Series de Fourier

Sea $f(t)$ una función periódica de periodo T, entonces existe una representación

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\frac{2\pi n}{T}t} \qquad (1)$$

donde

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-j\frac{2\pi n}{T}t}\, dt \qquad (2)$$

$\longrightarrow$ IMG04_SerieFourierCn.m

Empleando $e^{j\theta} = \cos\theta + j\sin\theta$ se obtiene "optional"

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left\{ a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n\left(\frac{2\pi n t}{T}\right)\right\}$$

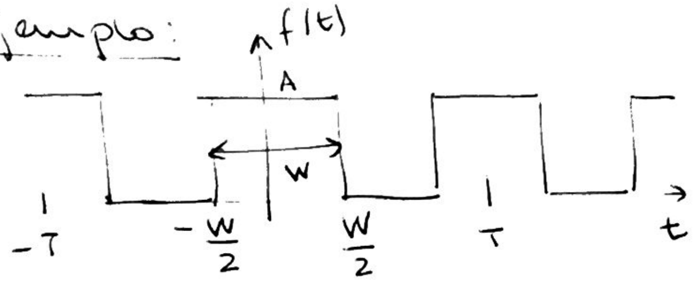
donde

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\, dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n t}{T}\right) dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi n t}{T}\right) dt$$

Ejemplo:



← función periódica de periodo T.

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{2}{T}\int_{-W/2}^{W/2} A\,dt = \frac{2A}{T}\, t\Big|_{-W/2}^{W/2} = \frac{2AW}{T}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n t}{T}\right)dt = \frac{2A}{T}\int_{-W/2}^{W/2}\cos\left(\frac{2\pi n t}{T}\right)dt$$

$$= \frac{2A}{T}\cdot\frac{T}{2\pi n}\sin\left(\frac{2\pi n t}{T}\right)\Big|_{-W/2}^{W/2}$$

$$= \frac{\cancel{2}AT}{\cancel{2}\pi n \cancel{T}}\left(\sin\left(\frac{\cancel{2}\pi n W}{\cancel{2}T}\right) - \sin\left(-\frac{\cancel{2}\pi n W}{\cancel{2}T}\right)\right)$$

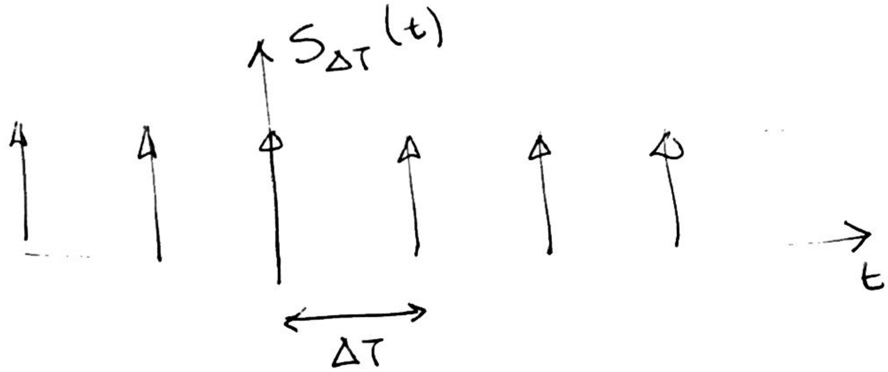
$$\boxed{a_n = \frac{2A}{\pi n}\sin\left(\frac{\pi n W}{T}\right)}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi n t}{T}\right)dt = 0$$ porque $f(t)\sin\left(\frac{2\pi n t}{T}\right)$ es impar.

$$f(t) = \frac{AW}{T} + \frac{2A}{\pi}\sum_{n=1}^{+\infty}\sin\left(\frac{\pi n W}{T}\right)\cos\left(\frac{2\pi n t}{T}\right)$$

→ IMG04_SerieFourier.m

TREN DE IMPULSOS



$$s_{\Delta T}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n\Delta T) \qquad (3)$$

Periodo = ΔT

$s_{\Delta T}(t)$ puede ser representada usando series de Fourier (1):

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

de (2)

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t)\, e^{-j\frac{2\pi n t}{\Delta T}}\, dt$$

$$= \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t)\, e^{-j\frac{2\pi n t}{\Delta T}}\, dt$$

$$\boxed{c_n = \frac{1}{\Delta T} e^{0} = \frac{1}{\Delta T}}$$

$$\Rightarrow s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{+\infty} e^{j\frac{2\pi n t}{\Delta T}} \qquad (4)$$

→ IMG04_TrenImpulsos.m

TRANSFORMADA DE FOURIER DE FUNCIONES IMPULSO:

$$\delta(t) \circ\!\!-\!\!-\!\!\bullet \int_{-\infty}^{+\infty} \delta(t)\, e^{-j2\pi w t}\, dt = e^{0} = 1 \qquad (5)$$

$$\delta(t-t_0) \circ\!\!-\!\!-\!\!\bullet \int_{-\infty}^{+\infty} \delta(t-t_0)\, e^{-j2\pi w t}\, dt = e^{-j2\pi w t_0} \qquad (6)$$

TRANSFORMADA INVERSA:

$$\delta(w) \bullet\!\!-\!\!-\!\!\circ \int_{-\infty}^{+\infty} \delta(w)\, e^{j2\pi w t}\, dw = e^{0} = 1 \qquad (7)$$

$$\delta(w-w_0) \bullet\!\!-\!\!-\!\!\circ \int_{-\infty}^{+\infty} \delta(w-w_0)\, e^{j2\pi w t}\, dw = e^{j2\pi w_0 t} \qquad (8)$$

EJEMPLO:

Calcular la Transformada de

$$f(t) = \frac{1}{2}\left[\delta(t-t_0) + \delta(t+t_0)\right]$$

$$f(t) \circ\!\!-\!\!-\!\!\bullet F(w) = \frac{1}{2}\left[e^{-j2\pi w t_0} + e^{j2\pi w t_0}\right]$$

$$= \cos(2\pi w t_0) \qquad (9)$$

de igual manera:

$$f(t) = \cos w_0 t \circ\!\!-\!\!-\!\!\bullet F(w) = \frac{1}{2}\left[\delta(w-w_0) + \delta(w+w_0)\right] \qquad (10)$$

-RANSFORMADA DE FOURIER DEL TREN DE IMPULSOS

$$\mathcal{F}\{S_{\Delta T}(t)\} = \int_{-\infty}^{+\infty} \frac{1}{\Delta T} \sum_{n=-\infty}^{+\infty} e^{j\frac{2\pi n t}{\Delta T}} e^{-j2\pi w t} dt$$

$$= \frac{1}{\Delta T} \sum_{n=-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} e^{j\frac{2\pi n t}{\Delta T}} e^{-j2\pi w t} dt}_{\text{Transformada de Fourier de } e^{j\frac{2\pi n t}{\Delta T}}}$$

de (8)

$$= \frac{1}{\Delta T} \sum_{n=-\infty}^{+\infty} \mathcal{F}\left\{e^{j\frac{2\pi n t}{\Delta T}}\right\} = \frac{1}{\Delta T} \sum_{n=-\infty}^{+\infty} \delta\left(w - \frac{n}{\Delta T}\right)$$

Esto quiere decir que la transformada de un tren de impulsos es un tren de impulsos!

$$S_{\Delta T}(t) \circ\!\!-\!\!\bullet \frac{1}{\Delta T} S_{1/\Delta T}(w) \qquad (11)$$

$$\sum \delta(t - n\Delta T) \circ\!\!-\!\!\bullet \frac{1}{\Delta T} \sum \delta\left(w - \frac{n}{\Delta T}\right) \qquad (12)$$

TRANSFORMADA DE LA CONVOLUCIÓN

$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$$

$$\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau\, e^{-j2\pi w t}\, dt$$

$$= \int_{-\infty}^{+\infty} f(\tau) \underbrace{\int_{-\infty}^{+\infty} g(t-\tau)\, e^{-j2\pi w t}\, dt}_{A}\, d\tau \quad (*)$$

con $u = t - \tau$

$$A = \int_{-\infty}^{+\infty} g(u)\, e^{-j2\pi w (u+\tau)}\, du$$

$$= e^{-j2\pi w \tau} \underbrace{\int_{-\infty}^{+\infty} g(u)\, e^{-j2\pi w u}\, du}_{G(w)\ :\ \text{Transformada de Fourier de } g}$$

$A = e^{-j2\pi w \tau}\, G(w)$ ... reemplazando A en $(*)$

$$\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{+\infty} f(\tau)\, e^{-j2\pi w \tau}\, G(w)\, d\tau$$

$$= \left[\int_{-\infty}^{+\infty} f(\tau)\, e^{-j2\pi w \tau}\, d\tau \right] G(w)$$

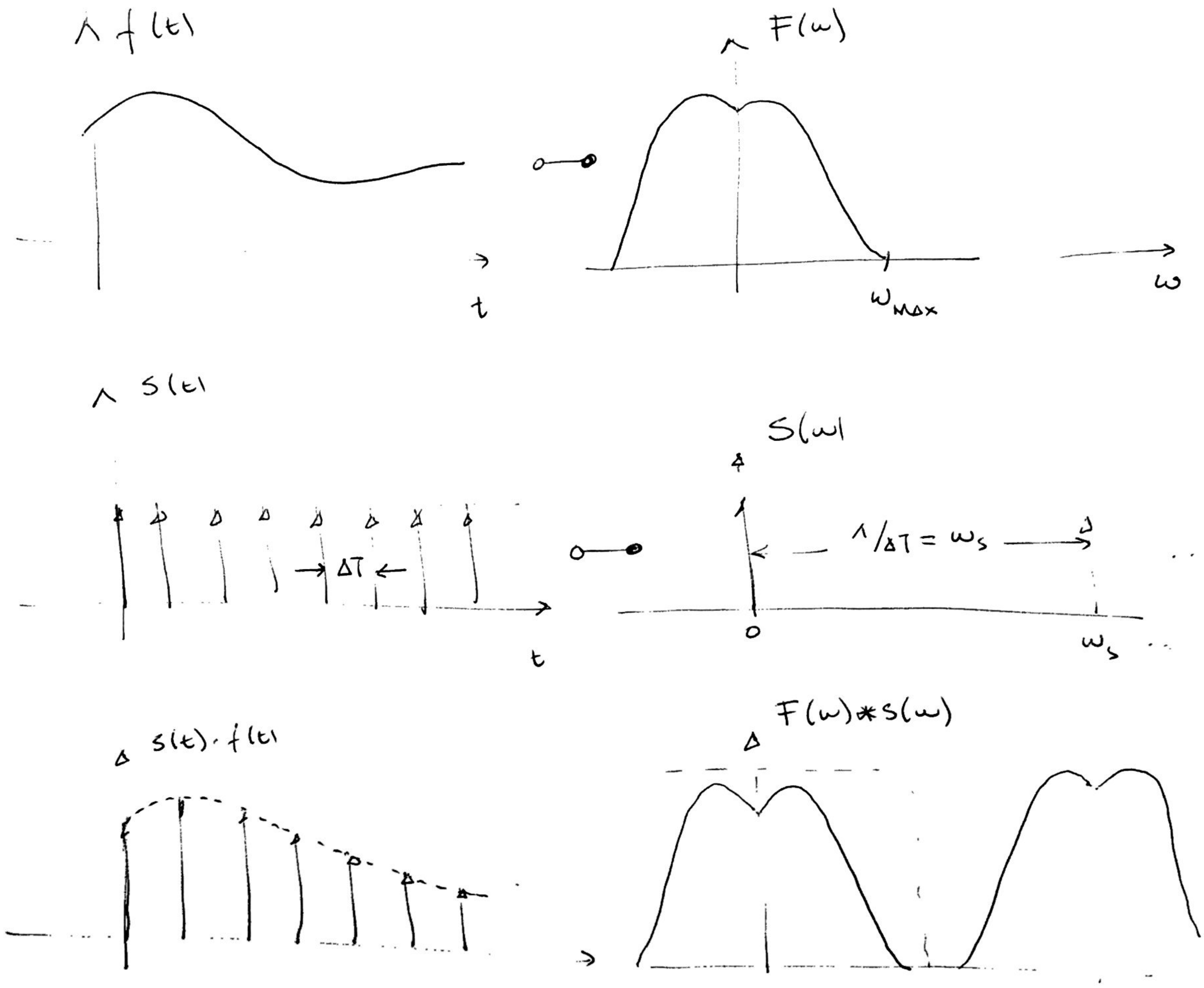
$$= F(w)\, G(w)$$

$$\boxed{f(t) * g(t) \circ\!\!-\!\!\bullet\ F(w)\, G(w)} \qquad (13)$$

La transformada de la convolución de dos funciones es la multiplicación de sus transformadas!

ver IMG04_Convolucion1D_Fourier.m

TEOREMA DEL MUESTREO



Para recuperar $f(t)$ a partir de un muestreo $s(t)f(t)$ es necesario que $W_s > 2W_{max}$ (Teorema de Nyquist). Se usa un filtro pasa bajos de ancho $W_{max}+\varepsilon$.