Balancing a cart-pole on uneven terrain

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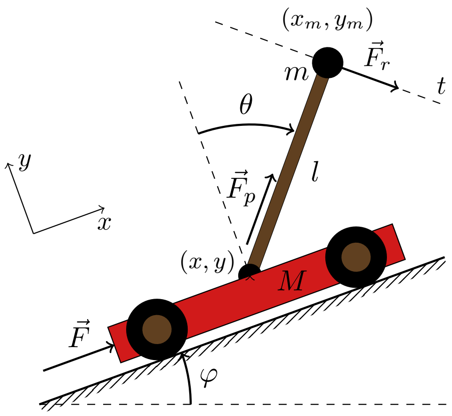
February 17, 2020

1 The cart-pole system

Our cart-pole system is depicted in Fig. 1. It is composed of a cart with mass M and a pendulum of length l and mass m. We presume that the center of the pendulum's mass m is at the top of the pole. Equivalently, l may represent half of the pole's length and m at its center. The position of the joint (x, y) is taken as the position of the cart. The cart is placed on a slope inclined at angle φ. The coordinate system is rotated so that the x-axis is parallel to the slope. The cart only moves along x, so we presume $y = 0$ at all times. The pendulum rotates freely, its inclination from the upward position in the clockwise direction is denoted by θ. The upward position means parallel to the y-axis. The cart is pushed by the control force F along the x-axis, a negative F pulls the cart.



F	– control force
F_r	– rotational force
M	– mass of the cart
m	– mass of the pole
l	– pole length
θ	– pole angle
φ	– slope angle
t	– pole motion tangent
(x, y)	– cart position
(x_m, y_m)	– pole top position

Figure 1: Forces working on the cart-pole composition while pushing (pulling) the cart with force $\vec{F}$.

2 Mathematical model

As the cart is pushed with some force F, it is opposed by the gravitational force working on the cart, as well as the tension between the cart and the pole at the joint. Let F_p denote the force of the cart on the pole and note that the pole is working on the cart with an equal force in the opposite direction. The motion of the cart can be expressed as:

$$M\ddot{x} = F - F_p \sin\theta - Mg\sin\varphi. \tag{1}$$

Now consider the motion of the pendulum, i.e. the motion of its top:

$$(x_m, y_m) = (x + l\sin\theta, l\cos\theta). \tag{2}$$

Forces working on the pendulum are the force F_p at the joint and the gravitational force. The resultant force $F_m = m \cdot (\ddot{x}_m, \ddot{y}_m)$ is therefore:

$$m\ddot{x}_m = F_p \sin\theta - mg\sin\varphi, \tag{3}$$

$$m\ddot{y}_m = F_p \cos\theta - mg\cos\varphi. \tag{4}$$

From (1) and (3) we get:

$$M\ddot{x} = F - m\ddot{x}_m - (M+m)g\sin\varphi. \tag{5}$$

Second order time derivate of (2) gives us

$$\ddot{x}_m = \ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta, \tag{6}$$

$$\ddot{y}_m = -l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta, \tag{7}$$

hence (5) reformulates as:

$$\ddot{x} = \frac{F + ml(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta)}{M+m} - g\sin\varphi. \tag{8}$$

Rotation of the pendulum is caused by the rotational force F_r, which is the projection of the resultant force $F_m = m(\ddot{x}_m, \ddot{y}_m)$ on tangent t. Its magnitude can therefore be computed as:

$$F_r = m\ddot{x}_m\cos\theta - m\ddot{y}_m\sin\theta. \tag{9}$$

This can further be evaluated either through (3)-(4) or (6)-(7). In the first case we get:

$$F_r = mg\sin(\theta - \varphi), \tag{10}$$

while in the second case we get:

$$F_r = m(\ddot{x}\cos\theta + l\ddot{\theta}). \tag{11}$$

Equality of (10) and (11) gives us:

$$\ddot{\theta} = \frac{g}{l}\sin(\theta - \varphi) - \frac{\ddot{x}}{l}\cos\theta. \tag{12}$$

Substituting $\ddot{x}$ with the right-hand side of (8) gives us:

$$\ddot{\theta} = \frac{(M+m)g\sin(\theta-\varphi) - \cos\theta\left(F + ml\dot{\theta}^2\sin\theta - (M+m)g\sin\varphi\right)}{(M+m)l - ml\cos^2\theta}. \tag{13}$$

The mathematical model of the cart-pole system on a slope is comprised of (8) and (13).