# OCELOT Program
# Physical Methods Manual

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**Abstract**

This work is compilation of what we know about Lagrangian and Hamiltonian approaches for the particle dynamics. Probably it will be basis of future document which will called ***"Physical Methods Manual"***.

The Hamiltonian for nonlinear coupled synchro-betatron oscillations of ultra-relativistic charge particle is derived. The equation of motions for various type of magnets (dipole, quadrupole, sextupole) and for cavity are then solved. The equations are used to develop a nonlinear, symplectic tracking module.

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# 1 Introduction

Hamiltonian approach is very powerful tool for accelerator physics. We will try to derive Hamiltonian for the relativistic particle in electro-magnetic field and we start from Lagrangian. I recommend read Feynman's treatment of *The Principle of least action* for example here [1].
You can find Hamilton derivatie for relativistic particle in wollowng works [2] (in Russian) and [3]. But most comprechensive derivation of the Hamiltonian for accelerator cases is in [4]. That is why the main part of section ***"Equations of motion"*** is almost exact copy of this work.
This document is based on following works [5], [6] and especially as I've mentioned before [4].

# 2 Equations of motion

## 2.1 The Lagrangian of a charged particle

Lagrangian for relativistic charged particle in an electromagnetic field is:

$$\begin{aligned} L &= -m_0c^2\sqrt{1-\frac{v^2}{c^2}} + \frac{e}{c}(\dot{\vec{r}}\vec{A}) - e\phi \ \textit{(in CGS system)} \\ L &= -m_0c^2\sqrt{1-\frac{v^2}{c^2}} + e(\dot{\vec{r}}\vec{A}) - e\phi \ \textit{(in SI system)} \end{aligned} \tag{2.1}$$

where $\vec{r}$ - is the position vector, $\vec{A}$ and $\phi$ vector and scalar potential, m and e mass and charge of particle, $v = \left|\dot{\vec{r}}\right|$.

$$\begin{aligned} \vec{E} &= -grad\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \ \textit{(in CGS system)} \\ \vec{E} &= -grad\phi - \frac{\partial\vec{A}}{\partial t} \ \textit{(in SI system)} \end{aligned} \tag{2.2}$$

$$\vec{B} = rot\vec{A} \tag{2.3}$$

The equations of motion are derived from the Euler-Lagrange equations and in cartesian coordinates we have:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\vec{r}}} - \frac{\partial L}{\partial\vec{r}} = 0 \tag{2.4}$$

## 2.2 The Moving frame

In accelerator physics the natural coordinates are x, z, s. We assume that the closed orbit comprises piecewise flat curves which lie either in the horizontal or vertical plane so that it has no torsion. The design orbit which will be used as the reference system will in the following be described by the vector $\vec{r}_0(s)$ where $s$ is the length along the design orbit. An arbitrary particle orbit $\vec{r}(s)$ is then described by the deviation $\delta\vec{r}(s)$ of the particle orbit $\vec{r}(s)$ from the design orbit $\vec{r}_0(s)$:

$$\vec{r}(s) = \vec{r}_0(s) + \delta\vec{r}(s)$$

To describe the vector $\delta\vec{r}$ we input the new orthonormal basis, Figure 2.1

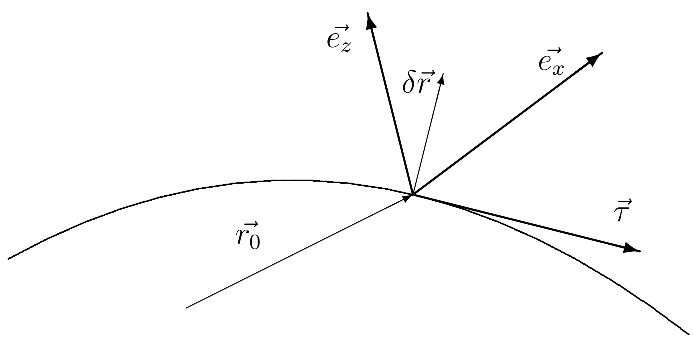


Figure 2.1: The Moving frame.

- $\vec{\tau}(s)$ is the unit vector tangent to the orbit, pointing in the direction of motion,
- $\vec{e_x}(s)$ is the normal unit vector,
- $\vec{e_z}(s) = \vec{\tau}(s) \times \vec{e_x}(s)$ is the binormal unit vector.

By definition (using Frenet-Serret formulas):

$$\vec{\tau}(s) = \frac{d}{ds}\vec{r_0}(s) \equiv \vec{r_0'}(s) \tag{2.5}$$

In the natural coordinate system we can represent $\delta\vec{r}(s)$ as:

$$\delta\vec{r}(s) = (\delta\vec{r} \cdot \vec{e_x}) \cdot \vec{e_x} + (\delta\vec{r} \cdot \vec{e_z}) \cdot \vec{e_z} \tag{2.6}$$

the $\vec{\tau}$-component of $\delta\vec{r}(s)$ is always zero by definition in frame accompanies the particle.
Thus, the orbit-vector $\vec{r}(s)$ can be written in the form

$$\vec{r}(s, x, z) = \vec{r_0}(s) + x(s) \cdot \vec{e_x}(s) + z(s) \cdot \vec{e_z}(s) \tag{2.7}$$

And the Frenet-Serret formulas now read:

$$\begin{aligned}
&\frac{d}{ds}\vec{e_x}(s) = h_x(s) \cdot \vec{\tau}(s) \\
&\frac{d}{ds}\vec{e_z}(s) = h_z(s) \cdot \vec{\tau}(s) \\
&\frac{d}{ds}\vec{\tau}(s) = -h_x(s) \cdot \vec{e_x}(s) - h_z(s) \cdot \vec{e_z}(s)
\end{aligned} \tag{2.8}$$

with

$$h_x(s) \cdot h_y(s) = 0 \tag{2.9}$$

where $h_x(s)$, $h_y(s)$ designate the curvatures in the x-direction and the z-direction respectively. Curvature in the vertical plane is negative if the centre of curvature lies above the reference trajectory.

The connections between the curvatures $h_x$, $h_z$ and the guide fields $B_z$, $B_x$ is given by

$$\begin{aligned}
h_x &= \frac{e}{E_0} \cdot B_z \\
h_y &= -\frac{e}{E_0} \cdot B_x
\end{aligned} \tag{2.10}$$

From Eq. 2.5, 2.7 and 2.8 one then has

$$\begin{aligned}
\dot{\vec{r}} &= \dot{s} \cdot \left[\frac{d\vec{r_0}}{ds} + x \cdot \frac{d\vec{e_x}}{ds} + z \cdot \frac{d\vec{e_z}}{ds}\right] + \dot{x} \cdot \vec{e_x} + \dot{z} \cdot \vec{e_z} \\
&= \vec{\tau} \cdot \dot{s}(1 + x \cdot h_x + z \cdot h_z) + \dot{x} \cdot \vec{e_x} + \dot{z} \cdot \vec{e_z}
\end{aligned} \tag{2.11}$$

so that for the expressions $\sqrt{1 - \frac{v^2}{c^2}}$ and $(\dot{\vec{r}} \cdot \vec{A})$ in Eq. 2.1 we have

$$\begin{aligned}
&\sqrt{1 - \frac{v^2}{c^2}} = \left(1 - \frac{1}{c^2}\left[\dot{x}^2 + \dot{z}^2 + (1 + h_x \cdot x + h_z \cdot z)^2 \cdot \dot{s}^2\right]\right)^{1/2}; \\
&(\dot{\vec{r}} \cdot \vec{A}) = \dot{x} \cdot A_x + \dot{z} \cdot A_z + \dot{s}(1 + h_x \cdot x + h_z \cdot z) \cdot A_s
\end{aligned}$$

where the component of a vector $\vec{A}$ with respect to the $(\vec{e_x}, \vec{e_z}, \vec{e_s})$ coordinate system are defined by the equation

$$\vec{A} = A_x \cdot \vec{e_x} + A_z \cdot \vec{e_z} + A_s \cdot \vec{\tau}$$

In the new coordinate system $x, z, s$, the Lagrangian in Eq. 2.1 then becomes

$$\begin{aligned}
L(x, z, s, \dot{x}, \dot{z}, \dot{s}, t) = &-m_0 c^2 \cdot \sqrt{1 - \frac{1}{c^2}\left[\dot{x}^2 + \dot{z}^2 + (1 + h_x \cdot x + h_z \cdot z)^2 \cdot \dot{s}^2\right]} + \\
&+ \frac{e}{c} \cdot (\dot{x} \cdot A_x + \dot{z} \cdot A_z + (1 + h_x \cdot x + h_z \cdot z) \cdot \dot{s} \cdot A_s) - e\phi
\end{aligned} \tag{2.12}$$

and the equations of motion take the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = \frac{\partial L}{\partial z} \tag{2.13}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = \frac{\partial L}{\partial s}$$

## 2.3 The Hamiltonian in the moving frame

In order to obtain the equations of motion in canonical form we now use the Lagrangian 2.12 to construct the corresponding Hamiltonian:

$$H = p_x \cdot \dot{x} + p_z \cdot \dot{z} + p_s \cdot \dot{s} - L \tag{2.14}$$

where $p_x, p_z, p_s$ are the canonical moments

$$p_x = \frac{\partial L}{\partial \dot{x}}; \quad p_z = \frac{\partial L}{\partial \dot{z}}; \quad p_s = \frac{\partial L}{\partial \dot{s}}$$

Using Eq.2.12 these are given by

$$\begin{aligned} p_x &= \frac{m_0 \dot{x}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c} A_x \\ p_z &= \frac{m_0 \dot{z}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c} A_z \\ p_x &= \frac{m_0 \dot{s}}{\sqrt{1 - \frac{v^2}{c^2}}} \cdot (1 + h_x \cdot x + h_z \cdot z)^2 + \frac{e}{c}(1 + h_x \cdot x + h_z \cdot z) \cdot A_s \end{aligned} \tag{2.15}$$

Putting now Eqs. 2.15 and 2.12 into 2.14 we have

$$H = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e \cdot \phi \tag{2.16}$$

thus, as is well know, H here is the sum of the mechanical and field energy. In this equation, the momenta are still written in term of the velocity. However, using the relation

$$(p_x - \frac{e}{c}A_x)^2 + (p_z - \frac{e}{c}A_z)^2 + \left(\frac{p_s}{1 + h_x \cdot x + h_z \cdot z} - \frac{e}{c}A_s\right)^2 + \\ + m_0^2c^2 = \frac{m_0^2c^2}{1 - \frac{v^2}{c^2}} \tag{2.17}$$

Now we may finally write the relativistic Hamiltonian for the motion of a particle of charge $e$ and mass $m_0$ in electromagnetic field given by potentials $\vec{A}$ and $\phi$ as

$$H(x, z, s, p_x, p_z, p_s, t) = c \cdot \left[ (m_0^2c^2 + (p_x - \frac{e}{c}A_x)^2 + (p_z - \frac{e}{c}A_z)^2 + \right. \\ \left. + \left(\frac{p_s}{1 + h_x \cdot x + h_z \cdot z} - \frac{e}{c}A_s\right)^2 \right]^{1/2} + e \cdot \phi \tag{2.18}$$

The equation of motion are then derived from the canonical equations

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x}; & \dot{p_x} &= -\frac{\partial H}{\partial x}; \\ \dot{z} &= \frac{\partial H}{\partial p_z}; & \dot{p_z} &= -\frac{\partial H}{\partial z}; \\ \dot{s} &= \frac{\partial H}{\partial p_s}; & \dot{p_s} &= -\frac{\partial H}{\partial s}. \end{aligned} \tag{2.19}$$

## 2.4 The arc length as independent variable

In Eq. 2.19 the time $t$ appeared as independent variable. In order, as is usual in accelerator physics, to introduce the arc length $s$ of the design orbit as independent variable we recall that Eq. 2.19 is equivalent to a version of Hamilton's principle.

$$\delta \int_{t1}^{t2} \mathrm{d}t \cdot (p_x \cdot \dot{x} + p_z \cdot \dot{z} + p_s \cdot \dot{s} - H(x, z, s, p_x, p_z, p_s, t)) = 0; \tag{2.20}$$

$$\begin{aligned} &\delta x(t_1) = \delta z(t_1) = \delta s(t_1) = 0; \quad \delta p_x(t_1) = \delta p_z(t_1) = \delta p_s(t_1) = 0; \\ &\delta x(t_2) = \delta z(t_2) = \delta s(t_2) = 0; \quad \delta p_x(t_2) = \delta p_z(t_2) = \delta p_s(t_2) = 0; \\ &\delta t(t_1) = \delta t(t_1) = 0, \end{aligned}$$

where the variables $x, z, s, p_x, p_z, p_s, t$ are varied independently of each other and are help constant at the end points.

Eq.2.20 can now be rewritten using $\mathrm{d}t = \frac{\mathrm{d}t}{\mathrm{d}s}\mathrm{d}s$ as:

$$
\begin{aligned}
&\delta \int_{s1}^{s2} \mathrm{d}s \cdot (p_x \cdot x' + p_z \cdot z' + t' \cdot (-H) + p_s(x, z, t, p_x, p_z, -H, s)) = 0; \\
&\delta x(s_1) = \delta z(s_1) = \delta t(s_1) = 0; \quad \delta p_x(s_1) = \delta p_z(s_1) = \delta H(s_1) = 0; \\
&\delta x(s_2) = \delta z(s_2) = \delta t(s_2) = 0; \quad \delta p_x(s_2) = \delta p_z(s_2) = \delta H(s_2) = 0; \\
&\delta s(s_1) = \delta s(s_1) = 0,
\end{aligned} \tag{2.21}
$$

with

$$y' \equiv \frac{dy}{ds}(y \equiv x, z, t)$$

where we make independent variations of the variables $x, z, t, p_x, p_z, (-H), t$; and $s$ is the independent variable. The required equations with $s$ as independent variable are then obtained from the Euler equations of the variational problem (2.21):

$$
\begin{aligned}
x' &= \frac{\partial K}{\partial p_x}; \quad p_x' = -\frac{\partial K}{\partial x}; \\
z' &= \frac{\partial K}{\partial p_z}; \quad p_z' = -\frac{\partial K}{\partial z}; \\
t' &= \frac{\partial K}{\partial(-H)}; \quad (-H)' = -\frac{\partial K}{\partial t};
\end{aligned} \tag{2.22}
$$

with

$$
\begin{aligned}
K &= -p_s \\
&= -(1 + h_x \cdot x + h_z \cdot z) \cdot \sqrt{\frac{(H - e\phi)^2}{c^2} - m_0^2c^2 - (p_x - \frac{e}{c}A_x)^2 - (p_z - \frac{e}{c}A_z)^2} \\
&\qquad - (1 + h_x \cdot x + h_z \cdot z) \cdot \frac{e}{c}A_s.
\end{aligned} \tag{2.23}
$$

Thus, we again have a set of equations with canonical structure but this time the Hamiltonian is $K = K(x, z, t, p_x, p_z, -H, s)$ and the canonical variables are

$$(x, p_x); \quad (z, p_z); \quad (t, -H).$$

In the following we put $\phi = 0$ then from Eq.2.16

$$H = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv E \quad \textit{the energy of particle} \tag{2.24}$$

and if we now use the variables $(-ct)$ and $\eta$

$$\eta = \frac{E - E_0}{E_0} \tag{2.25}$$

instead of $t$ and $H$, Eq.2.22 gives

$$\begin{aligned} x' &= \frac{\partial \overline{K}}{\partial \widehat{p_x}}; \quad \widehat{p_x}' = -\frac{\partial \overline{K}}{\partial x}; \\ z' &= \frac{\partial \overline{K}}{\partial p_z}; \quad \widehat{p_z}' = -\frac{\partial \overline{K}}{\partial z}; \\ (-ct)' &= \frac{\partial \overline{K}}{\partial \eta}; \quad \eta' = -\frac{\partial \overline{K}}{\partial (-ct)}; \end{aligned} \tag{2.26}$$

with

$$\begin{aligned} &\overline{K} = \frac{c}{E_0} \cdot K \\ &= -h \cdot \sqrt{(1+\eta)^2 - \left(\frac{m_0 c^2}{E_0}\right)^2 - (\widehat{p}_x - \frac{e}{E_0} A_x)^2 - (\widehat{p}_z - \frac{e}{E_0} A_z)^2} - h \cdot \frac{e}{E_0} A_s; \\ &\textit{where} \quad h = 1 + h_x \cdot x + h_z \cdot z \end{aligned} \tag{2.27}$$

$$\begin{aligned} \widehat{p_x} &= \frac{c}{E_0} p_x = \frac{c}{E_0} m\nu_x + \frac{e}{E_0} A_s; \\ \widehat{p_z} &= \frac{c}{E_0} p_z = \frac{c}{E_0} m\nu_z + \frac{e}{E_0} A_z. \end{aligned} \tag{2.28}$$

Since the variable $t(s)$ increase without limit, it is more useful to introduce the variable

$$\sigma = s - ct(s) \tag{2.29}$$

which describes the delay in arrival time at position $s$ of a particle travelling at the speed of light $c$

This further change of variables can be achieved using the generating function (more informations about generating functions you will find in Appendix ***Canonical Transformation***)

$$F_3(p, \overline{q}, s) \equiv F_3(p_x, p_z, \eta, \overline{x}, \overline{z}, \sigma, s) = \underbrace{-p_x \cdot \overline{x} - p_z \cdot \overline{z}}_{\text{identity transformation}} - \sigma \cdot \eta + s \cdot \eta + f(s) \tag{2.30}$$

The corresponding transformation equations:

$$
\begin{aligned}
&(-ct) = -\frac{\partial F_3}{\partial \eta} \qquad (\text{wich leads to } -ct = \sigma - s; \quad \sigma = s - ct);\\
&\overline{\eta} = -\frac{\partial F_3}{\partial \sigma} \qquad (\text{wich leads to } \overline{\eta} = \eta);
\end{aligned}
\tag{2.31}
$$

then immediately give (with $f(s) = s$) Hamiltonian equations of the form

$$
\begin{aligned}
x' &= \frac{\partial \widehat{K}}{\partial \widehat{p}_x}; \quad \widehat{p}_x{}' = -\frac{\partial \widehat{K}}{\partial x};\\
z' &= \frac{\partial \widehat{K}}{\partial p_z}; \quad \widehat{p}_z{}' = -\frac{\partial \widehat{K}}{\partial z};\\
\sigma' &= \frac{\partial \widehat{K}}{\partial \eta}; \quad \eta' = -\frac{\partial \widehat{K}}{\partial \sigma};
\end{aligned}
\tag{2.32}
$$

with

$$
\begin{aligned}
&\widehat{K} = (1+\eta)\cdot\left(1 - h\cdot\sqrt{1 - \frac{(\widehat{p}_x - \frac{e}{E_0}A_x)^2}{(1+\eta)^2} - \frac{(\widehat{p}_z - \frac{e}{E_0}A_z)^2}{(1+\eta)^2}}\right) - h\cdot\frac{e}{E_0}A_s\\
&\textit{with} \quad h = 1 + h_x\cdot x + h_z\cdot z
\end{aligned}
\tag{2.33}
$$

where the term $\left(\frac{m_0c^2}{E_0}\right)^2$ in Eq.2.27 has been dropped since we assume that $\left(\frac{m_0c^2}{E_0}\right)^2 \ll 1$ and can be neglected.

The canonical equations (2.32) together with Hamiltonian (2.33) give defining equations for non-linear coupled synchro-betatron motion and they will serve as the starting point for the developments to follow.

Finally we point out that since $\left|\widehat{p}_y - \frac{e}{E_0}A_y\right| = \left|\frac{c}{E_0}m\nu_y\right| \ll 1$ $(y \equiv x, z)$ the square root

$$
\sqrt{1 - \frac{(\widehat{p}_x - \frac{e}{E_0}A_x)^2}{(1+\eta)^2} - \frac{(\widehat{p}_z - \frac{e}{E_0}A_z)^2}{(1+\eta)^2}}
$$

in Eq.2.33 can be expanded in a series:

$$
\begin{aligned}
&\sqrt{1 - \frac{(\widehat{p}_x - \frac{e}{E_0}A_x)^2}{(1+\eta)^2} - \frac{(\widehat{p}_z - \frac{e}{E_0}A_z)^2}{(1+\eta)^2}} =\\
&= 1 - \frac{1}{2}\frac{(\widehat{p}_x - \frac{e}{E_0}A_x)^2}{(1+\eta)^2} - \frac{1}{2}\frac{(\widehat{p}_z - \frac{e}{E_0}A_z)^2}{(1+\eta)^2} + \ldots
\end{aligned}
\tag{2.34}
$$

# 3 Vector potential for various magnet type

In order to utilize the Hamiltonian of Eq.2.33, the vector potential $\vec{A} = \vec{A}(x, z, s, t)$, for the commonly occurring types of accelerator magnet must be given. Once $\vec{A}$ is know the fields $\vec{E}$ and $\vec{B}$ can be found using Eq.2.2 and 2.3. In the variables $x, z, s, \sigma$ these become (see Appendix ***Curl of the vector potential in the moving frame)***

$$\vec{E} = \frac{\partial}{\partial \sigma} \vec{A} \tag{3.1}$$

$$\begin{aligned} B_x &= \frac{1}{h} \cdot \left( \frac{\partial}{\partial z} (h \cdot A_s) - \frac{\partial}{\partial s} A_z \right) \\ B_z &= \frac{1}{h} \cdot \left( \frac{\partial}{\partial s} A_x - \frac{\partial}{\partial x} (h \cdot A_s) \right) \\ B_s &= \frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x \end{aligned} \tag{3.2}$$

with

$$h = 1 + h_x \cdot x + h_z \cdot z \tag{3.3}$$

Using freedom of gauge, we can choose any vector potential which leads to the correct form of the fields. Suitable vector potentials are ad follows and have been chosen for their simplicity.

## 3.1 Cavity

For a longitudinal electric field ...... .... ....

## 3.2 Transverse magnetic fields

### 3.2.1 Transverse magnetic fields in a stright section

$$\begin{aligned} &A_x = 0; \quad A_z = 0; \\ &h_x = h_z = 0; \\ &\widehat{K} = (1+\eta) \cdot \left( 1 - \sqrt{1 - \frac{\widehat{p}_x^2}{(1+\eta)^2} - \frac{\widehat{p}_z^2}{(1+\eta)^2}} \right) - \frac{e}{E_0} A_s; \\ &B_x = \frac{\partial}{\partial z} \cdot A_s; \\ &B_z = -\frac{\partial}{\partial x} \cdot A_s; \\ &B_s = 0. \end{aligned} \tag{3.4}$$

#### 3.2.1.1 Quadrupole

The quadrupole fields are

$$B_x = z \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0};$$
$$B_z = x \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0}; \tag{3.5}$$

so that we may use the vector potential

$$A_s = \frac{1}{2}(z^2 - x^2) \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0}. \tag{3.6}$$

In the following we rewrite the term $\frac{e}{E_0}A_s$ in (3.4) as

$$\frac{e}{E_0}A_s = \frac{1}{2}g_0 \cdot (z^2 - x^2)$$
$$g_0 = \frac{e}{E_0} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \tag{3.7}$$

#### 3.2.1.2 Skew Quadrupole

in the process

#### 3.2.1.3 Sextupole

$$B_x = \left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=0} \cdot xz$$
$$B_z = \left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=0} \cdot \frac{1}{2}(x^2 - z^2)$$

so that

$$\frac{e}{E_0}A_s = -\lambda_0 \cdot \frac{1}{6}(x^3 - 3xz^2) \tag{3.8}$$

with

$$\lambda_0 = \frac{e}{E_0}\left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=0}$$

#### 3.2.1.4 Octupole

$$
\begin{aligned}
&B_x = \frac{1}{6}\left(\frac{\partial^3 B_x}{\partial z^3}\right)_{x=z=0} \cdot (z^3 - 3x^2 z) \\
&B_z = \frac{1}{6}\left(\frac{\partial^3 B_x}{\partial z^3}\right)_{x=z=0} \cdot (3xz^2 - x^3) \\
&\text{so that} \\
&\frac{e}{E_0} A_s = \mu_0 \cdot \frac{1}{24}(z^4 - 6x^2 z^2 + x^4) \\
&\text{with} \\
&\mu_0 = \frac{e}{E_0}\left(\frac{\partial^3 B_x}{\partial z^3}\right)_{x=z=0}
\end{aligned}
\tag{3.9}
$$

#### 3.2.1.5 Dipole

in the process

### 3.2.2 Synchrotron-Magnet

in the process

## 3.3 Solenoid

in the process

# 4 Solution of the equations of motion

Now that the potential $A(x, z, s, t)$ for each magnet type is know in is possible to derive the equations of motion for the various magnets. For this purpose, we truncate the series expansion of the Hamiltonian at second order in the canonical momenta:

$$
\begin{aligned}
&\sqrt{1 - \frac{(\widehat{p}_x - \frac{e}{E_0} A_x)^2}{(1+\eta)^2} - \frac{(\widehat{p}_z - \frac{e}{E_0} A_z)^2}{(1+\eta)^2}} = \\
&= 1 - \frac{1}{2}\frac{(\widehat{p}_x - \frac{e}{E_0} A_x)^2}{(1+\eta)^2} - \frac{1}{2}\frac{(\widehat{p}_z - \frac{e}{E_0} A_z)^2}{(1+\eta)^2}
\end{aligned}
\tag{4.1}
$$

so that it is possible to solve the equations of motion exactly as shown in the following.

## 4.1 Cavity

in the process ...

## 4.2 Transverse magnetic fields

### 4.2.1 Transverse magnetic fields in a straight section

**4.2.1.1 Quadrupole** From Eq.3.4, 3.7 and 4.1, the Hamiltonian for a quadrupole is given by

$$\widehat{K} = \frac{1}{2}\frac{\widehat{p}_x^2}{1+\eta} + \frac{1}{2}\frac{\widehat{p}_z^2}{1+\eta} + \frac{1}{2}g_0 \cdot (x^2 - z^2) \tag{4.2}$$

The corresponding canonical equations are then (see Eq.2.32)

$$\begin{aligned}
x' &= \frac{\widehat{p}_x}{1+\eta} \\
\widehat{p}_x' &= -g_0 x \\
z' &= \frac{\widehat{p}_z}{1+\eta} \\
\widehat{p}_z' &= g_0 z \\
\sigma' &= -\frac{1}{2}\left(\frac{\widehat{p}_x^2}{(1+\eta)^2} + \frac{\widehat{p}_z^2}{(1+\eta)^2}\right) \equiv -\frac{1}{2}\left((x')^2 + (z')^2\right) \\
\eta' &= 0
\end{aligned} \tag{4.3}$$

by eliminating $\widehat{p}_x$ (and $\widehat{p}_z$) in Eq.4.3 one has

$$\begin{aligned}
x'' &= -g \cdot x \\
z'' &= g \cdot z
\end{aligned} \tag{4.4}$$

where (see Eq.3.7)

$$g = \frac{g_0}{1+\eta} = \frac{e}{E_0} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \tag{4.5}$$

Writing now the solution of Eq.4.3 in the form

$$\vec{y}(s) = M(s,0)\vec{y}(0) \tag{4.6}$$

with $\vec{y}^T = (x, \widetilde{p}_x, z, \widetilde{p}_z)$

$$\begin{aligned}
\widetilde{p}_x &\equiv x' \\
\widetilde{p}_z &\equiv z'
\end{aligned} \tag{4.7}$$

we obtain for the 4-dimensional transfer matrix $M(s,0)$.
For $g > 0$

$$M(s,0) = \begin{pmatrix} \cos\phi & \frac{1}{\sqrt{g}}\sin\phi & 0 & 0 \\ -\sqrt{g}\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cosh\phi & \frac{1}{\sqrt{g}}\sinh\phi \\ 0 & 0 & \sqrt{g}\sinh\phi & \cosh\phi \end{pmatrix} \tag{4.8}$$

$$\text{with} \quad \phi = \sqrt{g}\cdot s$$

For $g < 0$

$$M(s,0) = \begin{pmatrix} \cosh\phi & \frac{1}{\sqrt{g}}\sinh\phi & 0 & 0 \\ \sqrt{g}\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \frac{1}{\sqrt{g}}\sin\phi \\ 0 & 0 & -\sqrt{g}\sin\phi & \cos\phi \end{pmatrix} \tag{4.9}$$

$$\text{with} \quad \phi = \sqrt{g}\cdot s$$

From Eq.4.3

$$\eta(s) = \eta(0) \tag{4.10}$$

and finally form the last equation of Eq.4.3

$$\begin{aligned} \sigma(s) &= \sigma(0) - \frac{1}{2}\int_0^s \mathrm{d}\widetilde{s}\cdot([x'(\widetilde{s})]^2 + [z'(\widetilde{s})]^2) = \\ &= \sigma(0) - \frac{g}{4}[x^2(0)\cdot(s - M_{11}\cdot M_{12})) - z^2(0)\cdot(s - M_{33}\cdot M_{34})] - \\ &- \frac{1}{4}[x'^2(0)\cdot(s + M_{11}\cdot M_{12}) + z'^2(0)\cdot(s + M_{33}\cdot M_{43})] - \\ &- \frac{1}{2}x'(0)\cdot x'(0)\cdot M_{12}\cdot M_{21} - \frac{1}{2}z'(0)\cdot z'(0)\cdot M_{34}\cdot M_{43}. \\ &\text{with } M \equiv M(s,0) \end{aligned} \tag{4.11}$$

#### 4.2.1.2 Skew Quadrupole

in the process

#### 4.2.1.3 Sextupole

From Eq.3.4 together with Eq.3.8 the Hamiltonian for a sextupole is given by

$$\widehat{K} = \frac{1}{2}\frac{\widehat{p}_x^2}{1+\eta} + \frac{1}{2}\frac{\widehat{p}_z^2}{1+\eta} + \frac{\lambda_0}{6}g_0\cdot(x^3 - 3xz^2) \tag{4.12}$$

And the canonical equations of motion are

$$\begin{aligned}
x' &= \frac{\widehat{p}_x}{1+\eta} \\
\widehat{p}_x' &= -\frac{1}{2}\lambda_0(x^2 - z^2) \\
z' &= \frac{\widehat{p}_z}{1+\eta} \\
\widehat{p}_z' &= \lambda_0 \cdot xz \\
\sigma' &= -\frac{1}{2}\left((x')^2 + (z')^2\right) \\
\eta' &= 0
\end{aligned} \tag{4.13}$$

by eliminating $\widehat{p}_x$ (and $\widehat{p}_z$) in Eq.4.13 one has

$$\begin{aligned}
x'' &= -\frac{1}{2}\lambda \cdot (x^2 - z^2) \\
z'' &= \lambda \cdot zx
\end{aligned} \tag{4.14}$$

where

$$\lambda = \frac{\lambda_0}{1+\eta} = \frac{e}{E_0} \cdot \left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=0} \tag{4.15}$$

IN the case where the sextupole is a thin lens with

$$\lambda = \widehat{\lambda} \cdot \delta(s - s_0) \tag{4.16}$$

Eq.4.13 may be easily integrated

$$\begin{aligned}
\sigma(s_0+0) &= \sigma(s_0-0) \\
\eta(s_0+0) &= \eta(s_0-0)
\end{aligned} \tag{4.17}$$

Then from Eq.4.14 one obtains

$$\begin{aligned}
x(s_0+0) &= x(s_0-0) \\
x'(s_0+0) &= x'(s_0-0) - \frac{\widehat{\lambda}}{2}(x^2(s_0-0) + z^2(s_0-0)) \\
z(s_0+0) &= z(s_0-0) \\
z'(s_0+0) &= z'(s_0-0) - \widehat{\lambda} \cdot x(s_0-0)z(s_0-0)
\end{aligned} \tag{4.18}$$

#### 4.2.1.4 Octupole

in the process

#### 4.2.1.5 Dipole

in the process

### 4.2.2 Synchrotron-Magnet

in the process

## 4.3 Solenoid

in the process

# 5 Lie algebraic methods

For writing this chapter we used almost all available works for this topic. But most useful for start was Dragt's lectures [7] and works F.C. Iselin [8] [9] and J.Irwin [10]

## 5.1 Matrix exponential

One of the reasons for the importance of the matrix exponential is that it can be used to solve systems of linear ordinary differential equations. The solution of

$$\dot{\vec{y}}(t) = A \cdot \vec{y}(t), \quad \vec{y}(0) = \vec{y}_0 \tag{5.1}$$

where A is constant matrix $n \times n$, is given by

$$\vec{y}(t) = e^{A \cdot t} \cdot \vec{y}_0 \tag{5.2}$$

The exponential of $A$, denoted by $e^A$ or $exp(A)$, is the $n \times n$ matrix given by the power series

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \tag{5.3}$$

The above series always converges, so the exponential of $A$ is well-defined. Note that if $A$ is a $1 \times 1$ matrix the matrix exponential of $A$ is a $1 \times 1$ matrix whose single element is the ordinary exponential of the single element of $A$.

Abstractly, the matrix exponential gives the connection between a matrix Lie algebra and the corresponding Lie group.

## 5.2 Poisson Brackets

Let $H(q,p,t)$ be the Hamiltonian for some dynamical system and let $f$ be any dynamical variable. That is, let $f(q,p,t)$ be any function of the phase space variables $q,p$ and the time $t$. Consider the problem of computing the total time rate of change of $f$ along a trajectory generated by $H$. According to the chain rule, this derivative is given by the expression

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left( \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i \right) \tag{5.4}$$

However, the $\dot{q}_i$ and $\dot{p}_i$ are given by Hamiltonian's equations of motion 2.19. Consequently, the expression for $\frac{df}{dt}$ can also be written in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left( \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \tag{5.5}$$

The second quantity appearing on the right of the last equation occur so often that it is given a special symbol and a special name in honour of Poisson. Let $f$ and $g$ be any two function of the variables $q,p,t$. Then the Poisson bracket of $f$ and $g$, denoted by the symbol $[f,g]$ is defined by equation

$$[f,g] = \sum_i \left( \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right).^{1} \tag{5.6}$$

With this new notation 5.5 can be writen in the compact form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f,H] \tag{5.8}$$

The Poisson bracket operation has several remarkable properties. Upon calculation one finds the relations

1. Distribute property

$$[(af+bg),h] = a[f,h] + b[g,h] \tag{5.9}$$

   for arbitrary constants $a,b$

[1] Sometimes you can meet other way to write the definition of the Poisson bracket, but it means the same as is easily seen

$$[f,g] = (\vec{\nabla} g)^T \cdot J \cdot (\vec{\nabla} g) \tag{5.7}$$

where $J$ is "symplectic unit matrix" (see The Symplectic Form of Hamiltons Equations of Motion).

2. Antisymmetry condition

$$[f,g] = -[g,f] \tag{5.10}$$

3. Jacobi identify

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0. \tag{5.11}$$

4. Derivation with respect to ordinary multiplication

$$[f,gh] = [f,g]h + g[f,h] \tag{5.12}$$

Using Poisson brackets Hamiltonian's equations of motion 2.19 can be written as

$$\begin{aligned} \dot{x} &= [-H,x]; & \dot{p_x} &= [-H,p_x]; \\ \dot{z} &= [-H,z]; & \dot{p_z} &= [-H,p_z]; \\ \dot{s} &= [-H,s]; & \dot{p_s} &= [-H,p_s]. \end{aligned} \tag{5.13}$$

Or in the compact form

$$\dot{\vec{y}} = [-H,\vec{y}] \tag{5.14}$$

where $\vec{y} = [x,y,s,p_x,p_y,p_s]^T$ or $\vec{y} = [x,p_x,y,p_y,s,p_s]^T$

## 5.3 Lie operators

Consider the Lie algebra with the Poison bracket defining the Lie product. For any function $f$ we define the Lie operator by its effect on another arbitrary function $g$

$$:f:g = [f,g]. \tag{5.15}$$

In analogous way, powers of $:f:$ are defined by taking repeated Poisson brackets, for example, $:f:^2$ is defined by the relation

$$:f:^2 g = [f,[f,g]]. \tag{5.16}$$

Finally, $:f:$ to the zero power is defined to be the identity operator,

$$:f:^0 g = g \tag{5.17}$$

A Lie operator, as well as its power, is a linear operator because of Eq.5.9. For the same reason, the sum of two Lie operators is again a Lie operator. Specifically, one finds the relation

$$a:f:+b:g:=:(af+bg): \tag{5.18}$$

for any two scalars $a, b$ and any two functions $f, g$. Therefore, the set of Lie operators forms a linear vector space.

Expressed in Lie operators the Jacobi identity can be written as

$$: f :: g : h + : g :: h : f + : h :: f : g = 0 \tag{5.19}$$

And other properties:

1. Derivation with respect to ordinary multiplication

$$: f : (gh) = (: f : g)h + g(: f : h) \tag{5.20}$$

2. Moving a Lie operator in or out of function

$$: f : g(y) = g(: f : z) \tag{5.21}$$

3. Other look of the Jacobi identity

$$: f : [g, h] = [: f : g, h] + [g, : f : h] \tag{5.22}$$

## 5.4 Lie transformations as Symplectic Maps

We define the Lie transformation for a function $f$ as the formal operator series

$$F = e^{:f:} = \sum_{k=0}^{\infty} \frac{: f :^k}{k!} \tag{5.23}$$

It represents an operator acting on a function of the phase space vector $y$. Several authors have shown [7] and others, that a convergent Lie series acting on the phase space vector $y$ defines a symplectic map. Given a time independent Hamiltonian $H$, the equation of motion is

$$\dot{\vec{y}} = - : H : \vec{y} \tag{5.24}$$

Formally its solution can be written as

$$M\vec{y} = e^{:-\int_0^L H:}\vec{y} = e^{:-LH:}\vec{y} \tag{5.25}$$

# 6 Appendix I

## 6.1 Derivation of the Hamiltonian equations

We recommend to read [1], [6] and [3] for deeper undurstanding of this topic. Function $L(q_k, \dot{q_k}, t)$ called the Lagrangian, where $q_k$ means a coordinate, $\dot{q_k}$ the derivative (i.e. the velocity) and $t$ is the time. **The principle of least action** says that integral of the Lagrangian must be minimum:

$$\int_1^2 L(q_k, \dot{q_k}, t)\mathrm{d}t = min \tag{6.1}$$

Using variational analysis we obtained:

$$\delta \int_1^2 L(q_k, \dot{q_k}, t)\mathrm{d}t = 0 \tag{6.2}$$

and remember one of the properties of the minimum is that if we go away from the minimum in the **first** order, the deviation of the function from its minimum value is only **second** order. At any place else on the curve, if we move a small distance the value of the function changes also in the first order (see Figure 6.1).

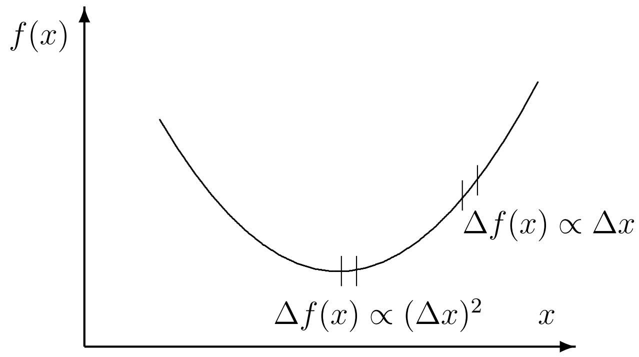


Figure 6.1: Variational principle.

Accordant to the variational principle we can write:

$$\delta \int_1^2 L(q_k, \dot{q_k}, t)dt = \int_1^2 \left[L(q_k + \delta q_k, \dot{q_k} + \delta \dot{q_k}, t) - L(q_k, \dot{q_k}, t)\right] \mathrm{d}t \tag{6.3}$$

where first term on the right side of equation we can write

$$L(q_k + \delta q_k, \dot{q_k} + \delta \dot{q_k}, t) = L(q_k, \dot{q_k}, t) + \frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q_k}}\delta \dot{q_k}. \tag{6.4}$$

So we obtained

$$\delta \int_1^2 L(q_k, \dot{q_k}, t)\mathrm{d}t = \int_1^2 \left( \frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q_k}}\delta \dot{q_k} \right) \mathrm{d}t. \tag{6.5}$$

The last term we can rewrite using integration by parts $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$, where $\mathrm{d}v = \delta\dot{q_k}\mathrm{d}t = \mathrm{d}\delta q_k$ and $u = \frac{\partial L}{\partial \dot{q_k}}$.

$$\int_1^2 \frac{\partial L}{\partial \dot{q_k}}\mathrm{d}\delta q_k = \frac{\partial L}{\partial \dot{q_k}}\delta q_k \bigg|_1^2 - \int_1^2 \frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}}\delta q_k \mathrm{d}t \tag{6.6}$$

As we know we can change $\delta q_k(t)$ and $\delta \dot{q_k}(t)$ everywhere but at the points $(q_k(t_1), \dot{q_k}(t_1))$ and $(q_k(t_2), \dot{q_k}(t_2))$ variance must be zero

$$\begin{aligned} \delta q_k(t_1) = \delta \dot{q_k}(t_1) = 0; \\ \delta q_k(t_2) = \delta \dot{q_k}(t_2) = 0. \end{aligned} \tag{6.7}$$

Thus $\frac{\partial L}{\partial \dot{q_k}}\delta q_k \Big|_1^2 = 0$ we can write

$$\delta \int_1^2 L(q_k, \dot{q_k}, t)dt = \int_1^2 \left( \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}} \right) \delta q_k \mathrm{d}t = 0$$

and finally the Euler-Lagrange equation is:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}} - \frac{\partial L}{\partial q_k} = 0 \tag{6.8}$$

The Lagrange equation of motion can be rewritten with the definition:

$$p_k = \frac{\partial L}{\partial \dot{q_k}} \tag{6.9}$$

to

$$\frac{d}{dt}p_k - \frac{\partial L}{\partial q_k} = 0 \tag{6.10}$$

where $p_k$ is called the canonical momentum.
A function $H(p_k, q_k, t)$, called the Hamiltonian function or the Hamiltonian, is defined as:

$$H(p_k, q_k, t) = p_k\dot{q_k} - L(q_k, \dot{q_k}, t) \tag{6.11}$$

Note that in a product like $p_k\dot{q_k}$, we mean the summation over the index k. Taking variations at both sides of Eq. 6.11 we get

$$\begin{aligned} \delta H &= \frac{\partial H}{\partial q_k}\delta q_k + \frac{\partial H}{\partial p_k}\delta p_k + \frac{\partial H}{\partial t}\delta t = \\ &= p_k\delta\dot{q_k} + \dot{q_k}\delta p_k - \frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q_k}}\delta \dot{q_k} + \frac{\partial L}{\partial t}\delta t \end{aligned} \tag{6.12}$$

We had already, by definition (Eq.6.9), $p_k = \frac{\partial L}{\partial \dot{q_k}}$. Therefore the coefficient of $\delta \dot{q_k}$ cancels. Futher the coefficients of $\delta q_k, \delta p_k, \delta t$ must be equal on both sides. This yelds

$$\dot{q_k} = \frac{\partial H}{\partial p_k}, \quad \frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial q_k}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{6.13}$$

The Lagrangian equations 6.10 show that $\dot{p_k} = \frac{\partial L}{\partial q_k}$. Thus

$$\dot{q_k} = \frac{\partial H}{\partial p_k}, \quad \dot{p_k} = -\frac{\partial H}{\partial q_k} \tag{6.14}$$

These are the canonical equations of motion.

# 7 Appendix II

## 7.1 Curl of the vector potential in the moving frame

Any (static) scalar field $u$ may be considered to be a function of the coordinates $x, z$, and $s$. The value of $u$ changes by an infinitesimal amount $du$ when the point of observation is changed by d$\vec{r}$. That change may be determined from the partial derivatives as:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial s}ds \tag{7.1}$$

But we also define the gradient in such a way as to obtain the result

$$du = \vec{\nabla} u \cdot d\vec{r} \tag{7.2}$$

Therefore,

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial s}ds = \vec{\nabla} u \cdot d\vec{r} \tag{7.3}$$

using expression for $d\vec{r}$ (see Eq.2.7 and 2.11)

$$d\vec{r} = \vec{\tau} \cdot ds \cdot (1 + x \cdot h_x + z \cdot h_z) + dx \cdot \vec{e_x} + dz \cdot \vec{e_z}$$

we can write in coordinates of the moving frame

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial s}ds = \left(\vec{\nabla} u\right)_x dx + \left(\vec{\nabla} u\right)_z dz + \left(\vec{\nabla} u\right)_s (1 + x \cdot h_x + z \cdot h_z)ds \tag{7.4}$$

and we demand that this hold for any choice of $dx, dz$ and $ds$. Thus,

$$\left(\vec{\nabla} u\right)_x = \frac{\partial u}{\partial x}, \quad \left(\vec{\nabla} u\right)_z = \frac{\partial u}{\partial z}, \quad \left(\vec{\nabla} u\right)_s = \frac{1}{h}\frac{\partial u}{\partial s}. \tag{7.5}$$

where $h = 1 + x \cdot h_x + z \cdot h_z$, and finally:

$$\vec{\nabla} = \vec{e_x}\frac{\partial}{\partial x} + \vec{e_z}\frac{\partial}{\partial z} + \frac{\vec{\tau}}{h}\frac{\partial}{\partial s} \tag{7.6}$$

where $h = 1 + x \cdot h_x + z \cdot h_z$.

The curl $\vec{\nabla} \times \vec{A}$ is carried out taking into account that the unit vectors themselves are functions of the coordinates. The derivatives must be taken before the cross product so that

$$\begin{aligned}\vec{\nabla} \times \vec{A} =& \widehat{e_x} \times \left(\frac{\partial A_x}{\partial x}\widehat{e_x} + \frac{\partial A_z}{\partial x}\widehat{e_z} + \frac{\partial A_s}{\partial x}\widehat{\tau} + A_x\frac{\partial \widehat{e_x}}{\partial x} + A_z\frac{\partial \widehat{e_z}}{\partial x} + A_s\frac{\partial \widehat{\tau}}{\partial x}\right) + \\ &+ \widehat{e_z} \times \left(\frac{\partial A_x}{\partial z}\widehat{e_x} + \frac{\partial A_z}{\partial z}\widehat{e_z} + \frac{\partial A_s}{\partial z}\widehat{\tau} + A_x\frac{\partial \widehat{e_x}}{\partial z} + A_z\frac{\partial \widehat{e_z}}{\partial z} + A_s\frac{\partial \widehat{\tau}}{\partial z}\right) + \\ &+ \frac{\widehat{\tau}}{h} \times \left(\frac{\partial A_x}{\partial s}\widehat{e_x} + \frac{\partial A_z}{\partial s}\widehat{e_z} + \frac{\partial A_s}{\partial s}\widehat{\tau} + A_x\frac{\partial \widehat{e_x}}{\partial s} + A_z\frac{\partial \widehat{e_z}}{\partial s} + A_s\frac{\partial \widehat{\tau}}{\partial s}\right)\end{aligned} \tag{7.7}$$

Using (2.8)

$$\begin{aligned}&\frac{d}{ds}\vec{e_x}(s) = h_x(s) \cdot \vec{\tau}(s) \\ &\frac{d}{ds}\vec{e_z}(s) = h_z(s) \cdot \vec{\tau}(s) \\ &\frac{d}{ds}\vec{\tau}(s) = -h_x(s) \cdot \vec{e_x}(s) - h_z(s) \cdot \vec{e_z}(s)\end{aligned} \tag{7.8}$$

we obtain

$$\begin{aligned}\vec{\nabla} \times \vec{A} =& \widehat{e_x} \times \left(\frac{\partial A_x}{\partial x}\widehat{e_x} + \frac{\partial A_z}{\partial x}\widehat{e_z} + \frac{\partial A_s}{\partial x}\widehat{\tau} + 0 + 0 + 0\right) + \\ &+ \widehat{e_z} \times \left(\frac{\partial A_x}{\partial z}\widehat{e_x} + \frac{\partial A_z}{\partial z}\widehat{e_z} + \frac{\partial A_s}{\partial z}\widehat{\tau} + 0 + 0 + 0\right) + \\ &+ \frac{\widehat{\tau}}{h} \times \left(\frac{\partial A_x}{\partial s}\widehat{e_x} + \frac{\partial A_z}{\partial s}\widehat{e_z} + \frac{\partial A_s}{\partial s}\widehat{\tau} + A_x h_x\widehat{\tau} + A_z h_z\widehat{\tau} + A_s(-h_x\widehat{e_x} - h_z\widehat{e_z})\right) \\ &= \left(\frac{\partial A_z}{\partial x}\widehat{\tau} - \frac{\partial A_s}{\partial x}\widehat{e_z}\right) - \left(\frac{\partial A_x}{\partial z}\widehat{\tau} - \frac{\partial A_s}{\partial z}\widehat{e_x}\right) + \\ &+ \left(\left(\frac{\partial A_x}{\partial s} - A_s h_x\right)\widehat{e_x} - \left(\frac{\partial A_x}{\partial s} - A_s h_x\right)\widehat{e_x}\right) = \\ &= \frac{1}{h}\left(\frac{\partial (h \cdot A_s)}{\partial z} - \frac{\partial A_z}{\partial s}\right)\widehat{e_x} + \frac{1}{h}\left(\frac{\partial A_x}{\partial s} - \frac{\partial (h \cdot A_s)}{\partial x}\right)\widehat{e_z} + \\ &+ \left(\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial z}\right)\widehat{\tau}\end{aligned} \tag{7.9}$$

# 8 Appendix III

## 8.1 Canonical Transformation

The canonical transformations are in many cases a powerful method for getting analytical solutions of rather comlicated particle trajectories. The aim generally is to transform the coordinates and momenta, often via several steps, such that the final Hamiltonian is time independent, easy to understand (e.g. via the study of flow lines in phase space) and/or shows originally coupled motion as a system where uncoupling exists for well chosen variables.

The Hamiltonian equations follow from Hamiltonian's principle

$$\delta \int_1^2 L(q_k, \dot{q_k}, t)\mathrm{d}t = 0 \tag{8.1}$$

In a transformed system we want to have the same principle

$$\delta \int_1^2 \overline{L}(q_k, \dot{\overline{q}}_k, t)\mathrm{d}t = 0 \tag{8.2}$$

The difference between the two Lagrangians must be a function that does not depend on the special path between the points (1) and (2). So we may put

$$L(q_k, \dot{q_k}, t) - \overline{L}(q_k, \dot{\overline{q}}_k, t) = \frac{dF_1(q_k, \overline{q}_k, t)}{dt} \tag{8.3}$$

The integral $\int_1^2 \frac{dF_1}{dt}dt$ only depends on the initial and final coordinates only and thus

$$\overline{L} = L - \frac{dF_1}{dt}, \tag{8.4}$$

satisfies Hamiltonian's principle.

Let us make small variations $\delta q, \delta \dot{q}, \delta \overline{q}, \delta \dot{\overline{q}}$. These variations are not independent of each other (only two of them can be chosen freely).

$$\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial t}\delta t - \frac{\partial \overline{L}}{\partial \overline{q}}\delta \overline{q} - \frac{\partial \overline{L}}{\partial \dot{\overline{q}}}\delta \dot{\overline{q}} - \frac{\partial \overline{L}}{\partial t}\delta t = \frac{d}{dt}\left(\frac{\partial F_1}{\partial q}\delta q + \frac{\partial F_1}{\partial \overline{q}}\delta \overline{q} + \frac{\partial F_1}{\partial t}\delta t\right) \tag{8.5}$$

Knowing that

$$p = \frac{\partial L}{\partial \dot{q}}, \qquad \overline{p} = \frac{\partial \overline{L}}{\partial \dot{\overline{q}}} \tag{8.6}$$

we find that, after equating the coefficients of the same variations, the coefficients for $\delta \dot{q}$ and $\delta \dot{\overline{q}}$ yelds

$$p = \frac{\partial F_1}{\partial q} \quad \textit{and} \quad \overline{p} = -\frac{\partial F_1}{\partial \overline{q}} \tag{8.7}$$

The other coefficients do not give new information.

Bothe system follow the Hamiltonian mechanics. Thus

$$\overline{H}(\overline{q},\overline{p},t) = \overline{p}\cdot\dot{\overline{q}} - \overline{L} = \overline{p}\cdot\dot{\overline{q}} - L + \frac{\partial F_1}{\partial t} = \overline{p}\cdot\dot{\overline{q}} - L + \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial \overline{q}}\dot{\overline{q}} + \frac{\partial F_1}{\partial t} \quad (8.8)$$

remember about 8.7 thus it follows that

$$\overline{H}(\overline{q},\overline{p},t) = p\cdot\dot{q} - L + \frac{\partial F_1}{\partial t} \quad (8.9)$$

or

$$\overline{H} = H + \frac{\partial F_1(q,\overline{q},t)}{\partial t} \quad (8.10)$$

The relation 8.7 and 8.10 together form the rules of the canonical transformations. Knowing $F$ the relation between the old and new variables follow from Eq.8.7. The function $F$ is called "the generating function".

An easier method is writing the principal fuction as $\int(pdq - Hdt)$.

$$p\cdot\dot{q} - H(p,q,t) = \overline{p}\cdot\dot{\overline{q}} - \overline{H}(\overline{p},\overline{q},t) + \frac{dF}{dt} \quad (8.11)$$

Then for

- $F \equiv F_1(q,\overline{q},t)$

$$p\cdot\dot{q} - H(p,q,t) = \overline{p}\cdot\dot{\overline{q}} - \overline{H}(\overline{p},\overline{q},t) + \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial \overline{q}}\dot{\overline{q}} + \frac{\partial F_1}{\partial t} \quad (8.12)$$

  Then

$$p = \frac{\partial F_1}{\partial q}, \quad \overline{p} = -\frac{\partial F_1}{\partial \overline{q}}, \quad \overline{H} = H + \frac{\partial F_1}{\partial t} \quad (8.13)$$

- $F \equiv -\overline{p}\cdot\overline{q} + F_2(q,\overline{p},t)$

$$p\cdot\dot{q} - H(p,q,t) = -\overline{q}\cdot\dot{\overline{p}} - \overline{H}(\overline{p},\overline{q},t) + \frac{\partial F_2}{\partial q}\dot{q} + \frac{\partial F_2}{\partial \overline{p}}\dot{\overline{p}} + \frac{\partial F_2}{\partial t} \quad (8.14)$$

  Then

$$p = \frac{\partial F_2}{\partial q}, \quad \overline{q} = \frac{\partial F_2}{\partial \overline{p}}, \quad \overline{H} = H + \frac{\partial F_2}{\partial t} \quad (8.15)$$

- $F \equiv p\cdot q + F_3(p,\overline{q},t)$

$$\begin{aligned} p\cdot\dot{q} - H(p,q,t) = \overline{p}\cdot\dot{\overline{q}} - \overline{H}(\overline{p},\overline{q},t) + \\ + p\cdot\dot{q} + q\cdot\dot{p} + \frac{\partial F_3}{\partial p}\dot{p} + \frac{\partial F_3}{\partial \overline{q}}\dot{\overline{q}} + \frac{\partial F_3}{\partial t} \end{aligned} \quad (8.16)$$

  Then

$$q = -\frac{\partial F_3}{\partial p}, \quad \overline{p} = -\frac{\partial F_3}{\partial \overline{q}}, \quad \overline{H} = H + \frac{\partial F_3}{\partial t} \quad (8.17)$$

- $F \equiv p \cdot q - \overline{p} \cdot \overline{q} + F_4(p, \overline{p}, t)$

$$p \cdot \dot{q} - H(p,q,t) = \overline{p} \cdot \dot{\overline{q}} - \overline{H}(\overline{p}, \overline{q}, t) + \\ + p \cdot \dot{q} + q \cdot \dot{p} - \overline{p} \cdot \dot{\overline{q}} - \overline{q} \cdot \dot{\overline{p}} + \frac{\partial F_4}{\partial p}\dot{p} + \frac{\partial F_4}{\partial \overline{q}}\dot{\overline{p}} + \frac{\partial F_3}{\partial t} \tag{8.18}$$

Then

$$q = -\frac{\partial F_4}{\partial p}, \quad \overline{q} = \frac{\partial F_4}{\partial \overline{p}}, \quad \overline{H} = H + \frac{\partial F_4}{\partial t} \tag{8.19}$$

The following forms of $F$ also obey a canonical transformations: The transformations keep the equations of motion Hamiltonian. Therefore Liouville is preserved in each system.

# 9 Appendix IV

## 9.1 The Symplectic Form of Hamiltons Equations of Motion

This matearial is taken from these works [11] and [7] and we think it will be usefull to collect different chapters to one place.

Definitions:

Let $z_1, z_2, ...z_{2n}$ be a sen of canonical veriables for some Hamiltonian dynamical system. Suppose a transformation is made to some new set of variables $\overline{z}_1(z,t)...\overline{z}_{2n}(z,t)$. Such a transformation will be called mapping, and will be denoted by the symbol $\mathsf{M}$,

$$\mathsf{M}: \quad z \to \overline{z}(z,t). \tag{9.1}$$

Also, let $M(z,t)$ be the Jacobian matrix of the map $M$. It is defined by the equation

$$M_{a,b}(z,t) = \frac{\partial \overline{z}_a}{\partial z_b} \tag{9.2}$$

The map $\mathsf{M}$ is said to be symplectic if its Jacobian matrix $M$ is a symplectic matrix for all value of $z$ and $t$,

$$M^T J M = J \qquad \text{or} \qquad M J M^T = J \tag{9.3}$$

or for our case (as we see below):

$$M^T S M = S \qquad \text{or} \qquad M S M^T = S \tag{9.4}$$

Hamiltons equations of motion can be written in a compact matrix formulation, and recasting them in that way is a good first step to introduce the study of symplecticity.[1] We begin with the equations themselves, which are normally written

$$\dot{q_k} = \frac{\partial H}{\partial p_k}, \quad \dot{p_k} = \frac{\partial H}{\partial q_k}, \quad k = 1, 2, \ldots, N \tag{9.5}$$

where overhead dots denote total differentiation with respect to time. These equations carry the implicit understanding that time is the independent vari- able. It is possible under certain circumstances to interchange the role of inde- pendent variable between time and one of the coordinates or momenta.The essential requirement for the feasibility of this procedure is that the variable chosen to replace time as independent variable must vary monotonically and smoothly with time. In particle-beam dynamics this interchange is customarily made between time and the longitudinal coordinate of the particle, called $s$, which is promoted to the status of independent variable, time becoming the third coordinate. This change of variables, its justification, and the derivation of the resulting Hamiltonian and equations of motion are discussed fully in the appendix. The resulting equations of motion, generalized to N degrees of freedom, are

$$q'_k = \frac{\partial \widetilde{H}}{\partial p_k}, \quad p'_k = \frac{\partial \widetilde{H}}{\partial q_k}, \quad k = 1, 2, \ldots, N \tag{9.6}$$

where primes denote total differentiation with respect to $s$, and the $\widetilde{H}$ denotes the Hamiltonian in which $s$, the longitudinal coordinate, is the independent variable (for our case $\widetilde{H} = \overline{K}$ see 2.27 or $\widetilde{H} = \widehat{K}$ see 2.33).

To cast these equations in matrix form, we first define $\vec{x}$, a vector of the canonical variables

$$\vec{x} = \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ \vdots \\ q_N \\ p_N \end{pmatrix} \tag{9.7}$$

The arrangement of this matrix has been choosen to accord with typical usage in accelerator beam dynamics.

Hamiltons equations of motion equate $q'$ to derivatives of $\widetilde{H}$ with respect to $p$, and they equate $p'$ to negative derivatives with respect to $q$. In order to express the equations in terms of $\vec{x}$, a rearranging matrix is needed. To help construct the rearranging matrix, we introduce an antisymmetric two-by-two constant matrix, $\mathbf{s}$.

$$\mathbf{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{9.8}$$

The rearranging matrix, which we shall denote by a capital $\mathbf{S}$, can be constructed of $\mathbf{s}$ and the null matrix by placing $\mathbf{s}$-matrix partitions along the diagonal up to the desired dimension and null matrix partitions everywhere else to form a $2N$ by $2N$ matrix.

$$\mathbf{S} = \begin{pmatrix} \mathbf{s} & 0 & \cdots & 0 \\ 0 & \mathbf{s} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{s} \end{pmatrix} \tag{9.9}$$

It is a very simple, sparse, square, even-dimensioned, antisymmetric matrix with only one entry in each row and one entry in each column. $\mathbf{S}$-matrices have the following obvious properties:

$$\mathbf{S} = \mathbf{S}^T = -\mathbf{S}, \quad \mathbf{S}^2 = -\mathbf{I}, \quad |\mathbf{S}| = 1 \tag{9.10}$$

where $\mathbf{I}$ is the identity matrix of appropriate order and $|\mathbf{S}|$ signifies the determinant of the matrix $\mathbf{S}$.

Using the matrix $\mathbf{S}$, we can now write Hamiltons equations of motion in compact form in terms of the vector $\vec{x}$.

$$\vec{x}' = \mathbf{S}\frac{\partial \widetilde{H}}{\partial \vec{x}} \tag{9.11}$$

where we define the derivative of a scalar function, $f$, with respect to the vector $\vec{x}$ as another vector according to

$$\left(\frac{\partial f}{\partial \vec{x}}\right)_i \equiv \frac{\partial f}{\partial x_i} \tag{9.12}$$

Eq. 9.11 is called the *symplectic form of Hamiltons canonical equations of motion.* The term, symplectic, comes from the Greek $\sigma\upsilon\mu\pi\lambda\epsilon\kappa\tau\iota\kappa o s$ which, according to the Oxford English Dictionary, means twining, plaited together or copulative.

The form of the matrix $\mathbf{S}$ was dictated entirely by the arbitrary organization of the vector $\vec{x}$. Our choice  alternating coordinates and momentaalthough usually used in beam dynamics, differs from the usual practice in classical mechanics textbooks of grouping the coordinates together at the top and the momenta at the bottom. In the latter case, the matrix $\mathbf{S}$, is replaced by another, often called $\mathbf{J}$, which has a different form:

$$\mathbf{J} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} \tag{9.13}$$

where $\mathbf{I}$ is the unit matrix. Of course, these are merely notational differences; the dynamics are the same.

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