$$\sqrt{R}\left(\bar{h}_{\tau_R} - \mathbb{E}^\pi[h(X)]\right) \overset{\mathcal{L}}{\rightsquigarrow} \mathcal{N}(0, \sigma_h^2)\,, \tag{6.36}$$

where

$$\sigma_h^2 = \frac{\mathbb{E}^Q\left[(\tilde{S}_1 - N_1\mathbb{E}^\pi[h(X)])^2\right]}{\{\mathbb{E}^Q[N_1]\}^2}\,.$$

While it seems that (6.33) and (6.36) are very similar, the advantage in using this approach is that σ_h^2 can be estimated much more easily due to the underlying independent structure. For instance,

$$\hat{\sigma}_h^2 = \frac{\sum_{t=1}^R (\tilde{S}_t - \bar{h}_{\tau_R} N_t)^2}{R\bar{N}^2}$$

is a consistent estimator of σ_h^2.

In addition, the conditions on $\mathbb{E}^Q[\tilde{S}_1^2]$ and $\mathbb{E}^Q[N_1^2]$ appearing in Theorem 6.68 are minimal in that they hold when the conditions of Theorem 6.67 hold (see Hobert et al. 2002, for a proof).

6.8 Problems

6.1 Examine whether a Markov chain (X_t) may always be represented by the deterministic transform $X_{t+1} = \psi(X_t, \epsilon_t)$, where (ϵ_t) is a sequence of iid rv's. (*Hint:* Consider that ϵ_t can be of infinite dimension.)

6.2 Show that if (X_n) is a time-homogeneous Markov chain, the transition kernel does not depend on n. In particular, if the Markov chain has a finite state-space, the transition matrix is constant.

6.3 Show that an ARMA(p, q) model, defined by

$$X_n = \sum_{i=1}^p \alpha_i X_{n-i} + \sum_{j=1}^q \beta_j \varepsilon_{n-j} + \varepsilon_n,$$

does not produce a Markov chain. (*Hint:* Examine the relation with an AR(q) process through the decomposition

$$Z_n = \sum_{i=1}^p \alpha_i Z_{n-i} + \varepsilon_n, \qquad Y_n = \sum_{j=1}^q \beta_j Z_{n-j} + Z_n,$$

since (Y_n) and (X_n) are then identically distributed.)

6.4 Show that the resolvant kernel of Definition 6.8 is truly a kernel.

6.5 Show that the properties of the resolvant kernel are preserved if the geometric distribution $\mathcal{G}eo(\epsilon)$ is replaced by a Poisson distribution $\mathcal{P}(\lambda)$ with arbitrary parameter λ.

6.6 Derive the strong Markov property from the decomposition

$$\begin{aligned}&\mathbb{E}_\mu[h(X_{\zeta+1}, X_{\zeta+2}, \ldots)|x_\zeta, x_{\zeta-1}, \ldots]\\ &= \sum_{n=1}^\infty \mathbb{E}_\mu[h(X_{n+1}, X_{n+2}, \ldots)|x_n, x_{n-1}, \ldots, \zeta = n]P(\zeta = n|x_n, x_{n-1}, \ldots)\end{aligned}$$

and from the weak Markov property.

6.7 Given the transition matrix

$$\mathbb{P} = \begin{pmatrix} 0.0 & 0.4 & 0.6 & 0.0 & 0.0 \\ 0.65 & 0.0 & 0.35 & 0.0 & 0.0 \\ 0.32 & 0.68 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.12 & 0.88 \\ 0.0 & 0.0 & 0.0 & 0.56 & 0.44 \end{pmatrix},$$

examine whether the corresponding chain is irreducible and aperiodic.

6.8 Show that irreducibility in the sense of Definition 6.13 coincides with the more intuitive notion that two arbitrary states are connected when the Markov chain has a discrete support.

6.9 Show that an aperiodic Markov chain on a finite state-space with transition matrix $\mathbb{P}$ is irreducible if and only if there exists $N \in \mathbb{N}$ such that $\mathbb{P}^N$ has no zero entries. (The matrix is then called *regular.*)

6.10 (Kemeny and Snell 1960) Show that for a regular matrix $\mathbb{P}$:
(a) The sequence $(\mathbb{P}^n)$ converges to a stochastic matrix A.
(b) Each row of A is the same probability vector π.
(c) All components of π are positive.
(d) For every probability vector μ, $\mu\mathbb{P}^n$ converges to π.
(e) π satisfies $\pi = \pi\mathbb{P}$.
(*Note:* See Kemeny and Snell 1960, p. 71 for a full proof.)

6.11 Show that for the measure ψ given by (6.9), the chain (X_n) is irreducible in the sense of Definition 6.13. Show that for two measures φ_1 and φ_2, such that (X_n) is φ_i-irreducible, the corresponding ψ_i's given by (6.9) are equivalent.

6.12 Let $Y_1, Y_2, \ldots$ be iid rv's concentrated on $\mathbb{N}_+$ and Y_0 be another rv also concentrated on $\mathbb{N}_+$. Define

$$Z_n = \sum_{i=0}^{n} Y_i.$$

(a) Show that (Z_n) is a Markov chain. Is it irreducible?
(b) Define the forward recurrence time as

$$V_n^+ = \inf\{Z_m - n; Z_m > n\}.$$

Show that (V_n^+) is also a Markov chain.
(c) If $V_n^+ = k > 1$, show that $V_{n+1}^+ = k - 1$. If $V_n^+ = 1$, show that a renewal occurs at $n+1$. (*Hint:* Show that $V_{n+1}^+ \sim Y_i$ in the latter case.)

6.13 Detail the proof of Theorem 6.15. In particular, show that the fact that K_ϵ includes a Dirac mass does not invalidate the irreducibility. (*Hint*: Establish that

$$\mathbb{E}_x[\eta_A] = \sum_n P_x^n(A) > P_x(\tau_A < \infty) \,,$$

$$\lim_{\epsilon \to 1} K_\epsilon(x, A) > P_x(\tau_A < \infty) \,,$$

$$K_\epsilon(x, A) = (1-\epsilon) \sum_{i=1}^{\infty} \epsilon^i P^i(x, A) > 0$$

imply that there exists n such that $K^n(x, A) > 0$. See Meyn and Tweedie 1993, p. 87.)

6.14 Show that the multiplicative random walk

$$X_{t+1} = X_t \epsilon_t$$

is not irreducible when $\epsilon_t \sim \mathcal{E}xp(1)$ and $x_0 \in \mathbb{R}$. (*Hint:* Show that it produces two irreducible components.)

6.15 Show that in the setup of Example 6.17, the chain is not irreducible when ϵ_n is uniform on $[-1, 1]$ and $|\theta| > 1$.

6.16 In the spirit of Definition 6.25, we can define a *uniformly transient set* as a set A for which there exists $M < \infty$ with

$$\mathbb{E}_x[\eta_A] \leq M, \qquad \forall x \in A\,.$$

Show that *transient* sets are denumerable unions of *uniformly transient* sets.

6.17 Show that the split chain defined on $\mathcal{X} \times \{0, 1\}$ by the following transition kernel:

$$\begin{aligned}
&P(\check{X}_{n+1} \in A \times \{0\}|(x_n, 0)) \\
&= \mathbb{I}_C(x_n) \left\{ \frac{P(X_n, A \cap C) - \epsilon\nu(A \cap C)}{1-\epsilon}(1-\epsilon) \right. \\
&\quad \left. + \frac{P(X_n, A \cap C^c) - \epsilon\nu(A \cap C^c)}{1-\epsilon} \right\} \\
&\quad + \mathbb{I}_{C^c}(X_n)\ \{P(X_n, A \cap C)(1-\epsilon) + P(X_n, A \cap C^c)\}\ , \\
&P(\check{X}_{n+1} \in A \times \{1\}|(x_n, 0)) \\
&\quad = \mathbb{I}_C(X_n)\ \frac{P(X_n, A \cap C) - \epsilon\nu(A \cap C)}{1-\epsilon}\epsilon + \mathbb{I}_{C^c}(X_n)\ P(X_n, A \cap C)\epsilon,
\end{aligned}$$

$$\begin{aligned}
P(\check{X}_{n+1} \in A \times \{0\}|(x_n, 1)) &= \nu(A \cap C)(1-\epsilon) + \nu(A \cap C^c), \\
P(\check{X}_{n+1} \in A \times \{1\}|(x_n, 1)) &= \nu(A \cap C)\epsilon\ ,
\end{aligned}$$

satisfies

$$\begin{aligned}
P(\check{X}_{n+1} \in A \times \{1\}|\check{x}_n) &= \varepsilon\nu(A \cap C), \\
P(\check{X}_{n+1} \in A \times \{0\}|\check{x}_n) &= \nu(A \cap C^c) + (1-\varepsilon)\nu(A \cap C)
\end{aligned}$$

for every $\check{x}_n \in C \times \{1\}$. Deduce that $C \times \{1\}$ is an atom of the split chain $(\check{X}_n)$.

6.18 If C is a small set and $B \subset C$, under which conditions on B is B a small set?

6.19 If C is a small set and $D = \{x; P^m(x, D) > \delta\}$, show that D is a small set for δ small enough. (*Hint:* Use the Chapman–Kolmogorov equations.)

6.20 Show that the period d given in Definition 6.23 is independent of the selected small set C and that this number characterizes the chain (X_n).

6.21 Given the transition matrix

$$\mathbb{P} = \begin{pmatrix} 0.0 & 0.4 & 0.6 & 0.0 & 0.0 \\ 0.6 & 0.0 & .35 & 0.0 & 0.05 \\ 0.32 & .68 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.12 & 0.0 & 0.88 \\ 0.14 & 0.3 & 0.0 & 0.56 & 0.0 \end{pmatrix},$$

show that the corresponding chain is aperiodic, despite the null diagonal.

The *random walk* (Examples 6.40 and 6.39) is a useful probability model and has been given many colorful interpretations. (A popular one is the description of an inebriated individual whose progress along a street is composed of independent steps in random directions, and a question of interest is to describe where the individual will end up.) Here, we look at a simple version to illustrate a number of the Markov chain concepts.

6.22 A random walk on the non-negative integers $I = \{0, 1, 2, \ldots\}$ can be constructed in the following way. For $0 < p < 1$, let $Y_0, Y_1, \ldots$ be iid random variables with $P(Y_i = 1) = p$ and $P(Y_i = -1) = 1 - p$, and $X_k = \sum_{i=0}^{k} Y_i$. Then, (X_n) is a Markov chain with transition probabilities

$$P(X_{i+1} = j + 1 | X_i = j) = p, \qquad P(X_{i+1} = j - 1 | X_i = j) = 1 - p,$$

but we make the exception that $P(X_{i+1} = 1 | X_i = 0) = p$ and $P(X_{i+1} = 0 | X_i = 0) = 1 - p$.

(a) Show that (X_n) is a Markov chain.
(b) Show that (X_n) is also irreducible.
(c) Show that the invariant distribution of the chain is given by

$$a_k = \left(\frac{p}{1-p} \right)^k a_0, \quad k = 1, 2, \ldots,$$

where a_k is the probability that the chain is at k and a_0 is arbitrary. For what values of p and a_0 is this a probability distribution?
(d) If $\sum a_k < \infty$, show that the invariant distribution is also the stationary distribution of the chain; that is, the chain is ergodic.

6.23 If (X_t) is a random walk, $X_{t+1} = X_t + \epsilon_t$, such that ϵ_t has a moment generating function f, defined in a neighborhood of 0, give the moment generating function of X_{t+1}, g_{t+1} in terms of g_t and f, when $X_0 = 0$. Deduce that there is no invariant distribution with a moment generating function in this case.

Although the property of aperiodicity is important, it is probably less important than properties such as recurrence and irreducibility. It is interesting that Feller (1971, Section XV.5) notes that the classification into periodic and aperiodic states "represents a nuisance." However, this is less true when the random variables are continuous.

6.24 (Continuation of Problem 6.22)
(a) Using the definition of periodic given here, show that the random walk of Problem 6.22 is periodic with period 2.
(b) Suppose that we modify the random walk of Problem 6.22 by letting $0 < p + q < 1$ and redefining

$$Y_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p - q \\ -1 & \text{with probability } q. \end{cases}$$

Show that this random walk is irreducible and aperiodic. Find the invariant distribution, and the conditions on p and q for which the Markov chain is positive recurrent.

6.25 (Continuation of Problem 6.22) A Markov chain that is not positive recurrent may be either null recurrent or *transient*. In either of these latter two cases, the invariant distribution, if it exists, is not a probability distribution (it does not have a finite integral), and the difference is one of expected return times. For any integer j, the probability of returning to j in k steps is $p_{jj}^{(k)} = P(X_{i+k} = j|X_i = j)$, and the expected return time is thus $m_{jj} = \sum_{k=1}^{\infty} kp_{jj}^{(k)}$.

(a) Show that since the Markov chain is irreducible, $m_{jj} = \infty$ either for all j or for no j; that is, for any two states x and y, x is transient if and only if y is transient.

(b) An irreducible Markov chain is *transient* if $m_{jj} = \infty$; otherwise it is recurrent. Show that the random walk is positive recurrent if $p < 1/2$ and transient if $p > 1/2$.

(c) Show that the random walk is null recurrent if $p = 1/2$. This is the interesting case where each state will be visited infinitely often, but the expected return time is infinite.

6.26 Explain why the resolvant chain is necessarily strongly irreducible.

6.27 Consider a random walk on $\mathbb{R}_+$, defined as

$$X_{n+1} = (X_n + \epsilon)^+.$$

Show that the sets $(0, c)$ are small, provided $P(\epsilon < 0) > 0$.

6.28 Consider a random walk on $\mathbb{Z}$ with transition probabilities

$$P(Z_t = n + 1|Z_{t-1} = n) = 1 - P(Z_t = n - 1|Z_{t-1} = n) \propto n^{-\alpha}$$

and

$$P(Z_t = 1|Z_{t-1} = 0) = 1 - P(Z_t = -1|Z_{t-1} = 0) = 1/2 .$$

Study the recurrence properties of the chain in terms of α.

6.29 Establish (i) and (ii) of Theorem 6.28.

(a) Use

$$K^n(x, A) \geq K^r(x, \alpha)K^s(\alpha, \alpha)K^t(\alpha, A)$$

for $r + s + t = n$ and r and s such that

$$K^r(x, \alpha) > 0 \qquad \text{and} \qquad K^s(\alpha, A) > 0$$

to derive from the Chapman–Kolmogorov equations that $\mathbb{E}_x[\eta_A] = \infty$ when $\mathbb{E}_\alpha[\eta_\alpha] = \infty$.

(b) To show (ii):

a) Establish that transience is equivalent to $P_\alpha(\tau_\alpha < \infty) < 1$.

b) Deduce that $\mathbb{E}_x[\eta_\alpha] < \infty$ by using a generating function as in the proof of Proposition 6.31.

c) Show that the covering of $\mathcal{X}$ is made of the

$$\overline{\alpha}_j = \{y; \sum_{n=1}^{j} K^n(y, \alpha) > j^{-1}\}.$$

6.30 Referring to Definition 6.32, show that if $P(\eta_A = \infty) \neq 0$ then $\mathbb{E}_x[\eta_A] = \infty$, but that $P(\eta_A = \infty) = 0$ does not imply $\mathbb{E}_x[\eta_A] < \infty$.

6.31 In connection with Example 6.42, show that the chain is null recurrent when $f'(1) = 1$.

6.32 Referring to (6.21):
(a) Show that $\mathbb{E}_\nu[\tau_C] < \infty$;
(b) show that $\sum_\nu P(\tau_C \geq t) = \mathbb{E}_\nu[\tau_C]$.

6.33 Let $\Gamma = \{Z_n : n = 0, 1, \ldots\}$ be a discrete time homogeneous Markov chain with state space $\mathcal{Z}$ and Markov transition kernel

$$M(z, \cdot) = \omega\nu(\cdot) + (1-\omega)K(z, \cdot) \,, \tag{6.37}$$

where $\omega \in (0, 1)$ and ν is a probability measure.
(a) Show that the measure

$$\phi(\cdot) = \sum_{i=1}^{\infty} \omega(1-\omega)^{i-1} K^{i-1}(\nu, \cdot)$$

is an invariant probability measure for Γ.
(b) Deduce that Γ is positive recurrent.
(c) Show that, when Φ satisfies a minorization condition with $C = \mathcal{X}$, (6.10) holds for all $x \in \mathcal{X}$ and is thus a mixture of the form (6.37).

(*Note:* Even if the Markov chain associated with K is badly behaved, e.g., transient, Γ is still positive recurrent. Breyer and Roberts (2000b) propose another derivation of this result, through the functional identity

$$\int \phi(x) M(x, z) dx = \pi(z) \,.\Big)$$

6.34 Establish the equality (6.14).

6.35 Consider the simple Markov chain (X_n), where each X_i takes on the values -1 and 1 with $P(X_{i+1} = 1 | X_i = -1) = 1$, $P(X_{i+1} = -1 | X_i = 1) = 1$, and $P(X_0 = 1) = 1/2$.
(a) Show that this is a stationary Markov chain.
(b) Show that $\text{cov}(X_0, X_k)$ does not go to zero.
(c) The Markov chain is not strictly positive. Verify this by exhibiting a set that has positive unconditional probability but zero conditional probability.

(*Note:* The phenomenon seen here is similar to what Seidenfeld and Wasserman 1993 call a *dilation*.)

6.36 In the setup of Example 6.5, find the stationary distribution associated with the proposed transition when $\pi_i = \pi_j$ and in general.

6.37 Show the decomposition of the "first entrance and last exit" equation (6.23).

6.38 If (a_n) is a sequence of real numbers converging to a, and if $b_n = (a_1 + \cdots + a_n)/n$, then show that

$$\lim_n b_n = a \,.$$

(*Note:* The sum $(1/n)\sum_{i=1}^n a_i$ is called a *Cesàro average*; see Billingsley 1995, Section A30.)

6.39 Consider a sequence (a_n) of positive numbers which is converging to a^* and a convergent series with running term b_n. Show that the convolution

$$\sum_{j=1}^{n-1} a_j b_{n-j} \stackrel{n\to\infty}{\longrightarrow} a^* \sum_{j=1}^{\infty} b_j \,.$$

(*Hint:* Use the Dominated Convergence Theorem.)

6.40 (a) Verify (6.26), namely, that $\|\mu\|_{TV} = (1/2) \sup_{|h|\leq 1} \left|\int h(x)\mu(dx)\right|$.

(b) Show that (6.26) is compatible with the definition of the total variation norm. Establish the relation with the alternative definition

$$||\mu||_{TV} = \sup_A \mu(A) - \inf_A \mu(A).$$

6.41 Show that if (X_n) and (X'_n) are coupled at time N_0 and if $X_0 \sim \pi$, then $X'_n \sim \pi$ for $n > N_0$ for any initial distribution of X'_0.

6.42 Using the notation of Section 6.6.1, set

$$u(n) = \sum_{j=0}^{\infty} p^{j\star}(n)$$

with $p^{j\star}$ the distribution of the sum $S_1 + \cdots + S_j$, $p^{0\star}$ the Dirac mass at 0, and

$$Z(n) = \mathbb{I}_{\exists j; S_j = n}.$$

(a) Show that $P_q(Z(n) = 1) = q \star u(n)$.

(b) Show that

$$|q \star u(n) - p \star u(n)| \leq 2\, P(T_{pq} > n).$$

(This bound is often called *Orey's inequality*, from Orey 1971. See Problem 7.10 for a slightly different formulation.)

(c) Show that if m_p is finite,

$$e(n) = \frac{\sum_{n+1}^{\infty} p(j)}{m_p}$$

is the invariant distribution of the renewal process in the sense that $P_e(Z(n) = 1) = 1/m_p$ for every n.

(d) Deduce from Lemma 6.49 that

$$\lim_n \left| q \star u(n) - \frac{1}{m_p} \right| = 0$$

when the mean renewal time is finite.

6.43 Consider the so-called "forward recurrence time" process V_n^+, which is a Markov chain on $\mathbb{N}_+$ with transition probabilities

$$P(1, j) = p(j), \qquad j \geq 1,$$
$$P(j, j-1) = 1, \qquad j > 1\,,$$

where p is an arbitrary probability distribution on $\mathbb{N}_+$. (See Problem 6.12.)

(a) Show that (V_n^+) is recurrent.

(b) Show that

$$P(V_n^+ = j) = p(j + n - 1).$$

(c) Deduce that the invariant measure satisfies

$$\pi(j) = \sum_{n \geq j} p(n)$$

and show it is finite if and only if

$$m_p = \sum_n np(n) < \infty.$$

6.44 (Continuation of Problem 6.43) Consider two independent forward recurrence time processes (V_n^+) and (W_n^+) with the same generating probability distribution p.

(a) Give the transition probabilities of the joint process $V_n^* = (V_n^+, W_n^+)$.

(b) Show that (V_n^*) is irreducible when p is aperiodic. (*Hint:* Consider r and s such that g.c.d.$(r,s) = 1$ with $p(r) > 0$, $p(q) > 0$, and show that if $nr - ms = 1$ and $i \geq j$, then

$$P_{(i,j)}(V^*_{j+(i-j)nr} = (1,1)) > 0\ .)$$

(c) Show that $\pi^* = \pi \times \pi$, with π defined in Problem 6.43 is invariant and, therefore, that (V_n^*) is positive Harris recurrent when $m_p < \infty$.

6.45 (Continuation of Problem 6.44) Consider V_n^* defined in Problem 6.44 associated with (S_n, S_n') and define $\tau_{1,1} = \min\{n; V_n^* = (1,1)\}$.

(a) Show that $T_{pq} = \tau_{1,1} + 1$.

(b) Use (c) in Problem 6.44 to show Lemma 6.49.

6.46 (Kemeny and Snell 1960) Establish (directly) the Law of Large Numbers for a finite irreducible state-space chain (X_n) and for $h(x_n) = \mathbb{I}_j(x_n)$, if j is a possible state of the chain; that is,

$$\frac{1}{N}\sum_{n=1}^{N} \mathbb{I}_j(x_n) \longrightarrow \pi_j\ ,$$

where $\pi = (\pi_1, \ldots, \pi_j, \ldots)$ is the stationary distribution.

6.47 (Kemeny and Snell 1960) Let $\mathbb{P}$ be a regular transition matrix, that is, $\mathbb{P}A = A\mathbb{P}$ (see Problem 6.9), with limiting (stationary) matrix A; that is, each column of A is equal to the stationary distribution.

(a) Show that the so-called *fundamental matrix* $\mathbb{Z} = (I - (\mathbb{P} - A))^{-1}$ exists.

(b) Show that $\mathbb{Z} = I + \sum_{n=1}^{\infty}(\mathbb{P}^n - A)$.

(c) Show that $\mathbb{Z}$ satisfies $\pi\mathbb{Z} = \pi$ and $\mathbb{P}\mathbb{Z} = \mathbb{Z}\mathbb{P}$, where π denotes a row of A (this is the stationary distribution).

6.48 (Continuation of Problem 6.47) Let $N_j(n)$ be the number of times the chain is in state j in the first n instants.

(a) Show that for every initial distribution μ,

$$\lim_{n\to\infty} \mathbb{E}_\mu[N_j(n)] - n\pi_j = \mu(\mathbb{Z} - A).$$

(*Note:* This convergence shows the strong stability of a recurrent chain since each term in the difference goes to infinity.)

(b) Show that for every pair of initial distributions, (μ, ν),

$$\lim_{n\to\infty} \mathbb{E}_\mu[N_j(n)] - \mathbb{E}_\nu[N_j(n)] = (\mu - \nu)\mathbb{Z}.$$

(c) Deduce that for every pair of states, (u, v),

$$\lim_{n\to\infty} \mathbb{E}_u[N_j(n)] - \mathbb{E}_v[N_j(n)] = z_{uj} - z_{vj},$$

which is called the *divergence* $\operatorname{div}_j(u, v)$.

6.49 (Continuation of Problem 6.47) Let f_j denote the number of steps before entering state j.

(a) Show that for every state i, $\mathbb{E}_i[f_j]$ is finite.
(b) Show that the matrix M with entries $m_{ij} = \mathbb{E}_i[f_j]$ can be written $M = \mathbb{P}(M - M_d) + E$, where M_d is the diagonal matrix with same diagonal as M and E is the matrix made of 1's.
(c) Deduce that $m_{ii} = 1/\pi_i$.
(d) Show that πM is the vector of the z_{ii}/π_i's.
(e) Show that for every pair of initial distributions, (μ, ν),

$$\mathbb{E}_\mu[f_i] - \mathbb{E}_\nu[f_i] = (\mu - \nu)(I - \mathbb{Z})D,$$

where D is the diagonal matrix $\text{diag}(1/\pi_i)$.

6.50 If h is a function taking values on a finite state-space $\{1, \ldots, r\}$, with $h(i) = h_i$, and if (X_n) is an irreducible Markov chain, show that

$$\lim_{n\to\infty} \frac{1}{n}\text{var}\left(\sum_{t=1}^{n} h(x_t)\right) = \sum_{i,j} h_i c_{ij} h_j,$$

where $c_{ij} = \pi_i z_{ij} + \pi_j z_{ji} - \pi_i \delta_{ij} - \pi_i \pi_j$ and δ_{ij} is Kroenecker's 0–1 function.

6.51 For the two-state transition matrix $\mathbb{P} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$, show that
(a) the stationary distribution is $\pi = (\beta/(\alpha+\beta), \alpha/(\alpha+\beta))$;
(b) the mean first passage matrix is

$$M = \begin{pmatrix} (\alpha+\beta)/\beta & 1/\alpha \\ 1/\beta & (\alpha+\beta)/\alpha \end{pmatrix};$$

(c) and the limiting variance for the number of times in state j is $\alpha\beta(2-\alpha-\beta)/(\alpha+\beta)^3$, for $j = 1, 2$.

6.52 Show that a finite state-space chain is always geometrically ergodic.

6.53 (Kemeny and Snell 1960) Given a finite state-space Markov chain, with transition matrix $\mathbb{P}$, define a second transition matrix by

$$p_{ij}(n) = \frac{P_\mu(X_{n-1} = j)P(X_n = i|X_{n-1} = j)}{P_\mu(X_n = j)}.$$

(a) Show that $p_{ij}(n)$ does not depend on n if the chain is stationary (i.e., if $\mu = \pi$).
(b) Explain why, in this case, the chain with transition matrix $\tilde{\mathbb{P}}$ made of the probabilities

$$\tilde{p}_{ij} = \frac{\pi_j p_{ji}}{\pi_i}$$

is called the *reverse* Markov chain.
(c) Show that the limiting variance C is the same for both chains.

6.54 (Continuation of Problem 6.53) A Markov chain is *reversible* if $\tilde{\mathbb{P}} = \mathbb{P}$. Show that every two-state ergodic chain is reversible and that an ergodic chain with symmetric transition matrix is reversible. Examine whether the matrix

$$\mathbb{P} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

is reversible. (*Hint:* Show that $\pi = (0.1, 0.2, 0.4, 0.2, 0.1)$.)

6.55 (Continuation of Problem 6.54) Show that an ergodic random walk on a finite state-space is reversible.

6.56 (Kemeny and Snell 1960) A Markov chain (X_n) is *lumpable* with respect to a nontrivial partition of the state-space, $(A_1, \ldots, A_k)$, if, for every initial distribution μ, the process

$$Z_n = \sum_{i=1}^{k} i \mathbb{I}_{A_i}(X_n)$$

is a Markov chain with transition probabilities independent of μ.

(a) Show that a necessary and sufficient condition for lumpability is that

$$p_{uA_j} = \sum_{v \in A_j} p_{uv}$$

is constant (in n) on A_i for every i.

(b) Examine whether

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0.5 & 0 \end{pmatrix}$$

is lumpable for $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5\}$.

6.57 Consider the random walk on $\mathbb{R}^+$, $X_{n+1} = (X_n + \epsilon_n)^+$, with $\mathbb{E}[\epsilon_n] = \beta$.

(a) Establish Lemma 6.70. (*Hint:* Consider an alternative V to V^* and show by recurrence that

$$\begin{aligned} V(x) &\geq \int_C K(x,y)V(y)dy + \int_{C^c} K(x,y)V(y)dy \\ &\geq \cdots \geq V^*(x) \text{ .)} \end{aligned}$$

(b) Establish Theorem 6.72 by assuming that there exists x^* such that $P_{x^*}(\tau_C < \infty) < 1$, choosing M such that $M \geq V(x^*)/[1 - P_{x^*}(\tau_C < \infty)]$ and establishing that $V(x^*) \geq M[1 - P_{x^*}(\tau_C < \infty)]$.

6.58 Show that

(a) a time-homogeneous Markov chain (X_n) is stationary if the initial distribution is the invariant distribution;

(b) the invariant distribution of a stationary Markov chain is also the marginal distribution of any X_n.

6.59 Referring to Section 6.7.1, let X_n be a Markov chain and $h(\cdot)$ a function with $\mathbb{E}h(X_n) = 0$, $\text{Var}h(X_n) = \sigma^2 > 0$, and $\mathbb{E}h(X_{n+1}|x_n) = h(x_n)$, so $h(\cdot)$ is a nonconstant harmonic function.

(a) Show that $\mathbb{E}h(X_{n+1}|x_0) = h(x_0)$.

(b) Show that $\text{Cov}(h(x_0), h(X_n)) = \sigma^2$.

(c) Use (6.52) to establish that $\text{Var}\left(\frac{1}{n+1}\sum_{i=0}^{n} h(X_i)\right) \to \infty$ as $n \to \infty$, showing that the chain is not ergodic.

6.60 Show that if an irreducible Markov chain has a σ-finite invariant measure, this measure is unique up to a multiplicative factor. (*Hint:* Use Theorem 6.63.)

6.61 (Kemeny and Snell 1960) Show that for an aperiodic irreducible Markov chain with finite state-space and with transition matrix $\mathbb{P}$, there always exists a stationary probability distribution which satisfies

$$\pi = \pi \mathbb{P}.$$

(a) Show that if $\beta < 0$, the random walk is recurrent. (*Hint:* Use the drift function $V(x) = x$ as in Theorem 6.71.)
(b) Show that if $\beta = 0$ and $\mathrm{var}(\epsilon_n) < \infty$, (X_n) is recurrent. (*Hint:* Use $V(x) = \log(1+x)$ for $x > R$ and $V(x) = 0$, otherwise, for an adequate bound R.)
(c) Show that if $\beta > 0$, the random walk is transient.

6.62 Show that if there exist a finite potential function V and a small set C such that V is bounded on C and satisfies (6.40), the corresponding chain is Harris positive.

6.63 Show that the random walk on $\mathbb{Z}$ is transient when $\mathbb{E}[W_n] \neq 0$.

6.64 Show that the chains defined by the kernels (6.46) and (6.48) are either both recurrent or both transient.

6.65 Referring to Example 6.66, show that the AR(1) chain is reversible.

6.66 We saw in Section 6.6.2 that a stationary Markov chain is *geometrically ergodic* if there is a non-negative real-valued function M and a constant $r < 1$ such that for any $A \in \mathcal{X}$,

$$|P(X_n \in A | X_0 \in B) - P(X_n \in A)| \leq M(x) r^n.$$

Prove that the following Central Limit Theorem (due to Chan and Geyer 1994) can be considered a corollary to Theorem 6.82 (see Note 6.9.4):

Corollary 6.69. *Suppose that the stationary Markov chain $X_0, X_1, X_2, \ldots$ is geometrically ergodic with $M^* = \int |M(x)| f(x) dx < \infty$ and satisfies the moment conditions of Theorem 6.82. Then*

$$\sigma^2 = \lim_{n \to \infty} n \,\mathrm{var} \bar{X}_n < \infty$$

and if $\sigma^2 > 0$, $\sqrt{n} \bar{X}_n / \sigma$ tends in law to $\mathcal{N}(0, \sigma^2)$.

(*Hint*: Integrate (with respect to f) both sides of the definition of geometric ergodicity to conclude that the chain has exponentially fast α-mixing, and apply Theorem 6.82.)

6.67 Suppose that $X_0, X_1, \ldots, X_n$ have a common mean ξ and variance σ^2 and that $\mathrm{cov}(X_i, X_j) = \rho_{j-i}$. For estimating ξ, show that

(a) $\bar{X}$ may not be consistent if $\rho_{j-i} = \rho \neq 0$ for all $i \neq j$. (*Hint*: Note that $\mathrm{var}(\bar{X}) > 0$ for all sufficiently large n requires $\rho \geq 0$ and determine the distribution of $\bar{X}$ in the multivariate normal case.)
(b) $\bar{X}$ is consistent if $|\rho_{j-i}| \leq M \gamma^{j-i}$ with $|\gamma| < 1$.

6.68 For the situation of Example 6.84:

(a) Prove that the sequence (X_n) is stationary provided $\sigma^2 = 1/(1 - \beta^2)$.
(b) Show that $\mathbb{E}(X_k | x_0) = \beta^k x_0$. (*Hint*: Consider $\mathbb{E}[(X_k - \beta X_{k-1}) | x_0]$.)
(c) Show that $\mathrm{cov}(X_0, X_k) = \beta^k / (1 - \beta^2)$.

6.69 Under the conditions of Theorem 6.85, it follows that $\mathbb{E}[\mathbb{E}(X_k | X_0)]^2 \to 0$. There are some other interesting properties of this sequence.

(a) Show that

$$\mathrm{var}[\mathbb{E}(X_k|X_0)] = \mathbb{E}[\mathbb{E}(X_k|X_0)]^2,$$
$$\mathrm{var}[\mathbb{E}(X_k|X_0)] \geq \mathrm{var}[\mathbb{E}(X_{k+1}|X_0)]] \ .$$

(*Hint*: Write $f_{k+1}(y|x) = \int f_k(y|x')f(x'|x)dx'$ and use Fubini and Jensen.)

(b) Show that

$$\mathbb{E}[\mathrm{var}(X_k|X_0)] \leq \mathbb{E}[\mathrm{var}(X_{k+1}|X_0)]$$

and that

$$\lim_{k\to\infty} \mathbb{E}[\mathrm{var}(X_k|X_0)] = \sigma^2.$$

6.9 Notes

6.9.1 Drift Conditions

Besides atoms and small sets, Meyn and Tweedie (1993) rely on another tool to check or establish various stability results, namely, *drift criteria*, which can be traced back to Lyapunov. Given a function V on $\mathcal{X}$, the *drift of* V is defined by

$$\Delta V(x) = \int V(y)\ P(x, dy) - V(x) \ .$$

(Functions V appearing in this setting are often referred to as *potentials*; see Norris 1997.) This notion is also used in the following chapters to verify the convergence properties of some MCMC algorithms (see, e.g., Theorem 7.15 or Mengersen and Tweedie 1996).

The following lemma is instrumental in deriving drift conditions for the transience or the recurrence of a chain (X_n).

Lemma 6.70. *If $C \in \mathcal{B}(\mathcal{X})$, the smallest positive function which satisfies the conditions*

$$\Delta V(x) \leq 0 \quad \text{if} \quad x \notin C, \ \ V(x) \geq 1 \quad \text{if} \quad x \in C \tag{6.38}$$

is given by

$$V^*(x) = P_x(\sigma_C < \infty) \ ,$$

where σ_C denotes

$$\sigma_C = \inf\{n \geq 0; x_n \in C\} \ .$$

Note that, if $x \notin C$, $\sigma_C = \tau_C$, while $\sigma_C = 0$ on C. We then have the following necessary and sufficient condition.

Theorem 6.71. *The ψ-irreducible chain (X_n) is transient if and only if there exist a bounded positive function V and a real number $r \geq 0$ such that for every x for which $V(x) > r$, we have*

$$\Delta V(x) > 0 \ . \tag{6.39}$$