t = 0$ to 10 years. Solve this problem using a 4^{th} order Runge-Kutta method and plot $N_1(t)$ and $N_2(t)$ versus time on the same graph. Experiment with the time-step size that you use in your calculations and by trying successively smaller values, convince yourself that the solution converges as the time-step size shrinks. For the results to be included in the write-up, use a time-step size that is small enough so that the computed answers are independent of the actual value used. Report the value of the time-step size that you have empirically determined is appropriate for this problem. Using this solution as the *exact* answer, now increase the time-step size by a factor of 2, 4, 8, 16 and compare the errors in $N_1(10)$ and $N_2(10)$ relative to the *exact* value you have determined. Plot the logarithm of these errors as a function of $\log_{10}(h^{-1})$. Is the convergence rate you observe in this empirical experiment in agreement with what you would expect theoretically?. [`5 points`]

[1] Posted on: `October 30, 2024`.