Introduction

This paper contains the computation of the motive of the irreducible $\mathrm{SL}_3(k)$-character variety of torus knots for any algebraically closed field k of zero characteristic. The calculation is based on the methods introduced in the paper [1].

The notations used in this paper are the following:

- R_3^{irr} is the irreducible $\mathrm{SL}_3(k)$-representation variety of torus knots, that is, the variety of irreducible representations $\rho : \Gamma \to \mathrm{SL}_3(k)$ where $\Gamma = \Gamma_{n,m}$ is the fundamental group of the complement of the (n, m)-torus knot (see section 3 of [1]).
- $\mathfrak{M}_3^{\mathrm{irr}} = R_3^{\mathrm{irr}} /\!\!/ \mathrm{SL}_3(k)$ is the irreducible $\mathrm{SL}_3(k)$-character variety of torus knots, that is, the moduli space of representations (see section 3 of [1]).
- $\kappa = (\boldsymbol{\epsilon}, \boldsymbol{\varepsilon})$ is a configuration of eigenvalues, that is a collection of possible eigenvalues for the matrices A and B of a torus knot representation $\rho = (A, B)$ (see section 3 of [1]).
- τ is the type of a semi-simple filtration of a torus knot representation (see section 2.1 of [1]).
- ξ is the shape of the type τ, that is the collection of dimensions and multiplicities of each isotypic component (see section 2.1 of [1]).
- σ_A are the collections of eigenvalues of A for each isotypic component of a torus knot representation $\rho = (A, B)$ (see section 7.1 of [1]).
- σ_B are the collections of eigenvalues of B for each isotypic component of a torus knot representation $\rho = (A, B)$ (see section 7.1 of [1]).
- $\mathcal{M}_\tau$ is the space parametrizing possible completions of a semi-simple representation to a general representation of type τ (see section 4 of [1])
- $\mathcal{G}_\tau$ is the gauge group acting on $\mathcal{M}_\tau \times \mathrm{SL}_3(k)$ that identifies isomorphic completions (see section 4 of [1]).
- $\mathfrak{M}_\tau^{\mathrm{irr}}$ is the variety of possible semi-simplifications of a representation of type τ (see section 4 of [1]).
- $R(\tau)$ is the variety of representations of type τ.
- $m_\kappa(\tau)$ is the multiplicity of the type τ, that is the number of isomorphic components $R(\tau')$ of types τ' with the same shape as τ but whose eigenvalues are given by a permutation of the ones of τ that preserves their multiplicity (see section 5 of [1]).
- $C_{\pi,\pi'}$ are the number of isomorphic components given by configurations of eigenvalues with the same structure of repeated eigenvalues (see Section 6 of [1]). Here, π, π' are two partitions of 3 that determine the number of repeated eigenvalues of the matrices A and B of a representation $\rho = (A, B)$. If $\pi = \{1^{e_1}, \dots, 3^{e_3}\}$ and $\pi' = \{1^{e'_1}, \dots, 3^{e'_3}\}$ we have the following characterization in terms of multinomial numbers (Theorem 6.8 of [1])
$$C_{\pi,\pi'} = \frac{3}{nm} \binom{n}{e_1, \dots, e_3} \binom{m}{e'_1, \dots, e'_3}.$$

Combinatorial formulas for the motives $[\mathcal{M}_\tau], [\mathcal{G}_\tau]$ and $[\mathfrak{M}_\tau^{\mathrm{irr}}]$ are described in section 5 of [1] in terms of the structure of the type τ.

The structure of the paper is as follows. Each section describes the count of the motive $[\mathfrak{M}_\kappa]$ for a possible configuration of eigenvalues κ. For that purpose, we analyze all the types τ compatible with κ and compute the motives $[R(\tau)]$. A configuration of eigenvalues κ not appearing as a section of the paper means that $R_\kappa^{\mathrm{irr}} = \emptyset$ (see Remark 3.5 and Proposition 8.1 of [1]). In the final section of this paper, we summarize the results for each configuration κ and we provide the final result depending on the combinatorial coefficients $C_{\pi,\pi'}$.

Warning: The script generating this paper is only valid for rank ≤ 4. The result for higher rank may not be correct.

1. Configuration $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_1, \epsilon_2)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$

$$\xi = ((2,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_1), \quad \sigma_B = (\epsilon_1, \epsilon_3, \epsilon_2).$$

- $[\mathcal{M}_\tau] = 1.$
- $[\mathcal{G}_\tau] = (q-1)^2.$
- $[\mathfrak{M}_\tau^{\text{irr}}] = q - 2.$
- $[R(\tau)] = q^8 - q^7 - 2\,q^6 - q^5 + q^4 + 2\,q^3$
- $m_\kappa(\tau) = 3.$

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_1, \epsilon_2), \quad \sigma_B = (\epsilon_1, \epsilon_2, \epsilon_3).$$

- $[\mathcal{M}_\tau] = 1.$
- $[\mathcal{G}_\tau] = (q-1)^3.$
- $[\mathfrak{M}_\tau^{\text{irr}}] = 1.$
- $[R(\tau)] = q^6 + 2\,q^5 + 2\,q^4 + q^3$
- $m_\kappa(\tau) = 3.$

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_1, \epsilon_2), \quad \sigma_B = (\epsilon_1, \epsilon_2, \epsilon_3).$$

- $[\mathcal{M}_\tau] = q^2 - 1.$
- $[\mathcal{G}_\tau] = (q-1)^3.$
- $[\mathfrak{M}_\tau^{\text{irr}}] = 1.$
- $[R(\tau)] = q^8 + 2\,q^7 + q^6 - q^5 - 2\,q^4 - q^3$
- $m_\kappa(\tau) = 3.$

$$\xi = ((2,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_1), \quad \sigma_B = (\epsilon_1, \epsilon_3, \epsilon_2).$$

- $[\mathcal{M}_\tau] = q^2 - q.$
- $[\mathcal{G}_\tau] = (q-1)^2 q.$
- $[\mathfrak{M}_\tau^{\text{irr}}] = q - 2.$
- $[R(\tau)] = q^9 - 2\,q^8 - q^7 + q^6 + 2\,q^5 + q^4 - 2\,q^3$
- $m_\kappa(\tau) = 3.$

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_1), \quad \sigma_B = (\epsilon_1, \epsilon_3, \epsilon_2).$$

- $[\mathcal{M}_\tau] = q^2 - q.$
- $[\mathcal{G}_\tau] = (q-1)^3 q.$
- $[\mathfrak{M}_\tau^{\text{irr}}] = 1.$
- $[R(\tau)] = q^7 + q^6 - q^4 - q^3$
- $m_\kappa(\tau) = 6.$

$$\xi = ((1,1),(2,1)), \quad \sigma_A = (\epsilon_1, \epsilon_1, \epsilon_2), \quad \sigma_B = (\epsilon_1, \epsilon_2, \epsilon_3).$$

- $[\mathcal{M}_\tau] = q^2 - q.$
- $[\mathcal{G}_\tau] = (q-1)^2 q.$
- $[\mathfrak{M}_\tau^{\text{irr}}] = q - 2.$
- $[R(\tau)] = q^9 - 2\,q^8 - q^7 + q^6 + 2\,q^5 + q^4 - 2\,q^3$
- $m_\kappa(\tau) = 3.$

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_2, \epsilon_1, \epsilon_1), \quad \sigma_B = (\epsilon_3, \epsilon_1, \epsilon_2).$$

- $[\mathcal{M}_\tau] = (q-1)^2$.
- $[\mathcal{G}_\tau] = (q-1)^3$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1$.
- $[R(\tau)] = q^8 - q^6 - q^5 + q^3$
- $m_\kappa(\tau) = 3$.

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_1), \quad \sigma_B = (\epsilon_1, \epsilon_3, \epsilon_2).$$

- $[\mathcal{M}_\tau] = (q-1)^2 q$.
- $[\mathcal{G}_\tau] = (q-1)^3 q$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1$.
- $[R(\tau)] = q^8 - q^6 - q^5 + q^3$
- $m_\kappa(\tau) = 6$.

Total count of $\kappa = ((\epsilon_1, \epsilon_1, \epsilon_2), (\varepsilon_1, \varepsilon_2, \varepsilon_3))$

$[R_\kappa^{\mathrm{red}}] = 6\,q^9 + 3\,q^8 + 3\,q^7 + 3\,q^6 + 3\,q^5 + 3\,q^4 - 3\,q^3$,
$[R_\kappa^{\mathrm{irr}}] = q^{10} - 3\,q^9 + 2\,q^8 + 2\,q^7 - 2\,q^5 - 3\,q^4 + 3\,q^3$,
$[R_\kappa] = q^{10} + 3\,q^9 + 5\,q^8 + 5\,q^7 + 3\,q^6 + q^5$,
$[\mathfrak{M}_\kappa] = q^2 - 3\,q + 3$.

2. Configuration $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_1, \varepsilon_2)$

$$\xi = ((2,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_2, \epsilon_1, \epsilon_1).$$

- $[\mathcal{M}_\tau] = 1.$
- $[\mathcal{G}_\tau] = (q-1)^2.$
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = q - 2.$
- $[R(\tau)] = q^8 - q^7 - 2\,q^6 - q^5 + q^4 + 2\,q^3$
- $m_\kappa(\tau) = 3.$

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_1, \epsilon_1, \epsilon_2).$$

- $[\mathcal{M}_\tau] = 1.$
- $[\mathcal{G}_\tau] = (q-1)^3.$
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1.$
- $[R(\tau)] = q^6 + 2\,q^5 + 2\,q^4 + q^3$
- $m_\kappa(\tau) = 3.$

$$\xi = ((2,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_2, \epsilon_1, \epsilon_1).$$

- $[\mathcal{M}_\tau] = q - 1.$
- $[\mathcal{G}_\tau] = (q-1)^2.$
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = q - 2.$
- $[R(\tau)] = q^9 - 2\,q^8 - q^7 + q^6 + 2\,q^5 + q^4 - 2\,q^3$
- $m_\kappa(\tau) = 3.$

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_1, \epsilon_1, \epsilon_2).$$

- $[\mathcal{M}_\tau] = q^2 - 1.$
- $[\mathcal{G}_\tau] = (q-1)^3.$
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1.$
- $[R(\tau)] = q^8 + 2\,q^7 + q^6 - q^5 - 2\,q^4 - q^3$
- $m_\kappa(\tau) = 3.$

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_2, \epsilon_1, \epsilon_1).$$

- $[\mathcal{M}_\tau] = q - 1.$
- $[\mathcal{G}_\tau] = (q-1)^3.$
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1.$
- $[R(\tau)] = q^7 + q^6 - q^4 - q^3$
- $m_\kappa(\tau) = 6.$

$$\xi = ((1,1),(2,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_1, \epsilon_1, \epsilon_2).$$

- $[\mathcal{M}_\tau] = q - 1.$
- $[\mathcal{G}_\tau] = (q-1)^2.$
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = q - 2.$
- $[R(\tau)] = q^9 - 2\,q^8 - q^7 + q^6 + 2\,q^5 + q^4 - 2\,q^3$
- $m_\kappa(\tau) = 3.$

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_2, \epsilon_1, \epsilon_1).$$

- $[\mathcal{M}_\tau] = (q-1)^2$.
- $[\mathcal{G}_\tau] = (q-1)^3$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1$.
- $[R(\tau)] = q^8 - q^6 - q^5 + q^3$
- $m_\kappa(\tau) = 3$.

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_1, \epsilon_2, \epsilon_1).$$

- $[\mathcal{M}_\tau] = (q-1)^2$.
- $[\mathcal{G}_\tau] = (q-1)^3$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1$.
- $[R(\tau)] = q^8 - q^6 - q^5 + q^3$
- $m_\kappa(\tau) = 6$.

Total count of $\kappa = ((\epsilon_1, \epsilon_2, \epsilon_3), (\varepsilon_1, \varepsilon_1, \varepsilon_2))$

$[R_\kappa^{\mathrm{red}}] = 6\,q^9 + 3\,q^8 + 3\,q^7 + 3\,q^6 + 3\,q^5 + 3\,q^4 - 3\,q^3,$
$[R_\kappa^{\mathrm{irr}}] = q^{10} - 3\,q^9 + 2\,q^8 + 2\,q^7 - 2\,q^5 - 3\,q^4 + 3\,q^3,$
$[R_\kappa] = q^{10} + 3\,q^9 + 5\,q^8 + 5\,q^7 + 3\,q^6 + q^5,$
$[\mathfrak{M}_\kappa] = q^2 - 3\,q + 3.$

3. Configuration $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$

$$\xi = ((2,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_1, \epsilon_2, \epsilon_3).$$

- $[\mathcal{M}_\tau] = 1$.
- $[\mathcal{G}_\tau] = (q-1)^2$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = q-2$.
- $[R(\tau)] = q^8 - q^7 - 2\,q^6 - q^5 + q^4 + 2\,q^3$
- $m_\kappa(\tau) = 9$.

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_1, \epsilon_2, \epsilon_3).$$

- $[\mathcal{M}_\tau] = 1$.
- $[\mathcal{G}_\tau] = (q-1)^3$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1$.
- $[R(\tau)] = q^6 + 2\,q^5 + 2\,q^4 + q^3$
- $m_\kappa(\tau) = 6$.

$$\xi = ((2,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_1, \epsilon_2, \epsilon_3).$$

- $[\mathcal{M}_\tau] = q^2 - 1$.
- $[\mathcal{G}_\tau] = (q-1)^2$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = q-2$.
- $[R(\tau)] = q^{10} - q^9 - 3\,q^8 + 3\,q^6 + 3\,q^5 - q^4 - 2\,q^3$
- $m_\kappa(\tau) = 9$.

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_1, \epsilon_2, \epsilon_3).$$

- $[\mathcal{M}_\tau] = q^2 - 1$.
- $[\mathcal{G}_\tau] = (q-1)^3$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1$.
- $[R(\tau)] = q^8 + 2\,q^7 + q^6 - q^5 - 2\,q^4 - q^3$
- $m_\kappa(\tau) = 18$.

$$\xi = ((1,1),(2,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_1, \epsilon_2, \epsilon_3).$$

- $[\mathcal{M}_\tau] = q^2 - 1$.
- $[\mathcal{G}_\tau] = (q-1)^2$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = q-2$.
- $[R(\tau)] = q^{10} - q^9 - 3\,q^8 + 3\,q^6 + 3\,q^5 - q^4 - 2\,q^3$
- $m_\kappa(\tau) = 9$.

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2, \epsilon_3), \quad \sigma_B = (\epsilon_1, \epsilon_2, \epsilon_3).$$

- $[\mathcal{M}_\tau] = (q-1)^2$.
- $[\mathcal{G}_\tau] = (q-1)^3$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1$.
- $[R(\tau)] = q^8 - q^6 - q^5 + q^3$
- $m_\kappa(\tau) = 18$.

$$\xi = ((1,1),(1,1),(1,1)), \quad \sigma_A = (\epsilon_1,\epsilon_2,\epsilon_3), \quad \sigma_B = (\epsilon_1,\epsilon_2,\epsilon_3).$$

- $[\mathcal{M}_\tau] = (q-1)^2 q$.
- $[\mathcal{G}_\tau] = (q-1)^3$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1$.
- $[R(\tau)] = q^9 - q^7 - q^6 + q^4$
- $m_\kappa(\tau) = 36$.

Total count of $\kappa = ((\epsilon_1,\epsilon_2,\epsilon_3),(\varepsilon_1,\varepsilon_2,\varepsilon_3))$

$$[R_\kappa^{\mathrm{red}}] = 18\,q^{10} + 18\,q^9 - 9\,q^8 - 9\,q^7 + 6\,q^6 + 21\,q^5 + 3\,q^4 - 12\,q^3,$$
$$[R_\kappa^{\mathrm{irr}}] = q^{12} + 4\,q^{11} - 10\,q^{10} - 8\,q^9 + 17\,q^8 + 13\,q^7 - 5\,q^6 - 21\,q^5 - 3\,q^4 + 12\,q^3,$$
$$[R_\kappa] = q^{12} + 4\,q^{11} + 8\,q^{10} + 10\,q^9 + 8\,q^8 + 4\,q^7 + q^6,$$
$$[\mathfrak{M}_\kappa] = q^4 + 4\,q^3 - 9\,q^2 - 3\,q + 12.$$

Summary

$[R^{\mathrm{irr}}_{(\epsilon_1,\epsilon_1,\epsilon_2),(\varepsilon_1,\varepsilon_2,\varepsilon_3)}] = q^{10} - 3\,q^9 + 2\,q^8 + 2\,q^7 - 2\,q^5 - 3\,q^4 + 3\,q^3.$
$[R^{\mathrm{irr}}_{(\epsilon_1,\epsilon_2,\epsilon_3),(\varepsilon_1,\varepsilon_1,\varepsilon_2)}] = q^{10} - 3\,q^9 + 2\,q^8 + 2\,q^7 - 2\,q^5 - 3\,q^4 + 3\,q^3.$
$[R^{\mathrm{irr}}_{(\epsilon_1,\epsilon_2,\epsilon_3),(\varepsilon_1,\varepsilon_2,\varepsilon_3)}] = q^{12} + 4\,q^{11} - 10\,q^{10} - 8\,q^9 + 17\,q^8 + 13\,q^7 - 5\,q^6 - 21\,q^5 - 3\,q^4 + 12\,q^3.$

Final result representations.

$[R^{\mathrm{irr}}_3] = (q^{10} - 3\,q^9 + 2\,q^8 + 2\,q^7 - 2\,q^5 - 3\,q^4 + 3\,q^3)C_{(1,2),(1,1,1)} + (q^{10} - 3\,q^9 + 2\,q^8 + 2\,q^7 - 2\,q^5 - 3\,q^4 + 3\,q^3)C_{(1,1,1),(1,2)} + (q^{12} + 4\,q^{11} - 10\,q^{10} - 8\,q^9 + 17\,q^8 + 13\,q^7 - 5\,q^6 - 21\,q^5 - 3\,q^4 + 12\,q^3)C_{(1,1,1),(1,1,1)}.$

Final result characters.

$[\mathfrak{M}^{\mathrm{irr}}_3] = (q^2 - 3\,q + 3)C_{(1,2),(1,1,1)} + (q^2 - 3\,q + 3)C_{(1,1,1),(1,2)} + (q^4 + 4\,q^3 - 9\,q^2 - 3\,q + 12)C_{(1,1,1),(1,1,1)}.$

References

[1] Á. González-Prieto and V. Muñoz, *Motive of the* $\mathrm{SL}_4(\mathbb{C})$*-character variety of torus knots*, arXiv.