What is Machine Learning?

$$ \text{data} + \text{model} \rArr \text{prediction}$$

- data : observations, could be actively or passively acquired (meta-data).
- model : assumptions, based on previous experience (other data! transfer learning etc), or beliefs about the regularities of the universe. Inductive bias.
- prediction : an action to be taken or a categorization or a quality score.

Two important Gaussian Properties

- Sum of Gaussianv

  Sum of Gaussian variables is also Gaussian.

$$ y_{i} \sim \mathcal{N}(\mu_{i},\sigma_{i}^{2})$$

And the sum is distributed as

$$\sum y_{i} \sim \mathcal{N}(\sum \mu_{i},\sum \sigma_{i}^{2})$$

Aside: As sum increase, sum of non-Gaussian, finite variance variables is also Gaussian because of **central limit theorem**.

- Scaling a Gaussian Scaling a Gaussian leads to a Gaussian.

  $$\omega*y \sim \mathcal{N}(\omega \mu, \omega^{2}\sigma^{2})$$

The **central limit theorem** (CLT) establishes that, in some situations, when independent random variables are added, their properly normalized sum tends toward a normal distribution (informally a "bell curve") even if the original variables themselves are not normally distributed.

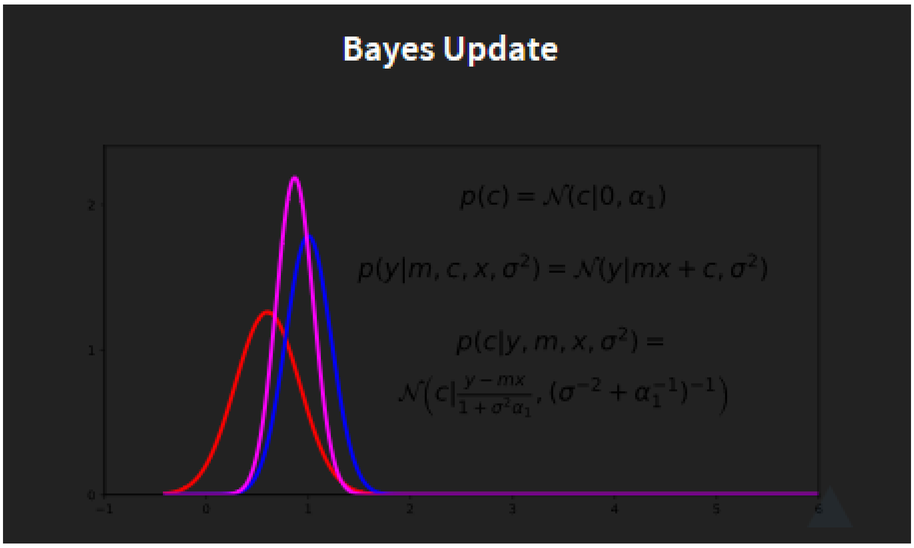
Prior Distribution

- Bayesian inference requires a prior on the parameters.
- The prior represents your belief *before* you see the data of the likely value of the parameters.
- For linear regression, consider a Gaussian prior on the intercept:

  $$ c\sim \mathcal{N}(0, \alpha_1)$$

Posterior Distribution

- Posterior distribution is found by combining the prior with the likelihood.
- Posterior distribution is your belief *after* you see the data of the likely value of the parameters.
- The posterior is found through **Bayes' Rule** $$p(c|y) = \frac{p(y|c)p(c)}{p(y)}$$ The $p(y|c)$ likelihood is not a density over c, it's a function of c. Here, c is a parameter of this density. The normalization step, e.g. find the suitable way to compute the $p(y)$ is the most difficult setp.



The red line describe the probablity for c with: $$p(c) = \mathcal{N}(c | 0,\alpha_{1})$$ the blue line stands for the observation, with: $$p(y|m,c,x,\sigma^{2}) = \mathcal{N}(y|mx+c,\sigma^{2})$$ Note that, this is a likelihood function over c not a distribution. based on the bayes'rule, the posterior could be written as: $$P(c|y,m,x,\sigma^{2}) = \mathcal{N}(c|\frac{y-mx}{1+\sigma^{2}}\alpha, (\sigma^{-2}+\alpha_{1}^{-1})^{-1})$$ Math Trick: $$p(c) = \frac{1}{\sqrt{2\pi\alpha_1}} \exp\left(-\frac{1}{2\alpha_1}c^2\right)$$ $$p(\mathbf{y}|\mathbf{x}, c, m, \sigma^2) = \frac{1}{\left(2\pi\sigma^2\right)^{\frac{n}{2}}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n(y_i - mx_i - c)^2\right)$$ $$p(c| \mathbf{y}, \mathbf{x}, m, \sigma^2) = \frac{p(\mathbf{y}|\mathbf{x}, c, m, \sigma^2)p(c)}{p(\mathbf{y}|\mathbf{x}, m, \sigma^2)}$$

$$p(c| \mathbf{y}, \mathbf{x}, m, \sigma^2) = \frac{p(\mathbf{y}|\mathbf{x}, c, m, \sigma^2)p(c)}{\int p(\mathbf{y}|\mathbf{x}, c, m, \sigma^2)p(c) \text{d} c}$$ $$p(c| \mathbf{y}, \mathbf{x}, m, \sigma^2) \propto p(\mathbf{y}|\mathbf{x}, c, m, \sigma^2)p(c)$$

$$\begin{aligned} \log p(c | \mathbf{y}, \mathbf{x}, m, \sigma^2) =&-\frac{1}{2\sigma^2} \sum_{i=1}^n(y_i-c - mx_i)^2-\frac{1}{2\alpha_1} c^2 + \text{const}\ = &-\frac{1}{2\sigma^2}\sum_{i=1}^n(y_i-mx_i)^2 - \left(\frac{n}{2\sigma^2} + \frac{1}{2\alpha_1}\right)c^2\ & + c\frac{\sum_{i=1}^n(y_i-mx_i)}{\sigma^2}, \end{aligned}$$

complete the square of the quadratic form to obtain $$\log p(c | \mathbf{y}, \mathbf{x}, m, \sigma^2) = -\frac{1}{2\tau^2}(c - \mu)^2 +\text{const}$$ where $$\tau^2 = \left(n\sigma^{-2} +\alpha_1^{-1}\right)^{-1}$$ and

$\mu = \frac{\tau^2}{\sigma^2} \sum_{i=1}^n(y_i-mx_i)$.

Piror comes from the model, where we think about it. And likelihood is coming from the data.

Stages to Derivation of the Posterior

- Multiply likelihood by prior
    - they are "exponentiated quadratics", the answer is always also an exponentiated quadratic because $$\exp(a^2)\exp(b^2) = \exp(a^2 + b^2)$$
- Complete the square to get the resulting density in the form of a Gaussian.
- Recognise the mean and (co)variance of the Gaussian. This is the estimate of the posterior.

Multivariate Regression Likelihood

- Noise corrupted data point $$y_{i} = \mathcal{w}^{T}X_{i,:}+\epsilon_{i}$$
- Multivariate regression likelihood: $$p(y|X,w)=\frac{1}{(2\pi\sigma^{2})}exp(-\frac{1}{2\sigma^{2}}\sum(y_{i}-w^{T}x_{i,:})^2)$$
- Multivariate Gaussian piroi: $$p(w)=\frac{1}{(2\pi\sigma^{2})^{p/2}}exp(-\frac{1}{2\sigma^{2}}w^{T}w)$$

The independent multivariate Gaussian could be seen as the independent Gaussian and multiple them and rotate the results.

Independent Gaussians:

$$p(w,h)=\frac{1}{\sqrt{2\pi\alpha_1}\sqrt{2\pi\alpha_2}} \exp(-\frac{1}{2}(\frac{(w-\mu_1)^2}{\sigma_1^2}+\frac{(h-\mu_2)^2}{\sigma_2^2}))$$

and we can write it with linear algebra form: $$p(w,h)=\frac{1}{\sqrt{2\pi\alpha_1}\sqrt{2\pi\alpha_2}} \exp(-\frac{1}{2}(\begin{bmatrix} w\ h \end{bmatrix}-\begin{bmatrix} u_1\ u_2 \end{bmatrix})^T(\begin{bmatrix} \sigma_1 & 0\ 0 & \sigma_2 \end{bmatrix}\begin{bmatrix} w\ h \end{bmatrix}-\begin{bmatrix} u_1\ u_2 \end{bmatrix}))$$

and then, rename it: $$p(y)=\frac{1}{|2 \pi*D|^\frac{1}{2}} \exp(-\frac{1}{2}(y-\mu)^TD^{-1}(Y-\mu))$$ $|D|$ means the determinant of the matrix.

Correlated Gaussian

Form correlated from original by rotating the data space using matrx R. $$p(y)=\frac{1}{|2 \pi*D|^\frac{1}{2}} \exp(-\frac{1}{2}(R^Ty-R^T\mu)^TD^{-1}(R^TY-R^T\mu))$$ $$p(y)=\frac{1}{|2 \pi*D|^\frac{1}{2}} \exp(-\frac{1}{2}(y-\mu)^TRD^{-1}R^T(Y-\mu))$$ this gives a covariance matric: $$C^{-1}=RD^{-1}R^T$$ which in some view is the result of the principal compoment.

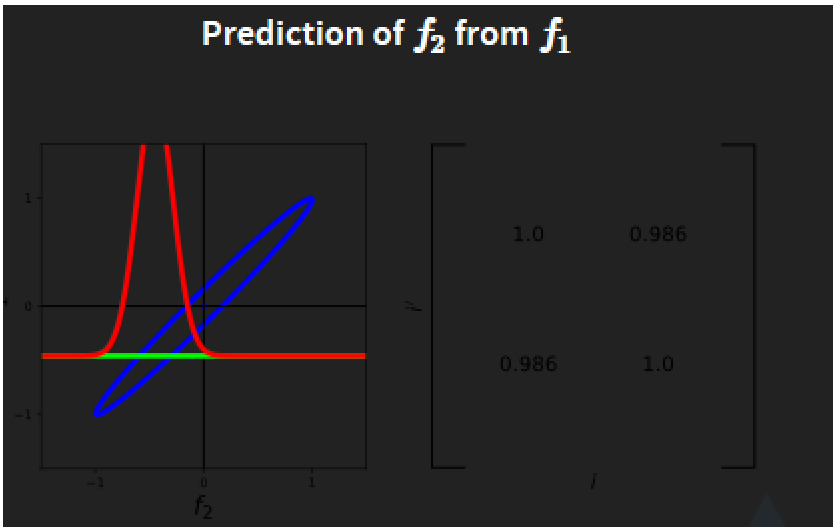
Multivariate Consequence

if $$x \sim \mathcal{N}(\mu,\varSigma)$$ and $$y=Wx$$ then $$y\sim \mathcal{N}(W\mu,W\varSigma W^T)$$

we can say the first equaption is the prior of x, the second is the likelihood, the last is the marginal of y. If we set $\mu = 0, \varSigma = I$, so it is just the inverse of PCA.

Prediction with Correlated Gaussians

- Prediction of $\mathbf{f}_*$ from $\mathbf{f}$ requires multivariate *conditional density*.
- Multivariate conditional density is *also* Gaussian. $$ p(\mathbf{f}*|\mathbf{f}) = \mathcal{N}\left(\mathbf{f}|\mathbf{K}_{,\mathbf{f}}\mathbf{K}{\mathbf{f},\mathbf{f}}^{-1}\mathbf{f},\mathbf{K}{,}-\mathbf{K}{*,\mathbf{f}} \mathbf{K}{\mathbf{f},\mathbf{f}}^{-1}\mathbf{K}_{\mathbf{f},*}\right) $$
- Here covariance of joint density is given by $$ \mathbf{K} = \begin{bmatrix} \mathbf{K}{\mathbf{f}, \mathbf{f}} & \mathbf{K}{, \mathbf{f}}\ \mathbf{K}{\mathbf{f}, *} & \mathbf{K}{, *}\end{bmatrix} $$



Take the picture as the example:

since that example is in 1D, all the values are scalar. $f=f1=-0.4$, $K_{f,f}=1,K_{,f}=0.98,K_{,}=1$, so $$p(p_2|p_1)=\mathcal{N}(p_2|0.981*(-0.4),1-0.98 10.98)$$ $$p(p_2|p_1)= \mathcal{N}(p_2|-0.392,0.0396)$$ the variance is 0.0396, and the standard variance is 0.2.