What is Machine Learning?

$\text{data} + \text{model} \Rightarrow \text{prediction}$

- data : observations, could be actively or passively acquired (meta-data).
- model : assumptions, based on previous experience (other data! transfer learning etc), or beliefs about the regularities of the universe. Inductive bias.
- prediction : an action to be taken or a categorization or a quality score.

Two important Gaussian Properties

- Sum of Gaussianv

  Sum of Gaussian variables is also Gaussian.

$y_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$

And the sum is distributed as

$\sum y_i \sim \mathcal{N}(\sum \mu_i, \sum \sigma_i^2)$

Aside: As sum increase, sum of non-Gaussian, finite variance variables is also Gaussian because of **central limit theorem**.

- Scaling a Gaussian Scaling a Gaussian leads to a Gaussian.

$\omega * y \sim \mathcal{N}(\omega\mu, \omega^2\sigma^2)$

The **central limit theorem** (CLT) establishes that, in some situations, when independent random variables are added, their properly normalized sum tends toward a normal distribution (informally a "bell curve") even if the original variables themselves are not normally distributed.

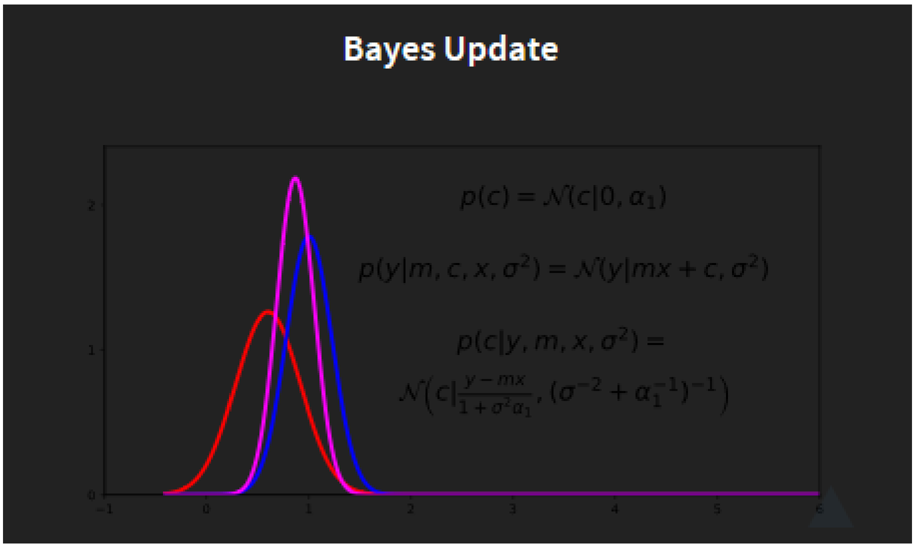
Prior Distribution

- Bayesian inference requires a prior on the parameters.
- The prior represents your belief *before* you see the data of the likely value of the parameters.
- For linear regression, consider a Gaussian prior on the intercept:

$$c \sim \mathcal{N}(0, \alpha_1) \tag{3}$$

Posterior Distribution

- Posterior distribution is found by combining the prior with the likelihood.
- Posterior distribution is your belief *after* you see the data of the likely value of the parameters.
- The posterior is found through **Bayes' Rule** $p(c|y) = \frac{p(y|c)p(c)}{p(y)}$ The $p(y|c)$ likelihood is not a density over c, it's a function of c. Here, c is a parameter of this density. The normalization step, e.g. find the suitable way to compute the $p(y)$ is the most difficult setp.



The red line describe the probablity for c with: $p(c) = \mathcal{N}(c|0, \alpha_1)$ the blue line stands for the observation, with: $p(y|m, c, x, \sigma^2) = \mathcal{N}(y|mx + c, \sigma^2)$ Note that, this is a likelihood function over c not a distribution. based on the bayes'rule, the posterior could be written as: $P(c|y, m, x, \sigma^2) = \mathcal{N}(c|\frac{y-mx}{1+\sigma^2}\alpha, (\sigma^{-2} + \alpha_1^{-1})^{-1})$

Math Trick: $p(c) = \frac{1}{\sqrt{2\pi\alpha_1}}\exp\left(-\frac{1}{2\alpha_1}c^2\right)$ $p(\mathbf{y}|\mathbf{x}, c, m, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - mx_i - c)^2\right)$

$p(c|\mathbf{y}, \mathbf{x}, m, \sigma^2) = \frac{p(\mathbf{y}|\mathbf{x}, c, m, \sigma^2)p(c)}{p(\mathbf{y}|\mathbf{x}, m, \sigma^2)}$

$p(c|\mathbf{y}, \mathbf{x}, m, \sigma^2) = \frac{p(\mathbf{y}|\mathbf{x}, c, m, \sigma^2)p(c)}{\int p(\mathbf{y}|\mathbf{x}, c, m, \sigma^2)p(c)\mathrm{d}c}$ $p(c|\mathbf{y}, \mathbf{x}, m, \sigma^2) \propto p(\mathbf{y}|\mathbf{x}, c, m, \sigma^2)p(c)$

$$\begin{aligned}\log p(c|\mathbf{y}, \mathbf{x}, m, \sigma^2) &= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - c - mx_i)^2 - \frac{1}{2\alpha_1}c^2 + \text{const} \\ &= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - mx_i)^2 - \left(\frac{n}{2\sigma^2} + \frac{1}{2\alpha_1}\right)c^2 \\ &\quad + c\frac{\sum_{i=1}^{n}(y_i - mx_i)}{\sigma^2},\end{aligned}$$

complete the square of the quadratic form to obtain $\log p(c|\mathbf{y}, \mathbf{x}, m, \sigma^2) = -\frac{1}{2\tau^2}(c - \mu)^2 + \text{const}$ where $\tau^2 = \left(n\sigma^{-2} + \alpha_1^{-1}\right)^{-1}$ and

$\mu = \frac{\tau^2}{\sigma^2}\sum_{i=1}^{n}(y_i - mx_i)$.

Piror comes from the model, where we think about it. And likelihood is coming from the data.

Stages to Derivation of the Posterior

- Multiply likelihood by prior
    - they are "exponentiated quadratics", the answer is always also an exponentiated quadratic because $\exp(a^2)\exp(b^2) = \exp(a^2 + b^2)$
- Complete the square to get the resulting density in the form of a Gaussian.
- Recognise the mean and (co)variance of the Gaussian. This is the estimate of the posterior.

Multivariate Regression Likelihood

- Noise corrupted data point $y_i = w^T X_{i,:} + \epsilon_i$
- Multivariate regression likelihood: $p(y|X, w) = \frac{1}{(2\pi\sigma^2)}exp(-\frac{1}{2\sigma^2}\sum(y_i - w^T x_{i,:})^2)$

- Multivariate Gaussian piroi: $p(w) = \frac{1}{(2\pi\sigma^2)^{p/2}} exp(-\frac{1}{2\sigma^2} w^T w)$

The independent multivariate Gaussian could be seen as the independent Gaussian and multiple them and rotate the results.

Independent Gaussians:

$p(w, h) = \frac{1}{\sqrt{2\pi\alpha_1}\sqrt{2\pi\alpha_2}} \exp(-\frac{1}{2}(\frac{(w-\mu_1)^2}{\sigma_1^2} + \frac{(h-\mu_2)^2}{\sigma_2^2}))$

and we can write it with linear algebra form:
$p(w, h) = \frac{1}{\sqrt{2\pi\alpha_1}\sqrt{2\pi\alpha_2}} \exp(-\frac{1}{2}(\begin{bmatrix} w \\ h \end{bmatrix} - \begin{bmatrix} u_1 \\ u_2 \end{bmatrix})^T (\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} w \\ h \end{bmatrix} - \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}))$
and then, rename it: $p(y) = \frac{1}{|2\pi * D|^{\frac{1}{2}}} \exp(-\frac{1}{2}(y-\mu)^T D^{-1}(Y-\mu))$ $|D|$ means the determinant of the matrix.

Correlated Gaussian

Form correlated from original by rotating the data space using matrx R.
$p(y) = \frac{1}{|2\pi * D|^{\frac{1}{2}}} \exp(-\frac{1}{2}(R^T y - R^T \mu)^T D^{-1}(R^T Y - R^T \mu))$ $p(y) = \frac{1}{|2\pi * D|^{\frac{1}{2}}} \exp(-\frac{1}{2}(y-\mu)^T R D^{-1} R^T (Y-\mu))$
this gives a covariance matric: $C^{-1} = RD^{-1}R^T$ which in some view is the result of the principal compoment.

Multivariate Consequence

if $x \sim \mathcal{N}(\mu, \Sigma)$ and $y = Wx$ then $y \sim \mathcal{N}(W\mu, W\Sigma W^T)$

we can say the first equaption is the prior of x, the second is the likelihood, the last is the marginal of y. If we set $\mu = 0, \Sigma = I$, so it is just the inverse of PCA.

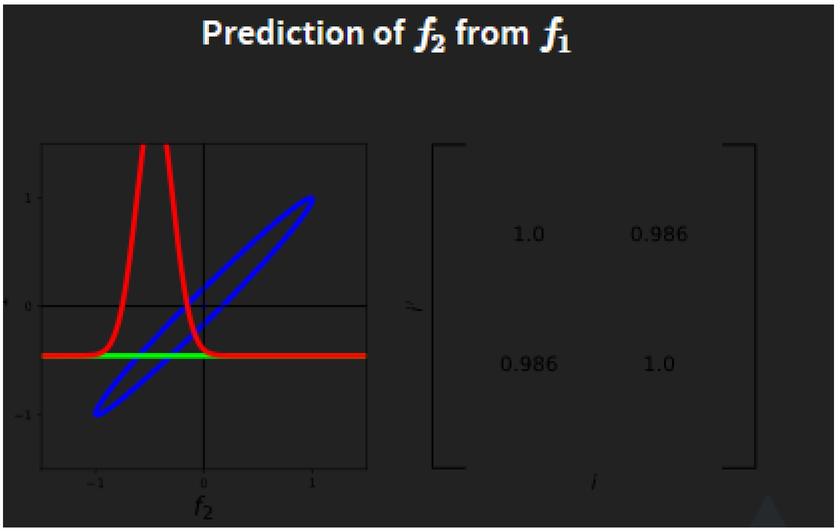
Prediction with Correlated Gaussians

- Prediction of $\mathbf{f}_*$ from $\mathbf{f}$ requires multivariate *conditional density*.
- Multivariate conditional density is *also* Gaussian.

$$p(\mathbf{f}_*|\mathbf{f}) = \mathcal{N}\left(\mathbf{f}_*|\mathbf{K}_{*,\mathbf{f}}\mathbf{K}_{\mathbf{f},\mathbf{f}}^{-1}\mathbf{f}, \mathbf{K}_{*,*} - \mathbf{K}_{*,\mathbf{f}}\mathbf{K}_{\mathbf{f},\mathbf{f}}^{-1}\mathbf{K}_{\mathbf{f},*}\right) \tag{1}$$

- Here covariance of joint density is given by

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{\mathbf{f},\mathbf{f}} & \mathbf{K}_{*,\mathbf{f}} \\ \mathbf{K}_{\mathbf{f},*} & \mathbf{K}_{*,*} \end{bmatrix} \tag{2}$$



Take the picture as the example:

since that example is in 1D, all the values are scalar. $f = f1 = -0.4$, $K_{f,f} = 1, K_{*,f} = 0.98, K_{*,*} = 1$, so $p(p_2|p_1) = \mathcal{N}(p_2|0.98 * 1 * (-0.4), 1 - 0.98 * 1 * 0.98)$ $p(p_2|p_1) = \mathcal{N}(p_2| - 0.392, 0.0396)$ the variance is 0.0396, and the standard variance is 0.2.