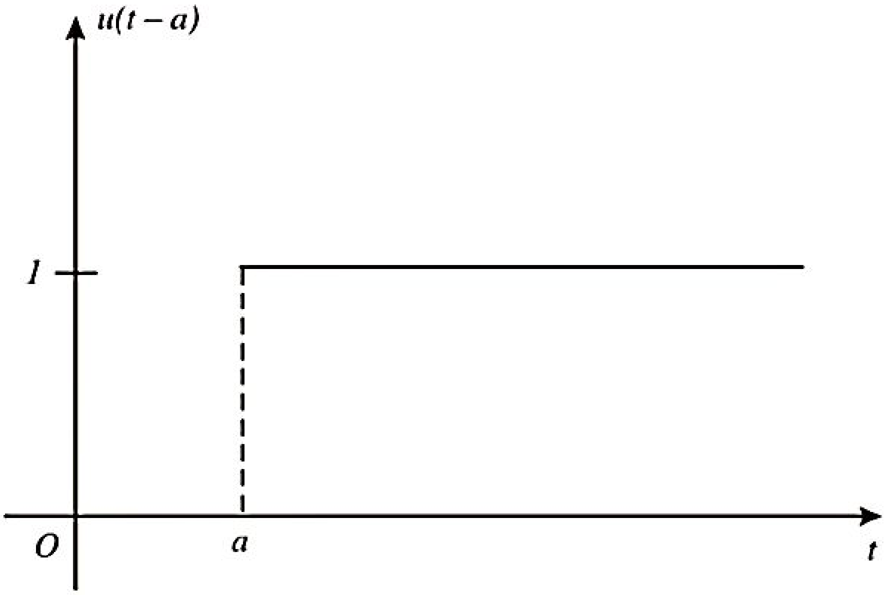
1.8 UNIT STEP FUNCTION

Definition: The unit step function or Heaviside's unit function is denoted by $u(t-a)$ or $H(t-a)$

and is defined as $u(t-a)$ or $H(t-a)=\begin{cases}0, & \text{for } t<a\\ 1, & \text{for } t\geq a\end{cases}$

Graph:



1.8.1 Laplace Transform of Unit Step Function

By definition, we have $L[f(t)]=\int_0^\infty e^{-st}f(t)\,dt$

$$\therefore \quad L[u(t-a)] = \int_0^\infty e^{-st}u(t-a)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt+\int_a^\infty e^{-st}(1)\,dt$$

$$= 0+\int_a^\infty e^{-st}\,dt=\left[\frac{e^{-st}}{-s}\right]_a^\infty=\left[\frac{e^{-\infty}-e^{-as}}{-s}\right]=\frac{e^{-as}}{s}$$

$$\therefore \quad L[u(t-a)] = \frac{e^{-as}}{s}$$

In particular, $L[u(t)]=\frac{1}{s}$

Second Shifting Property

If $L[f(t)] = \overline{f}(s)$ then $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

Proof: By definition, we have $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$\begin{aligned}
L[f(t-a)u(t-a)] &= \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt \\
&= \int_0^a e^{-st} f(t-a)(0)\,dt + \int_a^\infty e^{-st} f(t-a)(1)\,dt \\
&= 0 + \int_a^\infty e^{-st} f(t-a)\,dt
\end{aligned}$$

Put $t-a=v$

Diff w.r.t 't'

$1-0=\dfrac{dv}{dt} \Rightarrow dt = dv$

When $t=a,\ v=0$

$t=\infty,\ v=\infty$

$$\therefore \quad L[f(t-a)u(t-a)] = \int_0^\infty e^{-s(a+v)} f(v)\,dv = e^{-as}\int_0^\infty e^{-sv} f(v)\,dv = e^{-as}\,\overline{f}(s)$$

$$\therefore \quad L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$$

In particular, $\quad L[f(t)u(t)] = \overline{f}(s) = L[f(t)]$

Corollary:

If $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$ then $L^{-1}\left[e^{-as}\,\overline{f}(s)\right] = f(t-a)u(t-a)$

Remarks:

1. If $f(t)=\begin{cases} f_1(t) & \text{if } t\le a \\ f_2(t) & \text{if } t>a \end{cases}$ then $f(t)$ **in terms of unit step function is**

   $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

2. If $f(t)=\begin{cases} f_1(t) & \text{if } t\le a \\ f_2(t) & \text{if } a<t\le b \\ f_3(t) & \text{if } t>b \end{cases}$ **then** $f(t)$ **in terms of unit step function is**

   $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

Find the Laplace transforms of the following functions

(*i*) $(t-1)^2 u(t-1)$ (*ii*) $\sin t\, u(t-\pi)$ (*iii*) $e^{-3t} u(t-2)$

Solution:

(*i*) Let $f(t-a)u(t-a) = (t-1)^2 u(t-1)$

$$\text{Here, } a = 1 \text{ and } f(t-a) = (t-1)^2$$

$$\Rightarrow \quad f(t-1) = (t-1)^2$$

$$\Rightarrow \quad f(t) = (t+1-1)^2 = t^2$$

$$L[f(t)] = L[t^2]$$

$$\Rightarrow \quad \bar{f}(s) = \frac{2}{s^3}$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$$\therefore \quad L\{(t-1)^2 u(t-1)\} = e^{-s}\left(\frac{2}{s^3}\right) = \frac{2e^{-s}}{s^3}$$

(*ii*) Let $f(t-a)u(t-a) = \sin t\; u(t-\pi)$

$$\text{Here, } a = \pi \text{ and } f(t-a) = \sin t$$

$$\Rightarrow \quad f(t-\pi) = \sin t$$

$$\Rightarrow \quad f(t) = \sin(t+\pi) = -\sin t$$

$$L[f(t)] = L[-\sin t]$$

$$\Rightarrow \quad \bar{f}(s) = -\frac{1}{s^2+1}$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$$\therefore \quad L\{\sin t\; u(t-\pi)\} = e^{-\pi s}\left(-\frac{1}{s^2+1}\right) = -\frac{e^{-\pi s}}{s^2+1}$$

(*iii*) Let $f(t-a)u(t-a) = e^{-3t}u(t-2)$

$$\text{Here, } a = 2 \text{ and } f(t-a) = e^{-3t}$$

$$\Rightarrow \quad f(t-2) = e^{-3t}$$

$$\Rightarrow \quad f(t) = e^{-3(t+2)} = e^{-6}e^{-3t}$$

$$L[f(t)] = L[e^{-6}e^{-3t}] = e^{-6}L[e^{-3t}]$$

$$\Rightarrow \quad \bar{f}(s) = e^{-6}\left(\frac{1}{s+3}\right)$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$$\therefore \quad L\{e^{-3t}u(t-2)\} = e^{-2s}\left(e^{-6}\left(\frac{1}{s+3}\right)\right) = \frac{e^{-2s-6}}{s+3} = \frac{e^{-2(s+3)}}{s+3}$$

Express the function $f(t)=\begin{cases} t^2 & \text{if } 1<t<2 \\ 4t & \text{if } t>2 \end{cases}$ in terms of unit step function and hence find its Laplace transform. **(VTU 2004, 2005)**

Solution:

Let $\quad f_1(t) = t^2,\ f_2(t)=4t$

We have, $\quad f(t) = f_1(t)+[f_2(t)-f_1(t)]u(t-a)$

$\Rightarrow \quad f(t) = t^2+[4t-t^2]u(t-2)$

Its Laplace transform is

$$L[f(t)] = L(t^2)+L[(4t-t^2)u(t-2)] \quad \text{---- (1)}$$

$$L(t^2) = \frac{2}{s^3} \quad \text{---- (2)}$$

Consider, $L[(4t-t^2)u(t-2)]$

It is in the form $L[f(t-a)u(t-a)]$

Here, $a = 2$ and $f(t-a)=(4t-t^2)$

$\Rightarrow \quad f(t-2) = 4t-t^2$

$\Rightarrow \quad f(t) = 4(t+2)-(t+2)^2 = 4t+8-t^2-4-4t = 4-t^2$

$$L[f(t)] = L[4-t^2]$$

$\Rightarrow \quad \bar{f}(s) = \frac{4}{s}-\frac{2}{s^3}$

We have, $\quad L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$\therefore \quad L[(4t-t^2)u(t-2)] = e^{-2s}\left(\frac{4}{s}-\frac{2}{s^3}\right) \quad \text{---- (3)}$

Substituting (2) and (3) in (1), we get

$$L[f(t)] = \frac{2}{s^3}+e^{-2s}\left(\frac{4}{s}-\frac{2}{s^3}\right)$$

Express the function $f(t)=\begin{cases}\pi-t & \text{if } 0<t\le\pi \\ \sin t & \text{if } t>\pi\end{cases}$ **in terms of unit step function and hence find its Laplace transform.** **(VTU 2006)**

Solution:

Let $\qquad f_1(t) = \pi-t,\ f_2(t)=\sin t$

We have, $\qquad f(t) = f_1(t)+\left[f_2(t)-f_1(t)\right]u(t-a)$

$\Rightarrow \qquad f(t) = (\pi-t)+\left[\sin t-(\pi-t)\right]u(t-\pi)$

Its Laplace transform is

$$L\left[f(t)\right] = L(\pi-t)+L\left[(\sin t-\pi+t)u(t-\pi)\right] \qquad \text{---- (1)}$$

$$L(\pi-t) = \frac{\pi}{s}-\frac{1}{s^2} \qquad \text{---- (2)}$$

Consider, $L\left[(\sin t-\pi+t)u(t-\pi)\right]$

It is in the form $L\left[f(t-a)u(t-a)\right]$

Here, $\qquad a = \pi$ and $f(t-a)=(\sin t-\pi+t)$

$\Rightarrow \qquad f(t-\pi) = \sin t-\pi+t$

$\Rightarrow \qquad f(t) = \sin(t+\pi)-\pi+(t+\pi)=-\sin t+t$

$$L\left[f(t)\right] = L[-\sin t+t]$$

$\Rightarrow \qquad \overline{f}(s) = \left(-\frac{1}{s^2+1}+\frac{1}{s^2}\right)$

We have, $\qquad L\left[f(t-a)u(t-a)\right] = e^{-as}\,\overline{f}(s)$

$\therefore \qquad L\left[(4t-t^2)u(t-\pi)\right] = e^{-\pi s}\left(-\frac{1}{s^2+1}+\frac{1}{s^2}\right) \qquad \text{---- (3)}$

Substituting (2) and (3) in (1), we get

$$L\left[f(t)\right] = \left(\frac{\pi}{s}-\frac{1}{s^2}\right)+e^{-\pi s}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)$$

Express the function $f(t)=\begin{cases}\sin t & \text{if } 0<t\le \pi/2\\ \cos t & \text{if } t>\pi/2\end{cases}$ in terms of unit step function and hence find its Laplace transform.

Solution:

Let $f_1(t) = \sin t,\ f_2(t)=\cos t$

We have, $f(t) = f_1(t)+\left[f_2(t)-f_1(t)\right]u(t-a)$

$\Rightarrow$ $f(t) = \sin t+\left[\cos t-\sin t\right]u(t-\pi/2)$

Its Laplace transform is

$$L\left[f(t)\right] = L(\sin t)+L\left[(\cos t-\sin t)u(t-\pi/2)\right] \quad \text{---- (1)}$$

$$L(\sin t) = \frac{1}{s^2+1} \quad \text{---- (2)}$$

Consider, $L\left[(\cos t-\sin t)u(t-\pi/2)\right]$

It is in the form $L\left[f(t-a)u(t-a)\right]$

Here, $a = \pi/2$ and $f(t-a)=(\cos t-\sin t)$

$\Rightarrow$ $f(t-\pi/2) = \cos t-\sin t$

$\Rightarrow$ $f(t) = \cos t(t+\pi/2)-\sin(t+\pi/2)=-\sin t-\cos t$

$$L\left[f(t)\right] = -L\left[\sin t+\cos t\right]$$

$\Rightarrow$ $$\bar{f}(s) = -\left(\frac{1}{s^2+1}+\frac{s}{s^2+1}\right)=-\frac{s+1}{s^2+1}$$

We have, $L\left[f(t-a)u(t-a)\right] = e^{-as}\,\bar{f}(s)$

$\therefore$ $$L\left[(\cos t-\sin t)u(t-\pi/2)\right] = -e^{-(\pi/2)s}\left(\frac{s+1}{s^2+1}\right) \quad \text{---- (3)}$$

Substituting (2) and (3) in (1), we get

$$L\left[f(t)\right] = \frac{1}{s^2+1}-e^{-(\pi/2)s}\left(\frac{s+1}{s^2+1}\right)$$

Express the function $f(t)=\begin{cases}\cos t & \text{if} \quad 0<t<\pi \\ \cos 2t & \text{if} \quad \pi<t<2\pi \\ \cos 3t & \text{if} \quad t>2\pi\end{cases}$ **in terms of unit step function and hence find its Laplace transform.** **(VTU 2003)**

Solution:

Let $\quad f_1(t) = \cos t, \quad f_2(t)=\cos 2t, \quad f_3(t)=\cos 3t$

We have, $\quad f(t) = f_1(t)+[f_2(t)-f_1(t)]u(t-a)+[f_3(t)-f_2(t)]u(t-b)$

$\Rightarrow \quad f(t) = \cos t+[\cos 2t-\cos t]u(t-\pi)+[\cos 3t-\cos 2t]u(t-2\pi)$

Its Laplace transform is

$$L[f(t)]=L[\cos t]+L\{[\cos 2t-\cos t]u(t-\pi)\}+L\{[\cos 3t-\cos 2t]u(t-2\pi)\} \qquad \text{---- (1)}$$

$$L[\cos t] = \frac{s}{s^2+1} \qquad \text{---- (2)}$$

Consider, $L\{[\cos 2t-\cos t]u(t-\pi)\}$

It is in the form $L[f(t-a)u(t-a)]$

Here, $\quad a = \pi$ and $f(t-a)=[\cos 2t-\cos t]$

$$\Rightarrow \quad f(t-\pi) = [\cos 2t - \cos t]$$

$$\Rightarrow \quad f(t) = \cos 2(t+\pi) - \cos(t+\pi) = \cos(2t+2\pi) - \cos(t+\pi)$$

$$\Rightarrow \quad f(t) = \cos 2t + \cos t$$

$$L[f(t)] = L[\cos 2t + \cos t]$$

$$\Rightarrow \quad \overline{f}(s) = \frac{s}{s^2+4} + \frac{s}{s^2+1}$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$$\therefore \quad L\{[\cos 2t - \cos t]u(t-\pi)\} = e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) \qquad \text{---- (3)}$$

Consider, $L\{[\cos 3t - \cos 2t]u(t-2\pi)\}$

It is in the form $L[f(t-a)u(t-a)]$

Here, $a = 2\pi$ and $f(t-a) = [\cos 3t - \cos 2t]$

$$\Rightarrow \quad f(t-2\pi) = [\cos 3t - \cos 2t]$$

$$\Rightarrow \quad f(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi) = \cos(3t+3\times 2\pi) - \cos(2t+2\times 2\pi)$$

$$\Rightarrow \quad f(t) = \cos 3t - \cos 2t$$

$$L[f(t)] = L[\cos 3t - \cos 2t]$$

$$\Rightarrow \quad \overline{f}(s) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$$\therefore L\{[\cos 3t - \cos 2t]u(t-2\pi)\} = e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right) \qquad \text{---- (4)}$$

Substituting (2), (3) and (4) in (1), we get

$$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Express the function $f(t)=\begin{cases}\cos t & \text{if} \quad 0<t<\pi \\ 1 & \text{if} \quad \pi<t<2\pi \\ \sin t & \text{if} \quad t>2\pi\end{cases}$ **in terms of unit step function and hence find its Laplace transform.** **(VTU 2007)**

Solution:

Let $\quad f_1(t) = \cos t, \quad f_2(t)=1, \quad f_3(t)=\sin t$

We have, $\quad f(t) = f_1(t)+[f_2(t)-f_1(t)]u(t-a)+[f_3(t)-f_2(t)]u(t-b)$

$\Rightarrow \quad f(t) = \cos t+[1-\cos t]u(t-\pi)+[\sin t-1]u(t-2\pi)$

Its Laplace transform is

$$L[f(t)] = L[\cos t]+L\{[1-\cos t]u(t-\pi)\}+L\{[\sin t-1]u(t-2\pi)\} \quad \text{---- (1)}$$

$$L[\cos t] = \frac{s}{s^2+1} \quad \text{---- (2)}$$

Consider, $L\{[1-\cos t]u(t-\pi)\}$

It is in the form $L[f(t-a)u(t-a)]$

Here, $a = \pi$ and $f(t-a)=[1-\cos t]$

$$\Rightarrow \quad f(t-\pi) = 1-\cos t$$

$$\Rightarrow \quad f(t) = 1-\cos(t+\pi)$$

$$\Rightarrow \quad f(t) = 1+\cos t$$

$$L[f(t)] = L[1+\cos t]$$

$$\Rightarrow \quad \overline{f}(s) = \frac{1}{s}+\frac{s}{s^2+1}$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$$\therefore \quad L\{[1-\cos t]u(t-\pi)\} = e^{-\pi s}\left(\frac{1}{s}+\frac{s}{s^2+1}\right) \qquad \text{---- (3)}$$

Consider, $L\{[\sin t-1]u(t-2\pi)\}$

It is in the form $L[f(t-a)u(t-a)]$

Here, $a = 2\pi$ and $f(t-a)=[\sin t-1]$

$$\Rightarrow \quad f(t-2\pi) = \sin t-1$$

$$\Rightarrow \quad f(t) = \sin(t+2\pi)-1$$

$$\Rightarrow \quad f(t) = \sin t-1$$

$$L[f(t)] = L[\sin t-1]$$

$$\Rightarrow \quad \overline{f}(s) = \frac{1}{s^2+1}-\frac{1}{s}$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$$\therefore \quad L\{[\sin t-1]u(t-2\pi)\} = e^{-2\pi s}\left(\frac{1}{s^2+1}-\frac{1}{s}\right) \qquad \text{---- (4)}$$

Substituting (2), (3) and (4) in (1), we get

$$L[f(t)] = \frac{s}{s^2+1}+e^{-\pi s}\left(\frac{1}{s}+\frac{s}{s^2+1}\right)+e^{-2\pi s}\left(\frac{1}{s^2+1}-\frac{1}{s}\right)$$

Express the function $f(t)=\begin{cases} 1 & if \quad 0<t<3 \\ t & if \quad 3<t<6 \\ t^2 & if \quad t>6 \end{cases}$ **in terms of unit step function and hence find its Laplace transform.** **(VTU 2003)**

Solution:

Let $f_1(t) = 1,\ f_2(t)=t,\ f_3(t)=t^2$

We have, $f(t) = f_1(t)+[f_2(t)-f_1(t)]u(t-a)+[f_3(t)-f_2(t)]u(t-b)$

$\Rightarrow \quad f(t) = 1+[t-1]u(t-3)+[t^2-t]u(t-6)$

Its Laplace transform is

$$L[f(t)] = L[1]+L[(t-1)u(t-3)]+L[(t^2-t)u(t-6)] \quad \text{---- (1)}$$

$$L[1] = \frac{1}{s} \quad \text{---- (2)}$$

Consider, $L[(t-1)u(t-3)]$

It is in the form $L[f(t-a)u(t-a)]$

Here, $a = 3$ and $f(t-a)=(t-1)$

$\Rightarrow \quad f(t-3) = t-1$

$\Rightarrow \quad f(t) = (t+3)-1=t+2$

$$L[f(t)] = L[t+2]$$

$\Rightarrow \quad \overline{f}(s) = \frac{1}{s^2}+\frac{2}{s}$

We have, $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$$\therefore \quad L[(t-1)u(t-3)] = e^{-3s}\left(\frac{1}{s^2}+\frac{2}{s}\right) \quad \text{---- (3)}$$

Consider, $L[(t^2-t)u(t-6)]$

It is in the form $L[f(t-a)u(t-a)]$

Here, $a = 6$ and $f(t-a)=(t^2-t)$

$$\Rightarrow \qquad f(t-6) = t^2 - t$$

$$\Rightarrow \qquad f(t) = (t+6)^2 - (t+6) = t^2 + 36 + 12t - t - 6$$

$$\Rightarrow \qquad f(t) = t^2 + 11t + 30$$

$$L[f(t)] = L[t^2 + 11t + 30]$$

$$\Rightarrow \qquad \overline{f}(s) = \frac{2}{s^3} + \frac{11}{s^2} + \frac{30}{s}$$

We have, $\quad L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$$\therefore \qquad L[(t^2-t)u(t-6)] = e^{-6s}\left(\frac{2}{s^3} + \frac{11}{s^2} + \frac{30}{s}\right) \qquad \text{---- (4)}$$

Substituting (2), (3) and (4) in (1), we get

$$L[f(t)] = \frac{1}{s} + e^{-3s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-6s}\left(\frac{2}{s^3} + \frac{11}{s^2} + \frac{30}{s}\right)$$

Express the function $f(t)=\begin{cases} t^2 & if \quad 0<t<2 \\ 4t & if \quad 2<t<4 \\ 8 & if \quad t>4 \end{cases}$ **in terms of unit step function and hence find its Laplace transform.** **(VTU 2011, 2014)**

Solution:

Let $\qquad f_1(t) = t^2, \; f_2(t)=4t, \; f_3(t)=8$

We have, $\qquad f(t) = f_1(t)+[f_2(t)-f_1(t)]u(t-a)+[f_3(t)-f_2(t)]u(t-b)$

$\Rightarrow \qquad f(t) = t^2+[4t-t^2]u(t-2)+[8-4t]u(t-4)$

Its Laplace transform is

$$L[f(t)] = L(t^2)+L[(4t-t^2)u(t-2)]+L[(8-4t)u(t-4)] \qquad \text{---- (1)}$$

$$L(t^2) = \frac{2}{s^3} \qquad \text{---- (2)}$$

Consider, $L[(4t-t^2)u(t-2)]$

It is in the form $L[f(t-a)u(t-a)]$

Here, $\qquad a = 2$ and $f(t-a)=(4t-t^2)$

$\Rightarrow \qquad f(t-2) = 4t-t^2$

$\Rightarrow \qquad f(t) = 4(t+2)-(t+2)^2 = 4t+8-t^2-4-4t = 4-t^2$

$$L[f(t)] = L[4-t^2]$$

$\Rightarrow \qquad \overline{f}(s) = \frac{4}{s}-\frac{2}{s^3}$

We have, $\qquad L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$\therefore \qquad L[(4t-t^2)u(t-2)] = e^{-2s}\left(\frac{4}{s}-\frac{2}{s^3}\right) \qquad$ ---- (3)

Consider, $L[(8-4t)u(t-4)]$

It is in the form $L[f(t-a)u(t-a)]$

Here, $a = 4$ and $f(t-a) = (8-4t)$

$$\Rightarrow \quad f(t-4) = 8-4t$$

$$\Rightarrow \quad f(t) = 8-4(t+4) = 8-4t-16$$

$$\Rightarrow \quad f(t) = -4t-8$$

$$L[f(t)] = -4L[t] - L(8)$$

$$\Rightarrow \quad \overline{f}(s) = -4\left(\frac{1}{s^2}\right) - \frac{8}{s} = \frac{-4}{s^2} - \frac{8}{s}$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$$\therefore \quad L[(8-4t)u(t-4)] = e^{-4s}\left(\frac{-4}{s^2} - \frac{8}{s}\right) \qquad \text{---- (4)}$$

Substituting (2), (3) and (4) in (1), we get

$$L[f(t)] = \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right) + e^{-4s}\left(\frac{-4}{s^2} - \frac{8}{s}\right)$$