Module-2

Chapter - 1

PROPERTIES OF INTEGERS

THE WELL ORDERING PRINCIPLE

The **well-ordering principle** says that the positive integers are **well-ordered**. An **ordered** set is said to be **well-ordered** if each and every nonempty subset has a smallest or least element. So the **well-ordering principle** is the following statement:

"Every nonempty subset S of the positive integers has a least element."

MATHEMATICAL INDUCTION

Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number. This technique involves two steps to prove a statement, as stated below:

- **Step 1: (Basis step)**: It proves that a statement is true for the initial value.
- **Step 2: (Inductive step)**: It proves that if the statement is true for the n^{th} iteration (or number n), then it is also true for $(n + 1)^{th}$ iteration (or number $n + 1$).

Principle of Mathematical induction

Let $P(n)$ be the given statement or formula involving only natural numbers. Then $P(n)$ is true for all positive integers n if

- $P(1)$ is true
- $P(k)$ is true $\Rightarrow P(k + 1)$ is true

Prove by Mathematical Induction $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}\ \forall\, n \in \mathbb{N}.$

Solution:

$$\text{Let } P(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}\ \forall\, n \in \mathbb{N}.$$

Step 1: For $n = 1$,

$$\text{LHS} = 1^3 = 1 \text{ and } \quad \text{RHS} = \frac{1^2(1+1)^2}{4} = \frac{1(4)}{4} = 1$$

So, LHS = RHS

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., $P(k)$ is true

$$\text{Hence, } P(k) = 1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}\ \forall\, k \in \mathbb{N}.$$

To show that $P(k + 1)$ is true

$$\text{i.e., } P(k+1) = 1^3 + 2^3 + 3^3 + \ldots + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$\Rightarrow \quad P(k+1) = 1^3 + 2^3 + 3^3 + \ldots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

Consider, LHS = $1^3 + 2^3 + 3^3 + \dots + (k+1)^3$

$= 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 \qquad \text{(by using P}(k)\text{)}$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2\left[k^2 + 4(k+1)\right]}{4}$$

$$= \frac{(k+1)^2\left[k^2 + 4k + 4\right]}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$= \text{RHS}$

$\therefore$ P(k + 1) is true

Hence by the principle of induction P(n) is true for all $n \in$ N.

Prove by Mathematical Induction $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}\ \forall\ n \in N.$

Solution:

$$\text{Let } P(n) = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}\ \forall\ n \in N.$$

Step 1: For $n = 1$,

$$LHS = 1^2 = 1 \text{ and } RHS = \frac{1(1+1)(2+1)}{6} = \frac{1(2)(3)}{6} = 1$$

So, LHS = RHS

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., $P(k)$ is true

$$\text{Hence, } P(k) = 1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}\ \forall\ k \in N.$$

To show that $P(k + 1)$ is true

$$\text{i.e., } P(k+1) = 1^2 + 2^2 + 3^2 + ... + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\Rightarrow \quad P(k+1) = 1^2 + 2^2 + 3^2 + ... + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Consider, LHS = $1^2 + 2^2 + 3^2 + \ldots + (k + 1)^2$

$$= 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k + 1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2 \qquad \text{(by using P}(k))$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6} = \frac{(k+1)\left[2k^2+7k+6\right]}{6}$$

$$= \frac{(k+1)\left[2k^2+4k+3k+6\right]}{6} = \frac{(k+1)\left[2k(k+2)+3(k+2)\right]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} = \text{RHS}$$

$\therefore$ P(k + 1) is true

Hence by the principle of induction P(n) is true for all $n \in$ N.

Prove by Mathematical Induction $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2} \ \forall\ n \in \mathbf{N}.$

Prove by Mathematical Induction $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ **for** $n = 1, 2, 3\ldots$

Prove by mathematical induction for all positive integers n, that

$$1.3 + 2.4 + 3.5 + \ldots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

Solution:

Let $P(n) = 1.3 + 2.4 + 3.5 + \ldots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6} \quad \forall\, n \in N.$

Step 1: For $n = 1$,

$$\text{LHS} = 1.3 = 3 \quad \text{and RHS} = \frac{1(1+1)(2+7)}{6} = \frac{(2)(9)}{6} = 3$$

So, LHS = RHS

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., $P(k)$ is true

Hence, $P(k) = 1.3 + 2.4 + 3.5 + \ldots + k(k+2) = \dfrac{k(k+1)(2k+7)}{6} \quad \forall\, k \in N.$

To show that $P(k+1)$ is true

i.e., $P(k+1) = 1.3 + 2.4 + 3.5 + \ldots + (k+1)((k+1)+2) = \dfrac{(k+1)((k+1)+1)(2(k+1)+7)}{6}$

$\Rightarrow \quad P(k+1) = 1.3 + 2.4 + 3.5 + \ldots + (k+1)(k+3) = \dfrac{(k+1)(k+2)(2k+9)}{6} \ \forall\, k \in \mathbb{N}.$

Consider, LHS $= 1.3 + 2.4 + 3.5 + \ldots + (k+1)(k+3)$

$= 1.3 + 2.4 + 3.5 + \ldots + k(k+2) + (k+1)(k+3)$

$= \dfrac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$ (by using $P(k)$)

$= \dfrac{k(k+1)(2k+7)+6(k+1)(k+3)}{6}$

$= \dfrac{(k+1)\left[k(2k+7)+6(k+3)\right]}{6}$

$= \dfrac{(k+1)\left[2k^2+13k+18\right]}{6}$

$$= \frac{(k+1)\left[2k^2+4k+9k+18\right]}{6}$$

$$= \frac{(k+1)\left[2k(k+2)+9(k+2)\right]}{6}$$

$$= \frac{(k+1)(k+2)(2k+9)}{6} \qquad = \text{RHS}$$

$\therefore$ P(k + 1) is true

Hence by the principle of induction P(n) is true for all $n \in$ N.

Prove by mathematical induction for all positive integers n, that

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}.$$

Solution:

$$\text{Let } P(n) = \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}\ \forall\, n \in \mathrm{N}.$$

Step 1: For n =1,

$$\text{LHS} = \frac{1}{1.4}=\frac{1}{4} \text{ and } \text{RHS} = \frac{1}{3+1}=\frac{1}{4}$$

So, LHS = RHS

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., P(k) is true

$$\text{Hence, } P(k) = \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}\ \forall\, k \in \mathrm{N}.$$

To show that P(k + 1) is true

i.e., $P(k+1) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3(k+1)-2)(3(k+1)+1)} = \frac{(k+1)}{3(k+1)+1}$

$$\Rightarrow \quad P(k+1) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3k+1)(3k+4)} = \frac{k+1}{3k+4} \ \forall\ k \in \mathbf{N}.$$

Consider, LHS $= \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3k+1)(3k+4)}$

$$= \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} \qquad \text{(by using P}(k))$$

$$= \frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{3k^2+3k+k+1}{(3k+1)(3k+4)} = \frac{3k(k+1)+(k+1)}{(3k+1)(3k+4)}$$

$$= \frac{(k+1)(3k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4} = \text{RHS}$$

$\therefore$ P(k + 1) is true

Hence by the principle of induction P(n) is true for all $n \in$ N.

Establish each of the following by Mathematical Induction

(a) $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

(b) $\displaystyle\sum_{i=1}^{n} 2^{i-1} = 2^{n} - 1$

(c) $\displaystyle\sum_{i=1}^{n} i\left(2^{i}\right) = 2 + (n-1)2^{n+1}$

(d) $\displaystyle\sum_{i=1}^{n} i(i!) = (n+1)! - 1$

Solution:

(a) Let $P(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

Step 1: For $n = 1$,

$$\text{LHS} = \sum_{i=1}^{1} \frac{1}{1(1+1)} = \frac{1}{2} \text{ and } \quad \text{RHS} = \frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$$

So, LHS = RHS

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., $P(k)$ is true

$$\text{Hence, } P(k) = \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1} \ \forall\, k \in \mathbb{N}.$$

To show that $P(k + 1)$ is true

$$\text{i.e., } P(k+1) = \sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{(k+1)+1}$$

$$\Rightarrow \qquad P(k+1) = \sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2} \ \forall\, k \in \mathbb{N}.$$

$$\text{Consider, LHS} = \sum_{i=1}^{k+1} \frac{1}{i(i+1)}$$

$$= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)((k+1)+1)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad \text{(by using } P(k)\text{)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \text{RHS}$$

$\therefore$ P(k + 1) is true

Hence by the principle of induction P(n) is true for all $n \in$ N.

(b) Let $P(n) = \sum_{i=1}^{n} 2^{i-1} = 2^n - 1$ for all $n \in$ N.

Step 1: For $n = 1$,

$$\text{LHS} = \sum_{i=1}^{1} 2^{1-1} = 2^0 = 1 \quad \text{and RHS} = 2^1 - 1 = 2 - 1 = 1$$

So, LHS = RHS

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., P(k) is true

Hence, $P(k) = \sum_{i=1}^{k} 2^{i-1} = 2^k - 1$

To show that P(k + 1) is true

i.e., $P(k+1) = \sum_{i=1}^{k+1} 2^{i-1} = 2^{k+1} - 1$

Consider, LHS $= \sum_{i=1}^{k+1} 2^{i-1}$

$= \sum_{i=1}^{k} 2^{i-1} + 2^{(k+1)-1}$

$= 2^k - 1 + 2^k$ (by using P(k))

$= 2 \times 2^k - 1$ $= 2^{k+1} - 1$ = RHS

$\therefore$ P(k + 1) is true

Hence by the principle of induction P(n) is true for all $n \in$ N.

(c) Let P(n) = $\sum_{i=1}^{n} i\left(2^{i}\right) = 2+\left(n-1\right)2^{n+1}$ for all $n \in$ N.

Step 1: For n =1,

$$\text{LHS} = \sum_{i=1}^{1} i\left(2^{i}\right) = 1\left(2^{1}\right) = 2$$

$$\text{RHS} = 2+\left(1-1\right)2^{1+1} = 2+0 = 2$$

So, LHS = RHS

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., P(k) is true

Hence, $P(k) = \sum_{i=1}^{k} i\left(2^{i}\right) = 2+\left(k-1\right)2^{k+1}$

To show that P(k + 1) is true

i.e., $P(k+1) = \sum_{i=1}^{k+1} i\left(2^i\right) = 2 + \left(\left(k+1\right)-1\right)2^{(k+1)+1}$

$\Rightarrow \quad P(k+1) = \sum_{i=1}^{k+1} i\left(2^i\right) = 2 + \left(k\right)2^{k+2}$

Consider, LHS $= \sum_{i=1}^{k+1} i\left(2^i\right)$

$= \sum_{i=1}^{k} i\left(2^i\right) + (k+1)\left(2^{k+1}\right)$

$= 2 + \left(k-1\right)2^{k+1} + \left(k+1\right)\left(2^{k+1}\right)$ (by using $P(k)$)

$= 2 + 2^{k+1}\left[\left(k-1\right) + \left(k+1\right)\right]$

$= 2 + 2^{k+1}\left(2k\right)$ $= 2 + k\left(2^{k+2}\right)$ $=$ RHS

$\therefore$ $P(k+1)$ is true

Hence by the principle of induction $P(n)$ is true for all $n \in$ N.

(d) Let $P(n) = \sum_{i=1}^{n} i(i!) = (n+1)! - 1$ for all $n \in \mathbb{N}$.

Step 1: For $n = 1$,

$$\text{LHS} = \sum_{i=1}^{1} i(i!) = 1(1)! = 1$$

$$\text{RHS} = (1+1)! - 1 = 2 - 1 = 1$$

So, LHS = RHS

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., $P(k)$ is true

$$\text{Hence, } P(k) = \sum_{i=1}^{k} i(i!) = (k+1)! - 1$$

To show that $P(k + 1)$ is true

$$\text{i.e., } P(k+1) = \sum_{i=1}^{k+1} i(i!) = ((k+1)+1)! - 1$$

$$\Rightarrow \quad P(k+1) = \sum_{i=1}^{k+1} i(i!) = (k+2)! - 1$$

$$\text{Consider, LHS} = \sum_{i=1}^{k+1} i(i!)$$

$$= \sum_{i=1}^{k} i(i!) + (k+1)(k+1)!$$

$$= (k+1)! - 1 + (k+1)(k+1)! \qquad \text{(by using } P(k)\text{)}$$

$$= (k+1)!\left[1+(k+1)\right] - 1$$

$$= (k+1)!\,[k+2] - 1 = (k+2)! - 1 = \text{RHS}$$

$\therefore$ P(k + 1) is true

Hence by the principle of induction P(n) is true for all $n \in$ N.

Prove by Mathematical Induction $n! \geq 2^{n-1}$ for all integers $n \geq 1$.

Solution:

Let $P(n) = n! \geq 2^{n-1}$ for all integers $n \geq 1$.

Step 1: For $n = 1$,

$P(1) = 1! \geq 2^{1-1} = 1 \geq 1$ which is true.

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., $P(k)$ is true

Hence, $P(k) = k! \geq 2^{k-1}$ or $P(k) = 2^{k-1} \leq k! \quad \forall\, k \geq 1$

To show that $P(k+1)$ is true

i.e., $P(k+1) = (k+1)! \geq 2^{(k+1)-1}$

$$\Rightarrow \qquad P(k + 1) = (k + 1)! \geq 2^k$$

Now, $2^k = 2 \times 2^{k-1}$

$\leq 2 \times k!$ (by using $P(k)$)

$\Rightarrow \qquad 2^k \leq (k + 1) \times k!$ because $2 \leq k + 1$ for $k \geq 1$

$\Rightarrow \qquad 2^k \leq (k + 1)!$

$\Rightarrow \qquad (k + 1)! \geq 2^k$

$\therefore$ $P(k + 1)$ is true

Hence by the principle of induction $P(n)$ is true for all $n \geq 1$.

Prove that for all $n \in Z^+$, n > 3 $\Rightarrow 2^n < n!$

Solution:

Let $P(n) = 2^n < n!$ for all integers $n \geq 4$.

Step 1: For $n = 4$,

$P(4) = \mathbf{2^4 < 4!} = 16 < 24$ which is true.

$\therefore$ P(4) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., $P(k)$ is true

Hence, $P(k) = P(k) = 2^k < k! \;\; \forall\, k \geq 4$

To show that $P(k + 1)$ is true

i.e., $P(k + 1) = 2^{(k + 1)} < (k + 1)!$

Now, $2^{(k+1)} = 2 \times 2^k$

$< 2 \times k!$ (by using $P(k)$)

$\Rightarrow \quad 2^{(k+1)} < (k+1) \times k!$ because $2 < k + 1$ for $k \geq 4$

$\Rightarrow \quad 2^{(k+1)} < (k+1)!$

$\therefore$ P(k + 1) is true

Hence by the principle of induction P(n) is true for all $n \geq 4$.

Prove by Mathematical Induction $4n < (n^2 - 7)$ for all integers $n \geq 6$.

Solution:

Let $P(n) = 4n < (n^2 - 7)$ for all integers $n \geq 6$.

Step 1: For $n = 6$,

$P(6) = 4(6) < (6^2 - 7) = 24 < 29$ which is true.

$\therefore$ P(6) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., $P(k)$ is true

Hence, $P(k) = 4k < (k^2 - 7) \ \forall \ k \geq 6$

To show that $P(k + 1)$ is true

i.e., $P(k + 1) = 4(k + 1) < ((k + 1)^2 - 7)$

Now, $4(k+1) = 4k+4$

$< (k^2 - 7) + 4$ (by using $P(k)$)

$< (k^2 - 7) + (2k+1)$

because when $k \geq 6$, we have $2k + 1 \geq 13 > 4$

$= (k+1)^2 - 7$

$\Rightarrow \quad 4(k+1) < ((k+1)^2 - 7)$ for $k \geq 6$

$\therefore$ $P(k+1)$ is true

Hence by the principle of induction $P(n)$ is true for all $n \geq 6$.

Prove by Mathematical Induction $3^n - 1$ is a multiple of 2 for $n = 1, 2, 3...$

Solution:

Let $P(n) = 3^n - 1$ is a multiple of 2 for $n = 1, 2, 3...$

Step 1: For $n = 1$,

$P(1) = 3^1 - 1 = 3 - 1 = 2$ which is a multiple of 2.

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., $P(k)$ is true

Hence, $P(k) = 3^k - 1$ is a multiple of 2 for $k = 1, 2, 3...$

To show that $P(k + 1)$ is true

i.e., $P(k + 1) = 3^{k+1} - 1$ is a multiple of 2

Now, $3^{k+1} - 1 = 3 \times 3^k - 1 = (2 \times 3^k) + (3^k - 1)$

The first part $(2 \times 3k)$ is certain to be a multiple of 2 and the second part $(3^k - 1)$ is also true from our assumption $P(k)$.

Hence, $3^{k+1} - 1$ is a multiple of 2.

$\therefore$ $P(k + 1)$ is true

Hence by the principle of induction $P(n)$ is true for all $n \in N$.

Prove by mathematical induction that, for any positive integer n, the number $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Solution:

Let $P(n) = 11^{n+2} + 12^{2n+1}$ is divisible by 133 for $n = 1, 2, 3...$

Step 1: For $n = 1$,

$$P(1) = = 11^{1+2} + 12^{2+1} = 1331 + 1728 = 3059$$

which is divisible by 133.

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., $P(k)$ is true

Hence, $P(k) = 11^{k+2} + 12^{2k+1}$ is divisible by 133 for $k = 1, 2, 3...$

To show that $P(k + 1)$ is true

i.e., $P(k + 1) = 11^{(k+1)+2} + 12^{2(k+1)+1}$ is divisible by 133

$\Rightarrow \qquad P(k+1) = 11^{k+3} + 12^{2k+3}$ is divisible by 133

Now, $P(k+1) - P(k) = 11^{k+3} + 12^{2k+3} - 11^{k+2} - 12^{2k+1}$

$$= 11^{k+2}(11-1) + 12^{2k+1}(12^2 - 1)$$

$$= 10(11^{k+2} + 12^{2k+1}) + 12^{2k+1}(133)$$

$$= 10\,P(k) + 12^{2k+1}(133)$$

$\Rightarrow \qquad P(k+1) = 11\,P(k) + 12^{2k+1}(133)$

which is divisible by 133.

Hence, $11^{k+3} + 12^{2k+3}$ is divisible by 133

$\therefore$ $P(k+1)$ is true

Hence by the principle of induction $P(n)$ is true for all $n \in N$.

Prove by mathematical induction that, for every positive integer n,

$A_n = 5^n + 2.3^{n-1} + 1$ is a multiple of 8.

Solution:

Let P(n) – $A_n = 5^n + 2.3^{n-1} + 1$ is a multiple of 8 for n = 1, 2, 3...

Step 1: For $n = 1$,

$$P(1) = A_1 = 5^1 + 2.3^{1-1} + 1 = 5 + 2 + 1 = 8$$

which is a multiple of 8.

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., P(k) is true

Hence, P(k) = $A_k = 5^k + 2.3^{k-1} + 1$ is a multiple of 8 for k = 1, 2, 3...

To show that P($k + 1$) is true

i.e., $P(k+1) = A_{k+1} = 5^{k+1} + 2.3^{k} + 1$ is a multiple of 8

Now, $A_{k+1} - A_k = 5^{(k+1)} + 2.3^{(k+1)-1} + 1 - 5^{k} - 2.3^{k-1} - 1$

$= 5^{k}(5-1) + 2.3^{k-1}(3-1)$

$= 4(5^{k} + 3^{k-1})$

$= 4(\text{Even})$

$A_{k+1} = A_k + 4(\text{Even})$,

which is a multiple of 8.

Hence, $A_{k+1} = 5^{k+1} + 2.3^{k} + 1$ is a multiple of 8

$\therefore$ $P(k+1)$ is true

Hence by the principle of induction $P(n)$ is true for all $n \in N$.

Prove that every positive integer $n \geq 24$ can be written as a sum of 5's and/or 7's.

Solution:

Let P(n) = n can be written as a sum of 5's and/or 7's for $n \geq 24$

Step 1: For $n = 24$,

P(24) = 24 = 5 + 5 + 7 + 7 which is true

$\therefore$ P(24) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., P(k) is true

Hence, P(k) = k can be written as a sum of 5's and/or 7's for $k \geq 24$

To show that P(k + 1) is true

i.e., P(k + 1) = (k + 1) can be written as a sum of 5's and/or 7's for $k \geq 24$

Now, $k + 1 = (7 + 7 + \cdots \text{r times}) + (5 + 5 + \cdots \text{s times}) + 1$

$= (7 + 7 + \cdots (\text{r} - 2)\text{times}) + (5 + 5 + \cdots \text{s times}) + 7 + 7 + 1$

$= (7 + 7 + \cdots (\text{r} - 2)\text{times}) + (5 + 5 + \cdots \text{s times}) + 15$

$= (7 + 7 + \cdots (\text{r} - 2)\text{times}) + (5 + 5 + \cdots (\text{s} + 3) \text{ times})$

$=$ sum of 5's and 7's.

Hence, $(k + 1)$ can be written as a sum of 5's and/or 7's for $k \geq 24$

$\therefore$ P$(k + 1)$ is true

Hence by the principle of induction P(n) is true for $n \geq 24$.

Let $H_1 = 1,\ H_2 = 1+\frac{1}{2},\ H_3 = 1+\frac{1}{2}+\frac{1}{3},\ \ldots,\ H_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$

Prove that $\sum_{i=1}^{n} H_i = (n+1)H_n - n,$ **for all positive integers** $n \geq 1$**.**

Solution:

By given data, $H_k = H_{k+1} - \frac{1}{k+1}$

Let $P(n) = \sum_{i=1}^{n} H_i = (n+1)H_n - n$ for all $n \in \mathbb{N}$.

Step 1: For $n = 1$,

$\text{LHS} = \sum_{i=1}^{1} H_i = H_1 = 1$ and

$\text{RHS} = (1+1)H_1 - 1 = (2)(1) - 1 = 1$

So, LHS = RHS

$\therefore$ P(1) is true

Step 2: Let us assume that the result is true for $n = k$, i.e., P(k) is true

$$\text{Hence, } P(k) = \sum_{i=1}^{k} H_i = (k+1)H_k - k \;\; \forall\, k \in \mathbb{N}.$$

To show that P(k + 1) is true

$$\text{i.e., } P(k+1) = \sum_{i=1}^{k+1} H_i = ((k+1)+1)H_{(k+1)} - (k+1)$$

$$\Rightarrow \quad P(k+1) = \sum_{i=1}^{k+1} H_i = (k+2)H_{(k+1)} - (k+1)$$

$$\text{Consider, LHS} = \sum_{i=1}^{k+1} H_i$$

$$= \sum_{i=1}^{k} H_i + H_{k+1}$$

$$= (k+1) H_k - k + H_{k+1} \qquad \text{(by using P}(k))$$

$$= (k+1)\left(H_{k+1} - \frac{1}{k+1} \right) - k + H_{k+1} \qquad \text{(by given data)}$$

$$= (k+1) H_{k+1} - 1 - k + H_{k+1}$$

$$= (k+2) H_{k+1} - (k+1)$$

$$= \text{RHS}$$

$\therefore$ P(k + 1) is true

Hence by the principle of induction P(n) is true for all $n \in$ N.