Homogenous Functions, Euler's Theorem

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HOMOGENEOUS FUNCTIONS

A function $u(x,y)$ is said to be homogeneous of degree n, if it can be written as:

$$\boxed{\begin{gathered} u(x,y) = x^n \cdot \phi\left(\frac{y}{x}\right) \\ [\text{OR}] \\ u(x,y) = y^n \cdot \phi_1\left(\frac{x}{y}\right) \end{gathered}}$$

Eg: $u(x,y) = \dfrac{x^3+y^3}{\sqrt{x+y}} = \dfrac{x^3\left(1+\left(\frac{y}{x}\right)^3\right)}{\sqrt{x}\left(\sqrt{1+\left(\frac{y}{x}\right)}\right)}$

$u(x,y) = x^{\frac{5}{2}} \cdot \phi\left(\frac{y}{x}\right)$ ∴ It is homogeneous

Eg: $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

$= x^2\left[\tan^{-1}\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 \tan^{-1}\left(\frac{1}{y/x}\right)\right]$

$u = x^2 \cdot \phi\left(\frac{y}{x}\right)$

∴ It is homogeneous

NOTE:

Sum/difference of 2 homogeneous = Homogeneous w/ same degree

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The degrees of both are the same

NOTE:

Product of 2 homogeneous functions = Homogeneous w/ degree = sum of degrees of 2 functions

- Function w/ 3 variables

A function $u(x,y,z)$ is said to be homogenous with degree n if:

$$\boxed{\begin{gathered} u(x,y,z) = x^n \cdot \phi\left(\frac{y}{x}, \frac{z}{x}\right) \\ \text{[OR]} \\ u(x,y,z) = y^n \cdot \phi\left(\frac{x}{y}, \frac{z}{y}\right) \\ \text{[OR]} \\ u(x,y,z) = z^n \cdot \phi\left(\frac{x}{z}, \frac{y}{z}\right) \end{gathered}}$$

This can be extended for any number of independent variables

EULER'S THEOREM (for homogeneous functions)

If $u(x,y)$ is a homogeneous function of degree n,

$$\boxed{x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu}$$

PROOF

Given $u(x,y)$ is homogeneous fn. of degree n,

$$u(x,y) = x^n \cdot \phi\left(\frac{y}{x}\right) \quad \text{—— (1)}$$

Differentiating partially w.r.t. x,

$$\frac{\partial u}{\partial x} = nx^{n-1}\left[\phi\left(\frac{y}{x}\right)\right] + x^n\left[\phi'\left(\frac{y}{x}\right)\right]\left[\frac{-y}{x^2}\right]$$

Multiplying by x on both sides,

$$x\frac{\partial u}{\partial x} = nx^n\left[\phi\left(\frac{y}{x}\right)\right] - yx^{n-1}\phi'\left(\frac{y}{x}\right) \quad \text{—— (2)}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = x^n\left[\phi'\left(\frac{y}{x}\right)\right]\left(\frac{1}{x}\right)$$

Multiplying by y,

$$y\frac{\partial u}{\partial y} = yx^{n-1}\left[\phi'\left(\frac{y}{x}\right)\right] \quad \text{—— (3)}$$

(2) + (3)

②+③

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\left[\phi\left(\frac{y}{x}\right)\right]$$

$$= nu$$

NOTE:
If $u(x,y,z)$ is homogeneous w/ degree n,

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z} = nu$$

and so on for any no. of independent variables

Eg: $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\sin\left(\frac{x^2+y^2}{xy}\right)$

Find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Soln: $x^2\left[\tan^{-1}\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2\sin\left(\frac{x^2+y^2}{xy}\right)\right]$

$\Rightarrow$ It is homogeneous, $n=2$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2\tan^{-1}\left(\frac{y}{x}\right) - 2y^2\sin\left(\frac{x^2+y^2}{xy}\right)$$