3.4.3. EJERCICIOS

6. Considere el tensor de Maxwell definido como:

$$F_{\mu a} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -cB^z & cB^y \\ -E^y & cB^z & 0 & -cB^x \\ -E^z & -cB^y & cB^x & 0 \end{pmatrix}, \text{ otra vez con: } \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

donde $E = (E^x, E^y, E^z)$ y $B = (B^x, B^y, B^z)$ son los campos eléctricos y magnéticos (respectivamente), medidos por un observador O.

(a) Si un observador mide un campo eléctrico $E = E^x\hat{i}$ y ningún campo magnético. ¿Cuáles campos $F_{\mu a}$ medirá otro observador que viaja con una velocidad respecto al primero de $\beta = v\hat{i}$

$$F_{\mu a} = \begin{pmatrix} 0 & E^x & 0 & 0 \\ -E^x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$F_{\mu' a'} \longrightarrow$ Tensor de Maxwell del observador O'

$$F_{\mu' a'} = \Lambda^{\nu}_{\mu'} \cdot \Lambda^{B}_{a'} \cdot F_{\nu B}$$

$F_{\mu a} \longrightarrow$ antisimétrica $\longrightarrow F_{\mu a} = -F_{a\mu}$

$$F_{\mu' a'} = \Lambda^{0}_{\mu'} \Lambda^{1}_{a'} F_{01} + \Lambda^{1}_{\mu'} \Lambda^{0}_{a'} F_{10}$$

$$F_{\mu' a'} = (\Lambda^{0}_{\mu'} \Lambda^{1}_{a'} - \Lambda^{1}_{\mu'} \Lambda^{0}_{a'}) F_{01}$$

$\rightarrow \Lambda^{0}_{0'} = \gamma \ , \ \Lambda^{i}_{0'} = \gamma v^i \ , \ \Lambda^{0}_{i'} = \gamma v_{i'}$

$\rightarrow \Lambda^{i}_{j'} = \delta^{i}_{j'} + v^i v_{j'} \dfrac{\gamma - 1}{|v|^2} \quad i, j = 1, 2, 3 \longrightarrow$ cuando $\Lambda^{1}_{1'} = \gamma$

$v^i = (v, 0, 0)$

$$\Lambda^{a}_{B'} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Lambda^{0}_{0}, \Lambda^{1}_{0'}, \Lambda^{0}_{1'}, \Lambda^{1}_{1'}, \Lambda^{2}_{2'}, \Lambda^{3}_{3'}$$

$$F_{0'0'} = (\Lambda^{0}_{0'} \Lambda^{1}_{0'} - \Lambda^{1}_{0'} \Lambda^{0}_{0'}) F_{01} = (-\gamma^2 v + \gamma^2 v) F_{01} = 0$$
$$F_{1'1'} = (\Lambda^{0}_{1'} \Lambda^{1}_{0'} - \Lambda^{1}_{0'} \Lambda^{0}_{1'}) F_{01} = (-\gamma^2 v^2 + \gamma^2 v^2) F_{01} = 0$$
$$F_{1'0'} = (\Lambda^{0}_{1'} \Lambda^{1}_{0'} - \Lambda^{1}_{1'} \Lambda^{0}_{0'}) (-E^x) = (\gamma^2 v^2 - \gamma^2)(-E^x)$$
$$F_{0'1'} = (\Lambda^{0}_{0'} \Lambda^{1}_{1'} - \Lambda^{1}_{0'} \Lambda^{0}_{1'}) (E_x) = (\gamma^2 - \gamma^2 v^2)(E^x) \quad (?)$$

$$F_{\mu' a'} = \begin{pmatrix} 0 & (\gamma^2 v^2 - \gamma^2) E^x & 0 & 0 \\ -(\gamma^2 v^2 - \gamma^2) E^x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\longrightarrow \vec{E}' = (-(\gamma^2 v^2 - \gamma^2)E^x, 0, 0)$
$\vec{B}' = (0, 0, 0)$

El observador O' también sólo mide $\vec{E}'$ en $\hat{\imath}$ y ningún campo magnético. Y el módulo de E se incrementa v veces

(b) Muestre que las ecuaciones de Maxwell: $\nabla \times B - \frac{\partial}{\partial t} E = 4\pi J$ y $\nabla \cdot E = 4\pi\rho$ se pueden expresar como:

$$\frac{\partial}{\partial x^\nu} F^{\mu\nu} \equiv F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu, \text{ donde } J^\mu = (c\rho, J^1, J^2, J^3)$$

y $J = (J^1, J^2, J^3)$

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta} = \eta^{\mu\alpha} F_{\alpha\beta}\eta^{\beta\nu}$$

$$= \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & cB^z & -cB^y \\ -E^y & -cB^z & 0 & cB^x \\ -E^z & cB^y & -cB^x & 0 \end{pmatrix}$$

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = F^{\mu\nu}{}_{,\nu} = F^{\mu 0}{}_{,0} + F^{\mu 1}{}_{,1} + F^{\mu 2}{}_{,2} + F^{\mu 3}{}_{,3} \rightarrow \text{Divergencia de } F^{\mu\nu}$$

① $$F^{0\nu}{}_{,\nu} = F^{00}{}_{,0} + F^{01}{}_{,1} + F^{02}{}_{,2} + F^{03}{}_{,3}$$
$$= 0 + \frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z}$$
$$= \vec{\nabla}\cdot\vec{E} = \frac{c\rho}{c\varepsilon_0} = \frac{J^0}{c\,\frac{1}{\mu_0 c^2}} = c\mu_0 J^0$$

② $$F^{1\nu}{}_{,\nu} = F^{10}{}_{,0} + F^{11}{}_{,1} + F^{12}{}_{,2} + F^{13}{}_{,3}$$
$$= \underbrace{\left[-\frac{1}{c^2}\frac{\partial E^x}{\partial t} + \underbrace{\frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z}}_{\vec{\nabla}\times\vec{B}}\right]}_{\mu_0 J^{(1)} \rightarrow \text{componente } x} c$$

$$= c\mu_0 J^1$$

③ $$F^{2\nu}{}_{,\nu} = F^{20}{}_{,0} + F^{21}{}_{,1} + F^{22}{}_{,2} + F^{23}{}_{,3}$$
$$= \left[-\frac{1}{c^2}\frac{\partial E^y}{\partial t} - \frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z}\right] c$$
$$= c\mu_0 J^{(2)} \rightarrow \text{componente } y$$

(4) $F^{3\nu}{}_{,\nu} = F^{30}{}_{,0} + F^{31}{}_{,1} + F^{32}{}_{,2} + F^{33}{}_{,3}$

$$\left[-\frac{1}{c^2}\frac{\partial E^z}{\partial t} + \frac{\partial B^y}{\partial x} - \frac{\partial B^x}{\partial y}\right] c$$

$= c\,\mu_0 J^{(3)} \rightarrow$ componente z

Se obtiene como conclusión a partir de lo anterior que: $F^{\mu\nu}{}_{,\nu} = c\,\mu_0 J^{\mu}$

(c) Considere la identidad de Bianchi de la forma:

$$\frac{\partial F_{\mu\nu}}{\partial x^{\gamma}} + \frac{\partial F_{\nu\gamma}}{\partial x^{\mu}} + \frac{\partial F_{\gamma\mu}}{\partial x^{\nu}} = \partial_{\gamma} F_{\mu\nu} + \partial_{\mu} F_{\nu\gamma} + \partial_{\nu} F_{\gamma\mu}$$

$$\equiv F_{\mu\nu,\gamma} + F_{\nu\gamma,\mu} + F_{\gamma\mu,\nu} = 0. \quad (1)$$

y demuestre que las otras dos ecuaciones $\nabla \cdot B = 0$ y $\nabla \times E - \frac{\partial}{\partial t} B = 0$, también están contenidas en las expresiones $F^{\mu\nu}{}_{,\nu} = 4\pi J^{\mu}$

$$F_{13,2} + F_{32,1} + F_{21,3} = 0$$

$$-c\frac{\partial B^y}{\partial y} - c\frac{\partial B^x}{\partial x} - c\frac{\partial B^z}{\partial z} = 0$$

$$\frac{\partial B^x}{\partial x} + \frac{\partial B^y}{\partial y} + \frac{\partial B^z}{\partial z} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$F_{\mu\nu,0} + F_{\nu 0,\mu} + F_{0\mu,\nu} = 0$$

(1) $\mu = 2,\ \nu = 3$

$$\longrightarrow F_{23,0} + F_{30,2} + F_{02,3} = 0$$

$$\frac{\cancel{c}}{\cancel{c}}\frac{\partial B^x}{\partial t} + \frac{\partial E^z}{\partial y} - \frac{\partial E^y}{\partial z} = 0$$

$$\vec{B} + \vec{\nabla} \times \vec{E} = 0$$

$\hookrightarrow$ En x

(2) $\mu = 1,\ \nu = 3$

$$\longrightarrow F_{13,0} + F_{30,1} + F_{01,3} = 0$$

$$-\frac{\cancel{c}}{\cancel{c}}\frac{\partial B^y}{\partial t} + \frac{\partial E^z}{\partial x} - \frac{\partial E^x}{\partial z} = 0$$

$$-\vec{B} - \vec{\nabla} \times \vec{E} = 0$$

$\hookrightarrow$ En y

(3) $\mu = 2,\ \nu = 1$

$$\longrightarrow F_{21,0} + F_{10,2} + F_{02,1} = 0$$

$$-\frac{\cancel{c}}{\cancel{c}}\frac{\partial B^z}{\partial z} + \frac{\partial E^x}{\partial y} - \frac{\partial E^y}{\partial x} = 0$$

$$-B - \vec{\nabla} \times \vec{E} = 0$$

$\hookrightarrow$ En z

En conclusión, se tienen las dos ecuaciones

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{B} + \vec{\nabla} \times \vec{E} = 0$$