Introduction

This paper contains the computation of the motive of the irreducible $\mathrm{SL}_2(k)$-character variety of torus knots for any algebraically closed field k. The calculation is based on the methods introduced in the paper [1].

The notations used in this paper are the following:

- X_2^{irr} is the irreducible $\mathrm{SL}_2(k)$-representation variety of torus knots, that is, the variety of irreducible representations $\rho : \Gamma \to \mathrm{SL}_2(k)$ where $\Gamma = \Gamma_{n,m}$ is the fundamental group of the complement of the (n, m)-torus knot (see section 2 of [1]).
- $\mathfrak{M}_2^{\mathrm{irr}} = X_2^{\mathrm{irr}} /\!\!/ \mathrm{SL}_2(k)$ is the irreducible $\mathrm{SL}_2(k)$-character variety of torus knots, that is, the moduli space of representations (see section 2 of [1]).
- $\kappa = (\boldsymbol{\epsilon}, \boldsymbol{\varepsilon})$ is a configuration of eigenvalues, that is a collection of possible eigenvalues for the matrices A and B of a torus knot representation $\rho = (A, B)$ (see section 2 of [1]).
- τ is the type of a semi-simple filtration of a torus knot representation (see section 2.1 of [1]).
- ξ is the shape of the type τ, that is the collection of dimensions and multiplicities of each isotypic component (see section 2.1 of [1]).
- σ_A are the collections of eigenvalues of A for each isotypic component of a torus knot representation $\rho = (A, B)$ (see section 7.1 of [1]).
- σ_B are the collections of eigenvalues of B for each isotypic component of a torus knot representation $\rho = (A, B)$ (see section 7.1 of [1]).
- $\mathcal{M}_\tau$ is the space parametrizing possible completions of a semi-simple representation to a general representation of type τ (see section 4 of [1])
- $\mathcal{G}_\tau$ is the gauge group acting on $\mathcal{M}_\tau \times \mathrm{SL}_2(k)$ that identifies isomorphic completions (see section 4 of [1]).
- $\mathfrak{M}_\tau^{\mathrm{irr}}$ is the variety of possible semi-simplifications of a representation of type τ (see section 4 of [1]).
- $X(\tau)$ is the variety of representations of type τ.
- $m_\kappa(\tau)$ is the multiplicity of the type τ, that is the number of isomorphic components $X(\tau')$ of types τ' with the same shape as τ but whose eigenvalues are given by a permutation of the ones of τ that preserves their multiplicity (see section 5 of [1]).
- $C_{\pi,\pi'}$ are the number of isomorphic components given by configurations of eigenvalues with the same structure of repeated eigenvalues (see Section 6 of [1]). Here, π, π' are two partitions of 2 that determine the number of repeated eigenvalues of the matrices A and B of a representation $\rho = (A, B)$. If $\pi = \{1^{e_1}, \ldots, 2^{e_2}\}$ and $\pi' = \left\{1^{e'_1}, \ldots, 2^{e'_2}\right\}$ we have the following characterization in terms of multinomial numbers (Theorem 6.8 of [1])
$$C_{\pi,\pi'} = \frac{2}{nm} \binom{n}{e_1, \ldots, e_2} \binom{m}{e'_1, \ldots, e'_2}.$$

Combinatorial formulas for the motives $[\mathcal{M}_\tau], [\mathcal{G}_\tau]$ and $[\mathfrak{M}_\tau^{\mathrm{irr}}]$ are described in section 5 of [1] in terms of the structure of the type τ.

The structure of the paper is as follows. Each section describes the count of the motive $[\mathfrak{M}_\kappa]$ for a possible configuration of eigenvalues κ. For that purpose, we analyze all the types τ compatible with κ and compute the motives $[X(\tau)]$. A configuration of eigenvalues κ not appearing as a section of the paper means that $X_\kappa^{\mathrm{irr}} = \emptyset$ (see Remark 2.5 and Proposition 8.1 of [1]). In the final section of this paper, we summarize the results for each configuration κ and we provide the final result depending on the combinatorial coefficients $C_{\pi,\pi'}$.

Warning: The script generating this paper is only valid for rank ≤ 4. The result for higher rank may not be correct.

1. Configuration $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$

$$\xi = ((1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2), \quad \sigma_B = (\epsilon_1, \epsilon_2).$$

- $[\mathcal{M}_\tau] = 1$.
- $[\mathcal{G}_\tau] = (q-1)^2$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1$.
- $[X(\tau)] = q^2 + q$
- $m_\kappa(\tau) = 2$.

$$\xi = ((1,1),(1,1)), \quad \sigma_A = (\epsilon_1, \epsilon_2), \quad \sigma_B = (\epsilon_1, \epsilon_2).$$

- $[\mathcal{M}_\tau] = q - 1$.
- $[\mathcal{G}_\tau] = (q-1)^2$.
- $[\mathfrak{M}_\tau^{\mathrm{irr}}] = 1$.
- $[X(\tau)] = q^3 - q$
- $m_\kappa(\tau) = 4$.

Total count of $\kappa = ((\epsilon_1, \epsilon_2), (\varepsilon_1, \varepsilon_2))$

$[X_\kappa^{\mathrm{red}}] = 4\,q^3 + 2\,q^2 - 2\,q,$
$[X_\kappa^{\mathrm{irr}}] = q^4 - 2\,q^3 - q^2 + 2\,q,$
$[X_\kappa] = q^4 + 2\,q^3 + q^2,$
$[\mathfrak{M}_\kappa] = q - 2.$

Summary

$[X^{\mathrm{irr}}_{(\epsilon_1,\epsilon_2),(\varepsilon_1,\varepsilon_2)}] = q^4 - 2\,q^3 - q^2 + 2\,q.$

Final result representations.

$[X^{\mathrm{irr}}_2] = (q^4 - 2\,q^3 - q^2 + 2\,q)C_{(1,1),(1,1)}.$

Final result characters.

$[\mathfrak{M}^{\mathrm{irr}}_2] = C_{(1,1),(1,1)}(q-2).$

References

[1] Á. González-Prieto and V. Muñoz, *Motive of the* $\mathrm{SL}_4(\mathbb{C})$*-character variety of torus knots*, arXiv.