Module-2

Chapter - 2

FUNDAMENTAL PRINCIPLES OF COUNTING

BASIC COUNTING PRINCIPLES

There are two basic counting principles namely

1. **The product rule** and
2. **The sum rule.**

The Product Rule:

Consider an activity consisting of k different steps. Assume that the first step can occur in n_1 ways, the second step can occur in n_2 ways and so on up to the k^{th} step, which can be completed in n_k ways. Then, this activity can be completed in $n_1 \times n_2 \times n_3 \times \ldots \times n_k$ different number of ways. This is the **Fundamental principle of counting.**

Example: Suppose a person has 8 shirts and 5 ties and he wants choose a shirt and a tie.

By product rule, he can choose a shirt and a tie in $8 \times 5 = 40$ different ways.

The Sum Rule:

Consider an event A_1 can occur in n_1 ways and another event A_2 can occur in n_2 ways and A_1 and A_2 are mutually exclusive, then A_1 or A_2 can occur in $n_1 + n_2$ ways. It can be extended for any number of events.

Example: Suppose there are 16 boys and 18 girls in a class and if we select one of these students as the class representative.

Then, the no. of ways of selecting a boy = 16 and the no. of ways of selecting a girl = 18.

By sum rule, the no. of ways of selecting a student

$= 16 + 18 = 34.$

PROBLEMS

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:

By the product rule, there are $12 \times 11 = 132$ ways

There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

Solution:

The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the micro- computer and 24 ways to choose the port no matter which microcomputer has been selected.

By the product rule there are $32 \times 24 = 768$ ports.

There are 18 mathematics majors and 325 computer science majors at a college.

(a) In how many ways can two representatives be picked so that one is mathematics major and the other is a computer science major?

(b) In how many ways can one representative be picked who is either mathematics major or a computer science major?

Solution:

(a) By the product rule, there are $18 \times 325 = 5850$ ways

(b) By the sum rule, there are $18 + 325 = 343$ ways

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution:

There are 26 choices for each of the three uppercase English letters and ten choices for each of the three digits.

Hence, by the product rule there are a total of

$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17{,}576{,}000$ possible license plates.

How many license plates can be made using

(i) Either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

(ii) Either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

(iii) Either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?

Solution:

(i) $(10 \times 10 \times 10 \times 26 \times 26 \times 26) + (26 \times 26 \times 26 \times 10 \times 10 \times 10)$

$= 17576000 + 17576000 = 35152000$ ways.

(ii) $(26 \times 26 \times 10 \times 10 \times 10 \times 10) + (10 \times 10 \times 26 \times 26 \times 26 \times 26)$

$= 6760000 + 45697600 = 52457600$ ways.

(iii) $(26 \times 26 \times 26 \times 10 \times 10 \times 10) + (26 \times 26 \times 26 \times 26 \times 10 \times 10)$

$= 17576000 + 45697600 = 63273600$ ways.

Let a restaurant sells 6 South Indian dishes, 4 North Indian dishes, 3 hot beverages and 2 cold drinks. For breakfast, a student wishes to buy one south Indian dish and one hot beverage or one North Indian dish and one cold drink. In how many ways can he buy a breakfast?

Solution:

First choice = $6 \times 3 = 18$ ways

Second choice = $4 \times 2 = 8$ ways

$\therefore$ A student can buy a breakfast in $18 + 8 = 26$ ways.

PERMUTATIONS

- A Permutation is an **arrangement**.
- An arrangement of objects along a straight line is called a **Linear Permutation** and an arrangement of objects along a circle is called a **Circular permutation.**

The number of linear permutations of n objects taken r at a time when repetition is not allowed is given by ${}^nP_r$ or $P(n, r)$ and is defined by ${}^nP_r = \frac{n!}{(n-r)!}$

The number of linear permutations of n objects taken all at a time when repetition is not allowed is given by $n!$ And the number of circular permutations of n objects is given by $(n-1)!$

The number of linear permutations of n objects taken r at a time when repetition is allowed is given by n^r

The number of linear permutations of n objects in which p objects are alike of one kind, q objects are alike of second kind, r objects are alike of third kind and so on and the rest are distinct is given by $\dfrac{n!}{p!q!r!\ldots}$

PROBLEMS

Find the number of ways in which 6 friends may be seated in a row for a photograph.

Solution:

Number of arrangements = P(6, 6) = 6! = 720

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is

$$P(100, 3) = 100 \times 99 \times 98 = 970{,}200.$$

In how many ways can 6 men and 6 women be seated in a row (i) if any person may sit next to any other? (ii) If men and women must occupy alternative seats?

Solution:

(i) If there is no restriction, 12 persons in a row can sit in 12! Ways.

(ii) 6 men in odd places and 6 women in even places can be seated in $6! \times 6!$ ways. 6 men in even places and 6 women in odd places can be seated in $6! \times 6!$ ways.

Therefore, total number of arrangements $= 2 \times 6! \times 6!$

In how many ways can three men and three women be seated at a round table if (i) No restriction is imposed? (ii) Two particular women must not sit together? (iii) Each woman is to be between two men?

Solution:

(i) If no restriction imposed, 6 persons can be seated in a round table in $(6-1)! = 5! = 120$ ways.

(ii) Two women can sit together in 2 ways. Consider this as 1 unit. One unit and 4 remaining persons can sit in $(5-1)! = 4! = 24$ ways.

Therefore, If two women can sit together, total no. of arrangements is $2 \times 24 = 48$

If two women can't sit together, total no. of arrangements is

$120 - 48 = 72$

(iii) Three men can be seated in $(3-1)! = 2!$ ways by leaving one seat between them. Three women can be seated in the remaining 3 seats in $3!$ ways.

Therefore, total number of arrangements is $2! \times 3! = 12$.

Find the number of permutations of the letters of the word MASSASAUGA. In how many of these, all four A's together? How many of them begin with S?

Solution:

(i) In 10 alphabets, ‘S’ repeated three times and ‘A’ repeated 4 times.

Therefore, total number of permutations $= \frac{10!}{3! \times 4!} = 25{,}200$

(ii) Consider four A’s together as one unit. Consider the remaining 6 letters as 6 units. Now, we have 7 units. Out of 7 units, ‘S’ repeated three times.

Therefore, total number of permutations $= \frac{7!}{3!} = 840$

(iii) First alphabet is fixed as S. Now, 9 alphabets remaining. In 9 alphabets, ‘S’ repeated twice and ‘A’ repeated 4 times.

Therefore, total number of permutations $= \frac{9!}{2! \times 4!} = 7560$

How many arrangements are there for all letters in the word SOCIOLOGICAL? In how many of these arrangements (i) A and G are adjacent? (ii) All the vowels are adjacent?

Solution:

In 12 letters, 'O' repeated thrice and 'C', 'I', 'L' repeated twice each.

Therefore, total number of arrangements $= \frac{12!}{3!\times 2!\times 2!\times 2!} =$ 99,79,200

(i) A and G together can be arranged in 2 ways. Consider this as one unit. Consider the remaining 10 letters as 10 units. Now we have 11 units. In 11 units, 'O' repeated thrice and 'C', 'I', 'L' repeated twice each.

Therefore, total number of arrangements$= 2 \times \frac{11!}{3!\times 2!\times 2!\times 2!} = 16{,}63{,}200$

(ii) One A, Two I's and three O's together can be arranged in $\frac{6!}{2!\times 3!} = 60$ ways. Consider this as a single unit. Consider the remaining 6 letters as 6 units. In 7 units, 'C' and 'L' repeated twice each.

Therefore, total number of arrangements $= 60 \times \frac{7!}{2!\times 2!} = 75{,}600$

Find the number of permutations of the letters of the word MISSISSIPPI. (i) How many of these begin with I? (ii) How many of these begin and end with an S?

Solution:

In 11 letters, ‘S’ and ‘I’ repeated 4 times each and ‘P’ repeated twice.

Therefore, total number of permutations $= \frac{11!}{4! \times 4! \times 2!} = 34{,}650$

(i) First letter is fixed as I. Now, 10 letters remaining. In 10 letters, ‘S’ repeated four times, ‘I’ repeated thrice and ‘P’ repeated twice.

Therefore, total number of permutations $= \frac{10!}{4! \times 3! \times 2!} = 12{,}600$

(ii) Starting and ending letters are fixed as ‘S’. Now, 9 letters remaining. In 9 letters, ‘S’ and ‘P’ repeated twice each and ‘I’ repeated four times.

Therefore, total number of permutations $= \frac{9!}{2! \times 2! \times 4!} = 3780$

How many numbers greater than a million can be formed using the digits 0, 3, 4, 4, 5, 5, 5?

Solution:

Given digits are 0, 3, 4, 4, 5, 5, 5

We have to form the numbers greater than a million.

(i.e., seven digit numbers).

This can be done in $\frac{7!}{2!3!} - \frac{6!}{2!3!} = 360.$

(We have excluded the numbers begin with zero)

$\therefore$ 360 numbers are greater than a million can be formed by using the given digits.

How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

Solution:

We have 9 digits, out of which there are two 4's and two 5's.

Let $n = x_1x_2x_3x_4x_5x_6x_7$. x_1 must be 5 or 6 or 7.

Suppose $x_1 = 5$, remaining 6 digits can be arranged in $\frac{6!}{2!} = 360$ ways.

Suppose $x_1 = 6$, remaining 6 digits can be arranged in $\frac{6!}{2!\times 2!} = 180$ ways.

Suppose $x_1 = 7$, remaining 6 digits can be arranged in $\frac{6!}{2!\times 2!} = 180$ ways.

Therefore total number of arrangements =

$$360 + 180 + 180 = 720.$$

COMBINATIONS

A Combination is a **selection**. The number of combinations of n objects taken r at a time is given by ${}^{n}C_{r}$ or $C(n, r)$ or $\begin{pmatrix} n \\ r \end{pmatrix}$ and is defined by ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$

The number of combinations of n objects taken r at a time in which p particular objects always occur is given by ${}^{n-p}C_{r-p}$ or $C(n-p, r-p)$ or $\begin{pmatrix} n-p \\ r-p \end{pmatrix}$.

The number of combinations of n objects taken r at a time in which p particular objects do not occur is given by ${}^{n-p}C_{r}$ or $C(n-p, r)$ or $\begin{pmatrix} n-p \\ r \end{pmatrix}$.

The number of combinations of r objects from n distinct objects is given by ${}^{n+r-1}C_{r}$ or $C(n+r-1, r)$ or $\begin{pmatrix} n+r-1 \\ r \end{pmatrix}$.

Note: $\binom{n}{r} = \binom{n}{n-r}$, $\binom{n}{0} = \binom{n}{n} = 1$, $\binom{n}{1} = \binom{n}{n-1} = n.$

How many different committees of three students can be formed from a group of four students?

Solution:

From a group of 4 students 3 students can be selected in $C\,(4, 3) = 4$ ways.

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

Solution:

Five players can be selected from 10 member in $C\,(10, 5) = 252$ ways.

In how many different ways can a committee of 5 teachers and 4 students be selected from 9 teachers and 15 students?

Solution:

Number of ways of selecting 5 teachers from 9 teachers = $\binom{9}{5}$

Number of ways of selecting 4 students from 15 students = $\binom{15}{4}$.

By product rule, total number of different ways = $\binom{9}{5} \times \binom{15}{4} = 1{,}71{,}990.$

A bag contains 5 red marbles and 6 white marbles. Find the number of ways that 4 marbles can be drawn from the bag if the 4 marbles of the same colour?

Solution:

Number of ways of selecting 4 red marbles from 5 red marbles = $\binom{5}{4}$

Number of ways of selecting 4 white marbles from 6 white marbles = $\binom{6}{4}$

By sum rule, total number of different ways = $\binom{5}{4} + \binom{6}{4} = 75.$

How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Solution:

Five cards can be dealt from a standard deck of 52 cards in $C(52, 5) = 26 \times 17 \times 10 \times 49 \times 12 = 2{,}598{,}960$ ways.

Also, 47 cards can be selected from a standard deck of 52 cards in $C(52, 47) = 2{,}598{,}960$ ways.

How many arrangements of the letters can be made in the word MISSISSIPPI? How many have no consecutive S's?

Solution:

The number of letters in the given word is 11 of which 4 each are S's, I's, 2 P's and 1 M.

Therefore, the number of arrangements of the letters in the given word is $\dfrac{11!}{4!4!2!} = 34,650.$

If we ignore four S's, then the remaining 7 letters can be arranged in $\dfrac{7!}{4!2!} = 105$ ways.

In each of these arrangements, there are 8 possible locations for four S's. These locations can be chosen in

$$\binom{8}{4} = 70 \text{ ways}$$

Therefore, total number of arrangements having no adjacent S's = $105 \times 70 = 7350$.

Find the number of arrangements of all the letters in TALLAHASSEE. How many of these arrangements have no adjacent A's?

Solution:

The number of letters in the given word is 11 of which 3 are A's, 2 each are L's, S's, E's and 1 each are T and H.

Therefore, the number of arrangements of the letters in the given word is

$$\frac{11!}{3!2!2!2!} = 8,31,600.$$

If we ignore three A's, the remaining 8 letters can be arranged in $\frac{8!}{2!2!2!} = 5040$ ways.

In each of these arrangements, there are 9 possible locations for the three A's. These locations can be chosen in

$$\binom{9}{3} = 84 \text{ ways.}$$

$\therefore$ The number of arrangements having no adjacent A's is $5040 \times 84 = 4,23,360$.

A woman has 11 close relatives and she wishes to invite 5 of them to dinner. In how many ways can she invite them in the following situations:

(i) There is no restriction on the choice.

(ii) Two particular persons will not attend separately.

(iii) Two particular persons will not attend together.

Solution:

(i) Since there is no restriction on the choice of invitees, 5 out of 11 can be invited in

$$C(11, 5) = \frac{11!}{6!5!} = 462 \text{ ways.}$$

(ii) Since two particular persons will not attend separately, they should both be invited or not invited.

If both of them are invited, then three more invitees are to be selected from the remaining 9 relatives. This can be done in $C(9, 3) = \frac{9!}{6!3!} = 84$ ways.

If both of them are not invited, then 5 invitees are to be selected from 9 relatives. This can be done in

$$C(9, 5) = \frac{9!}{5!4!} = 126 \text{ ways.}$$

$\therefore$ The total number of ways in which the invitees can be selected in this case is

$$84 + 126 = 210 \text{ ways}$$

(iii) Since two particular persons (say P_1 and P_2) will not attend together, only one of them can be invited or none of them can be invited.

The number of ways of choosing the invitees with P_1 is invited and P_2 is not invited, is $C(9,4) = \dfrac{9!}{5!4!} = 126$ ways.

Similarly, the number of ways of choosing the invitees with P_1 is not invited and P_2 is invited, is 126.

If both P_1 and P_2 are not invited, then the number of ways of choosing the invitees is $C(9,5) = \dfrac{9!}{5!4!} = 126$ ways.

Thus, the total number of ways in which the invitees can be selected in this case is 126 + 126 + 126 = 378.

A certain question paper contains two parts A and B each containing 4 questions. How many different ways a student can answer 5 questions by selecting at least 2 questions from each part?

Solution:

There are 4 questions in Part A and 4 questions in Part B. The different ways a student can select his 5 questions are as follows:

Part A	Part B	Number of selections
2	3	$C\ (4, 2) \times C\ (4, 3) = 24$ ways
3	2	$C\ (4, 3) \times C\ (4, 2) = 24$ ways

Therefore, the total number of ways a student can answer 5 questions under the given restrictions is 24 + 24 = 48.

A certain question paper contains three parts A, B, C with four questions in part A, five questions in part B and six questions in part C. It is required to answer 7 questions selecting at least two questions from each part. In how many ways can a student select his seven questions for answering?

Solution:

There are 4 questions in part A, 5 questions in part B & 6 questions in part C.

The different ways a student can select his 7 questions are as follows:

Part A	Part B	Part C	Number of selections
2	2	3	$\binom{4}{2} \times \binom{5}{2} \times \binom{6}{3} = 1200$ ways
2	3	2	$\binom{4}{2} \times \binom{5}{3} \times \binom{6}{2} = 900$ ways
3	2	2	$\binom{4}{3} \times \binom{5}{2} \times \binom{6}{2} = 600$ ways

Therefore, the total number of ways a student can answer 7 questions under the given restrictions is

$1200 + 900 + 600 = 2700.$