Module-3

RELATIONS AND FUNCTIONS

PART - 4

EQUIVALENCE CLASSES AND PARTITION SET

Equivalence Classes

Let R be an equivalence relation on a set A and $a \in A$. Then the set of all those elements x of A which are related to a by R is called the equivalence class of a w.r.t 'R'. The equivalence class of a is denoted by $R(a)$ or $[a]$.

i.e., $R(a)$ or $[a] = \{x \in A \mid (x, a) \in R\}$

EXAMPLES

Let $R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$ be an equivalence relation on $A = \{1, 2, 3\}$ then

$R(1)$ or $[1] = \{(1, 1), (3, 1)\} = \{1, 3\}$

$R(2)$ or $[2] = \{(2, 2)\} = \{2\}$

$R(3)$ or $[3] = \{(1, 3), (3, 3)\} = \{1, 3\}$

Let $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$ be an equivalence relation on $A = \{1, 2, 3, 4\}$ then

$R(1)$ or $[1] = \{(1, 1), (2, 1)\} = \{1, 2\}$

$R(2)$ or $[2] = \{(1, 2), (2, 2)\} = \{1, 2\}$

$R(3)$ or $[3] = \{(3, 3), (4, 3)\} = \{3, 4\}$

$R(4)$ or $[4] = \{(3, 4), (4, 4)\} = \{3, 4\}$

Let $R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (c, a), (c, c), (d, d), (d, e), (e, d), (e, e)\}$ be an equivalence relation on $A = \{a, b, c, d, e\}$ then

$R(a)$ or $[a] = \{(a, a), (b, a), (c, a)\} = \{a, b, c\}$

$R(b)$ or $[b] = \{(a, b), (b, b)\} = \{a, b\}$

$R(c)$ or $[c] = \{(a, c), (c, c)\} = \{a, c\}$

$R(d)$ or $[d] = \{(d, d), (e, d)\} = \{d, e\}$

$R(e)$ or $[e] = \{(d, e), (e, e)\} = \{d, e\}$

PARTITION OF A SET

Let A be a non-empty set. Suppose there exist non-empty subsets $A_1, A_2, \ldots, A_k$ of A such that

i. $A = A_1 \cup A_2 \cup \ldots \cup A_k$ and

ii. $A_i \cap A_j = \phi$ for $i \neq j$

Then the set $P = \{A_1, A_2, \ldots, A_k\}$ is called a partition of A.

EXAMPLES

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A_1 = \{1, 3, 5, 7\}$, $A_2 = \{2, 4\}$, $A_3 = \{6, 8\}$ be the subsets of A. Determine whether the set $P = \{A_1, A_2, A_3\}$ is a partition of A or not?

i. $A = A_1 \cup A_2 \cup A_3$

ii. $A_1 \cap A_2 = \phi, A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$

$\therefore\ P = \{A_1, A_2, A_3\}$ is a partition of A.

Let $A = \{a, b, c, \ldots, h\}$ and $A_1 = \{d, e\}$, $A_2 = \{a, c, d\}$, $A_3 = \{f, h\}$. Determine whether the set $P = \{A_1, A_2, A_3\}$ is a partition of A or not?

i. $A \neq A_1 \cup A_2 \cup A_3$

$\therefore\ P = \{A_1, A_2, A_3\}$ is not a partition of A.

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A_1 = \{2, 3, 5\}$, $A_2 = \{4, 6\}$, $A_3 = \{6, 8\}$ and $A_4 = \{1, 7\}$ be the subsets of A. Determine whether the set $P = \{A_1, A_2, A_3, A_4\}$ is a partition of A or not?

Solution:

i. $A = A_1 \cup A_2 \cup A_3 \cup A_4$

ii. $A_1 \cap A_2 = \phi, A_1 \cap A_3 = \phi, A_1 \cap A_4 = \phi$ and $A_2 \cap A_3 \neq \phi$

$\therefore$ $P = \{A_1, A_2, A_3\}$ is not a partition of A.

Note:

- Any equivalence relation R on set A induces a partition of A.
- Any partitions of a set A gives rise to an equivalence relation R on A.

EXAMPLES

Let R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)} be an equivalence relation on A = {1, 2, 3, 4} then determine the partition induced by R.

First we find all the equivalence classes

$R(1)$ or $[1] = \{(1,1),(2,1)\} = \{1,2\}$

$R(2)$ or $[2] = \{(1,2),(2,2)\} = \{1,2\}$

$R(3)$ or $[3] = \{(3,3),(4,3)\} = \{3,4\}$

$R(4)$ or $[4] = \{(3,4),(4,4)\} = \{3,4\}$

$\therefore$ Partition set P induced by $R = \{[1],[3]\} = \{\{1,2\},\{3,4\}\}$.

Let $R = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$ be an equivalence relation on $A = \{1, 2, 3, 4, 5\}$ then determine the partition induced by R.

First we find all the equivalence classes

$R(1)$ or $[1] = \{(1, 1)\} = \{1\}$

$R(2)$ or $[2] = \{(2, 2), (3, 2)\} = \{2, 3\}$

$R(3)$ or $[3] = \{(2, 3), (3, 3)\} = \{2, 3\}$

$R(4)$ or $[4] = \{(4, 4), (5, 4)\} = \{4, 5\}$

$R(5)$ or $[5] = \{(4, 5), (5, 5)\} = \{4, 5\}$

$\therefore$ Partition set P induced by $R = \{[1], [2], [4]\}$

$= \{\{1\}, \{2, 3\}, \{4, 5\}\}.$

If $A = \{1, 2, 3, 4, 5\}$ and partition set $P = \{\{1, 2\}, \{3, 4\}, \{5\}\}$. Find an equivalence relation R on A.

From the partition set P, we find that

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$

If $A = \{1, 2, 3, 4, 5, 6, 7\}$ and partition set $P = \{\{1, 2\}, \{3\}, \{4, 5, 7\}, \{6\}\}$. Find an equivalence relation R on A.

From the partition set P, we find that

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4), (4, 5), (4, 7), (5, 4), (5, 5), (5, 7), (7, 4), (7, 5), (7, 7), (6, 6)\}$

Theorem:

Let R be an equivalence relation on set A and a, b $\in$ A then the following are equivalent:

(i) $a \in [a]$

(ii) a R b iff $[a] = [b]$

(iii) If $[a] \cap [b] \neq \phi$ then $[a] = [b]$

Proof:

Given that R is an equivalence relation.

i.e., R is reflexive, symmetric and transitive.

(i) $\forall\ a \in A \Rightarrow (a, a) \in R$ [by reflexive]

$\Rightarrow a \in [a]$

(ii) Let a R b

i.e., $(a, b) \in R$

Case 1:

Let $x \in [a]$

$\Rightarrow (x, a) \in R$

Now, $(x, a), (a, b) \in R \Rightarrow (x, b) \in R$ [by transitive]

$\Rightarrow x \in [b]$

$\therefore [a] \subseteq [b]$

Case 2:

Let $x \in [b]$

$\Rightarrow (x, b) \in R$

Now, $(x, b), (a, b) \in R$

$\Rightarrow (x, b), (b, a) \in R$ [by symmetric]

$\Rightarrow (x, a) \in R$ [by transitive]

$\Rightarrow x \in [a]$

$\therefore [b] \subseteq [a]$

By combining both the cases, we have

$[a] = [b]$

Conversely, let $[a] = [b]$

From (i), we have

$a \in [a]$

$\Rightarrow \quad a \in [b] \quad$ (by using (ii))

$\Rightarrow \quad (a, b) \in R$

$\Rightarrow \quad a\ R\ b$

(iii) Given that, $[a] \cap [b] \neq \phi$

Let $x \in [a] \cap [b]$

$\Rightarrow x \in [a]$ and $x \in [b]$

$\Rightarrow (x, a) \in R$ and $(x, b) \in R$

$\Rightarrow (a, x) \in R$ and $(x, b) \in R$ [by symmetric]

$\Rightarrow (a, b) \in R$

$\Rightarrow a R b$

$\Rightarrow [a] = [b]$ (from (ii))