Module-2

Chapter - 2

BINOMIAL AND MULTINOMIAL THEOREMS

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Binomial Theorem:

If x and y are any two real numbers and n is a positive integer then

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}y^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

Multinomial Theorem:

If $x_1, x_2, \ldots, x_k$ are any real numbers and n is a positive integer then

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_i} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1}\, x_2^{n_1} \ldots x_k^{n_k},$$

$$where\ n_1 + n_2 + \cdots + n_k = n.$$

Note: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1!\, n_2! \dots n_k!}$

What is the expansion of $(x + y)^4$?

Solution:

By the binomial theorem, we have $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$

$$(x+y)^4 = \sum_{r=0}^{4} \binom{4}{r} x^{4-r} y^r$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^{4-1} y + \binom{4}{2} x^{4-2} y^2 + \binom{4}{3} x^{4-3} y^3 + \binom{4}{4} x^{4-4} y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4 .$$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

Solution:

By the binomial theorem, we have $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$

Here, $x = x$, $y = y$ and $n = 25$.

$$\therefore \quad (x+y)^{25} = \sum_{r=0}^{25} \binom{25}{r} (x)^{25-r} (y)^r$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $r = 13$, namely,

$$\binom{25}{13} = \frac{25!}{13!12!} = 5,200,300 .$$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution:

By the binomial theorem, we have $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$

Here, $x = 2x$, $y = -3y$ and $n = 25$.

$$\therefore \quad (2x-3y)^{25} = \sum_{r=0}^{25} \binom{25}{r} (2x)^{25-r} (-3y)^r$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $r = 13$, namely,

$$\binom{25}{13} (2)^{25-13} (-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13}.$$

Evaluate the following:

i) $\binom{12}{5,3,2,2}$, *ii*) $\binom{7}{2,3,2}$, *iii*) $\binom{8}{4,2,2,0}$, *iv*) $\binom{10}{5,3,2,2}$

Solution:

i) $\binom{12}{5,3,2,2} = \frac{12!}{5!\times3!\times2!\times2!} 166320$

ii) $\binom{7}{2,3,2} = \frac{7!}{2!\times3!\times2!} = 210$

iii) $\binom{8}{4,2,2,0} = \frac{8!}{4!\times2!\times2!\times0!} = 420$

iv) $\binom{10}{5,3,2,2}$ is meaningless. Because, $5 + 3 + 2 + 2 > 10$.

Determine the coefficient of

(i) xyz^2 in the expansion of $(2x - y - z)^4$

(ii) $x^2y^2z^3$ in the expansion of $(3x - 2y - 4z)^7$

(iii) $x^{11}y^4$ in the expansion of $\left(2x^3 - 3xy^2 + z^2\right)^6$

(iv) $a^2b^3c^2d^5$ in the expansion of

$$(a + 2b - 3c + 2d + 5)^{16}.$$

Solution:

(i) By Multinomial theorem, general term in the expansion of $(2x - y - z)^4$ is

$$\binom{4}{n_1, \quad n_2, \quad n_3}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}$$

Put $n_1 = 1,\ n_2 = 1,\ n_3 = 2$

The coefficient of $xyz^2 = \binom{4}{1,\ 1,\ 2}(2)^1(-1)^1(-1)^2$

$$= \frac{4!}{2!} \times 2 \times -1 = -24$$

(ii) By Multinomial theorem, general term in the expansion of

$(3x-2y-4z)^7$ is

$$\binom{7}{n_1,\quad n_2,\quad n_3}(3x)^{n_1}(-2y)^{n_2}(-4z)^{n_3}$$

Put $n_1=2,\ n_2=2,\ n_3=3$

The coefficient of $xyz^2 = \binom{7}{2,\ 2,\ 3}(3)^2(-2)^2(-4)^3$

$$= \frac{7!}{2!\times 2!\times 3!}\times 9\times 4\times(-64)$$

$$= -4{,}83{,}840$$

(iii) By Multinomial theorem, general term in the expansion of $(2x^3 - 3xy^2 + z^2)^6$ is

$$\binom{6}{n_1,\quad n_2,\quad n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}$$

Put $n_1 = 3,\ n_2 = 2,\ n_3 = 0$

The coefficient of $x^{11}y^4 = \binom{6}{3,\ 2,\ 0}(2)^3(-3)^2$

$$= \frac{6!}{3! \times 2!} \times 8 \times 9 = 4320$$

(iv) By Multinomial theorem, general term in the expansion of $(a+2b-3c+2d+5)^{16}$ is

$$\binom{16}{n_1, n_2, n_3, n_4, n_5}(a)^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}$$

Put $n_1 = 2,\ n_2 = 3,\ n_3 = 2,\ n_4 = 5, n_5 = 4$

The coefficient of $a^2b^3c^2d^5$

$$= \binom{16}{2, \quad 3, \quad 2, \quad 5, \quad 4}(1)^2(2)^3(-3)^2(2)^5(5)^4$$

$$= 3 \times 2^5 \times 5^3 \times \frac{16!}{4! \times 4!}$$