1. Equations Reducible to Exact Form, Orthogonal Trajectories

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EQUATIONS REDUCIBLE TO EXACT FORM

Integrating Factor

non exact DE × IF = exact DE

Ways to find integrating factor

Case 1, 2

Given $M\,dx + N\,dy = 0$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

If:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \phi(x), \quad IF = e^{\int \phi(x)\,dx}$$

If:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \phi(y) \quad IF = e^{-\int \phi(y)\,dy}$$

Case 3

If M and N are homogeneous functions of same degree:

$$IF = \frac{1}{Mx + Ny}$$

Case 4

Given,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If it can be written as

$$\underbrace{y \cdot f_1(xy)}_{M}\,dx + \underbrace{x \cdot f_2(xy)}_{N}\,dy = 0$$

Then

$$IF = \frac{1}{Mx - Ny}$$

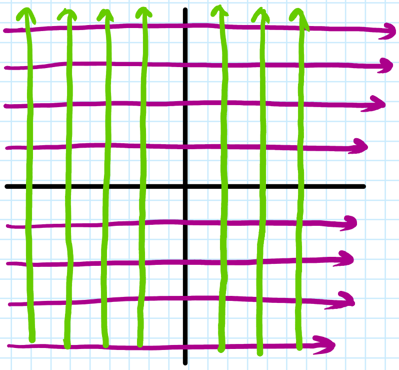
APPLICATIONS OF FIRST ORDER DERIVATIVES

ORTHOGONAL TRAJECTORIES

Two families of plane curves F_1 and F_2 are said to be orthogonal trajectories of

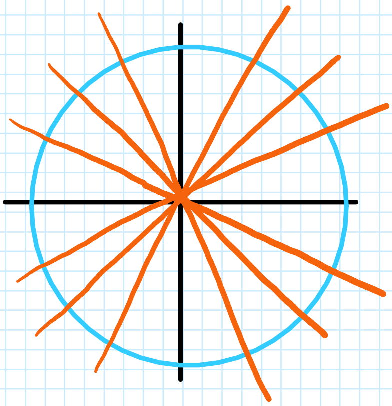
ORTHOGONAL TRAJECTORIES

Two families of plane curves F_1 and F_2 are said to be orthogonal trajectories of each other if every member of one family intersects every member of the other orthogonally (at 90°).



$x = c_1$

$y = c_2$



$x^2 + y^2 = c^2$

$y = mx$

Steps to find orthogonal trajectory of some F_1 in Cartesian form

Let $f(x,y) = c_1$ be the Cartesian equation of F_1

① Form differential equation of F_1

$$\frac{dy}{dx} = g(x,y)$$

② Let F_2 be the OT of F_1; $m_1 \longrightarrow$ slope of tangent drawn to some member of F_1

$m_2 \rightarrow$ slope of tangent drawn to some member of F_2

Since they cut at 90°,

$$m_1 \times m_2 = -1$$

$$m_2 = \frac{-1}{m_1} = \frac{-1}{g(x,y)}$$

$$\boxed{\frac{dy_2}{dx_2} = \frac{-1}{g(x,y)}} \longrightarrow \text{D.E. of } F_2$$

③ Solve DE of F_2.

GS of this eqn represents F_2.

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[OR]

① Form DE of F_1
② Replace $\frac{dy}{dx}$ by $\left(\frac{-dx_2}{dy_2}\right)$
y_1 by $\frac{-1}{y_1}$
or vice versa, to get DE for F_2
③ Solve to get eqn for F_2

→ Use this for problems

Self orthogonal curves

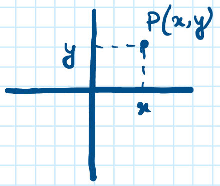
A family of curves is said to be self orthogonal if the members of the family intersect each other at 90°, if at all they intersect.

Eg: $y^2 = 4c(c+x)$

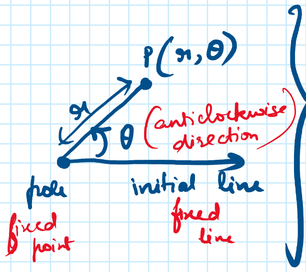
Algorithm to prove F_1 is self orthogonal

① Form DE
② Replace $\frac{dy}{dx}$ with $\frac{-dx}{dy}$ to get DE of orthogonal trajectory (DE_{orth})
③ If $DE_{orth} = DE$,
given family is self orthogonal

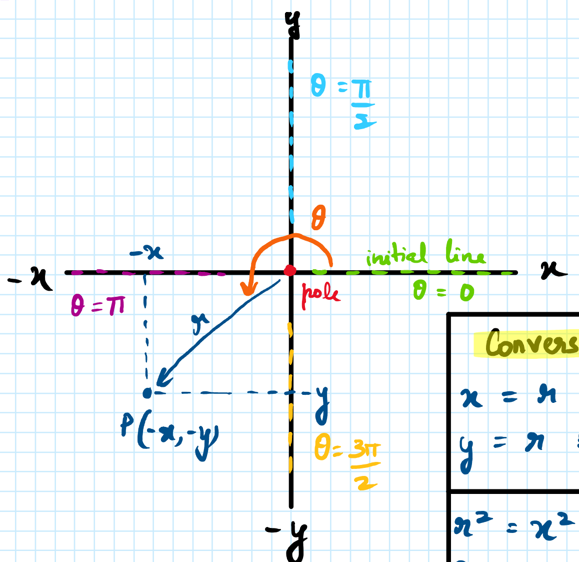
Orthogonal trajectories in polar form



} Cartesian



} Polar

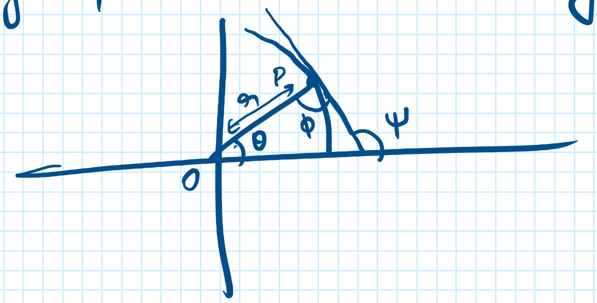


Conversion:

$x = r\cos\theta$
$y = r\sin\theta$

$r^2 = x^2 + y^2$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Angle b/w radius vector and tangent



$$\phi_1 + \phi_2 = \frac{\pi}{2}$$

$$\boxed{\tan\phi_1 \times \tan\phi_2 = -1}$$

condition to cut orthogonally in polar system