Univariate Gaussian joint probability

$$P(\mathbf{x}, \mu, \gamma) = \mathcal{N}(\mu|m, \beta^{-1}) Ga(\gamma|a, b) \prod_{n=1}^{N} \mathcal{N}(x_n|\mu, \gamma^{-1})$$

Variational Distribution

$$q(\mu, \gamma) = q_\mu(\mu) q_\gamma(\gamma)$$

Coordiante Ascent Variational Inference (CAVI)

$$q(\mu|\mu') \propto \exp(\mathbb{E}_{q(\gamma|\gamma')}[\log P(\mathbf{x}, \mu, \gamma)])$$
$$q(\gamma|\gamma') \propto \exp(\mathbb{E}_{q(\mu|\mu')}[\log P(\mathbf{x}, \mu, \gamma)])$$

Mean $q(\mu|\mu')$

$$\begin{aligned}
q(\mu|\mu') &\propto \exp\left(\mathbb{E}_{q(\gamma|\gamma')}\left[\log P(\mathbf{x}, \mu, \gamma)\right]\right) \\
&\propto \exp\left(\mathbb{E}_{q(\gamma|\gamma')}\left[\log \mathcal{N}(\mu|m, \beta^{-1}) + \log Ga(\gamma|a, b) + \sum_{n=1}^{N} \log \mathcal{N}(x_n|\mu, \gamma^{-1})\right]\right) \\
&\propto \exp\left(\log \mathcal{N}(\mu|m, \beta^{-1}) + \sum_{n=1}^{N} \mathbb{E}_{q(\gamma|\gamma')}\left[\log \mathcal{N}(x_n|\mu, \gamma^{-1})\right]\right) \\
&\propto \exp\left(\frac{1}{2}\log\beta - \frac{1}{2}\beta\left(\mu - m\right)^2 + \sum_{n=1}^{N} \mathbb{E}_{q(\gamma|\gamma')}\left[\frac{1}{2}\log\gamma - \frac{1}{2}\gamma\left(x_n - \mu\right)^2\right]\right) \\
&\propto \exp\left(\frac{1}{2}\log\beta - \frac{1}{2}\beta\left(\mu^2 - 2m\mu + m^2\right) + \frac{N}{2}\mathbb{E}_{q(\gamma|\gamma')}\left[\log\gamma\right] - \frac{1}{2}\mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right]\sum_{n=1}^{N}\left(x_n^2 - 2x_n\mu + \mu^2\right)\right) \\
&\propto \exp\left(-\frac{1}{2}\beta\left(\mu^2 - 2m\mu + m^2\right) - \frac{1}{2}\mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right]\sum_{n=1}^{N}\left(x_n^2 - 2x_n\mu + \mu^2\right)\right) \\
&\propto \exp\left(-\frac{1}{2}\mu^2\left(\beta + N\mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right]\right) + \mu\left(\beta m + \mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right]\sum_{n=1}^{N} x_n\right)\right) \\
q(\mu|m_\mu, \beta_\mu) &= \mathcal{N}\left(\mu|m_u = \frac{\beta m + \mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right]\sum_{n=1}^{N} x_n}{\beta + N\mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right]}, \beta_\mu = \beta + N\mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right]\right)
\end{aligned}$$

Precision $q(\gamma|\gamma')$

$$
\begin{aligned}
q(\gamma|\gamma') &\propto \exp\left(\mathbb{E}_{q(\mu|\mu')}\left[\log P(\mathbf{x},\mu,\gamma)\right]\right) \\
&\propto \exp\left(\mathbb{E}_{q(\mu|\mu')}\left[\log \mathcal{N}(\mu|m,\beta^{-1}) + \log Ga(\gamma|a,b) + \sum_{n=1}^{N}\log \mathcal{N}(x_n|\mu,\gamma^{-1})\right]\right) \\
&\propto \exp\left(\log Ga(\gamma|a,b) + \sum_{n=1}^{N}\mathbb{E}_{q(\mu|\mu')}\left[\log \mathcal{N}(x_n|\mu,\gamma^{-1})\right]\right) \\
&\propto \exp\left((a-1)\log\gamma - b\gamma + \sum_{n=1}^{N}\frac{1}{2}\log\gamma - \frac{1}{2}\gamma\mathbb{E}_{q(\mu|\mu')}\left[(x_n-\mu)^2\right]\right) \\
&\propto \exp\left((a-1+\frac{N}{2})\log\gamma - \left(b + \frac{1}{2}\sum_{n=1}^{N}\mathbb{E}_{q(\mu|\mu')}\left[(x_n-\mu)^2\right]\right)\gamma\right) \\
q(\gamma|\gamma') &= Ga\left(\gamma|a_\gamma = a+\frac{N}{2}, b_\gamma = b + \frac{1}{2}\sum_{n=1}^{N}x_n^2 - 2x_n\mathbb{E}_{q(\mu|\mu')}\left[\mu\right] + \mathbb{E}_{q(\mu|\mu')}\left[\mu^2\right]\right)
\end{aligned}
$$

LowerBound

$$
\begin{aligned}
ELBO(x,\mu',\gamma') =& \mathbb{E}_q\left[\log P(\mathbf{x},\mu,\gamma)\right] - \mathbb{E}_q\left[\log q(\mu,\gamma)\right] \\
=& \mathbb{E}_{q(\mu|\mu')}\left[\log \mathcal{N}(\mu|m,\beta^{-1})\right] + \mathbb{E}_{q(\gamma|\gamma')}\left[\log Ga(\gamma|a,b)\right] + \sum_{n=1}^{N}\mathbb{E}_q\left[\log \mathcal{N}(x_n|\mu,\gamma^{-1})\right] \\
& - \mathbb{E}_{q(\mu|\mu')}\left[\log \mathcal{N}(\mu|m_\mu,\beta_\mu)\right] - \mathbb{E}_{q(\gamma|\gamma')}\left[\log Ga(\gamma|a_\gamma,b_\gamma)\right] \\
=& \frac{1}{2}\log\beta - \frac{1}{2}\log 2\pi - \frac{1}{2}\beta\left(\mathbb{E}_{q(\mu|\mu')}\left[\mu^2\right] - 2m\mathbb{E}_{q(\mu|\mu')}\left[\mu\right] + m^2\right) \\
& + a\log b - \log\Gamma(a) + (a-1)\mathbb{E}_{q(\gamma|\gamma')}\left[\log\gamma\right] - b\mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right] \\
& + \sum_{n=1}^{N}\frac{1}{2}\mathbb{E}_{q(\gamma|\gamma')}\left[\log\gamma\right] - \frac{1}{2}\log 2\pi - \frac{1}{2}\mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right]\left(x_n^2 - 2x_n\mathbb{E}_{q(\mu|\mu')}\left[\mu\right] + \mathbb{E}_{q(\mu|\mu')}\left[\mu^2\right]\right) \\
& - \frac{1}{2}\log\beta_\mu + \frac{1}{2}\log 2\pi + \frac{1}{2}\beta_\mu\left(\mathbb{E}_{q(\mu|\mu')}\left[\mu^2\right] - 2m_\mu\mathbb{E}_{q(\mu|\mu')}\left[\mu\right] + m_\mu^2\right) \\
& - a_\gamma\log b_\gamma + \log\Gamma(a_\gamma) - (a_\gamma-1)\mathbb{E}_{q(\gamma|\gamma')}\left[\log\gamma\right] + b_\gamma\mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right] \\
ELBO(x,\mu',\gamma') =& \frac{1}{2}\log\frac{\beta}{\beta_\mu} + \frac{1}{2}\mathbb{E}_{q(\mu|\mu')}\left[\mu^2\right](\beta_\mu - \beta) - \mathbb{E}_{q(\mu|\mu')}\left[\mu\right](\beta_\mu m_\mu - \beta m) + \frac{1}{2}\left(\beta_\mu m_\mu^2 - \beta m^2\right) \\
& + a\log b - a_\gamma\log b_\gamma + \log\frac{\Gamma(a_\gamma)}{\Gamma(a)} + \mathbb{E}_{q(\gamma|\gamma')}\left[\log\gamma\right](a-a_\gamma) + \mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right](b_\gamma - b) \\
& + \frac{N}{2}\mathbb{E}_{q(\gamma|\gamma')}\left[\log\gamma\right] - \frac{N}{2}\log 2\pi - \frac{1}{2}\mathbb{E}_{q(\gamma|\gamma')}\left[\gamma\right]\sum_{n=1}^{N}\left(x_n^2 - 2x_n\mathbb{E}_{q(\mu|\mu')}\left[\mu\right] + \mathbb{E}_{q(\mu|\mu')}\left[\mu^2\right]\right)
\end{aligned}
$$