DS 215 : Assignment 1.

Aneesh Panchal

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Solution 1. Given $X_1, X_2, \ldots, X_n$ are iid

$$f(x) = \begin{cases} 1/2 & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

ie. $X_i \; \forall i$ is uniform distribution ($X_i \sim U(-1,1)$)

$m_{X_1} = \frac{-1+1}{2} = 0$ (expected value) $\Rightarrow E[X_i] = 0$

Variance, $\sigma^2_{X_i} = \frac{(1+1)^2}{12} = \frac{4}{12} = \frac{1}{3}$

$S_n = X_1 + X_2 + \ldots + X_n$

$E[S_n] = E[X_1] + E[X_2] + \ldots + E[X_n] = 0$

$E\left[\frac{S_n}{n}\right] = \frac{0}{n} = 0$

$$\sigma^2(S_n) = E[S_n^2] - [E(S_n)]^2 \qquad \{\text{here } \sigma^2(S_n) = Var(S_n)\}$$
$$= E[X_1^2] + E[X_2^2] + \ldots + E[X_n^2]$$
$$= \sigma^2_{X_1} + 0 + \sigma^2_{X_2} + 0 + \ldots + \sigma^2_{X_n} + 0$$
$$= \frac{1}{3} + \frac{1}{3} + \ldots + \frac{1}{3} = \frac{n}{3}$$

$$\sigma^2\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\sigma^2(S_n) = \frac{1}{n^2}\frac{n}{3} = \frac{1}{3n} \qquad \left\{\text{here } \sigma^2\left(\frac{S_n}{n}\right) = Var\left(\frac{S_n}{n}\right)\right\}$$

According to Chebyshev inequality,

$$P[|X - m_X| \ge a] \le \frac{\sigma_X^2}{a^2}$$

now put $X = \frac{S_n}{n}$, $m_X = 0$, $a = \varepsilon$, we get

$$P\left[\left|\frac{S_n}{n}\right| \ge \varepsilon\right] \le \frac{\sigma^2\left(\frac{S_n}{n}\right)}{\varepsilon^2} = \frac{1}{3n\varepsilon^2}$$

Hence, proved.

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Solution 2. Given $X_1, X_2, \dots, X_n$ with mean μ & $Cov(X_i, X_j) = \sigma^2 \rho^{|i-j|}$ & $|\rho| < 1$

(a) $S_n = X_1 + X_2 + \dots + X_n$

$$E[S_n] = E[X_1] + E[X_2] + \dots + E[X_n]$$
$$= \mu + \mu + \dots + \mu = n\mu$$

$$Cov(X_i, X_j) = \sigma^2 \rho^{|i-j|} = E[(X_i - \mu)(X_j - \mu)]$$
$$= E[X_i X_j] - 2\mu^2 + \mu^2 = E[X_i \cdot X_j] - \mu^2$$

$$E[X_i X_j] = \sigma^2 \rho^{|i-j|} + \mu^2$$

$$Var[S_n] = E[S_n^2] - (E[S_n])^2$$
$$= E\left[\sum_{i=1}^{n} X_i (X_1 + X_2 + \dots + X_n)\right] - \mu^2 n^2$$
$$= \sum_{i=1}^{n} E[X_i (X_1 + X_2 + \dots + X_n)] - n^2\mu^2$$
$$= n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1} (\rho + \rho^2 + \dots + \rho^i) + n^2\mu^2 - n^2\mu^2$$
$$= n\sigma^2 + 2\sigma^2\rho \sum_{i=1}^{n-1} \left(\frac{1-\rho^i}{1-\rho}\right) = n\sigma^2 + 2\sigma^2\rho\left[\frac{n-1}{1-\rho} - \frac{\rho}{1-\rho}\sum_{i=1}^{n-1}\frac{\rho^{i-1}}{1-\rho}\right]$$

$$Var[S_n] = n\sigma^2 + 2\sigma^2\rho\left[\frac{n-1}{1-\rho} - \left(\frac{\rho}{1-\rho}\right)\left(\frac{1-\rho^{n-1}}{1-\rho}\right)\right]$$

(b) Sample mean, $M_n = \frac{S_n}{n}$

Expected value, $E[M_n] = E\left[\frac{S_n}{n}\right] = \frac{E[S_n]}{n} = \frac{n\mu}{n} = \mu = E[X_i]$ ______ (i)

Variance, $Var[M_n] = Var\left[\frac{S_n}{n}\right] = \frac{Var[S_n]}{n^2}$

$$= \frac{\sigma^2}{n} + \frac{2\sigma^2\rho}{n}\left[\frac{1-1/n}{1-\rho} - \frac{1}{n}\frac{\rho(1-\rho^{n-1})}{(1-\rho)^2}\right]$$

$\lim_{n\to\infty} Var[M_n] = 0$ ______ (ii)

Hence, from (i) & (ii) we conclude that weak law of large no. holds true for sample mean.

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Solution 3. Given $X_1, X_2, \dots, X_n$ are iid

$$f(x) = \begin{cases} 1/2a & \text{for } |x| \le a \\ 0 & \text{otherwise} \end{cases}$$

ie. $X_i \; \forall i$ is uniform distribution ($X_i \sim U(-a,a)$)

(a) let $S_n = \dfrac{X_1 + X_2 + \dots + X_n}{\sqrt{n}}$ & Notation : $\exp\{x\} = e^x$

Characteristic function, $\phi_{S_n}(j\omega) = E[\exp\{-j\omega S_n\}]$

$$= E\left[\exp\left\{-j\omega\left(\frac{X_1+X_2+\dots+X_n}{\sqrt{n}}\right)\right\}\right]$$

as given $X_1, X_2, \dots, X_n$ are iid

$$\phi_{S_n}(j\omega) = \left[E\left[\exp\left\{-j\omega \frac{X_1}{\sqrt{n}}\right\}\right]\right]^n$$

$$E\left[\exp\left\{-j\omega \frac{X_1}{\sqrt{n}}\right\}\right] = \int_{-\infty}^{\infty} f_{X_1}(x) \exp\left\{\frac{-j\omega x}{\sqrt{n}}\right\} dx = \int_{-a}^{a} \frac{1}{2a} \exp\left\{\frac{-j\omega x}{\sqrt{n}}\right\} dx$$

$$= \left|\frac{\sqrt{n}}{-2aj\omega} \exp\left\{\frac{-j\omega x}{\sqrt{n}}\right\}\right|_{-a}^{a} = \frac{\sqrt{n}}{2aj\omega}\left[\exp\left\{\frac{j\omega a}{\sqrt{n}}\right\} - \exp\left\{\frac{-j\omega a}{\sqrt{n}}\right\}\right]$$

$\therefore$ Characteristic function, $\phi_{S_n}(j\omega) = \left[\dfrac{\sqrt{n}}{2aj\omega}\left[\exp\left\{\dfrac{j\omega a}{\sqrt{n}}\right\} - \exp\left\{\dfrac{-j\omega a}{\sqrt{n}}\right\}\right]\right]^n$

(b) As we know, $e^x - e^{-x} = 2\left[x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dots\right]$

put $x = \dfrac{j\omega a}{\sqrt{n}}$ & substitute value of $e^x - e^{-x}$ in $\phi_{S_n}(j\omega)$ we get,

$$\phi_{S_n}(j\omega) = \left[1 + \frac{(j\omega a)^2}{3!\, n} + \frac{(j\omega a)^4}{5!\, n^2} + \dots\right]^n = \left[1 + \sum_{i=1}^{\infty} \frac{(j\omega a)^{2i}}{(2i+1)!\, n^i}\right]^n$$

Now assume $k = j\omega a$ & put limit $n \to \infty$ on both sides we get,

$$\lim_{n\to\infty} \phi_{S_n}(j\omega) = \exp\left\{\lim_{n\to\infty} n \log\left[1 + \frac{k^2}{3!\, n} + \frac{k^4}{5!\, n^2} + \dots\right]\right\}$$

apply change of variable $t = 1/n$, (as $n \to \infty$ we get $t \to 0$)

$$= \exp\left\{\lim_{t\to 0} \frac{1}{t} \log\left[1 + \frac{k^2 t}{3!} + \frac{k^4 t^2}{5!} + \dots\right]\right\}$$

apply L'hopitals we get, (because it is $0/0$ form)

$$= \exp\left\{\lim_{t\to 0} \frac{\left[k^2/3! + 2tk^4/5! + \dots\right]}{\left[1 + k^2 t/3! + k^4 t^2/5! + \dots\right]}\right\} = \exp\left\{\frac{k^2}{3!}\right\}$$

$$\lim_{n\to\infty} \phi_{S_n}(j\omega) = \exp\left\{\frac{k^2}{3!}\right\} = \exp\left\{\frac{j^2 \omega^2 a^2}{6}\right\} = e^{-\omega^2 a^2/6}$$

Aneesh Panchal
25223

Solution 4. Given $U_0, U_1, \ldots \sim N(0,1)$ are iid

$$X_n = \frac{(U_n + U_{n-1})}{2} \quad \& \quad Y_n = \frac{(U_n - U_{n-1})}{2}$$

(a) X_n & X_{n-1} can be written in matrix form as,

$$\begin{bmatrix} X_{n-1} \\ X_n \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} U_{n-2} \\ U_{n-1} \\ U_n \end{bmatrix}$$

here it is equivalent to $X = AU$

$$\mu_X = A\mu_U = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Covariance matrix, $C_X = A\,C_U\,A^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$

$$|C_X| = \frac{1}{4} - \frac{1}{16} = \frac{3}{16} \quad \& \quad C_X^{-1} = \frac{16}{3} \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix}$$

$$(x-\mu_X)^T C_X^{-1} (x-\mu_X) = \begin{bmatrix} x_{n-1} & x_n \end{bmatrix} \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} \frac{16}{3} = \frac{8}{3} (x_n^2 + x_{n-1}^2 - x_n x_{n-1})$$

So, joint pdf of X_n & X_{n-1} can be written as,

$$f_{X_n X_{n-1}}(x_n, x_{n-1}) = \frac{1}{2\pi\sqrt{|C_X|}} \exp\left\{\frac{-1}{2}(x-\mu_X)^T C_X^{-1} (x-\mu_X)\right\}$$

$$= \frac{2}{\pi\sqrt{3}} \exp\left\{\frac{-4}{3}(x_n^2 + x_{n-1}^2 - x_n x_{n-1})\right\}$$

(b) Y_n & Y_{n+m} can be written in matrix form as,

$$\begin{bmatrix} Y_n \\ Y_{n+m} \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} U_{n-1} \\ U_n \\ \vdots \\ U_{n+m-1} \\ U_{n+m} \end{bmatrix}$$

for $m > 1$ ie. $m = 2, 3, \ldots$

here it is equivalent to $Y = AU$

$$\mu_Y = A\mu_U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_Y = AC_U A^T = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \frac{1}{2} I \quad \& \quad |C_Y| = \frac{1}{4} \quad \& \quad C_Y^{-1} = 2I$$

$$(y-\mu_Y)^T C_Y^{-1} (y-\mu_Y) = 2(y_n^2 + y_{n+m}^2)$$

So, joint pdf of Y_n & Y_{n+m} can be written as,

$$f_{X_n X_{n+m}}(y_n, y_{n+m}) = \frac{1}{2\pi\sqrt{|C_Y|}} \exp\left\{\frac{-1}{2}(y-\mu_Y)^T C_Y^{-1} (y-\mu_Y)\right\}$$

$$= \frac{1}{\pi} \exp\left\{-y_n^2 - y_{n+m}^2\right\}$$

Aneesh Panchal
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(c) Here 5 cases arise for X_n & Y_m

Case I: $n = m$

$$\begin{bmatrix} X_n \\ Y_n \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} U_{n-1} \\ U_n \end{bmatrix} \approx B = AU$$

$$\mu_B = A\mu_U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_B = A^T C_U A = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix} \ \& \ |C_B| = \frac{1}{4} \ \& \ C_B^{-1} = 4 \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$(B-\mu_B)^T C_B^{-1} (B-\mu_B) = 0$$

So, joint pdf of X_n & Y_n can be written as,

$$f_{X_n Y_n}(x_n, y_n) = \frac{1}{\pi}$$

Case II: $n = m+1$

$$\begin{bmatrix} Y_{n-1} \\ X_n \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} U_{n-2} \\ U_{n-1} \\ U_n \end{bmatrix} \approx B = AU$$

$$\mu_B = A\mu_U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_B = A^T C_U A = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \ \& \ |C_B| = \frac{3}{16} \ \& \ C_B^{-1} = \frac{16}{3} \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix}$$

$$(B-\mu_B)^T C_B^{-1} (B-\mu_B) = \frac{8}{3}(x_n^2 + y_{n-1}^2 - x_n y_{n-1})$$

So, joint pdf of X_n & Y_{n-1} can be written as,

$$f_{X_n Y_{n-1}}(x_n, y_{n-1}) = \frac{2}{\pi\sqrt{3}} \exp\left\{\frac{-4}{3}(x_n^2 + y_{n-1}^2 - x_n y_{n-1})\right\}$$

Case III: $m = n+1$

$$\begin{bmatrix} X_n \\ Y_{n+1} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} U_{n-1} \\ U_n \\ U_{n+1} \end{bmatrix} \approx B = AU$$

$$\mu_B = A\mu_U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_B = A^T C_U A = \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix} \ \& \ |C_B| = \frac{3}{16} \ \& \ C_B^{-1} = \frac{16}{3} \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$$

$$(B-\mu_B)^T C_B^{-1} (B-\mu_B) = \frac{8}{3}(x_n^2 + y_{n+1}^2 + x_n y_{n+1})$$

So, joint pdf of X_n & Y_{n+1} can be written as,

$$f_{X_n Y_{n+1}}(x_n, y_{n+1}) = \frac{2}{\pi\sqrt{3}} \exp\left\{\frac{-4}{3}(x_n^2 + y_{n+1}^2 + x_n y_{n+1})\right\}$$

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Case IV: $n > m+1$

$$\begin{bmatrix} Y_m \\ X_n \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} U_{m-1} \\ U_m \\ \vdots \\ U_{n-1} \\ U_n \end{bmatrix} \approx B = AU$$

$$\mu_B = A\mu_U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_B = A^T C_U A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \& \quad |C_B| = \frac{1}{4} \quad \& \quad C_B^{-1} = 2I$$

$$(B-\mu_B)^T C_B^{-1} (B-\mu_B) = 2(x_n^2 + y_m^2)$$

So, joint pdf of X_n & Y_m can be written as,

$$f_{X_n Y_m}(x_n, y_m) = \frac{1}{\pi} \exp\{-x_n^2 - y_m^2\}$$

Case V: $m > n+1$

$$\begin{bmatrix} X_n \\ Y_m \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} U_{n-1} \\ U_n \\ \vdots \\ U_{m-1} \\ U_m \end{bmatrix} \approx B = AU$$

$$\mu_B = A\mu_U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_B = A^T C_U A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \& \quad |C_B| = \frac{1}{4} \quad \& \quad C_B^{-1} = 2I$$

$$(B-\mu_B)^T C_B^{-1} (B-\mu_B) = 2(x_n^2 + y_m^2)$$

So, joint pdf of X_n & Y_m can be written as,

$$f_{X_n Y_m}(x_n, y_m) = \frac{1}{\pi} \exp\{-x_n^2 - y_m^2\}$$

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(d) Mean square convergence means/implies

$\lim_{n\to\infty} E[|X_n - X|]^2 = 0$ ~ converges to X

Now here we don't have X, hence we use Cauchy criteria

$\lim_{i,j\to\infty} E[|X_i - X_j|^2] = 0$

we have $X_i = \frac{U_i + U_{i-1}}{2}$ & $X_j = \frac{U_j + U_{j-1}}{2}$

$|X_i - X_j|^2 = \frac{1}{4}(U_i + U_{i-1} - U_j - U_{j-1})^2$

$= \frac{1}{4}[U_i^2 + U_{i-1}^2 + U_j^2 + U_{j-1}^2 + 2(-U_iU_j + U_iU_{i-1} - U_iU_{j-1} - U_{i-1}U_j - U_{i-1}U_{j-1} + U_jU_{j-1})]$

As we know U_i are iid & $\sim N(0,1)$ & hence we got,

$E(|X_i - X_j|^2) = \frac{1}{4}(1+1+1+1) = \frac{4}{4} = 1 \neq 0$

$\lim_{i,j\to\infty} E(|X_i - X_j|^2) \neq 0$

Hence, X_n do not converge in mean square sense.

(c) $X_n = \frac{(U_n + U_{n-1})}{2} \Rightarrow X_n = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} U_{n-1} \\ U_n \end{bmatrix}$

$\approx X_n = AU$

$\mu_X = A\mu_U = 0$

$C_X = AC_UA^T = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2}$

$C_X^{-1} = 2$ & $|C_X| = \frac{1}{2}$

$(x - \mu_x)^T C_X^{-1} (x - \mu_x) = 2x^2$

$f_{X_n}(x_n) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ which is pdf for X_n

ie. $X_n \sim N(0, 1/2)$ & also do not depend on n

$\therefore \lim_{n\to\infty} F_{X_n}(x) = F_{X^*}(x) \quad \forall x$

$\sim N(0, 1/2)$

ie. same distribution

$\therefore X_n$ converges in distribution.

Aneesh Panchal
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Solution 5. $M_n = \dfrac{X_1 + X_2 + \ldots + X_n}{n}$ where X_i are iid

(a) $E[M_n] = \dfrac{1}{n} E[X_1 + X_2 + \ldots + X_n] = \dfrac{\cancel{n}}{\cancel{n}} E[X] = E[X] = \mu$ (let)

let $Var(X) = \sigma^2$

$E[X_i^2] = \sigma^2 + \mu^2$

$E[X_i X_j] = E[X_i]\, E[X_j] = \mu^2$ (iid samples for $i \neq j$)

$Var[M_n] = \dfrac{1}{n^2} Var[X_1 + X_2 + \ldots + X_n] = \dfrac{n\sigma^2}{n^2} = \dfrac{\sigma^2}{n}$

Covariance, $Cov(M_n, M_k) = E[(M_n - \mu)(M_k - \mu)] = E[M_n M_k] - \mu^2 + \mu^2 - \mu^2$

$$= \frac{\min\{n,k\}(E(X^2) - \mu^2) + nk\mu^2}{nk} - \mu^2$$

$$= \min\{n,k\} \frac{\sigma^2}{nk} \quad (\text{because } E(X^2) - \mu^2 = \sigma^2)$$

(b) $M_{n+1} - M_n = \dfrac{X_1 + X_2 + \ldots + X_n + X_{n+1}}{n+1} - \dfrac{X_1 + X_2 + \ldots + X_n}{n} = -\dfrac{M_n}{n+1} + \dfrac{X_{n+1}}{n+1}$

ie. $M_{n+1} - M_n = \dfrac{X_{n+1} - M_n}{n+1}$

here it is clear that $M_{n+1} - M_n$ depends on M_n & hence not independent increment

$\therefore M_n$ does not have independent increments.

NOTE: $Cov((M_{n+2} - M_{n+1}), (M_{n+1} - M_n)) = E((M_{n+2} - M_{n+1})(M_{n+1} - M_n))$

because $E[M_{n+m} - M_n] = \mu - \mu = 0$

$Cov((M_{n+2} - M_{n+1}), (M_{n+1} - M_n)) = E[M_{n+2} M_{n+1}] - E[M_{n+2} M_n] - E[M_{n+1}^2] + E[M_{n+1} M_n]$

$$= \frac{\sigma^2}{n+2} - \frac{\sigma^2}{n+2} - \frac{\sigma^2}{n+1} + \frac{\sigma^2}{n+1} = 0$$

here we get covariance $= 0$ but not independent ie.

covariance does not imply independence

(c) For stationary increments, $M_{n+1} - M_n$ and M_1 have same distribution

$E[M_{n+1} - M_n] = \mu - \mu = 0$ ___ (i)

$E[M_1] = E[X_1] = \mu$ ___ (ii)

$Var[M_{n+1} - M_n] = Var(M_{n+1}) + Var(M_n) - 2\,Cov(M_{n+1}, M_n)$

$$= \frac{\sigma^2}{n+1} + \frac{\sigma^2}{n} - \frac{2\sigma^2}{n+1} = \frac{\sigma^2}{n(n+1)} \quad \text{___ (iii)}$$

$Var[M_1] = Var[X_1] = \sigma^2$ ___ (iv)

from (i), (ii), (iii) & (iv) we conclude that,

M_n does not have stationary increments.

Aneesh Panchal
25223

Solution 6. Probability distribution for X_n can be given as

$$Pr[X(n)=x] = \begin{cases} 1/2 & , x=1 \text{ (heads)} \\ 1/2 & , x=-1 \text{ (tails)} \end{cases}$$

As events doesnot depend on previous. So, independent events.

(a) Expected value, $m_{X(n)} = \frac{1}{2} - \frac{1}{2} = 0 \quad \forall n$ identical (not depend on n)

Covariance, $C_X(n_1,n_2) = E[(X(n_1)-m_{X(n_1)})(X(n_2)-m_{X(n_2)})]$

$$= \underbrace{E[X(n_1)X(n_2)]}_{R_X(n_1,n_2)} = E[X(n_1)]\,E[X(n_2)] \quad \text{(independent events)}$$

$$= 0 = 0\,(n_1-n_2)$$

& $R_X(n_1,n_2)$

As $C_X(n_1,n_2)$ & $R_X(n_1,n_2)$ is function of n_1-n_2.

Hence, X_n is Wide Sense Stationary random process.

(b) $Pr[X(n_1)=a_1, X(n_2)=a_2, \dots, X(n_k)=a_k] = Pr[X(n_1)=a_1]\,Pr[X(n_2)=a_2] \dots Pr[X(n_k)=a_k]$ ——— (i)

because independent events for $a_i \in \{-1,1\} \; \forall i$

As Pr value will not depend on n & hence,

$Pr[X(n_i+\tau)=a_i] = Pr[X(n_i)=a_i]$

$Pr[X(n_1+\tau)=a_1, X(n_2+\tau)=a_2, \dots, X(n_k+\tau)=a_k] = Pr[X(n_1+\tau)=a_1] \dots Pr[X(n_k+\tau)=a_k]$

$= Pr[X(n_1)=a_1]\,Pr[X(n_2)=a_2] \dots Pr[X(n_k)=a_k]$ ——— (ii)

from (i) & (ii) we conclude that,

$Pr[X(n_1+\tau)=a_1, X(n_2+\tau)=a_2, \dots, X(n_k+\tau)=a_k] = Pr[X(n_1)=a_1, X(n_2)=a_2, \dots, X(n_k)=a_k]$

Hence, X_n is stationary random process.

(c) if coin is biased then probability distribution for X_n can be given as

$$Pr[X(n)=x] = \begin{cases} p & , x=1 \text{ (heads)} \\ 1-p & , x=-1 \text{ (tails)} \end{cases}$$

As events neither depend on previous nor on n. So, identical & independent events.

$m_{X(n)} = p-(1-p) = 2p-1 \quad \forall n \qquad (\text{let } \mu = 2p-1)$

$C_X(n_1,n_2) = E[(X(n_1)-m_{X(n_1)})(X(n_2)-m_{X(n_2)})]$

$= E[X(n_1)X(n_2)] - \mu^2 + \cancel{\mu^2} - \cancel{\mu^2}$

$= E[X(n_1)]\,E[X(n_2)] - \mu^2 = \mu^2 - \mu^2 = 0 = 0\,(n_1-n_2)$

& $R_X(n_1,n_2)$

As $C_X(n_1,n_2)$ & $R_X(n_1,n_2)$ is function of n_1-n_2

Hence, X_n is WSS random process here also (in biased coin case).

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$$Pr[X(n_1)=a_1, X(n_2)=a_2, \dots, X(n_k)=a_k] = Pr[X(n_1)=a_1]\, Pr[X(n_2)=a_2] \dots Pr[X(n_k)=a_k] \quad \text{— (i)}$$

because independent events for $a_i \in \{-1,1\}\ \forall i$

As Pr value will not depend on n & hence,

$Pr[X(n_i+\tau)=a_i] = Pr[X(n_i)=a_i]$

$$Pr[X(n_1+\tau)=a_1, X(n_2+\tau)=a_2, \dots, X(n_k+\tau)=a_k] = Pr[X(n_1+\tau)=a_1] \dots Pr[X(n_k+\tau)=a_k]$$

$$= Pr[X(n_1)=a_1]\, Pr[X(n_2)=a_2] \dots Pr[X(n_k)=a_k] \quad \text{— (ii)}$$

from (i) & (ii) we conclude that,

$Pr[X(n_1+\tau)=a_1, X(n_2+\tau)=a_2, \dots, X(n_k+\tau)=a_k] = P[X(n_1)=a_1, X(n_2)=a_2, \dots, X(n_k)=a_k]$

Hence, X_n is stationary random process here also (in biased coin case)

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