Variational Inference

Variational inference optimizes $q(\theta|\Theta)$ so that,

$$argmin_{q(\theta|\Theta)}\ KL(q(\theta|\Theta),\ p(\theta|X;\Omega)\ \equiv argmax_{q(\theta|\Theta)}\ ELBO(p(X,\theta;\Omega),\ q(\theta|\Theta))$$

where the Evidence Lower BOund is defined as;

$$\begin{aligned} ELBO(p(X,\theta;\Omega),\ q(\theta|\Theta)) =& \int q(\theta|\Theta)\log p(X,\theta;\Omega)d\theta - \int q(\theta|\Theta)\log q(\theta|\Theta)d\theta \\ =& \mathbb{E}_{q(\theta|\Theta)}[\log p(X,\theta;\Omega)] + \mathbb{H}[q(\theta|\Theta)] \end{aligned}$$

and where:

$$\begin{aligned} \theta =& \{\pi, z, \mu\} \\ \Omega =& \{\alpha, m_o\} \\ X =& \{X\} \\ \Theta =& \{\lambda_\pi, \lambda_\mu, \phi\} \end{aligned}$$

Joint Distribution

$$p(x,\ z,\ \mu\,|\,\alpha,\ m_o, W_o, \Delta_o) = p(\pi|\alpha)\prod_{k=1}^{K} p(\mu_k|\,m_o,\ (\beta_o\Delta_o)^{-1})\prod_{n=1}^{N} p(z_n|\pi)p(x_n|\mu_{z_n},\ \Delta_o^{-1})$$

Mean-field Distribution

$$q(\pi,\ z,\ \mu\,|\lambda_\pi,\ \phi,\ \lambda_\mu) = q(\pi|\lambda_\pi)\prod_{n=1}^{N} q(z_n|\phi_n)\prod_{k=1}^{K} q(\mu_k|\lambda_{\mu_k})$$

Coordinate Ascent

Under the mean-field assumption, the variational distributions for each variable can be computed as follows:

$$\begin{aligned} q(\pi|\lambda_\pi) &\propto \exp(\mathbb{E}_{q(z_n|\phi_n)q(\mu|\lambda_\mu)}[\log p(x,\ z,\ \mu\,|\,\alpha,\ m_o, W_o, \Delta_o)]) \\ q(z_n|\phi_n) &\propto \exp(\mathbb{E}_{q(\pi|\lambda_\pi)q(\mu|\lambda_\mu)}[\log p(x,\ z,\ \mu\,|\,\alpha,\ m_o, W_o, \Delta_o)]) \\ q(\mu_k|\lambda_{\mu_k}) &\propto \exp(\mathbb{E}_{q(\pi|\lambda_\pi)q(z_n|\phi_n)}[\log p(x,\ z,\ \mu\,|\,\alpha,\ m_o, W_o, \Delta_o)]) \end{aligned}$$

The mixture proportions:

$$
\begin{aligned}
q(\pi|\lambda_\pi) &\propto \exp\left(\mathbb{E}_{q(z_n|\phi_n)q(\mu|\lambda_\mu)}[\log p(x,\, z,\, \mu \,|\, \alpha,\, m_o, W_o, \Delta_o)]\right) \\
&\propto \exp\left(\log p(\pi|\alpha) + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}[\log p(z_n|\pi)]\right) \\
&\propto \exp\left(\sum_{k=1}^{K}(\alpha_k - 1)\log \pi_k + \sum_{n=1}^{N}\sum_{k=1}^{K} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\log \pi_k\right) \\
&\propto \exp\left(\sum_{k=1}^{K}\left(\alpha_k - 1 + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right)\log \pi_k\right) \\
q(\pi|\lambda_\pi) =& Dir\left(\alpha_k + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right)
\end{aligned}
$$

The mixture assignments:

$$
\begin{aligned}
q(z_n|\phi_n) &\propto \exp\left(\mathbb{E}_{q(\pi|\lambda_\pi)q(\mu|\lambda_\mu)}[\log p(x,\, z,\, \mu \,|\, \alpha,\, m_o, W_o, \Delta_o)]\right) \\
&\propto \exp\left(\mathbb{E}_{q(\pi|\lambda_\pi)}[\log p(z_n|\pi)] + \mathbb{E}_{q(\mu|\lambda_\mu)}[\log p(x_n|\mu_{z_n},\, \Delta_o^{-1})]\right) \\
&\propto \exp\left(\sum_{k=1}^{K} \mathbb{I}(z_n = k)\mathbb{E}_{q(\pi|\lambda_\pi)}[\log \pi_k] + \mathbb{I}(z_n = k)\mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\log p(x_n|\mu_k,\, \Delta_o^{-1})]\right) \\
&\propto \exp\left(\sum_{k=1}^{K} \mathbb{I}(z_n = k)\left(\mathbb{E}_{q(\pi|\lambda_\pi)}[\log \pi_k] + \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[-\frac{1}{2}(x_n - \mu_k)^T\Delta_o(x_n - u_k)]\right)\right) \\
&\propto \exp\left(\sum_{k=1}^{K} \mathbb{I}(z_n = k)\left(\mathbb{E}_{q(\pi|\lambda_\pi)}[\log \pi_k] + \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[-\frac{1}{2}Tr\left(\Delta_o(x_n - \mu_k)^T(x_n - u_k)\right)]\right)\right) \\
&\propto \exp\left(\sum_{k=1}^{K} \mathbb{I}(z_n = k)\left(\mathbb{E}_{q(\pi|\lambda_\pi)}[\log \pi_k] - \frac{1}{2}Tr(\Delta_o\mathbb{E}_{q(\mu|\lambda_{\mu_k})}[(x_n - \mu_k)^T(x_n - u_k)])\right)\right) \\
&\propto \exp\left(\sum_{k=1}^{K} \mathbb{I}(z_n = k)\left(\mathbb{E}_{q(\pi|\lambda_\pi)}[\log \pi_k] - \frac{1}{2}Tr\left(\Delta_o(x_n - \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\mu_k])^T(x_n - \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\mu_k])\right.\right.\right. \\
&\left.\left.\left. +\Delta_o\left(\beta_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right)^{-1}\Delta_o^{-1}\right)\right)\right) \\
&\propto \exp\left(\sum_{k=1}^{K} \mathbb{I}(z_n = k)\left(\mathbb{E}_{q(\pi|\lambda_\pi)}[\log \pi_k] - \frac{1}{2}(x_n - \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\mu_k])^T\Delta_o(x_n - \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\mu_k])\right.\right. \\
&\left.\left. -\frac{1}{2}D\left(\beta_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right)^{-1}\right)\right) \\
q(z_n|\phi_n) &= Cat\left(\mathbb{E}_{q(\pi|\lambda_\pi)}[\log \pi_k] - \frac{1}{2}(x_n - \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\mu_k])^T\Delta_o(x_n - \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\mu_k])\right. \\
&\left. -\frac{1}{2}D\left(\beta_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right)^{-1}\right)
\end{aligned}
$$

The mixture means:

$$
\begin{aligned}
q(\mu_k|\lambda_{\mu_k}) &\propto \exp\left(\mathbb{E}_{q(\pi|\lambda_\pi)q(z_n|\phi_n)}[\log p(x,\, z,\, \mu \,|\, \alpha,\, m_o, W_o, \Delta_o)]\right) \\
&\propto \exp\left(\log p(\mu_k|\, m_o,\, (\beta_o\Delta_o)^{-1}) + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k) \log p(x_n|\mu_k,\, \Delta_o^{-1})\right) \\
&\propto \exp\left(-\frac{1}{2}{\mu_k}^T \beta_o \Delta_o u_k - \frac{1}{2}{\mu_k}^T \beta_o \Delta_o m_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)(-\frac{1}{2}{\mu_k}^T \Delta_o u_k - \frac{1}{2}{x_n}^T \Delta_o u_k)\right) \\
&\propto \exp\left(-\frac{1}{2}\left(\beta_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right) {\mu_k}^T \Delta_o \mu_k - \frac{1}{2}{\mu_k}^T \Delta_o \left(\beta_o m_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k) x_n^T\right)\right) \\
q(\mu_k|\lambda_{\mu_k}) &= N\left(\frac{\left(\beta_o m_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k) x_n^T\right)}{\left(\beta_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right)}, \left(\beta_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right)^{-1} \Delta_o^{-1}\right)
\end{aligned}
$$

The above expectactions based on the functional forms of q's

$$
\begin{aligned}
\mathbb{E}_{q(\pi|\lambda_\pi)}[\log \pi_k] =& \Psi\left(\alpha_k + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right) - \Psi\left(\sum_{k=1}^{K} \alpha_k + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right) \\
\mathbb{E}_{q(z_n|\phi_n)}[\mathbb{I}(z_n = k)] =& \mathbb{E}_{q(\pi|\lambda_\pi)}[\log \pi_k] - \frac{1}{2}(x_n - \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\mu_k])^T \Delta_o (x_n - \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\mu_k]) - \frac{1}{2} D(\beta_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k))^{-1} \\
\mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\mu_k] =& \frac{\left(\beta_o m_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k) x_n^T\right)}{\left(\beta_o + \sum_{n=1}^{N} \mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k)\right)}
\end{aligned}
$$

ELBO

$$
\begin{aligned}
ELBO(p(X, \theta; \Omega),\, q(\theta|\Theta)) =& \mathbb{E}[\log p(X, \theta; \Omega)] + \mathbb{H}[q(\theta|\Theta)] \\
=& \mathbb{E}[\log p(\pi|\alpha)] + \sum_{k=1}^{K} \mathbb{E}[\log p(\mu_k|\, m_o,\, (\beta_o\Delta_o)^{-1})] + \sum_{n=1}^{N} \mathbb{E}[\log p(z_n|\pi)] + \mathbb{E}[\log p(x_n|\mu_{z_n},\, \Delta_o^{-1})] \\
& - \mathbb{E}[\text{logq}(\pi|\lambda_\pi)]\text{-}\sum_{n=1}^{N} \mathbb{E}[\text{logq}(z_n|\phi_n)]\text{-}\sum_{k=1}^{K} \mathbb{E}[\text{logq}(\mu_k|\lambda_{\mu_{m_k}}, \lambda_{\mu_{\beta_k}})] \\
=& \log \frac{B(\lambda_\pi)}{B(\alpha)} + \sum_{k=1}^{K} \Big((\alpha_k - \lambda_{\pi_k})\mathbb{E}[\log \pi_k] + \mathbb{E}[\log p(\mu_k|m_o,\, (\beta_o\Delta_o)^{-1}) - \log q(\mu_k|\lambda_{\mu_{m_k}}, \lambda_{\mu_{\beta_k}})] \\
& \sum_{n=1}^{N} \mathbb{E}[\mathbb{I}(z_n = k)] \left(-\mathbb{E}[\log \phi_{n_k}] + \mathbb{E}[\log p(x_n|\mu_k,\, \Delta_o^{-1})]\right)\Big)
\end{aligned}
$$

The above expectations:

$$
\begin{aligned}
\mathbb{E}_{q(z_n|\phi_n)}\mathbb{I}(z_n = k) =& \phi_{n,k} \\
\sum_{n=1}^{N} \phi_{n,k} =& N_k \\
\mathbb{E}_{q(\mu|\lambda_{\mu_k})}[\mu_k] =& \lambda_{\mu_k} \\
\mathbb{E}_{q(\pi|\lambda_\pi)}[\log \pi_k] =& \Psi\left(\lambda_{\pi_k}\right) - \Psi\left(\sum_{k=1}^{K} \lambda_{\pi_k}\right) \\
\mathbb{E}[\log p(x_n|\mu_k,\, {\Delta_o}^{-1})] =& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\Delta_o| - \frac{1}{2}\mathbb{E}_{q(\mu|\lambda_{\mu_k})}[(x_n - \mu_k)^T \Delta_o (x_n - u_k)] \\
=& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\Delta_o| - \frac{1}{2} Tr\left(\Delta_o \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[(x_n - \mu_k)^T (x_n - u_k)]\right) \\
=& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\Delta_o| - \frac{1}{2} Tr\left(\Delta_o \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[(x_n - \mu_k)^T (x_n - u_k)]\right) \\
=& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\Delta_o| - \frac{1}{2} Tr\left(\Delta_o (x_n - \lambda_{\mu_k})^T (x_n - \lambda_{\mu_k}) + \Delta_o \lambda_{\beta_k}^{-1} \Delta_o^{-1}\right) \\
=& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\Delta_o| - \frac{1}{2}(x_n - \lambda_{\mu_k})\Delta_o^T (x_n - \lambda_{\mu_k}) - \frac{D}{2\lambda_{\beta_k}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[\log q(\mu_k|\lambda_{\mu_k},\, (\lambda_{\beta_k}\Delta_o)^{-1})] =& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\lambda_{\beta_k}\Delta_o| - \frac{\lambda_{\beta_k}}{2}\mathbb{E}_{q(\mu|\lambda_{\mu_k})}[(\mu_k - \lambda_{\mu_k})^T \Delta_o (u_k - \lambda_{\mu_k})] \\
=& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\lambda_{\beta_k}\Delta_o| - \frac{\lambda_{\beta_k}}{2} Tr\left(\Delta_o \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[(\mu_k - \lambda_{\mu_k})^T (u_k - \lambda_{\mu_k})]\right) \\
=& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\lambda_{\beta_k}\Delta_o| - \frac{D}{2}
\end{aligned}
$$

$$
\mathbb{E}[\log \phi_{n,k}] = \Psi\left(\phi_{n,k}\right) - \Psi\left(\sum_{k=1}^{K} \phi_{n,k}\right)
$$

$$
\begin{aligned}
\mathbb{E}[\log p(\mu_k|m_o, (\beta_o\Delta_o)^{-1})] =& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\beta_o\Delta_o| - \frac{\beta_o}{2}\mathbb{E}_{q(\mu|\lambda_{\mu_k})}[(\mu_k - m_o)^T \Delta_o (\mu_k - m_o)] \\
=& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\beta_o\Delta_o| - \frac{\beta_o}{2} Tr\left(\Delta_o \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[(\mu_k - m_o)^T (u_k - m_o)]\right) \\
=& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\beta_o\Delta_o| - \frac{\beta_o}{2} Tr\left(\Delta_o \mathbb{E}_{q(\mu|\lambda_{\mu_k})}[(\mu_k - m_o)^T (u_k - m_o)]\right) \\
& -\frac{1}{2}\log 2\pi + \frac{1}{2}\log|\beta_o\Delta_o| - \frac{\beta_o}{2}(\lambda_{\mu_k} - m_o)\Delta_o^T (\lambda_{\mu_k} - m_o) - \frac{D\beta_o}{2\lambda_{\beta_k}}
\end{aligned}
$$

$$
\begin{aligned}
ELBO(p(X,\theta;\Omega),\, q(\theta|\Theta)) =& \log\frac{B(\lambda_\pi)}{B(\alpha)} + \sum_{k=1}^{K}\left((\alpha_k - \lambda_{\pi_k})\left(\Psi\left(\lambda_{\pi_k}\right) - \Psi\left(\sum_{k=1}^{K}\lambda_{\pi_k}\right)\right)\right. \\
& + \frac{1}{2}\log|\beta_o\Delta_o| - \frac{\beta_o}{2}(\lambda_{\mu_k} - m_o)^T \Delta_o (\lambda_{\mu_k} - m_o) - \frac{D\beta_o}{2\lambda_{\beta_k}} - \frac{1}{2}\log|\lambda_{\beta_k}\Delta_o| + \frac{D}{2} \\
& \left. + \sum_{n=1}^{N}\phi_{n,k}\left(\Psi\left(\lambda_{\pi_k}\right) - \Psi\left(\sum_{k=1}^{K}\lambda_{\pi_k}\right) - \log\phi_{n,k} + \frac{1}{2}\log\frac{|\Delta_o|}{2\pi} - \frac{1}{2}(x_n - \lambda_{\mu_k})^T \Delta_o (x_n - \lambda_{\mu_k}) - \frac{D}{2\lambda_{\beta_k}}\right)\right)
\end{aligned}
$$