1. **Find the Laplace transform of the following functions:**

(1) $e^{2t}\, t^3$

Solution: Let $f(t) = e^{2t}\, t^3$

$$\therefore\ L\{f(t)\} = L\{e^{2t}\, t^3\} = \frac{3!}{(s-2)^4}$$

(2) $e^{3t}\, sin^2 t$

Solution: Let $f(t) = e^{3t}\, sin^2 t$

$$L\{\sin^2 t\} = L\left\{\frac{1-cos2t}{2}\right\} = \frac{1}{2}L\{1\} - \frac{1}{2}L\{cos2t\} = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2+4}\right)$$

$$\therefore\ L\{e^{3t}\, sin^2 t\} = \left[\frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2+4}\right)\right]_{s\to s-3} = \frac{1}{2(s-3)} - \frac{s-3}{2[(s-3)^2+4]}$$

(3) $e^{-3t}\, sin\, 5t\, sin\, 3t$

Solution: Let $f(t) = e^{-3t}\sin 5t \sin 3t$

$$L\{\sin 5t \sin 3t\} = L\left\{\frac{1}{2}(cos2t - cos8t)\right\} = \frac{1}{2}L\{cos2t\} - \frac{1}{2}L\{cos8t\} = \frac{1}{2}\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{s}{s^2+64}\right)$$

$$\therefore\ L\{e^{-3t}\sin 5t \sin 3t\} = \left[\frac{1}{2}\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{s}{s^2+64}\right)\right]_{s\to s+3} = \frac{s+3}{2[(s+3)^2+4]} - \frac{s+3}{2[(s+3)^2+64]}$$

(4) $e^{-4t}\, cos\, 5t\, cos\, 3t.$

Solution: Let $f(t) = e^{-4t}\cos 5t \cos 3t$

$$L\{\cos 5t \cos 3t\} = L\left\{\frac{1}{2}(cos8t + cos2t)\right\} = \frac{1}{2}L\{cos8t\} + \frac{1}{2}L\{cos2t\} = \frac{1}{2}\left(\frac{s}{s^2+64}\right) + \frac{1}{2}\left(\frac{s}{s^2+4}\right)$$

$$\therefore\ L\{e^{-4t}\cos 5t \cos 3t\} = \left[\frac{1}{2}\left(\frac{s}{s^2+64}\right) + \frac{1}{2}\left(\frac{s}{s^2+4}\right)\right]_{s\to s+4}$$

$$L\{e^{-4t}\cos 5t \cos 3t\} = \frac{s+4}{2[(s+4)^2+64]} - \frac{s+4}{2[(s+4)^2+4]}$$

Example 1.12 :

Find $L\left[e^{-3t}\left(2\cos 5t - 3\sin 5t\right)\right]$

Solution:

$$L\left[e^{-3t}\left(2\cos 5t - 3\sin 5t\right)\right] = 2L\left[e^{-3t}\cos 5t\right] - 3L\left[e^{-3t}\sin 5t\right]$$

$$= 2\left(\frac{s+3}{(s+3)^2+25}\right) - 3\left(\frac{5}{(s+3)^2+25}\right)$$

$$= \frac{2s+6-15}{s^2+9+6s+25} = \frac{2s-9}{s^2+6s+34}$$

Example 1.9 **:**

Find $L\left[t^5e^{4t}\cosh 3t\right]$ **(VTU 2003, 2014)**

Solution:

$$L\left[t^5e^{4t}\cosh 3t\right]=L\left[t^5e^{4t}\left(\frac{e^{3t}+e^{-3t}}{2}\right)\right]=\frac{1}{2}L\left[t^5e^{4t}e^{3t}+t^5e^{4t}e^{-3t}\right]=\frac{1}{2}L\left[t^5e^{7t}+t^5e^{t}\right]$$

We have, $L\left[e^{at}t^n\right]=\dfrac{n!}{(s-a)^{n+1}}$, where n is a positive integer.

$$\therefore\ L\left[t^5e^{4t}\cosh 3t\right]=\frac{1}{2}L\left[t^5e^{7t}+t^5e^{t}\right]=\frac{1}{2}\left\{\frac{5!}{(s-7)^{5+1}}+\frac{5!}{(s-1)^{5+1}}\right\}=\frac{5!}{2}\left\{\frac{1}{(s-7)^6}+\frac{1}{(s-1)^6}\right\}$$

$$=60\left\{\frac{1}{(s-7)^6}+\frac{1}{(s-1)^6}\right\}$$

Find the Laplace transform of the following functions:

(1) $t \cos at$

Solution:

(1) $t \cos at$

Let $f(t) = t \cos at$

$L\{cosat\} = \frac{s}{s^2+a^2}$

$$\therefore \; L\{t\ cosat\} = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{-1}{(s^2+a^2)^2}[(s^2+a^2)(1) - s(2s)] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$L\{t\ cosat\} = \frac{s^2-a^2}{(s^2+a^2)^2}$

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(2) $t(\sin^3 t - \cos^3 t)$

Solution: Let $f(t) = t(\sin^3 t - \cos^3 t)$

$$L\{\sin^3 t - \cos^3 t\} = L\left\{\frac{1}{4}[3sint - sin3t] - \frac{1}{4}[cos3t + 3cost]\right\}$$

$$= \frac{3}{4}L\{sint\} - \frac{1}{4}L\{sin3t\} - \frac{1}{4}L\{cos3t\} - \frac{3}{4}L\{cost\}$$

$$= \frac{3}{4}\left(\frac{1}{s^2+1}\right) - \frac{1}{4}\left(\frac{3}{s^2+9}\right) - \frac{1}{4}\left(\frac{s}{s^2+9}\right) - \frac{3}{4}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{3}{4}\left(\frac{1-s}{s^2+1}\right) - \frac{1}{4}\left(\frac{3+s}{s^2+9}\right)$$

$$\therefore \; L\{t\ (\sin^3 t - \cos^3 t)\} = -\frac{d}{ds}\left(\frac{3}{4}\left(\frac{1-s}{s^2+1}\right) - \frac{1}{4}\left(\frac{3+s}{s^2+9}\right)\right)$$

$$= \frac{-3}{4(s^2+1)^2}[(s^2+1)(-1) - (1-s)(2s)] + \frac{1}{4(s^2+9)^2}[(s^2+9)(1) - (3+s)(2s)]$$

$$L\{t\ (\sin^3 t - \cos^3 t)\} = \frac{[6-6s-s^2]}{4(s^2+9)^2} - \frac{3[s^2-2s-1]}{4(s^2+1)^2}$$

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(3) $t\, e^{-2t} \sin 4t$

Solution: Let $f(t) = t\, e^{-2t} \sin 4t$

$L\{\sin 4t\} = \frac{4}{s^2+16}$

$$L\{t\ (\sin 4t)\} = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right) = \frac{-4}{(s^2+16)^2}[(s^2+16)(0) - 2s] = \frac{8s}{(s^2+16)^2}$$

$$\therefore \; L\{e^{-2t}\ t\ (\sin 4t)\} = \left[\frac{8s}{(s^2+16)^2}\right]_{s\to s+2} = \frac{8(s+2)}{[(s+2)^2+16}$$

(4) $t^2\, e^{-2t} \cos t$

Solution: Let $f(t) = t^2\, e^{-2t} \cos t \;\Rightarrow\; L\{\cos t\} = \frac{s}{s^2+1}$

$$L\{t^2 \cos t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{d}{ds}\left(\frac{1}{(s^2+1)^2}[(s^2+1)(1) - (s)2s]\right)$$

$$= \frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right) = \frac{1}{(s^2+1)^4}[(s^2+1)^2(-2s) - (1-s^2)2s(s^2+1)] =$$

$$\therefore \; L\{\,e^{-2t} t^2 \cos t\} = \left[\frac{-4s}{(s^2+1)^3}\right]_{s\to s+2} = \frac{-4(s+2)}{\left((s+2)^2+1\right)^3}$$

***Example 1.18* :**

Find $L\left[2^t + t\sin t\right]$ **(VTU 2004)**

Solution:

$$L\left[2^t + t\sin t\right] = L\left[2^t\right] + L[t\sin t]$$

$$L\left[2^t\right] = L\left[e^{(\log 2)t}\right] = \frac{1}{s-\log 2}$$

Let $f(t) = \sin t$

$$\Rightarrow \quad L[f(t)] = L[\sin t]$$

$$\Rightarrow \quad \bar{f}(s) = \frac{1}{s^2+1}$$

We have, $L[t\,f(t)] = -\dfrac{d}{ds}\left[\bar{f}(s)\right]$

$$\therefore \quad L[t\,f(t)] = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{(s^2+1)(0)-(2s)}{(s^2+1)^2}\right] = -\frac{2s}{(s^2+1)^2}$$

$$\therefore \quad L\left[2^t + t\sin t\right] = \frac{1}{s-\log 2} - \frac{2s}{(s^2+1)^2}$$

Example 1.27 **:**

Find $L\left[\frac{\sin^2 3t}{t}\right]$.

Solution:

$$L\left[\frac{\sin^2 3t}{t}\right] = L\left[\frac{1-\cos 6t}{2t}\right] = \frac{1}{2}L\left[\frac{1-\cos 6t}{t}\right]$$

Let $$f(t) = 1-\cos 6t$$

$\Rightarrow$ $$L\left[f(t)\right] = L[1] - L[\cos 6t]$$

$\Rightarrow$ $$\bar{f}(s) = \frac{1}{s} - \frac{s}{s^2+6^2}$$

We have, $$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$$

$\therefore$ $$L\left[\frac{1-\cos 6t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+36}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2+36\right)\right]_s^\infty = \left[\log\left(\frac{s}{\sqrt{s^2+36}}\right)\right]_s^\infty = \underset{s\to\infty}{Lt}\log\left(\frac{s}{\sqrt{s^2+36}}\right) - \log\left(\frac{s}{\sqrt{s^2+36}}\right)$$

$$= \underset{s\to\infty}{Lt}\log\left(\frac{s}{s\sqrt{1+36/s^2}}\right) - \log\left(\frac{s}{\sqrt{s^2+36}}\right) = 0 - \log\left(\frac{s}{\sqrt{s^2+36}}\right) = \log\left(\frac{\sqrt{s^2+36}}{s}\right)$$

$\therefore$ $$L\left[\frac{\sin^2 3t}{t}\right] = \frac{1}{2}\log\left(\frac{\sqrt{s^2+36}}{s}\right)$$

Example 1.24 :

Find $L\left[\frac{1-\cos 3t}{t}\right]$. **(VTU 2006)**

Solution:

Let $$f(t) = 1-\cos 3t$$

$\Rightarrow$ $$L\left[f(t)\right] = L[1]-L[\cos 3t]$$

$\Rightarrow$ $$\bar{f}(s) = \frac{1}{s}-\frac{s}{s^2+9}$$

We have, $$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$$

$\therefore$ $$L\left[\frac{1-\cos 3t}{t}\right] = \int_s^\infty \left(\frac{1}{s}-\frac{s}{s^2+9}\right)ds$$

$$= \left[\log s-\frac{1}{2}\log\left(s^2+9\right)\right]_s^\infty = \left[\log\left(\frac{s}{\sqrt{s^2+9}}\right)\right]_s^\infty$$

$$= \left[\underset{s\to\infty}{Lt}\log\left(\frac{s}{\sqrt{s^2+9}}\right)-\log\left(\frac{s}{\sqrt{s^2+9}}\right)\right]$$

$$= \left[\underset{s\to\infty}{Lt}\log\left(\frac{s}{s\sqrt{1+\left(9/s^2\right)}}\right)-\log\left(\frac{s}{\sqrt{s^2+9}}\right)\right] = \left[0-\log\left(\frac{s}{\sqrt{s^2+9}}\right)\right]$$

$$= \log\left(\frac{\sqrt{s^2+9}}{s}\right)$$

Find $L\left[\frac{\cos at - \cos bt}{t} + t\sin at\right]$. (VTU 2010)

Solution:

$$L\left[\frac{\cos at-\cos bt}{t}+t\sin at\right] = L\left[\frac{\cos at-\cos bt}{t}\right]+L[t\sin at]$$

Let $\qquad f(t) = \cos at - \cos bt$

$\Rightarrow \qquad L\,[f(t)] = L[\cos at] - L[\cos bt]$

$\Rightarrow \qquad \overline{f}(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

We have, $\qquad L\left[\frac{f(t)}{t}\right] = \int_s^\infty \overline{f}(s)\,ds$

$\therefore \qquad L\left[\frac{\cos at-\cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds$

$$= \frac{1}{2}\left[\log\left(s^2+a^2\right)-\log\left(s^2+b^2\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\underset{s\to\infty}{Lt}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[\underset{s\to\infty}{Lt}\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[0-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Let $\qquad f(t) = \sin at$

$\Rightarrow \qquad L[f(t)] = L[\sin at]$

$\Rightarrow \qquad \overline{f}(s) = \frac{a}{s^2+a^2}$

We have, $\qquad L[t\,f(t)] = -\frac{d}{ds}\left[\overline{f}(s)\right]$

$\therefore \qquad L[t\,f(t)] = -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right] = -\left[\frac{\left(s^2+a^2\right)(0)-a(2s)}{\left(s^2+a^2\right)^2}\right]$

$$= -\frac{2as}{\left(s^2+a^2\right)^2}$$

$$\therefore\ L\left[\frac{\cos at-\cos bt}{t}+t\sin at\right] = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)-\frac{2as}{\left(s^2+a^2\right)^2}$$

Example 1.22 :

Find $L\left[\frac{\cos 2t - \cos 3t}{t}\right]$. **(VTU 2004)**

Solution:

Let $f(t) = \cos 2t - \cos 3t$

$\Rightarrow \quad L[f(t)] = L[\cos 2t] - L[\cos 3t]$

$$\Rightarrow \quad \bar{f}(s) = \frac{s}{s^2+4} - \frac{s}{s^2+9}$$

We have, $$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$$

$$\therefore \quad L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$$

$$= \frac{1}{2}\left[\log\left(s^2+4\right) - \log\left(s^2+9\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\underset{s\to\infty}{Lt}\log\left(\frac{s^2+4}{s^2+9}\right) - \log\left(\frac{s^2+4}{s^2+9}\right)\right] = \frac{1}{2}\left[\underset{s\to\infty}{Lt}\log\left(\frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}}\right) - \log\left(\frac{s^2+4}{s^2+9}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+4}{s^2+9}\right)\right] = \frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right)$$