



Indian Institute of Science, Bangalore
Department of Computational and Data Sciences (CDS)

DS284: Numerical Linear Algebra

Assignment 0 [Posted Aug 2, 2024]

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This assignment helps you to refresh some of the concepts useful for this course

Notations: (i) Vectors and matrices are denoted by bold-faced lower case and upper case alphabets, respectively. (ii) Set of all real numbers is denoted by \mathbb{R} (iii) Set of all n dimensional vectors is denoted by \mathbb{R}^n and set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m \times n}$

1. **Vector Space:** A non-empty set \mathbb{V} is said to form a real vector space if it satisfies the following conditions:

1. $\mathbf{u} + \mathbf{v} \in \mathbb{V}, \forall \mathbf{u}, \mathbf{v} \in \mathbb{V}$ [Closure under addition property]
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}, \forall \mathbf{u}, \mathbf{v} \in \mathbb{V}$ [Commutative property]
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}, \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$ [Associative property]
4. $\mathbf{0} \in \mathbb{V}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}, \forall \mathbf{v} \in \mathbb{V}$ [Zero vector]
5. Each $\mathbf{v} \in \mathbb{V}$ has a $\mathbf{w} \in \mathbb{V}$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$ [Additive inverse]
6. $c\mathbf{u} \in \mathbb{V}, \forall \mathbf{u} \in \mathbb{V}$ and $c \in \mathbb{R}$ [Closure under scalar Multiplication]
7. $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}, \forall \mathbf{v} \in \mathbb{V}$ and $c, d \in \mathbb{R}$ [Distributive property for scalar addition]
8. $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}, \forall \mathbf{v}, \mathbf{w} \in \mathbb{V}$ and $c \in \mathbb{R}$ [Distributive property for vector addition]
9. Multiplication by the scalar 1, $1\mathbf{v} = \mathbf{v}, \forall \mathbf{v} \in \mathbb{V}$ [Identity operation]
10. $(cd)\mathbf{v} = c(d\mathbf{v}), \forall \mathbf{v} \in \mathbb{V}$ and $c, d \in \mathbb{R}$ [Compatibility of scalar multiplication with field multiplication]

Now, verify if the following form a vector space.

$$(a) \mathbb{V} = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \mid w - x - y + z = 0; w, x, y, z \in \mathbb{R} \right\}$$

$$(b) \mathbb{M}^{2 \times 2} = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

$$(c) \mathbb{N} = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid \frac{df}{dx} + 2f = 1\}$$

2. **Subspace:** A non-empty subset \mathbb{W} of a vector space \mathbb{V} over a field F is a subspace of \mathbb{V} if and only if

1. $\mathbf{a} \in \mathbb{W}, \mathbf{b} \in \mathbb{W} \Rightarrow \mathbf{a} + \mathbf{b} \in \mathbb{W}$
2. $\mathbf{a} \in \mathbb{W}, \alpha \in F \Rightarrow \alpha \mathbf{a} \in \mathbb{W}$

Note: In the problems below we will consider $F = \mathbb{R}$.
Verify the following for \mathbb{V} defined as below:.

(a) Does $\mathbb{V} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x, y, z \geq 0 \right\}$ form a subspace of \mathbb{R}^3 ?

(b) Does $\mathbb{V} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x, y, z \in \mathbb{R}, x + z = 0 \right\}$ form a subspace of \mathbb{R}^3 ?

(c) Does $\mathbb{V} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}$ form a subspace of $\mathbb{R}^{2 \times 2}$?

3. **Linear Dependence and Linear Independence:** A finite set vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of a vector space \mathbb{V} over a field F (in our case it is \mathbb{R} or \mathbb{C}) is said to be linearly dependent in \mathbb{V} if there exist scalars $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ not all zero in F such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n = \mathbf{0} \quad (1)$$

The set is said to be linearly independent in \mathbb{V} if the equality (1) is satisfied only when $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

Basis: Let \mathbb{V} be a vector space over a field F . A set \mathbb{S} of vectors in \mathbb{V} is said to be a basis of \mathbb{V} if

1. \mathbb{S} is linearly independent in \mathbb{V} , and
2. Any vector in \mathbb{V} can be written as linear combination of the vectors in \mathbb{S} .

Now answer the following questions:

- (a) Can $(0, -26, -9)$ and $(1, 3, 5)$ be expressed as a linear combinations of $(5, 3, 7)$ and $(2, -4, 1)$. Explain.
- (b) Find a basis of the given subspace.

$$\mathbf{U} = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4 \mid 3w + x - 7z = 0 \right\}$$

4. Solving linear system of equations multiple ways:

- (a) Solve the following set of linear system of equations graphically and comment on the solvability (no solution, infinite solutions or unique solution) of the systems.

- Set 1

$$x + 2y = 3$$

$$4x + 5y = 6$$

- Set 2

$$x + 2y = 3$$

$$4x + 8y = 6$$

- Set 3

$$x + 2y = 3$$

$$4x + 8y = 12$$

- (b) Solve the same set of linear system of equations algebraically. Confirm your observations obtained from part (a).
- (c) The matrix vector product $\mathbf{A}\mathbf{x}$ can also be viewed as linear combination of column vectors of \mathbf{A} .

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + y \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Using the above view point and the definition of linear independence/dependence of vectors, analyse the solvability of the above system of equations without having to solve the system of equations completely.

5. Review of Matrices:

Consider three vectors, $\mathbf{a} = \begin{pmatrix} -209/362 \\ -209/362 \\ 209/362 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ -408/577 \\ -408/577 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 396/485 \\ -198/485 \\ 198/485 \end{pmatrix}$

- (a) Evaluate the dot product/inner product $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \cdot \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{c}$. The dot product is given by $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.
- (b) Calculate the geometric length of the above vectors \mathbf{a} , \mathbf{b} , \mathbf{c} and also of \mathbf{x} where $\mathbf{x} = \begin{pmatrix} 2 \\ -40/57 \\ 8/77 \end{pmatrix}$. Note that the geometric length of the vector $\mathbf{a} \in \mathbb{R}^n$ is computed as $\sqrt{\mathbf{a} \cdot \mathbf{a}}$.

- (c) Construct a matrix \mathbf{A} with the above vectors \mathbf{a} , \mathbf{b} , \mathbf{c} as the first, second and third columns of \mathbf{A} . Calculate the geometric length of \mathbf{Ax} and compare with that of \mathbf{x} . What do you observe?
- (d) Evaluate $\mathbf{A}^T\mathbf{A}$ and \mathbf{AA}^T
- (e) Prove that for full rank square matrices with the property $\mathbf{AA}^T = \mathbf{I}$, your observation in part (c) will always be true for any \mathbf{x} .

Reference textbooks for review:

1. Linear Algebra and its Applications by Gilbert Strang.
2. Linear Algebra by S. Friedberg, A. Insel and L. Spence. (Solved examples)