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MODULE-2

Basic laws

Name	AND form	OR form
Identity law	$1A = A$	$0 + A = A$
Null law	$0A = 0$	$1 + A = 1$
Idempotent law	$AA = A$	$A + A = A$
Inverse law	$A\bar{A} = 0$	$A + \bar{A} = 1$
Commutative law	$AB = BA$	$A + B = B + A$
Associative law	$(AB)C = A(BC)$	$(A + B) + C = A + (B + C)$
Distributive law	$A + BC = (A + B)(A + C)$	$A(B + C) = AB + AC$
Absorption law	$A(A + B) = A$	$A + AB = A$
De Morgan's law	$\overline{AB} = \bar{A} + \bar{B}$	$\overline{A + B} = \bar{A}\bar{B}$

2.1 Minimum Forms of Switching Functions

A *minimum sum-of-products* expression for a function is defined as a sum of product terms which

(a) has a minimum number of terms which results in min no. of gates

(b) all expressions with same minimum number of terms, has a minimum number of literals which reduces number of inputs.

Given a minterm expansion, the minimum sum-of- products form can often be obtained by the following procedure:

1. Combine terms by using $XY' + XY = X$. Do this repeatedly to eliminate as many literals as possible. A given term may be used more than once because $X + X = X$.
2. Eliminate redundant terms by using the consensus theorem or other theorems.

Unfortunately, the result of this procedure may depend on the order in which terms are

Find a minimum sum-of-products expression for

$$F(a, b, c) = \Sigma\, m\,(0, 1, 2, 5, 6, 7)$$

$$F = a'b'c' + a'b'c + a'bc' + ab'c + abc' + abc$$

$$= a'b' + b'c + bc + ab \qquad (5\text{-}1)$$

None of the terms in the above expression can be eliminated by consensus. However, combining terms in a different way leads directly to a minimum sum of products:

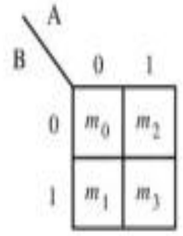
$$F = a'b'c' + a'b'c + a'bc' + ab'c + abc' + abc$$

$$= a'b' + bc' + ac \qquad (5\text{-}2)$$

2.1 Two- and Three-Variable Karnaugh Maps

Just like a truth table, the Karnaugh map of a function specifies the value of the function for every combination of values of the independent variables. A two-variable Karnaugh map is shown. The values of one variable are listed across the top of the map, and the values of the other variable are listed on the left side. Each square of the map corresponds to a pair of values for *A* and *B* as indicated.

Row #	A	B	minterm
0	0	0	m_0
1	0	1	m_1
2	1	0	m_2
3	1	1	m_3



(a) the truth table for a function *F* and

(b) **the corresponding Karnaugh map.**

- for $A = B = 0$ is plotted in the upper left square, and the other map entries are plotted in a similar way.
- Each 1 on the map corresponds to a minterm of *F*. We can read the minterms from the map just like we can read them from the truth table.

(c) A 1 in square 00 indicates that $A'B'$ is a minterm of *F*. Similarly, a 1 in square 01 indicates that $A'B$ is a minterm.

(d) Minterms in adjacent squares of the map can be combined since they differ in only one variable. Thus, $A'B'$ and $A'B$ combine to form A', and this is indicated by looping the corresponding 1's on the map in Figure 2-1(d).

Figure 2-1

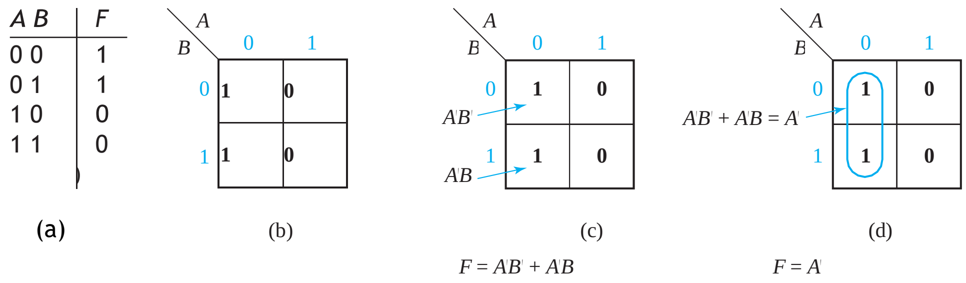


Figure 2-2 shows a three-variable truth table and the corresponding Karnaugh map.

- The value of one variable (*A*) is listed across the top of the map, and the values of the other two variables (*B*, *C*) are listed along the side of the map.
- The rows are labeled in the sequence 00, 01, 11, 10 so that values in adjacent rows differ in only one variable.
- For each combination of values of the variables, the value of *F* is read from the truth table and plotted in the appropriate map square.
- For example, for the input combination $ABC = 001$, the value $F = 0$ is plotted in the square for which $A = 0$ and $BC = 01$. For the combination $ABC = 110$, $F = 1$ is plotted in the $A = 1$, $BC = 10$ square.

2.1 Two- and Three-Variable Karnaugh Maps

Just like a truth table, the Karnaugh map of a function specifies the value of the function for every combination of values of the independent variables. A two-variable Karnaugh map is shown. The values of one variable are listed across the top of the map, and the values of the other variable are listed on the left side. Each square of the map corresponds to a pair of values for *A* and *B* as indicated.

A B C	F
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

(a)

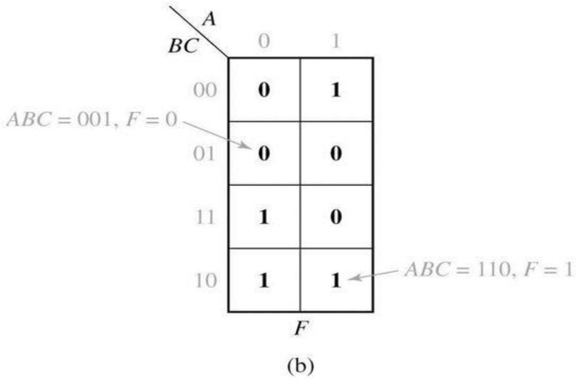


Figure 2.2 Truth table and karnaugh map for three-variable function

Figure 2-3 shows the location of the minterms on a three-variable map.

- Minterms in adjacent squares of the map differ in only one variable and therefore can be combined using the theorem $XY' + XY = X$.
- For example, minterm 011 ($a'bc$) is adjacent to the three minterms with which it can be combined—001 ($a'b'c$), 010 ($a'bc'$), and 111 (abc).
- In addition to squares which are physically adjacent, the top and bottom rows of the map are defined to be adjacent because the corresponding minterms in these rows differ in only one variable. Thus 000 and 010 are adjacent, and so are 100 and 110.

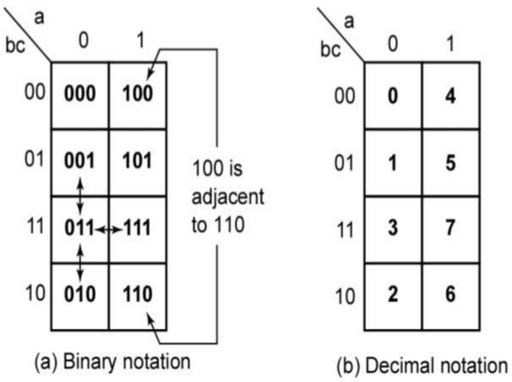


Figure 2.3 Location of Minterms on a Three-variable Karnaugh Map

Given the minterm expansion of a function, it can be plotted on a map by placing 1's in the squares which correspond to minterms of the function and 0's in the remaining squares (the 0's may be omitted if desired).

Figure 2-4 shows the plot of $F(a, b, c) = m_1 + m_3 + m_5$. If F is given as a maxterm expansion, the map is plotted by placing 0's in the squares which correspond to the maxterms and then by filling in the remaining squares with 1's. Thus, $F(a, b, c) = M_0 M_2 M_4 M_6 M_7$ gives the same map as **Figure 2-4.**

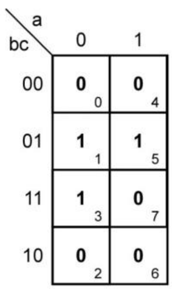


Figure 2-4 Karnaugh Map of $F(a, b, c) = \sum m(1, 3, 5) = \prod M(0, 2, 4, 6, 7)$

Figure 2-5 illustrates how product terms can be plotted on Karnaugh maps. To plot the term b, 1's are entered in the four squares of the map where $b = 1$. The term bc' is 1 when $b = 1$ and $c = 0$, so 1's are entered in the two squares in the $bc = 10$ row. The term ac' is 1 when $a = 1$ and $c = 0$, so 1's are entered in the $a = 1$ column in the rows where $c = 0$.

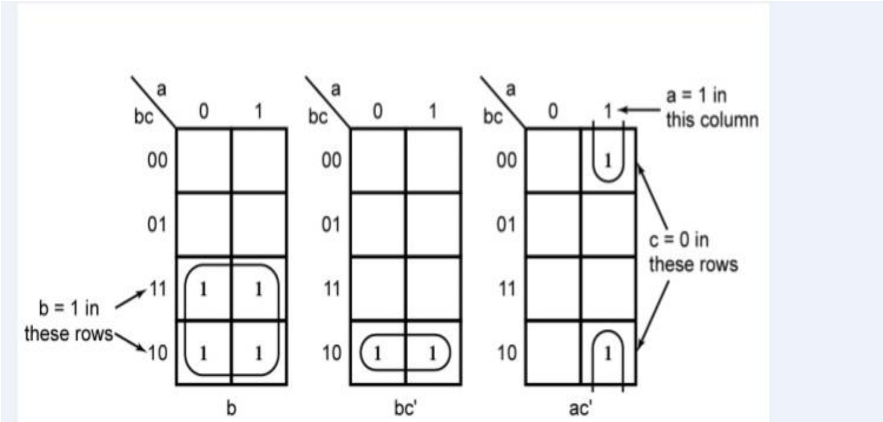


Figure 2-5 Karnaugh Maps for Product Terms

- If a function is given in algebraic form, it is unnecessary to expand it to minterm form before plotting it on a map.
- If the algebraic expression is converted to sum-of- products form, then each product term can be plotted directly as a group of 1's on the map.
- For example, given that

$$f(a, b, c) = abc' + b'c + a'$$

we would plot the map as follows:

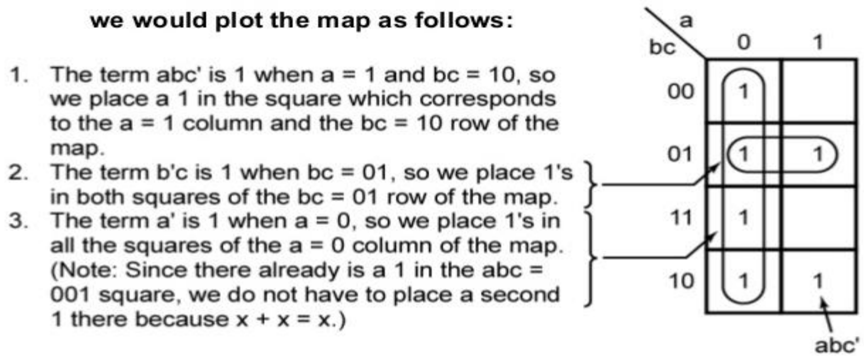


Figure 2-6 illustrates how a simplified expression for a function can be derived using a Karnaugh map.

- The function to be simplified is first plotted on a Karnaugh map in Figure 5-6(a). Terms in adjacent squares on the map differ in only one variable and can be combined using the theorem $XY' + XY = X$. Thus $a'b'c$ and $a'bc$ combine to form $a'c$, and $a'b'c$ and $ab'c$ combine to form $b'c$, as shown in Figure 5-6(b).
- A loop around a group of minterms indicates that these terms have been combined. The looped terms can be read directly off the map.
- Thus, for Figure 5-6(b), term T_1 is in the $a = 0$ (a') column, and it spans the rows where $c = 1$, so $T_1 = a'c$. Note that b has been eliminated because the two minterms in T_1 differ in the variable b.
- Similarly, the term T_2 is in the $bc = 01$ row so $T_2 = b'c$, and a has been eliminated because

T_2 spans the $a = 0$ and $a = 1$ columns. Thus, the minimum sum-of-products form for F is $a'c + b'c$.

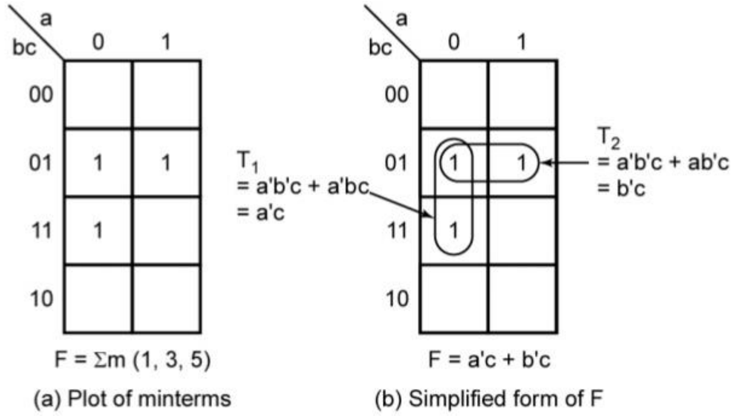


Figure 2-6 Simplification of a Three-Variable Function

- The map for the complement of F (Figure 5-7) is formed by replacing 0's with 1's and 1's with 0's on the map of F.
- To simplify F', note that the terms in the top row combine to form $b'c'$, and the terms in the bottom row combine to form bc'. Because $b'c'$ and bc' differ in only one variable, the top and bottom rows can then be combined to form a group of four 1's, thus eliminating two variables and leav- ing $T_1 = c'$.
- The remaining 1 combines, as shown, to form $T_2 = ab$, so the minimum sum-of-products form for F' is $c' + ab$.

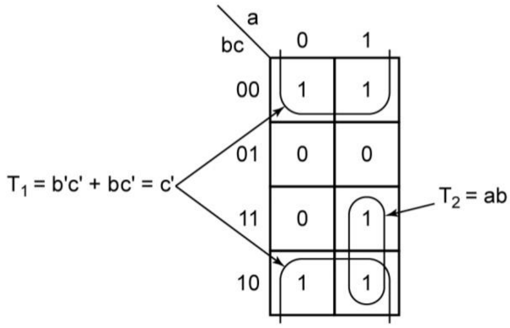


Figure 2-7 Complement of Map in Figure 2.6(a)

- ➢ The Karnaugh map can also illustrate the basic theorems of Boolean algebra.
- Figure 2-8 illustrates the consensus theorem, $XY + X'Z + YZ = XY + X'Z$. Note that the consensus term (YZ) is redundant because its 1's are covered by the other two terms.

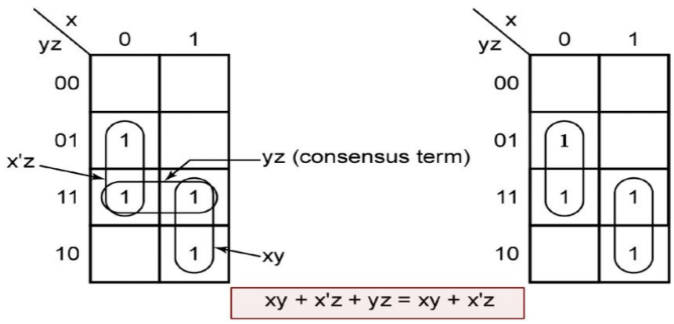


Figure 2-8 Karnaugh Maps that Illustrate the Consensus Theorem

If a function has two or more minimum sum-of-products forms, all of these forms can be determined from a map.

Figure 2-9 shows the two minimum solutions for $F = G\ m(0, 1, 2, 5, 6, 7)$.

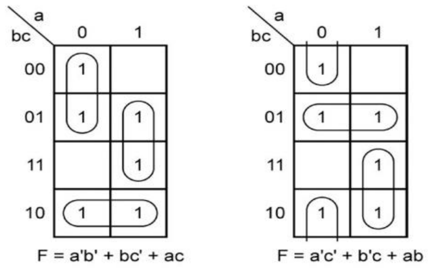


Figure 2-9 Function with Two Minimum Forms

2.3 Four-Variable Karnaugh Maps

Figure 2-10 shows the location of minterms on a four-variable map.

- Each minterm is located adjacent to the four terms with which it can combine. For example, m_5 (0101) could combine with m_1 (0001), m_4 (0100), m_7 (0111), or m_{13} (1101) because it differs in only one variable from each of the other minterms.

- The definition of adjacent squares must be extended so that not only are top and bottom rows adjacent as in the three-variable map, but the first and last columns are also adjacent. This requires numbering the columns in the sequence 00, 01, 11, 10 so that minterms 0 and 8, 1 and 9, etc., are in adjacent squares.

CD \ AB	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

Figure 2-10 Location of Minterms on Four-Variable Karnaugh Map

We will now plot the following four-variable expression on a Karnaugh map (Figure 2-11):

$$f(a, b, c, d) = acd + a'b + d'$$

- The first term is 1 when a = c = d = 1, so we place 1's in the two squares which are in the a = 1 column and cd = 11 row.
- The term $a'b$ is 1 when $ab = 01$, so we place four 1's in the $ab = 01$ column. Finally, d' is 1 when $d = 0$, so we place eight 1's in the two rows for which because $1 + 1 = 1$.)

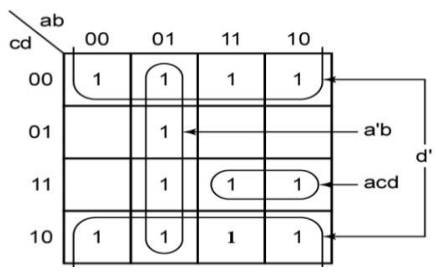


Figure 2-11 Plot of acd + a'b + d'

Next, we will simplify the functions f_1 and f_2 given in **Figure 2-12.**

- Because the functions are specified in minterm form, we can determine the locations of the 1's on the map by referring to **Figure 5-10**.
- After plotting the maps, we can then combine adjacent groups of 1's.
- Minterms can be combined in groups of two, four, or eight to eliminate one, two, or three variables, respectively.
- In **Figure 5-12(a)**, the pair of 1's in the $ab = 00$ column and also in the $d = 1$ rows represents $a'b'd$. The group of four 1's in the $b = 1$ columns and $c = 0$ rows represents bc'.
- In **Figure 5-12(b),** note that the four corner 1's span the $b = 0$ columns and $d = 0$ rows and, therefore, can be combined to form the term $b'd'$.
- The group of eight 1's covers both rows where c = 1 and, therefore, represents the term c.
- The pair of 1's which is looped on the map represents the term a'bd because it is in the ab = 01 column and spans the d = 1 rows.

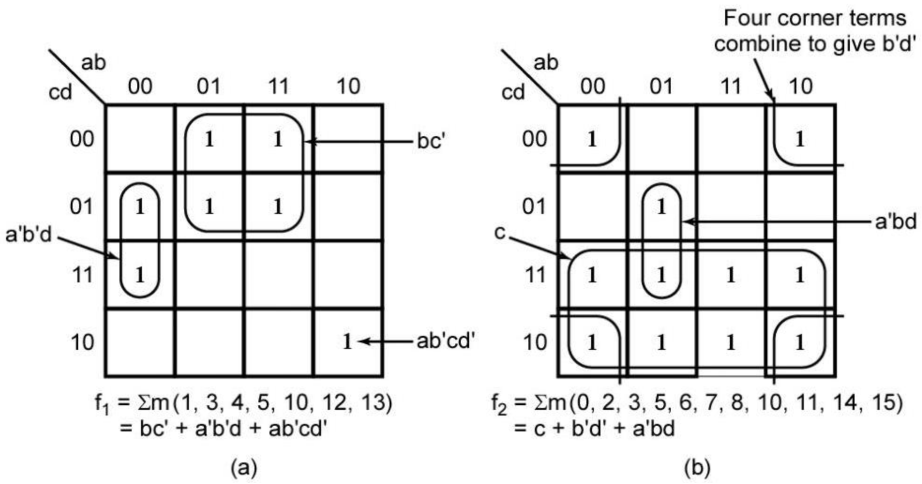


Figure 2-12 Simplification of Four-Variable Functions

The Karnaugh map method is easily extended to functions with **don't-care terms**.

- The required minterms are indicated by 1's on the map, and the **don't-care minterms are indicated by X's**. When choosing terms to form the minimum sum of products, all the 1's must be covered, but the X's are only used if they will simplify the resulting expression.
- In **Figure 2-13**, the only don't-care term used in forming the simplified expression is 13.

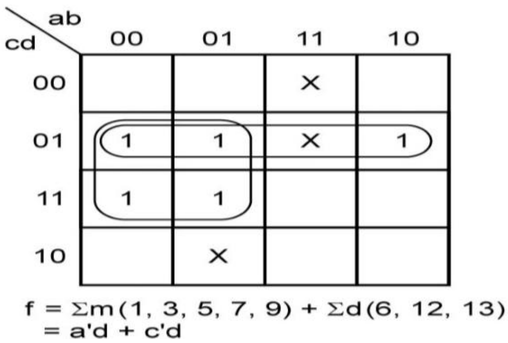


Figure 2-13 Simplification of an Incompletely Specified Function

The use of Karnaugh maps to find a minimum sum-of-products form for a function has been illustrated in **Figures 2-1, 2-6, and 2-12**. A minimum product of sums can also be obtained from the map by looping the 0's on a map of *f*. The complement of the minimum sum of products for f' is then the minimum product of sums for *f*. The following example illustrates this procedure for

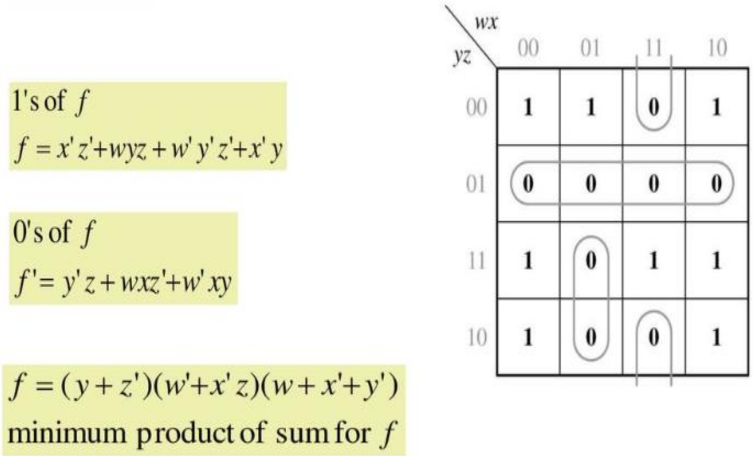


Figure 2-14

Note:

- Pairs will elimate 1 variable
- Quards will elimate 2 variable
- Octets will eliminate 3 variable

2.4 Determination of Minimum Expressions Using Essential Prime Implicants

- **Implicants:** Any single 1 or any group of 1's which can be combined together on a map of the function *F* represents a product term which is called an *implicant* of *F*.
- **Prime Implicant:** A product term implicant is called a *prime implicant* if it cannot be combined with another term to eliminate a variable.
- **Essential prime Implicants:** It is a prime implicant which is having atleast single 1 which is covered by only one group known as essential prime implicant.
- **Redundant prime Implicants**: It is the prime implicant with or without it there is no effect on the circuit. i.e all 1's of this group is covered by other prime implicants.

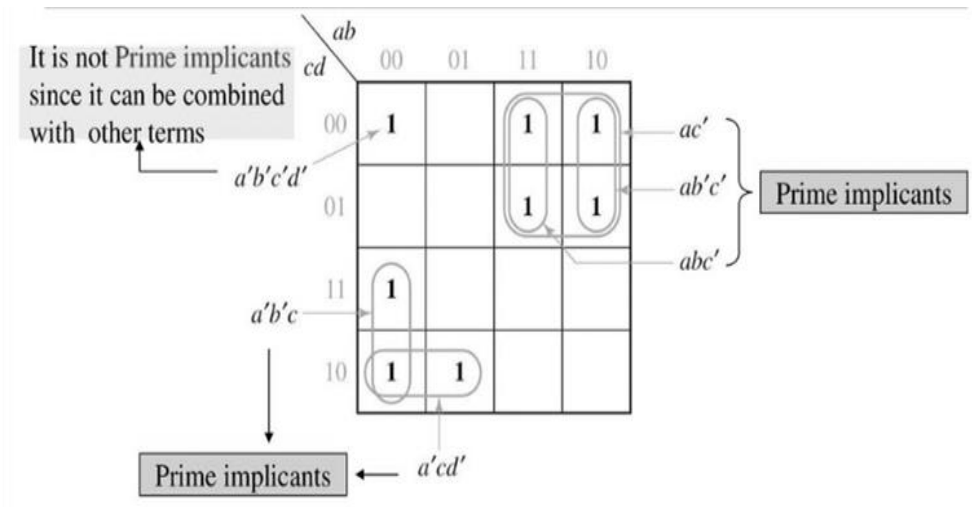


Figure 2-15

- In **Figure 2-15**, a'b'c, a'cd', and ac ' are prime implicants because they cannot be combined with other terms to eliminate a variable.

- On the other hand, a'b'c'd' is not a prime implicant because it can be combined with a'b'cd' or ab'c'd'. Neither abc', nor ab'c' is a prime implicant because these terms can be combined together to form ac'.
- All of the prime implicants of a function can be obtained from a Karnaugh map.
- A single 1 on a map represents a prime implicant if it is not adjacent to any other 1's.
- Two adjacent 1's on a map form a prime implicant if they are not contained in a group of four 1's; four adjacent 1's form a prime implicant if they are not contained in a group of eight 1's, etc.

➢ The minimum sum-of-products expression for a function consists of some (but not necessarily all) of the prime implicants of a function. In other words, a sum-of-products expression containing a term which is not a prime implicant cannot be minimum. This is true because if a nonprime term were present, the expression could be simplified by combining the nonprime term with additional minterms.

➢ In order to find the minimum sum of products from a map, we must find a minimum number of prime implicants which cover all of the 1's on the map.

➢ The function plotted in Figure 2-16 has six prime implicants. Three of these prime implicants cover all of the 1's on the map, and the minimum solution is the sum of these three prime implicants. The shaded loops represent prime implicants which are not part of the minimum solution.

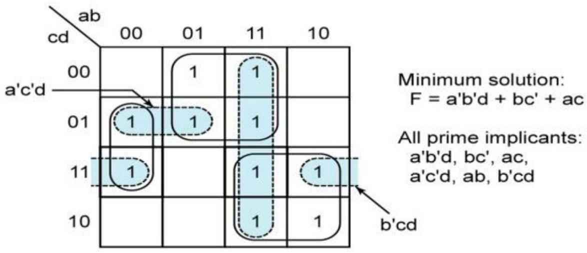


Figure 2-16 Determination of All Prime Implicants

➢ When writing down a list of *all* of the prime implicants from the map, note that there are often prime implicants which are not included in the minimum sum of products. Even though all of the 1's in a term have already been covered by prime implicants, that term may still be a prime implicant provided that it is not included in a larger group of 1's.

➢ For example, in **Figure 2-16**, *a'c'd* is a prime implicant because it cannot be combined with other 1's to eliminate another variable. However, *abd* is not a prime implicant because it can be combined with two other 1's to form *ab*.

➢ The term *b'cd* is also a prime implicant even though both of its 1's are already covered by other prime implicants.

➢ In the process of finding prime implicants, **don't-cares are treated just like 1's**. However, a prime implicant composed entirely of don't-cares can never be part of the minimum solution.

➢ Because all of the prime implicants of a function are generally not needed in forming the minimum sum of products, a systematic procedure for selecting prime implicants is needed.

➢ If prime implicants are selected from the map in the wrong order, a non minimum solution may result.
  - For example, in Figure 5-17, if *CD* is chosen first, then *BD*, *B'C*, and *AC* are needed to cover the remaining 1's, and the solution contains four terms.
  - However, if the prime implicants indicated in Figure 2-17(b) are chosen first, all 1's are covered and *CD* is not needed.

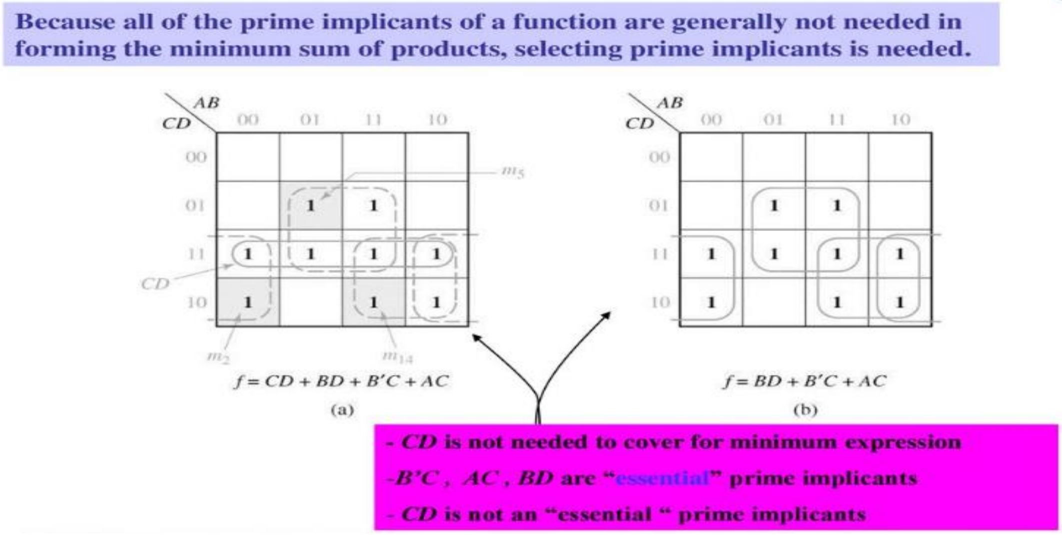


Figure: 2-17

- Note that some of the minterms on the map of Figure 2-17(a) can be covered by only a single prime implicant, but other minterms can be covered by two different prime implicants.
- For example, m_2 is covered only by $B'C$, but m_3 is covered by both $B'C$ and CD. If a minterm is covered by only one prime implicant, that prime implicant is said to be *essential*, and it must be included in the minimum sum of products.
- Thus, $B'C$ *is* an essential prime implicant because m_2 is not covered by any other prime implicant. However, CD is *not* essential because each of the 1's in CD can be covered by another prime implicant.
- The only prime implicant which covers m_5 is BD, so BD is essential. Similarly, AC is essential because no other prime implicant covers m_{14}. In this example, if we choose all of the essential prime implicants, all of the 1's on the map are covered and the nonessential prime impli- cant CD is not needed.

In general, in order to find a minimum sum of products from a map, we should first loop all of the essential prime implicants.

- One way of finding essential prime implicants on a map is simply to look at each 1 on the map that has not already been covered, and check to see how many prime implicants cover that 1. If there is only one prime implicant which covers the 1, that prime implicant is essential.
- If there are two or more prime implicants which cover the 1, we cannot say whether these prime implicants are essential or not without checking the other minterms.
- For simple problems, we can locate the essential prime implicants in this way by inspection of each 1 on the map.
- For example, in Figure 2-16, m_4 is covered only by the prime implicant bc', and m_{10} is covered only by the prime implicant ac. All other 1's on the map are covered by two prime implicants; therefore, the only essential prime implicants are bc' and ac.

- ✓ When checking a minterm to see if it is covered by only one prime implicant, we must look at all squares adjacent to that minterm.
- ✓ If the given minterm and all of the 1's adjacent to it are covered by a single term, then that term is an *essential* prime implicant. If all of the 1's adjacent to a given minterm are *not* covered by a single term, then there are two or more prime implicants which cover that minterm, and we cannot say whether these prime implicants are essential or not without checking the other minterms. Figure 2-18 illustrates this principle.

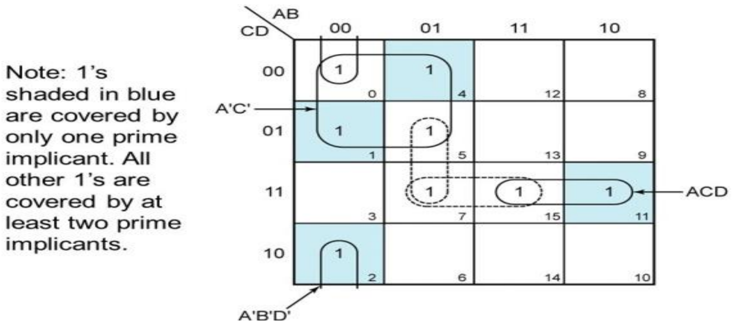


Figure 2-18

The adjacent 1's for minterm m_0 (1_0) are 1_1, 1_2, and 1_4. Because no single term covers these four 1's, no essential prime implicant is yet apparent. The adjacent 1's for 1_1 are 1_0 and 1_5, so the term which covers these three 1's ($A'C'$) is an essential prime implicant. Because the only 1 adjacent to 1_2 is 1_0, $A'B'D'$ is also essential. Because the 1's adjacent to 1_7 (1_5 and 1_{15}) are not covered by a single term, neither $A'BD$ nor BCD is essential at this point. However, because the only 1 adjacent to 1_{11} is 1_{15}, ACD is essential. To complete the minimum solution, one of the nonessen- tial prime implicants is needed. Either $A'BD$ or BCD may be selected. The final solution is

$$A'C' + A'B'D' + ACD + \{A'BD \text{ or } BCD\}$$

2.5 Quine-McCluskey Method

The Karnaugh map method described is an effective way to simplify switching functions which have a small number of variables. When the number of variables is large or if several functions must be simplified, the use of a digital computer is desirable.

- The Quine-McCluskey method is a systematic simplification procedure which can be readily programmed for a digital computer.
- The Quine-McCluskey method reduces the minterm expansion (standard sum- of-products form) of a function to obtain a minimum sum of products. The procedure consists of two main steps:

  1. Eliminate as many literals as possible from each term by systematically applying the theorem $XY + XY' = X$. The resulting terms are called prime implicants.
  2. Use a prime implicant chart to select a minimum set of prime implicants which, when ORed together, are equal to the function being simplified and which con- tain a minimum number of literals.

2.6 Determination of Prime Implicants

- In order to apply the Quine-McCluskey method to determine a minimum sum- of-products expression for a function, the function must be given as a sum of minterms.
- In the first part of the Quine-McCluskey method, all of the prime implicants of a function are systematically formed by combining minterms.
- The minterms are represented in binary notation and combined using

$$XY + XY' = X \qquad \textbf{(1)}$$

where X represents a product of literals and Y is a single variable. Two minterms will combine if they differ in exactly one variable.

The examples given below show both the binary notation and its algebraic equivalent.

$$AB'CD' + AB'CD = AB'C$$

$$\underbrace{1\ 0\ 1}_{X}\ \underset{Y}{0} + \underbrace{1\ 0\ 1}_{X}\ \underset{Y'}{1} = \underbrace{1\ 0\ 1}_{X}\ -$$

(the dash indicates a missing variable)

$$A'BC'D + A'BCD' \quad \text{(will not combine)}$$

$$0\ 1\ 0\ 1 + 0\ 1\ 1\ 0 \quad \text{(will not combine)}$$

- ✓ In order to find all of the prime implicants, all possible pairs of minterms should be compared and combined whenever possible.

- ✓ To reduce the required number of comparisons, the binary minterms are sorted into groups according to the number of 1's in each term. Thus,

$$f(a, b, c, d) = \sum m(0, 1, 2, 5, 6, 7, 8, 9, 10, 14) \qquad (2)$$

is represented by the following list of minterms:

To reduce the required number of comparisons, the binary minterms are sorted into groups according to the number of 1's in each term.

group 0	0	0000
group 1	1	0001
	2	0010
	8	1000
group 2	5	0101
	6	0110
	9	1001
	10	1010
group 3	7	0111
	14	1110

- In this list, the term in group 0 has zero 1's, the terms in group 1 have one 1, those in group 2 have two 1's, and those in group 3 have three 1's.
- Two terms can be combined if they differ in exactly one variable.
- Comparison of terms in nonadjacent groups is unnecessary because such terms will always differ in at least two variables and cannot be combined using $XY + XY' = X$.
- Similarly, the comparison of terms within a group is unnecessary because two terms with the same number of 1's must differ in at least two variables. Thus, only terms in adjacent groups must be compared.

1. First, we will compare the term in group 0 with all of the terms in group 1. Terms 0000 and 0001 can be combined to eliminate the fourth variable, which yields 000–. Similarly, 0 and 2 combine to form 00–0 ($a'b'd'$), and 0 and 8 combine to form –000 ($b'c'd'$).
2. The resulting terms are listed in Column II of Table 2-1.

- Whenever two terms combine, the corresponding decimal numbers differ by a power of 2 (1, 2, 4, 8, etc.).
- This is true because when the binary representations differ in exactly one column and if we subtract these binary representations, we get a 1 only in the column in which the difference exists.

- A binary number with a 1 in exactly one column is a power of 2.
- Because the comparison of group 0 with groups 2 and 3 is unnecessary, we proceed to compare terms in groups 1 and 2.

- Comparing term 1 with all terms in group 2, we find that it combines with 5 and 9 but not with 6 or 10. Similarly, term 2 combines only with 6 and 10, and term 8 only with 9 and 10.
- The resulting terms are listed in Column II. Each time a term is combined with another term, it is checked off. A term may be used more than once because $X + X = X$. Even though two terms have already been combined with other terms, they still must be compared and combined if possible.
- This is necessary because the resultant term may be needed to form the minimum sum solu- tion. At this stage, we may generate redundant terms, but these redundant terms will be eliminated later.
- We finish with Column I by comparing terms in groups 2 and 3. New terms are formed by combining terms 5 and 7, 6 and 7, 6 and 14, and 10 and 14.

  - ✓ Note that the terms in Column II have been divided into groups, according to the number of 1's in each term. Again, we apply $XY + XY' = X$ to combine pairs of terms in Column II.
  - ✓ In order to combine two terms, the terms must have the same variables, and the terms must differ in exactly one of these variables. Thus, it is necessary only to compare terms which have dashes (missing variables) in corresponding places and which differ by exactly one in the number of 1's.
  - ✓ Terms in the first group in Column II need only be compared with terms in the second group which have dashes in the same places. Term 000– (0, 1) combines only with term 100– (8, 9) to yield –00–.
  - ✓ This is algebraically equivalent to $a'b'c' + ab'c' = b'c'$. The resulting term is listed in Column III along with the designation 0, 1, 8, 9 to indicate that it was formed by combining minterms 0, 1, 8, and 9. Term (0, 2) combines only with (8, 10), and term (0, 8) combines with both (1, 9) and (2, 10).
  - ✓ Again, the terms which have been combined are checked off. Comparing terms from the second and third groups in Column II, we find that (2,6) combines with (10, 14), and (2, 10) combines with (6,14).

Column I

group 0	0	0000	✓
group 1	1	0001	✓
	2	0010	✓
	8	1000	✓
group 2	5	0101	✓
	6	0110	✓
	9	1001	✓
	10	1010	✓
group 3	7	0111	✓
	14	1110	✓

Column II

0, 1	000–	✓
0, 2	00–0	✓
0, 8	–000	✓
1, 5	0–01	
1, 9	–001	✓
2, 6	0–10	✓
2, 10	-010	✓
8, 9	100–	✓
8, 10	10–0	✓
5, 7	01–1	
6, 7	011–	
6, 14	–110	✓
10, 14	1–10	✓

Column III

0, 1, 8, 9	–00–
0, 2, 8, 10	–0–0
~~0, 8, 1, 9~~	~~–00–~~
~~0, 8, 2, 10~~	~~–0–0~~
2, 6, 10, 14	––10
~~2, 10, 6, 14~~	~~––10~~

Table 2-1 Determination of Prime Implicants

- ✓ Note that there are three pairs of duplicate terms in Column III.
- ✓ These duplicate terms were formed in each case by combining the same set of four minterms in a dif- ferent order. After deleting the duplicate terms, we compare terms from the two groups in Column III.
- ✓ Because no further combination is possible, the process ter- minates. In general, we would keep comparing terms and forming new groups of terms and new columns until no more terms could be combined.
- ✓ The terms which have not been checked off because they cannot be combined with other terms are called prime implicants. Because every minterm has been included in at least one of the prime implicants, the function is equal to the sum of its prime implicants. In this example we have

$$f = \underset{(1,5)}{a'c'd} + \underset{(5,7)}{a'bd} + \underset{(6,7)}{a'bc} + \underset{(0,1,8,9)}{b'c'} + \underset{(0,2,8,10)}{b'd'} + \underset{(2,6,10,14)}{cd'}$$

(3)

Next, we will define implicant and prime implicant and relate these terms to the Quine-McCluskey method.

Definition Given a function F of n variables, a product term P is an *implicant* of F iff for every combination of values of the n variables for which $P = 1$, F is also equal to 1.

In other words, if for some combination of values of the variables, $P = 1$ and $F = 0$, then P is *not* an implicant of F. For example, consider the function

$$F(a, b, c) = a'b'c' + ab'c' + ab'c + abc = b'c' + ac \quad (5)$$

- ➢ *If* $a'b'c' = 1$, then $F = 1$; if $ac = 1$, then $F = 1$; etc. Hence, the terms $a'b'c'$, ac, etc., are implicants of F. In this example, bc is *not* an implicant of F because when $a = 0$ and $b = c = 1$, $bc = 1$ and $F = 0$.
- ➢ In general, if F is written in sum-of-products form, every product term is an implicant.
    - o Every minterm of F is also an implicant of F, and so is any term formed by combining two or more minterms. For example, in Table 2-1, all of the terms listed in any of the columns are implicants of the function given in **Equation (2).**

Definition A *prime implicant* of a function F is a product term implicant which is no longer an implicant if any literal is deleted from it.

- ➢ In Equation (5), the implicant $a'b'c'$ *is not* a *prime* implicant because a' can be eliminated, and the resulting term ($b'c'$) is still an implicant of F.
- ➢ The implicants $b'c'$ and ac are *prime implicants* because *if* we delete a literal from either term, the term will no longer be an implicant of F. Each prime implicant of a function has a minimum number of literals in the sense that no more literals can be eliminated from it by combining it with other terms.
    - A minimum sum-of-products expression for a function consists of a sum of some (but not necessarily all) of the prime implicants of that function. In other words, a sum-of-products expression which contains a term which is not a prime implicant cannot be minimum.
    - This is true because the nonprime term does not contain a minimum number of literals—it can be combined with additional minterms to form a prime implicant which has fewer literals than the nonprime term.
    - Any nonprime term in a sum-of-products expression can thus be replaced with a prime implicant, which reduces the number of literals and simplifies the expression.

2.7 The Prime Implicant Chart

- The second part of the Quine-McCluskey method employs a prime implicant chart to select a minimum set of prime implicants.

- The minterms of the function are listed across the top of the chart, and the prime implicants are listed down the side. A prime implicant is equal to a sum of minterms, and the prime implicant is said to cover these minterms.

- If a prime implicant covers a given minterm, an X is placed at the inter- section of the corresponding row and column. Table 2 shows the prime implicant chart derived from Table 1. All of the prime implicants (terms which have not been checked off in Table 1) are listed on the left.
  - In the first row, X's are placed in columns 0, 1, 8, and 9, because prime implicant $b'c'$ was formed from the sum of minterms 0, 1, 8, and 9. Similarly, X's are placed in columns 0, 2, 8, and 10 opposite the prime implicant $b'd'$ and so forth.

Essential Prime Implicant : ⊗

		0	1	2	5	6	7	8	9	10	14
(0, 1, 8, 9)	$b'c'$	X	X					X	⊗		
(0, 2, 8, 10)	$b'd'$	X		X				X		X	
(2, 6, 10, 14)	cd'			X		X				X	⊗
(1, 5)	$a'c'd$		X		X						
(5, 7)	$a'bd$				X		X				
(6, 7)	$a'bc$					X	X				

Remaining cover

The resulting minimum sum of products is

$$f = b'c' + cd' + a'bd$$

Table 2 Prime Implicant Chart

- **If a minterm is covered by only one prime implicant, then that prime implicant is called an *essential* prime implicant**
- must be included in the minimum sum of products. Essential prime implicants are easy to find using the prime implicant chart. If a given column contains only one X, then the corresponding row is an essential prime implicant. In Table 6-2, columns 9 and 14 each contain one X, so prime implicants $b'c'$ and cd' are essential.
- <u>Each time a prime implicant is selected for inclusion in the minimum sum, the corresponding row should be crossed out.</u>
- After doing this, the columns which correspond to all minterms covered by that prime implicant should also be crossed out. Table -3 shows the resulting chart when the essential prime implicants and the corresponding rows and columns of Table 2 are crossed out.
- A minimum set of prime implicants must now be chosen to cover the remaining columns. In this example, $a'bd$ covers the remaining two columns, so it is chosen. The resulting minimum sum of products is

$$f = b'c' + cd' + a'bd$$

which is the same as Equation (4). Note that even though the term $a'bd$ is included in the minimum sum of products, $a'bd$ is *not* an *essential* prime implicant. It is the sum of minterms m_5 and m_7; m_5 is also covered by $a'c'd$, and m_7 is also covered by $a'bc$.

		0	1	2	5	6	7	8	9	10	14
(0, 1, 8, 9)	$b'c'$	X	X					X	X		
(0, 2, 8, 10)	$b'd'$	X		X				X		X	
(2, 6, 10, 14)	cd'			X		X				X	X
(1, 5)	$a'c'd$		X		X						
(5, 7)	$a'bd$				X		X				
(6, 7)	$a'bc$					X	X				

Table 3

Example: A prime implicant chart which has two or more X's in every column is called a *cyclic* prime implicant chart. The following function has such a chart:

$$F = \Sigma\, m(0, 1, 2, 5, 6, 7)$$

Derivation of prime implicants:

0	000 ✓		0, 1	00–
1	001 ✓		0, 2	0–0
2	010 ✓		1, 5	–01
5	101 ✓		2, 6	–10
6	110 ✓		5, 7	1–1
7	111 ✓		6, 7	11–

Table 4 shows the resulting prime implicant chart.

- All columns have two X's, so we will proceed by trial and error. Both (0, 1) and (0, 2) cover column 0, so we will try (0, 1).
- After crossing out row (0, 1) and columns 0 and 1, we examine column 2, which is covered by (0, 2) and (2, 6).
- The best choice is (2, 6) because it covers two of the remaining columns while (0, 2) covers only one of the remaining columns. After crossing out row (2, 6) and columns 2 and 6, we see that (5, 7) covers the remaining columns and completes the solution. Therefore, one solution is $F = a'b' + bc' + ac$.

A *cyclic* prime implicant chart is a prime implicant chart which has two or more X's in every column. We will find a solution by trial and error. We will start by trying (0, 1) to cover column 0.

			0	1	2	5	6	7
① →	(0, 1)	$a'b'$	X	X				
	(0, 2)	$a'c'$	X		X			
	(1, 5)	$b'c$		X		X		
② →	(2, 6)	bc'			X		X	
③ →	(5, 7)	ac				X		X
	(6, 7)	ab					X	X

Table -4

- However, we are not guaranteed that this solution is minimum.
- We must go back and solve the problem over again starting with the other prime implicant that covers column 0. The resulting table (Table 5) is

column 0.

			0	1	2	5	6	7
P_1	(0, 1)	$a'b'$	×	×				
P_2	(0, 2)	$a'c'$	×		×			
P_3	(1, 5)	$b'c$		×		×		
P_4	(2, 6)	bc'			×		×	
P_5	(5, 7)	ac				×		×
P_6	(6, 7)	ab					×	×

Table 2-5

- Finish the solution and show that $F = a'c' + b'c + ab$. Because this has the same number of terms and same number of literals as the expression for F derived in Table 2-4, there are two minimum sum-of-products solutions to this problem.
- Compare these two minimum solutions for Equation (6) with the solutions obtained in Figure 2-9 using Karnaugh maps.
- Note that each minterm on the map can be covered by two different loops. Similarly, each column of the prime implicant chart (Table 2-4) has two X's, indicating that each minterm can be covered by two different prime implicants.

2.8 Petrick's Method

Petrick's method is a technique for determining all minimum sum-of-products solutions from a prime implicant chart.

- The example shown in **Tables 2-4 and 2-5** has two minimum solutions.
- As the number of variables increases, the number of prime implicants and the complexity of the prime implicant chart may increase significantly.
- In such cases, a large amount of trial and error may be required to find the minimum solution(s).

Petrick's method is a more systematic way of finding all minimum solutions from a prime implicant chart than the method used previously.

- Before applying Petrick's method, all essential prime implicants and the minterms they cover should be removed from the chart.

1. We will illustrate Petrick's method using Table 2-5. First, we will label the rows of the table P_1, P_2, P_3, etc. We will form a logic function, P, which is true when all of the minterms in the chart have been covered.
2. Let P_1 be a logic variable which is true when the prime implicant in row P_1 is included in the solution, P_2 be a logic variable which is true when the prime implicant in row P_2 is included in the solution, etc.
3. Because column 0 has X's in rows P_1 and P_2, we must choose row P_1 or P_2 in order to cover minterm 0.
4. Therefore, the expression $(P_1 + P_2)$ must be true. In order to cover minterm 1, we must choose row P_1 or P_3; therefore, $(P_1 + P_3)$ must be true.
5. In order to cover minterm 2, $(P_2 + P_4)$ must be true. Similarly, in order to cover minterms 5, 6, and 7, the expressions $(P_3 + P_5)$, $(P_4 + P_6)$ and $(P_5 + P_6)$ must be true. Because we must cover all of the minterms, the following function must be true:

$$P = (P_1 + P_2)(P_1 + P_3)(P_2 + P_4)(P_3 + P_5)(P_4 + P_6)(P_5 + P_6) = 1$$

6. The expression for P in effect means that we must choose row P_1 or P_2, *and* row P_1

or P_3, *and* row P_2 or P_4, etc.

7. The next step is to reduce P to a minimum sum of products. This is easy because there are no complements.

First, we multiply out, using $(X + Y)(X + Z) = X + YZ$ and the ordinary distributive law:

$$P = (P_1 + P_2P_3)(P_4 + P_2P_6)(P_5 + P_3P_6)$$
$$= (P_1P_4 + P_1P_2P_6 + P_2P_3P_4 + P_2P_3P_6)(P_5 + P_3P_6)$$
$$= P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_2P_3P_5P_6 + P_1P_3P_4P_6$$
$$+ P_1P_2P_3P_6 + P_2P_3P_4P_6 + P_2P_3P_6$$

Next, we use $X + XY = X$ to eliminate redundant terms from P, which yields

$$P = P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_1P_3P_4P_6 + P_2P_3P_6$$

- Because P must be true ($P = 1$) in order to cover all of the minterms,
- we can translate the equation back into words as follows. In order to cover all of the minterms, we must choose rows P_1 and P_4 and P_5, or rows P_1 and P_2 and P_5 and P_6, or . . . or rows P_2 and P_3 and P_6.
- Although there are five possible solutions, only two of these have the minimum number of rows. Thus, the two solutions with the minimum number of prime implicants are obtained by choosing rows P_1, P_4, and P_5 or rows P_2, P_3, and P_6.

The first choice leads to $F = a'b' + bc' + ac$, and the second choice to $F = a'c' + b'c + ab$, which are the two minimum solutions derived .

In summary, Petrick's method is as follows:

1. Reduce the prime implicant chart by eliminating the essential prime implicant rows and the corresponding columns.
2. Label the rows of the reduced prime implicant chart P_1, P_2, P_3, etc.
3. Form a logic function P which is true when all columns are covered. P consists of a product of sum terms, each sum term having the form $(P_{i0} + P_{i1} + \ldots)$, where P_{i0}, P_{i1} . . . represent the rows which cover column i.
4. Reduce P to a minimum sum of products by multiplying out and applying
$$X + XY = X.$$
5. Each term in the result represents a solution, that is, a set of rows which covers all of the minterms in the table. To determine the minimum solutions (as defined in Section 5.1), find those terms which contain a minimum number of variables. Each of these terms represents a solution with a minimum number of prime implicants.
6. For each of the terms found in step 5, count the number of literals in each prime implicant and find the total number of literals. Choose the term or terms which correspond to the minimum total number of literals, and write out the corre- sponding sums of prime implicants.

The application of Petrick's method is very tedious for large charts, but it is easy to implement on a computer.

2.9 Simplification of Incompletely Specified Functions

- In this section, we will show how to modify the Quine-McCluskey method in order to obtain a minimum solution when don't-care terms are present.
- In the process of finding the prime implicants, we will treat the don't-care terms as if they were required minterms. In this way, they can be combined with other minterms to eliminate as many literals as possible.
- If extra prime implicants are generated because of the don't-cares, this is correct because the extra prime implicants will be eliminated in the next step anyway.
- When forming the prime implicant chart, the don't-cares are *not* listed at the top. This way,

when the prime implicant chart is solved, all of the required minterms will be covered by one of the selected prime implicants.

- However, the don't-care terms are not included in the final solution unless they have been used in the process of forming one of the selected prime implicants.

- The following example of simplifying an incompletely specified function should clarify the procedure.

$$F(A, B, C, D) = \Sigma\, m(2, 3, 7, 9, 11, 13) + \Sigma\, d(1, 10, 15)$$

(the terms following *d* are don't-care terms)

1	0001 ✓
2	0010 ✓
3	0011 ✓
9	1001 ✓
10	1010 ✓
7	0111 ✓
11	1011 ✓
13	1101 ✓
15	1111 ✓

(1, 3)	00–1 ✓
(1, 9)	–001 ✓
(2, 3)	001– ✓
(2, 10)	–010 ✓
(3, 7)	0–11 ✓
(3, 11)	–011 ✓
(9, 11)	10–1 ✓
(9, 13)	1–01 ✓
(10, 11)	101– ✓
(7, 15)	–111 ✓
(11, 15)	1–11 ✓
(13, 15)	11–1 ✓

(1, 3, 9, 11)	–0–1
(2, 3, 10, 11)	–01–
(3, 7, 11, 15)	--11
(9, 11, 13, 15)	1--1

The don't-care terms are treated like required minterms when finding the prime implicants.

$$F(A, B, C, D) = \Sigma\, m(2, 3, 7, 9, 11, 13) + \Sigma\, d(1, 10, 15)$$

The don't-care columns are omitted when forming the prime implicant chart:

	2	3	7	9	11	13
(1, 3, 9, 11)		×		×	×	
*(2, 3, 10, 11)	×	×			×	
*(3, 7, 11, 15)		×	×		×	
*(9, 11, 13, 15)				×	×	×

*indicates an essential prime implicant.

$$F = B'C + CD + AD$$

- Aalthough the original function was incompletely specified, the final simplified expression for *F* is defined for all combinations of values for *A*, *B*, *C*, and *D* and is therefore completely specified.
- In the process of simplification, we have automatically assigned values to the don't-cares in the original truth table for *F*. If we replace each term in the final expression for *F* by its corresponding sum of minterms, the result is

$$F = (m_2 + m_3 + m_{10} + m_{11}) + (\cancel{m_3} + m_7 + \cancel{m_{11}} + m_{15}) + (m_9 + \cancel{m_{11}} + m_{13} + \cancel{m_{15}})$$

- Because m_{10} and m_{15} appear in this expression and m_1 does not, this implies that the don't-care terms in the original truth table for *F* have been assigned as follows:

for ***ABCD* = 0001, *F* = 0;** **for 1010, *F* = 1;** **for 1111, *F* = 1**

2.10 Simplification Using Map-Entered Variables

- Quine-McCluskey method is not very efficient for functions that have many variables and relatively few terms.

- Karnaugh map techniques can be extended to simplify functions with more than four or five variables By using map-entered variables

- Figure 2-20 (a) shows a four-variable map with two additional variables entered in the squares in the map. When E appears in a square, this means that

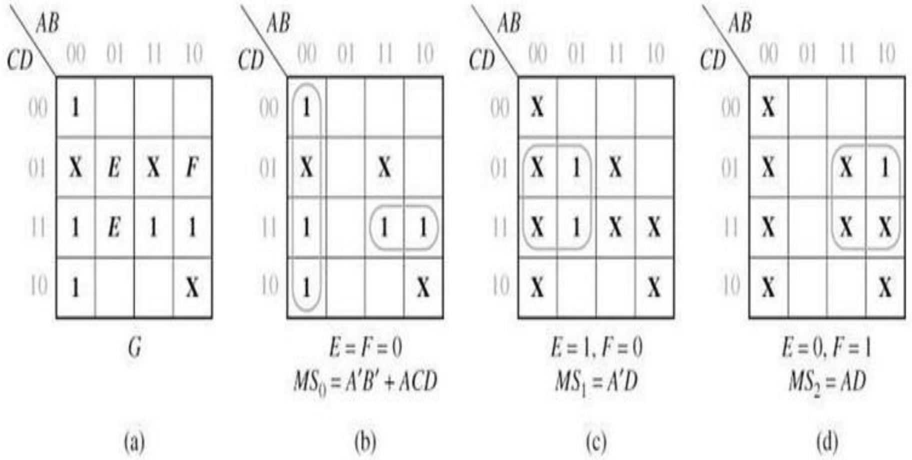


Figure 2.20 Use of Map entered variables

- if $E = 1$, the corresponding minterm is present in the function G, and if $E = 0$, the minterm is absent. Thus, the map represents the six-variable function

$$G(A, B, C, D, E, F) = m_0 + m_2 + m_3 + Em_5 + Em_7 + Fm_9 + m_{11} + m_{15}$$
$$(+ \text{ don't-care terms})$$

- where the minterms are minterms of the variables A, B, C, and D. Note that m_9 is present in G only when $F = 1$.

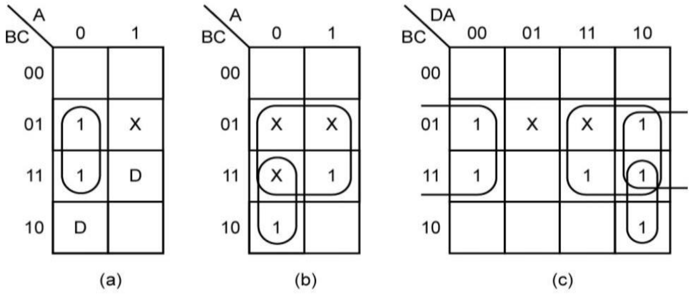


Figure 2.21 Simplification using a map-entered variable

We will now use a three-variable map to simplify the function:

$$F(A, B, C, D) = A'B'C + A'BC + A'BC'D + ABCD + (AB'C)$$

- where the $AB'C$ is a don't-care term.
- Because D appears in only two terms, we will choose it as a map-entered variable, which leads to Figure 2.21 (a).
- We will simplify F by first considering $D = 0$ and then $D = 1$.
- First set $D = 0$ on the map, and F reduces to $A'C$. Setting $D = 1$ leads to the map of Figure 2-21(b). The two 1's on the original map have already been covered by the term $A'C$, so they are changed to X's because we do not care whether they are covered again or not. From Figure 2.21(b), when $D = 1$.Thus, the expression

$$F = A'C + D(C + A'B) = A'C + CD + A'BD$$

- This is a minimum expression for F, as can be verified by plotting the original function on a four-variable map; see Figure 2.21(c).

- General method of simplifying functions using map-entered variables.
- In general, if a variable P_i is placed in square m_j of a map of function F, this means that $F = 1$ when $P_i = 1$, and the variables are chosen so that $m_j = 1$. Given a map with variables P_1, P_2, entered into some of the squares, the minimum sum- of-products form of F can be found as follows:

Find a sum-of-products expression for F of the form

$$F = MS_0 + P_1MS_1 + P_2MS_2 + \cdots$$

where

1. MS_0 is the minimum sum obtained by setting $P_1 = P_2 = \cdots = 0$.
2. MS_1 is the minimum sum obtained by setting $P_1 = 1$, $P_j = 0$ ($j \neq 1$), and replacing all 1's on the map with don't-cares.
3. MS_2 is the minimum sum obtained by setting $P_2 = 1$, $P_j = 0$ ($j \neq 2$) and replacing all 1's on the map with don't-cares.

   (Corresponding minimum sums can be found in a similar way for any remaining map-entered variables.)

- The resulting expression for F will always be a correct representation of F.
- This expression will be minimum provided that the values of the map-entered variables can be assigned independently.
- On the other hand, the expression will not general- ly be minimum if the variables are not independent (for example, if $P_1 = P_2'$).

For the example of Figure 2-20(a), maps for finding MS_0, MS_1 and MS_2 are shown in Figures 2.20(b), (c), and (d), where E corresponds to P_1 and F corresponds to P_2. The resulting expression is a minimum sum of products for G:

$$G = A'B' + ACD + EA'D + FAD$$