Math 2605. Week 2 Homework.

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1. Provide a set of vectors $\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_2, \overline{\mathbf{v}}_3, \overline{\mathbf{v}}_4$ in $\mathbb{R}^4$ that are linea rly independent, given that
$$\overline{\mathbf{v}}_1 = (0,1,2,1)^t.$$

2. Find a basis of the null space of A, $N(A)$, where
$$A = \begin{bmatrix} 3 & -1 & -1 & -3 \\ -2 & 2 & -2 & 2 \\ -1 & -1 & 3 & 1 \end{bmatrix}$$

3. Find the rank and nullity of A, if
$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ -1 & -2 & 6 & -7 \\ 2 & 4 & 3 & 5 \end{bmatrix}$$

4. Find the rank without any calculations:
$$A = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 7 & 5 \end{bmatrix}$$

5. For $\alpha = (2,4,5)^t$, what are the coordinates of α in the basis $\beta_1 = (1,1,1)^t$, $\beta_2 = (0,1,1)^t$, $\beta_3 = (1,1,0)^t$?

6. Take the bases of $\mathbb{R}^3$ given by
$$\{\overline{\mathbf{v}}\} = \left\{ \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 10 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 10 \\ 7 \end{pmatrix} \right\}, \qquad \{\overline{\mathbf{w}}\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$
Find the matrix that produces the change of basis ${}_{\{\overline{\mathbf{w}}\}}M_{\{\overline{\mathbf{v}}\}}$.

7. Find M, the matrix representation of the differentiation operator D, in the space $\mathcal{P}_3$ of polynomials of degree up to 3, with respect to the bases $\{\overline{\mathbf{v}}\} = \{2, x+1, x^2 - x, x^3 - x - 1\}$ of $\mathcal{P}_3$, and $\{\overline{\mathbf{w}}\} = \{2, x-1, x^2 + x\}$ of $\mathcal{P}_2$.

8. Find the matrix representation of a rotation in $\mathbb{R}^3$ with respect to the x-axis by an angle θ. Repeat with respect to the y-axis, z-axis.

9. Consider the transformation in $\mathbb{R}^3$ given by $\overline{\mathbf{x}} \mapsto A\overline{\mathbf{x}}$ for $A = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Describe the image of points (x,y,z) in $\mathbb{R}^3$. Is it linear?

10. Is the translation $(x, y, z) \mapsto (x - a, y - b, z - c)$ a linear transformation?

11. Let τ be the linear transformation on $\mathbb{R}^3$, which is represented, relative to the canonical basis, by

$$T = {}_{\{\overline{\mathrm{e}}\}}M_{\{\overline{\mathrm{e}}\}} = \begin{pmatrix} 7 & -8 & -8 \\ 9 & -16 & -18 \\ -5 & 11 & 13 \end{pmatrix}.$$

Find ${}_{\{\overline{\alpha}\}}M_{\{\overline{\alpha}\}}$, where $\alpha_1 = (0, -1, 1)^t$, $\alpha_2 = (1, 3, -2)^t$, $\alpha_3 = (2, 0, 1)^t$.

12. Consider the transformation $\tau : \mathbb{R}^3 \to \mathbb{R}^3$: $\tau \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ 3x_2 \\ 5x_3 \end{pmatrix}$. What is the matrix representation of τ^2? And of $\tau^3 + 7\tau + 1$? (Consider the standard basis, and remember that $\tau^2(\overline{\mathbf{x}}) = \tau \circ \tau(\overline{\mathbf{x}}) = \tau(\tau(\overline{\mathbf{x}}))$.)

13. Consider the standard basis of $\mathbb{R}^2$. What is the matrix representation H_θ of the reflection through the θ line? What is the matrix representation P_θ of the projection onto the θ line? Show that $H_\theta^2 = I$, $P_\theta^2 = I$, and $H_\theta = 2P_\theta - I$.

14. Consider $\alpha_1 = (1, 1, 2)^t$, $\alpha_2 = (-2, 0, 1)^t$. Find β_3 such that $\beta_1 = \alpha_1/||\alpha_1||$, $\beta_2 = \alpha_2/||\alpha_2||$ and β_3 are orthogonal.

15. Find an orthonormal basis for the null space of D, where

$$D = \begin{pmatrix} 1 & 0 & 5 & -2 & -1 \\ 0 & 1 & -2 & 4 & 2 \end{pmatrix}.$$

Solutions to some problems

2. $\{(1,2,1,0)^t, (1,0,0,1)^t\}$.

3. Nullity=2, Rank=2.

4. Rank is 2.

5. $\Psi_\beta(\alpha) = (3,2,-1)^t$.

6. $$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}.$$

7. $$\begin{pmatrix} 0 & 1/2 & 1/2 & -2 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

10. Not linear. For instance, show $T(\overline{\mathbf{x}} + \overline{\mathbf{y}}) \neq T(\overline{\mathbf{x}}) + T(\overline{\mathbf{y}})$.

11. $$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

14. $\beta_3 = (1,-5,2)$

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$$\overline{\mathbf{q}}_1 = \begin{pmatrix} -5 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{30}}, \qquad \overline{\mathbf{q}}_2 = \begin{pmatrix} -5 \\ -14 \\ 3 \\ 5 \\ 0 \end{pmatrix} \frac{1}{\sqrt{255}}, \qquad \overline{\mathbf{q}}_3 = \begin{pmatrix} -5/102 \\ -7/51 \\ 1/39 \\ 23/51 \\ 1 \end{pmatrix} \frac{1}{\sqrt{125/102}}.$$