28/3/24

Assignment 1

Q-1 Suggest sample spaces for the following experiments:

a) Three dice are rolled and their sum computed.

→ $S = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$

b) Two real numbers between 0 and 1 are chosen.

→ $S = \{(x, y) : 0 < x < 1,\ 0 < y < 1\}$

c) An American is chosen at random and is classified according to gender and age.

→ S = { Male-Young, Female-Young, Male-MiddleAge, Female-MiddleAge, Male-Senior, Female-Senior }

d) Two different integers are chosen between 1 and 10 and are listed in increasing order.

→ $S = \{(a, b) : 1 < a < b < 10 ;\ a, b \in Z^+\}$ (28 Pairs)

e) Two points are chosen at random on a yardstick and the distance between them is measured.

→ A standard Yardstick is typically 36 inches long, so, $S = \{d : 0 \leq d \leq 36\}$

Q.2 Consider the experiment to toss a coin three times and count the number of heads. Which sample spaces can be used to describe this experiment?

→ You can represent each outcome by the number of heads obtained in the three tosses.

$\therefore S = \{0, 1, 2, 3\}$

› Here, each outcome represents the count of heads obtained in the three coin tosses.

Q.3 If you roll a fair six-sided die, what is the probability of rolling an even number?

—4 Total number of possible outcomes when rolling a fair six-sided dice is,

$\therefore S = \{1, 2, 3, 4, 5, 6\}$

› So, the probability of rolling an even number on a fair six-sided die is:

$$\therefore \text{Probability (Even Number)} = \frac{\text{No. of Favourable Outcomes}}{\text{Total possible Outcomes}}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2} \,//$$

→ Therefore, the probability of rolling an even number is $\frac{1}{2}$ or 0.5.

Q.4 A bag contains 5 red balls and 3 blue balls. If one ball is drawn at random from the bag, what is the probability that it is red?

→ Given,

- No. of red balls = 5
- No. of blue balls = 3
- Total no. of balls = 5 + 3 = 8.

› So, the probability of drawing a red ball from the bag is :

∴ Probability (Red Ball) = $\frac{\text{No. of red balls}}{\text{Total balls}}$

$= \frac{5}{8}$ //

→ Therefore, the probability of drawing a red ball is $\frac{5}{8}$ or 0.625.

Q.5 If you flip a fair coin three times, what is the probability of getting exactly two heads ?

→ Total number of possible outcomes when flipping a fair coin three times is $2^3 = 8$.

$\therefore S = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,T,T), (T,T,H), (T,H,T), (T,H,H)\}$

- Here, 'H' = Heads and 'T' = Tails

› So, the probability of getting exactly two heads when flipping a fair coin three times is :

∴ Probability (Exactly two heads) = $\frac{\text{No. of outcomes with exactly 2 heads}}{\text{Total Outcomes}}$

$= \frac{3}{8}$ //

⇒ Therefore, the probability of getting exactly two heads is $\frac{3}{8}$.

Q.6 In a group of 30 people, 18 are men and 12 are women. If one person is selected at random from the group, what is the probability that is a man?

→ Given,

- No. of men = 18.
- No. of women = 12.
- Total no. of people = 30.

» So, the probability of selecting a man is:

∴ Probability (Man) $= \frac{\text{No. of men}}{\text{Total people}}$

$= \frac{18}{30}$

$= \frac{3}{5}$. //

⇒ Therefore, the probability that a randomly selected person from the group is a man is $\frac{3}{5}$ or 0.6.

Q.7 In a group of students, 60% are girls and 40% are boys. If 25% of the girls and 20% of the boys are left-handed, what is the probability that a left-handed student selected at random is a girl?

→ Given,

- $P(G)$ = Probability of selecting a girl = 0.60
- $P(B)$ = Probability of selecting a boy = 0.40
- $P(L|G)$ = Probability of being left-handed given that it is a girl = 0.25
- $P(L|B)$ = Probability of being left-handed given that it is a boy = 0.20

> Now, the probability of being a left-handed student, $P(L)$, using law of total probability is:

$$\therefore P(L) = P(L|G) \times P(G) + P(L|B) \times P(B)$$
$$= (0.25) \times (0.60) + (0.20) \times (0.40)$$
$$= 0.15 + 0.08$$
$$= 0.23$$

> So, the probability that a left-handed student selected is a girl is: (Bayes' Theorem)

$$\therefore P(G|L) = \frac{P(L|G) \times P(G)}{P(L)} = \frac{(0.25) \times (0.60)}{0.23}$$
$$= \frac{0.15}{0.23} \approx 0.6522 \;//$$

⇒ Therefore, the probability that a left-handed student selected is a girl is 0.6522.

Q.8 A company hired 60% of its employees from university A and 40% from university B. If 30% of the employees from university A are managers and 20% from university B are managers, what is the probability that a randomly selected manager is from university B?

→ Given,

- $P(A)$ = Probability of employees from University A $= 0.60$
- $P(B)$ = Probability of employees from University B $= 0.40$
- $P(M/A)$ = Probability of employees from University A are managers. $= 0.30$
- $P(M/B)$ = Probability of employees from University B are managers. $= 0.20$

> Now, the probability of being manager, $P(M)$ is:

$$\therefore P(M) = P(M/A) \times P(A) + P(M/B) \times P(B)$$

$$= (0.30) \times (0.60) + (0.20) \times (0.40)$$

$$= 0.26$$

> So, the probability that a randomly selected is from university B is: (Bayes' Theorem)

$$\therefore P(B/M) = \frac{P(M/B) \times P(B)}{P(M)} = \frac{(0.20) \times (0.40)}{0.26} = \frac{0.08}{0.26}$$

$$= 0.307$$

→ Therefore, the probability that a selected manager is from University B is 0.307.

Q.9 On average, 5 customers arrive at store every hour. What is the probability that exactly 3 customers will arrive in the next hour?

→ Here, we can use the Poisson distribution which is defined as:

$$\therefore P(X=k) = \frac{e^{-\lambda}\lambda^{k}}{k!} \quad \text{; Poisson Probability Mass Function} \quad (1)$$

where,

- $P(X=k)$ is the probability of observing k events
- λ is the average rate of arrivals per unit time
- k is the number of events observed.

> Given, $k=3$ $\lambda=5$, so substitute the values in eq (1), so that the probability that exactly 3 customers will arrive in the next hour is:

$$\therefore P(X=3) = \frac{e^{-5}5^{3}}{3!} = \frac{(0.006737946)\times(125)}{3\times2\times1}$$

$$= \frac{0.08422375}{6}$$

$$= 0.01403729 \;//$$

⇒ Therefore, the probability that exactly 3 customers will arrive in the next hour is 0.01403729 or 14.04%.

Q.10 The lifetime of a certain type of battery follows an exponential distribution with a mean lifetime of 500 hours. What is the probability that a randomly selected battery lasts less than 400 hours?

→ Here, we can use the probability density function (PDF) of exponential distribution which is defined as:

$$\therefore P(X) = \lambda e^{-\lambda x} \quad \text{--- (1)}$$

; x is random variable
λ is the inverse of mean time i.e. $\frac{1}{\mu}$

› Given, that the mean lifetime μ is 500 hours, so substitute the value in eq (1)

$$\therefore P(x) = \frac{1}{\mu} e^{-x/\mu} = \frac{1}{500} e^{-x/500}$$

› So, the probability that a randomly selected battery lasts less than 400 hours is:

Integrate the above eq from 0 to 400.

$$\therefore P(X < 400) = \int_0^{400} \frac{1}{500} e^{-x/500} \cdot dx$$

$$= \left[-500 \times \frac{1}{500} e^{-x/500}\right]_0^{400}$$

$$= \left[-e^{-x/500}\right]_0^{400}$$

$$= -e^{-400/500} - (-e^{-0/500}) = -e^{-0.8} + 1$$

$$= 1 - 0.4493$$

$$= 0.5507$$

⇒ Therefore, the probability that a randomly selected battery last less than 400 hours is 0.5507 or 55.07%.

Q.11 The time between two consecutive arrivals at a shop follows an exponential distribution with a mean inter-arrival time of 10 minutes. What is the probability that a customer will arrive within the next 5 minutes?

→ Here, we can use the PDF of exponential distribution which is defined as:

$\therefore P(X) = \lambda e^{-\lambda x}$; x is random variable
λ is the inverse of mean time i.e. $1/\mu$

> Given, μ is 10 min, so the probability that a customer will arrive within the next 5 min:

Integrate the above eq from 0 to 5:

$$\therefore P(X \leq 5) = \int_0^5 \frac{1}{10} e^{-x/10} \cdot dx$$

$$= \left[-10 \times \frac{1}{10} e^{-x/10}\right]_0^5$$

$$= \left[-e^{-x/10}\right]_0^5$$

$$= -e^{-5/10} - (-e^{-0/10}) = -e^{-0.5} + 1$$

$$= 1 - 0.6065$$

$$= 0.3935 //$$

⇒ Therefore, the probability that a customer will arrive within the next 5 minutes is 0.3935 or 39.35 %.

~~~~ X ~~~~ X ~~~~
~~~~