Name:	Aneesh Panchal	**Homework No:**	Assignment 4
SR No:	06-18-01-10-12-24-1-25223	**Course Code:**	DS284
Email ID:	aneeshp@iisc.ac.in	**Course Name:**	Numerical Linear Algebra
Date:	October 26, 2024	**Term:**	AUG 2024

Solution 3

Solution 3 (a)

For 2×2 matrix, Classical Gram Schmidt (CGS) and Modified Gram Schmidt (MGS) gives same result. Given matrix **A** is,

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$$

Hence, we get 1^{st} orthogonal vector q_1 as follows,

$$\|a_1\|_2 = \sqrt{(0.70000)^2 + (0.70001)^2} \approx 0.98996$$

$$q_1 = \frac{1}{\|a_1\|_2} a_1 = \frac{1}{0.98996} \begin{bmatrix} 0.70000 \\ 0.70001 \end{bmatrix} \approx \begin{bmatrix} 0.70710 \\ 0.70711 \end{bmatrix}$$

Now we calculate, the 2^{nd} orthogonal vector q_2 as follows,

$$q_1^T a_2 = (0.70710 \times 0.70711) + (0.70711 \times 0.70711) \approx 1.00000$$

$$a_2 - (q_1^T a_2) q_1 = \begin{bmatrix} 0.70711 \\ 0.70711 \end{bmatrix} - (1.00000) \times \begin{bmatrix} 0.70710 \\ 0.70711 \end{bmatrix} \approx \begin{bmatrix} 0.00001 \\ 0.00000 \end{bmatrix}$$

$$\|a_2 - (q_1^T a_2) q_1\|_2 = 0.00001$$

$$q_2 = \frac{1}{\|a_2 - (q_1^T a_2) q_1\|_2} (a_2 - (q_1^T a_2) q_1) \approx \frac{1}{0.00001} \begin{bmatrix} 0.00001 \\ 0.00000 \end{bmatrix} \approx \begin{bmatrix} 1.00000 \\ 0.00000 \end{bmatrix}$$

Hence, the resulting **Q** matrix is as follows,

$$\mathbf{Q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 0.70710 & 1.00000 \\ 0.70711 & 0.00000 \end{bmatrix}$$

Now, check the orthogonality of the hence obtained **Q** matrix,

$$\mathbf{Q^T Q} = \begin{bmatrix} 0.70710 & 0.70711 \\ 1.00000 & 0.00000 \end{bmatrix} \begin{bmatrix} 0.70710 & 1.00000 \\ 0.70711 & 0.00000 \end{bmatrix} \approx \begin{bmatrix} 0.99999 & 0.70710 \\ 0.70710 & 1.00000 \end{bmatrix}$$

Therefore, **Q is not orthogonal**.

Solution 3 (b)

Now for 2×2 matrix **A**, apply Householder Triangularization,

$$\mathbf{Q_2 Q_1 A} = \mathbf{R}$$

where,

$$\mathbf{Q_1} = \mathbf{F_{2\times2}} \qquad \mathbf{Q_2} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{F_{1\times1}} \end{bmatrix}$$

Matrix **A** can be written as,

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$$

Now we first find $\mathbf{Q_1} = \mathbf{F_{2\times2}}$,

$$\|a_1\|_2 = \sqrt{(0.70000)^2 + (0.70001)^2} \approx 0.98996$$

$$v = sign(a_{11}) \|a_1\|_2 e_1 + a_1 = (+1)(0.98996) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.70000 \\ 0.70001 \end{bmatrix} = \begin{bmatrix} 1.68996 \\ 0.70001 \end{bmatrix}$$

Now, we compute the matrix $\mathbf{F_{2\times2}}$ as follows,

$$\mathbf{Q_1} = \mathbf{F_{2\times2}} = I - 2\frac{vv^T}{\|v\|_2^2} = \begin{bmatrix} -0.70710 & -0.70711 \\ -0.70711 & 0.70710 \end{bmatrix}$$

Now, assume $x = [0.70711]$ and hence, $\|x\|_2 = 0.70711$ we have,

$$v = sign(a_{22})\|x\|_2 e_1 + x = 0.70711 + 0.70711 = 1.41422$$

Now, we compute the matrix $\mathbf{F_{1\times 1}}$ as follows,

$$\mathbf{F_{1\times 1}} = I - 2\frac{vv^T}{\|v\|_2^2} = 1 - 2 = -1$$

And hence, the final matrix $\mathbf{Q_2}$ can be written as,

$$\mathbf{Q_2} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{F_{1\times 1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hence, the resulting $\mathbf{Q}$ matrix can be written as,

$$\mathbf{Q} = \mathbf{Q_1}\mathbf{Q_2} = \begin{bmatrix} -0.70710 & -0.70711 \\ -0.70711 & 0.70710 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -0.70710 & 0.70711 \\ -0.70711 & -0.70710 \end{bmatrix}$$

Now, check the orthogonality of the hence obtained $\mathbf{Q}$ matrix,

$$\mathbf{Q^T}\mathbf{Q} = \begin{bmatrix} -0.70710 & -0.70711 \\ 0.70711 & -0.70710 \end{bmatrix} \begin{bmatrix} -0.70710 & 0.70711 \\ -0.70711 & -0.70710 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, **Q is orthogonal**.

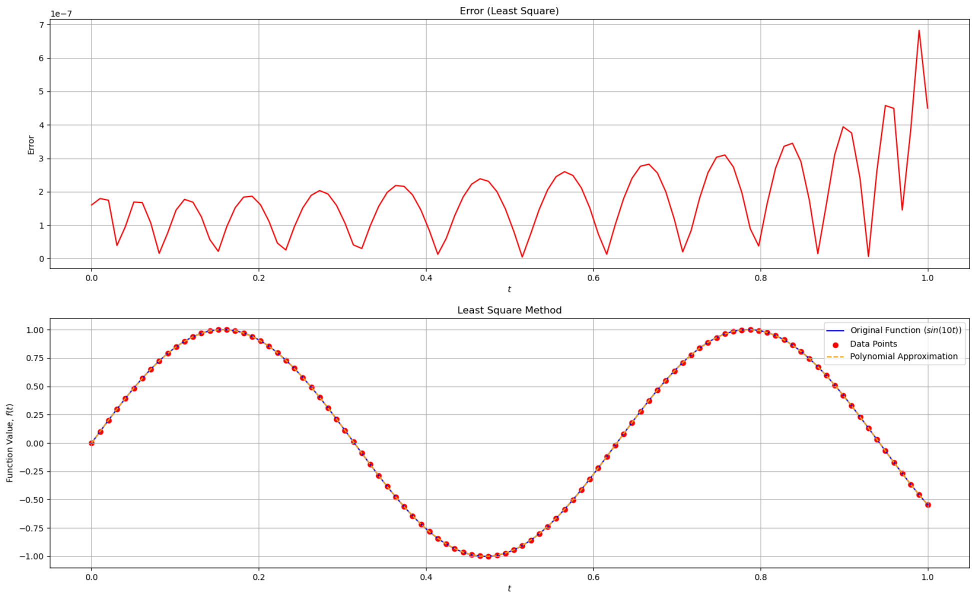
If we compare the orthogonal matrix $\mathbf{Q}$ obtained through Gram Schmidt and Householder Traingularization method, we can conclude that the matrix obtained through Gram Schmidt may not be orthogonal always but is orthogonal in case of Householder Traingularization method.
Loss of orthogonality in case of **Gram Schmidt Method** is due to numerical instability of the method and as we also know the error is also proportional to $O(K(A)\epsilon_M)$ (if we use Modified Gram Schmidt Method). Hence, when the matrix is ill conditioned then the error may be very large along with loss of orthogonality. However, in case of **Householder Triangularization method** the error is proportional to $O(\epsilon_M)$. Hence, when matrix is ill conditioned then also the error is of the range of machine epsilon. Also, Householder Triangularization method is numerically more stable than Gram Schmidt because it uses global transformations or, reflections instead of local transformations or, reflections which leads to better preservation of orthogonality properties even in finite precision arithmetic.

Name: Aneesh Panchal
SR No: 06-18-01-10-12-24-1-25223
Email ID: aneeshp@iisc.ac.in
Date: October 26, 2024

Solution 5

Absolute Error is calculated using actual value (ground truth) of the function $f(t) = \sin(10t)$ and approximated solution. Approximated solution is calculated using Ax when x is found for the linear system of equations $Ax = b$.

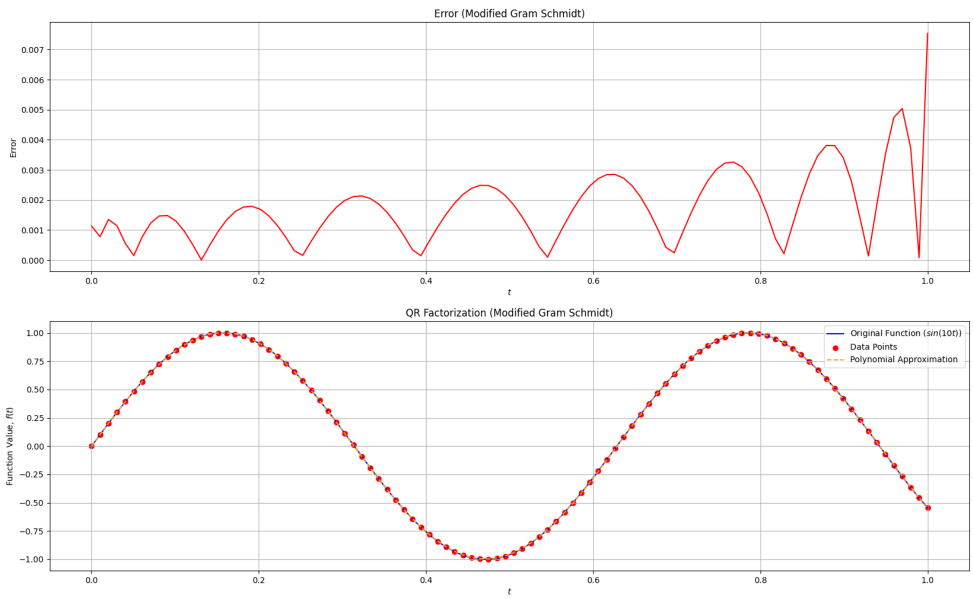
1. **Least Square Method (Inbuillt Function)**
   Using inbuilt function of Numpy library of python function **numpy.linalg.lstsq**.
   It is clearly observable that absolute error is of $O(10^{-7})$.



2. **QR Factorization using Modified Gram Schmidt Method**
   Using Modified Gram Schmidt Method i.e. for $A = QR$,

$$q_i = P_i a_i \qquad \text{where, } P_i = P_{\perp_{q_{i-1}}} P_{\perp_{q_{i-2}}} \dots P_{\perp_{q_2}} P_{\perp_{q_1}} \quad \text{and, } P_{\perp_{q_j}} = I - q_j q_j^T$$

   After finding Q, we can easily find R using, $R = Q^T A$.
   It is clearly observable that absolute error is of $O(10^{-3})$.

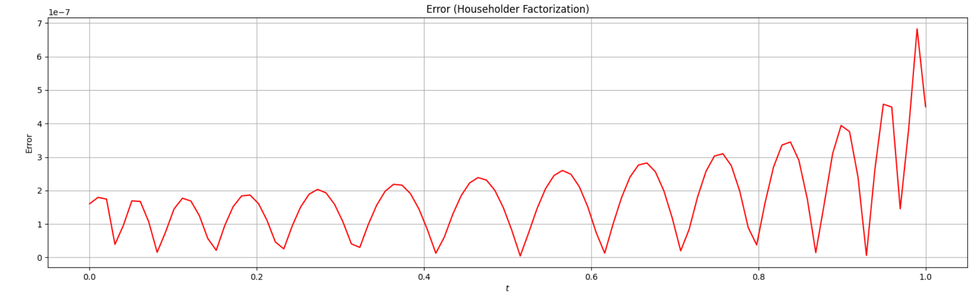


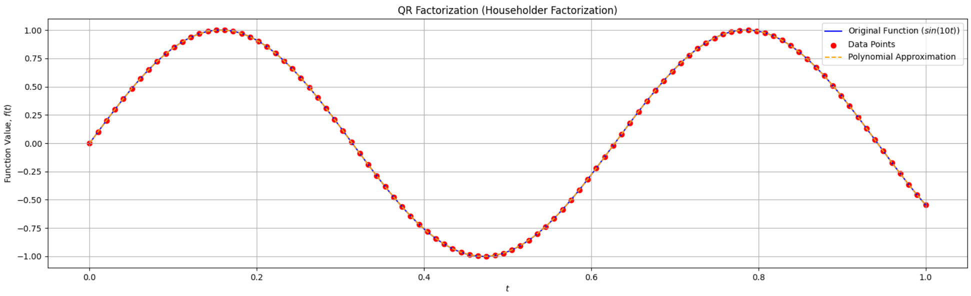
3. **QR Factorization using Householder Factorization**
   Using QR Factorization (Householder Factorization Method) i.e. for $A = Q_1 Q_2 \dots Q_n R$,

$$Q_k = \begin{bmatrix} I_{(k-1)\times(k-1)} & 0 \\ 0 & F_{(n-k+1)\times(n-k+1)} \end{bmatrix} \qquad \text{where, } F = I - 2\frac{uu^T}{\|u\|_2^2}$$

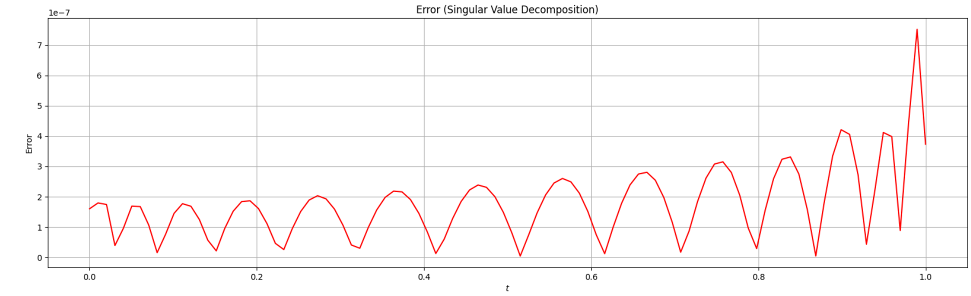
$$u = sign(x(1))\|x\|_2 e_1 + x \qquad \text{where, } x = A(k:m, k)$$

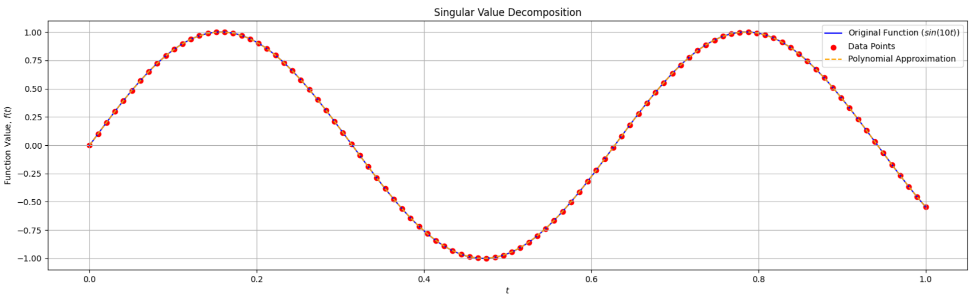
   After finding Q, we can easily find R using, $R = Q^T A$.
   It is clearly observable that absolute error is of $O(10^{-7})$.



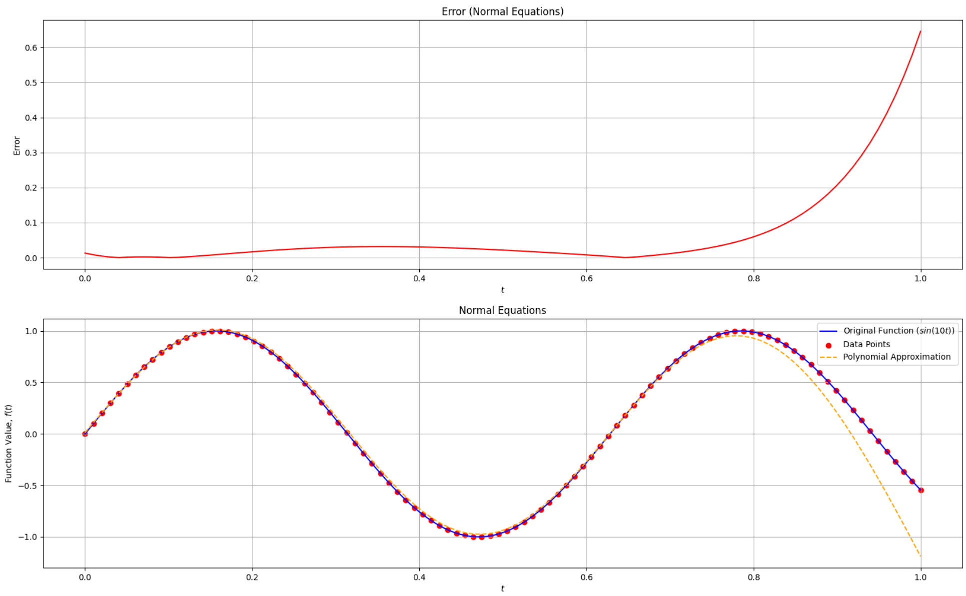


4. **Singular Value Decomposition**
   Using Singular Value Decomposition (SVD) i.e. for $A = U\Sigma V^T$.
   First Solve the system, $\Sigma w = U^T b$ where, $w = V^T x$. Then, final solution is given by, $x = Vw$.
   It is clearly observable that absolute error is of $O(10^{-7})$.





5. **Normal Equations**
   Using Normal Equations, i.e. for $A^{m\times n}$, $m > n$ full rank matrix then we have $A^T A$ as non singular matrix.
   Hence, the system $Ax = b$ can be solved as,
   $$x = (A^T A)^{-1} A^T b$$
   It is clearly observable that absolute error is of $O(10^{-1})$.



Analysis of the Results

Method Name	Order of Absolute Error	Theoretical Order of Error
Inbuilt Function	$\mathbf{O(10^{-7})}$	–
Modified Gram Schmidt	$\mathbf{O(10^{-3})}$	$\mathbf{O(K(A)\epsilon_M)}$
Householder Factorization	$\mathbf{O(10^{-7})}$	$\mathbf{O(\epsilon_M)}$
Singular Value Decomposition	$\mathbf{O(10^{-7})}$	$\mathbf{O(\epsilon_M)}$
Normal Equations	$\mathbf{O(10^{-1})}$	$\mathbf{O(K(A^TA)\epsilon_M)}$

Along with these error ranges, we also know the following condition numbers,

$$K(A) = 22717773709.14235$$

$$K(A^T A) = 1.1020061528999347 \times 10^{18}$$

From above results, it can be easily seen that, Householder Factorization method and Singular Value Decomposition method performs best with order of error $O(10^{-7})$. Also, the inbuilt function performs the same as the Householder Factorization method and Singular Value Decomposition method. Hence, we can conclude the inbuilt function numpy.linalg.lstsq have one of these methods to solve the least square problem.
Following which Modified Gram Schmidt method (with order of error $O(10^{-3})$) which performs better than Normal equations which performs worst (with order of error $O(10^{-1})$) because Modified Gram Schmidt method decomposes the matrix A into QR matrices to solve the system of equation which do not include any inversion of matrix which is included in normal equations method. And, as we know inversion of matrix is not suitable for ill conditioned problems (High condition number of $A^T A$).
Due to above reasons, Normal equations performs worst due to condition number of $A^T A$ and Modified Gram Schmidt method performs better than Normal equations but worse than Singular Value Decomposition and Householder Factorization due to condition number of A. Hence, best method to solve $Ax = b$ when A is rank deficient is either Singular Value Decomposition or, Householder Factorization.

Solution 7

Solution 7 (a) $-$ True

If λ is an eigenvalue of $\mathbf{A}$ and $\mu \in \mathcal{C}$, then $\lambda - \mu$ is an eigenvalue of $\mathbf{A} - \mu\mathbf{I}$.
Proof:
Let $\mathbf{u}$ be the eigenvector corresponding to eigenvalue λ for matrix $\mathbf{A}$.
Then, we have, $\mathbf{Au} = \lambda\mathbf{u}$
Subtracting both side by $\mu\mathbf{u}$ we get,
$\mathbf{Au} - \mu\mathbf{u} = \lambda\mathbf{u} - \mu\mathbf{u}$
$(\mathbf{A} - \mu\mathbf{I})\mathbf{u} = (\lambda - \mu)\mathbf{u}$
Hence, we can conclude that if λ is an eigenvalue of $\mathbf{A}$ and $\mu \in \mathcal{C}$, then $\lambda - \mu$ is an eigenvalue of $\mathbf{A} - \mu\mathbf{I}$.
Hence, proved.

Solution 7 (b) $-$ False

If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$ then so is $-\lambda$.
Contradictory Example:
Let us assume a real matrix, $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
Matrix $\mathbf{A}$ have two eigenvalues which are 1 and 2.

Solution 7 (c) $-$ True

If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$ then so is λ^*. (λ^* is complex conjugate of λ).
Proof:
Let $\mathbf{u}$ be the eigenvector corresponding to eigenvalue λ for matrix $\mathbf{A}$.
Then, we have, $\mathbf{Au} = \lambda\mathbf{u}$
Take complex conjugate on both sides we get,
$(\mathbf{Au})^* = (\lambda\mathbf{u})^*$
$\mathbf{A}^*\mathbf{u}^* = \lambda^*\mathbf{u}^*$
As $\mathbf{A}$ is real $\mathbf{A} = \mathbf{A}^*$, we get,
$\mathbf{Au}^* = \lambda^*\mathbf{u}^*$
Hence, we can conclude that if $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$ then so is λ^*.
Hence, proved.

Solution 7 (d) $-$ True

If λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{A}$ is non-singular, then λ^{-1} is the eigenvalue of $\mathbf{A}^{-1}$.
Proof:
Let $\mathbf{u}$ be the eigenvector corresponding to eigenvalue λ for matrix $\mathbf{A}$.
Then, we have, $\mathbf{Au} = \lambda\mathbf{u}$
As we know, $\mathbf{A}$ is non-singular and hence invertible, so pre-multiply the equation above by $\mathbf{A}^{-1}$, we get,
$\mathbf{A}^{-1}\mathbf{Au} = \lambda\mathbf{A}^{-1}\mathbf{u}$
As A is non-singular, hence, $\lambda \neq 0$,
$\lambda^{-1}\mathbf{u} = \mathbf{A}^{-1}\mathbf{u}$
Hence, we can conclude that if λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{A}$ is non-singular, then λ^{-1} is the eigenvalue of $\mathbf{A}^{-1}$.
Hence, proved.

Solution 7 (e) $-$ False

If all the eigenvalues of $\mathbf{A}$ are zero, then $\mathbf{A} = 0$.
Contradictory Example:
Let us assume a matrix, $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
Eigenvalue of matrix $\mathbf{A}$ is 0 with algebraic multiplicity of 2.
But it is clear that $\mathbf{A} \neq 0$.

Name:	Aneesh Panchal	**Homework No:**	Assignment 4
SR No:	06-18-01-10-12-24-1-25223	**Course Code:**	DS284
Email ID:	aneeshp@iisc.ac.in	**Course Name:**	Numerical Linear Algebra
Date:	October 26, 2024	**Term:**	AUG 2024

Solution 7 (f) $-$ True

If **A** is diagonalizable and all its eigenvalues are equal, then **A** is diagonal.
Proof:
As **A** is diagonalizable it can be written as,
$\mathbf{A} = \mathbf{X}\Lambda\, \mathbf{X}^{-1}$
Also, all eigenvalues (say, λ) are equal and hence,
$\Lambda = \lambda\mathbf{I}$
From above 2 equations, we get,
$\mathbf{A} = \mathbf{X}\Lambda\, \mathbf{X}^{-1} = \mathbf{X}\lambda\mathbf{I}\, \mathbf{X}^{-1} = \lambda\, \mathbf{X}\mathbf{X}^{-1} = \lambda\, \mathbf{I} = \Lambda$
Hence, we can conclude that if **A** is diagonalizable and all its eigenvalues are equal, then **A** is diagonal.
Hence, proved.