Classroom Problems -2

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Q. Find the maximum and minimum values of $F(x,y) = x^3 - 12x + y^3 - 3y^2 - 9y$

Soln: To get critical points, equate first order derivative to zero

$F_x = 3x^2 - 12$ $\quad\quad$ $F_y = 3y^2 - 6y - 9$

Equating to 0,

$3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = \frac{12}{3} = 4$

$x = 2$ (or) $x = -2$

$3(y^2 - 2y - 3) = 0$

$y^2 + y - 3y - 3 = 0$

$y(y+1) - 3(y+1) = 0$

$(y-3)(y+1) = 0$

$y = 3$ (or) $y = -1$

Critical points $= (2,3), (2,-1), (-2,3), (-2,-1)$

To classify critical points:

$r = F_{xx} = 6x$ $\quad\quad$ $s = F_{xy} = 0$ $\quad\quad$ $t = F_{yy} = 6y - 6$

At $(2,3)$:

$r = 12$ $\quad\quad$ $s = 0$ $\quad\quad$ $t = 12$

$rt - s^2 = 144 > 0$

$(2,3)$ is a minimum point

At $(2,-1)$

$r = 12$ $\quad\quad$ $s = 0$ $\quad\quad$ $t = -12$

$rt - s^2 = -144 < 0$

$(2,-1)$ is a saddle point

At $(-2,3)$

$r = -12$ $\quad\quad$ $s = 0$ $\quad\quad$ $t = 12$

$rt - s^2 < 0$

$(-2,3)$ is a saddle point

At $(-2,-1)$

$r = -12 \quad s = 0 \quad t = -12$

$rt - s^2 > 0$

$(-2,-1)$ is a maximum point

Maximum value $= 21$

Minimum value $= -43$

Q: $F(x,y) = 4x^2 + 2y^2 + 4xy - 10x - 2y - 3$

Minima, maxima?

Soln: $F_x = 8x + 4y - 10 \qquad F_y = 4y + 4x - 2$

Equating to 0,

$8x + 4y = 10 \qquad 4y + 4x = 2$

$4x = 2 - 4y$

$2(2-4y) + 4y = 10$

$4 - 8y + 4y = 10$

$4 - 4y = 10$

$-4y = 6$

$y = \frac{-6}{4} = \frac{-3}{2}$

$\Rightarrow \cancel{4}^{2}\left(\frac{-3}{\cancel{2}}\right) + 4x = 2$

$-6 + 4x = 2$

$4x = 8$

$x = 2$

$r = F_{xx} = 8 \qquad s = F_{xy} = 4 \qquad t = 4$

$rt - s^2 = 8(4) - 16 > 0$

$\Rightarrow \left(2, \frac{-3}{2}\right)$ is a minimum point

Minimum value $= -11.5$

Q. Locate the stationary points of $F(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$. Discuss their nature.

Soln. $F_x = 3x^2 + 3y^2 - 30x + 72$

$F_y = 3x(2y) - 30y = 6xy - 30y$

Equating to 0.

$6xy - 30y = 0$

$6y(x-5) = 0$

$y = 0$ [or] $x = 5$

Put $y=0$ in $F_x = 0$

$\Rightarrow 3x^2 - 30x + 72 = 0$

$x^2 - 10x + 24 = 0$

$(x-6)(x-4) = 0$

$x = 6$ (or) $x = 4$ $\Rightarrow$ Two critical points = $(4,0), (6,0)$

Put $x=5$ in $F_x = 0$

$\Rightarrow 3(5)^2 + 3y^2 - 30(5) + 72 = 0$

$3y^2 = 3$

$y^2 = 1$

$y = 1$ (or) $y = -1$ $\Rightarrow$ Two crit points $= (5,1) (5,-1)$

$\therefore$ 4 critical points : $(4,0), (6,0), (5,1), (5,-1)$

$F_{xx} = 6x - 30 = r$

$F_{xy} = 6y = s$

$F_{yy} = 6x - 30 = t$

For $(4,0)$	For $(6,0)$
$r = -6,\ s = 0,\ t = -6$	$r = 6,\ s = 0,\ t = 6$
$rt - s^2 > 0,\ r < 0$	$rt - s^2 > 0,\ r > 0$

$r = -6,\ s = 0,\ t = -6$

$rt - s^2 > 0,\ r < 0$

$\Rightarrow (4,0)$ is maxima

$r = 6,\ s = 0,\ t = 6$

$rt - s^2 > 0,\ r > 0$

$\Rightarrow (6,0)$ is minima

For $(5,1)$

$r = 0,\ s = 1$

$rt - s^2 = -1 < 0$

$\Rightarrow (5,1)$ is saddle point

$\Rightarrow (5,-1)$ is saddle point

Q: Find the extreme value of $a^3x^2 + b^3y^2 + c^3z^2$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Soln: $\frac{\partial F}{\partial x} = a^3(2x)$ $\qquad \frac{\partial F}{\partial y} = b^3(2y)$ $\qquad \frac{\partial F}{\partial z} = c^3(2z)$

$\frac{\partial \phi}{\partial x} = \frac{-1}{x^2}$ $\qquad \frac{\partial \phi}{\partial y} = \frac{-1}{y^2}$ $\qquad \frac{\partial \phi}{\partial z} = \frac{-1}{z^2}$