Module- I
II Part

PREDICATES OR OPEN SENTENCES

An expression P(x) in the variable x is called an **open sentence or a predicate** on a set S, if P (a), $\forall$ a $\in$ S is a proposition. The set S is called a replacement set or domain of the open sentence.

i.e., P(x) is called open sentence if p(a) is true or p(a) is false $\forall$ a $\in$ S.

A Predicate in two variables is denoted by P(x, y) and in three variables is denoted by P (x, y, z). In general, a predicate involving the *n* variables $x_1, x_2, x_3 \ldots x_n$ can be denoted by P $(x_1, x_2, x_3, \ldots, x_n)$.

EXAMPLE:

1. **Let P(x) denote the statement “x > 3.” What are the truth values of P(4) and P(2)?**

Solution:

We obtain the statement P (4) by setting x = 4 in the statement “x > 3.”

Hence, P (4), which is the statement “4 > 3,” is true.

However, P (2), which is the statement “2 > 3,” is false.

2. Let P(x) denotes the statement "$x \leq 4$." What are the truth values of P(0), P(4) and P(6) ?

Solution:

We obtain the statement P (0) by setting $x = 0$ in the statement "$x \leq 4$"

Hence, P(0), which is the statement "$0 \leq 4$", is true.

P(4) is the statement "$4 \leq 4$", is true.

However, P (6), which is the statement "$6 \leq 4$", is false.

3. Let Q(x, y) denote the statement “x = y + 3.” What is the truth values of the propositions Q (1, 2) and Q (3, 0)?

Solution:

To obtain Q(1, 2), set x = 1 and y = 2 in the statement Q(x, y).

Hence, Q(1, 2) is the statement “1 = 2 + 3,” which is false.

The statement Q(3, 0) is the proposition “3 = 0 + 3,” which is true.

3. **Let R(x, y, z) denote the statement `"x + y = z." Find the truth values of the propositions R (1, 2, 3) and R (0, 0, 1)?**

Solution:

The proposition R (1, 2, 3) is obtained by setting x = 1, y = 2 and z = 3 in the statement R(x, y, z).

We see that R(1, 2, 3) is the statement "1 + 2 = 3," which is true.

Also note that R(0, 0, 1), which is the statement "0 + 0 = 1," is false.

5. Let A(x) denote the statement "Computer x is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of A(CS1), A(CS2), and A(MATH1)?

Solution:

We obtain the statement A(CS1) by setting x = CS1 in the statement "Computer x is under attack by an intruder." Because CS1 is not on the list of computers currently under attack, we conclude that A(CS1) is false.

Similarly, because CS2 and MATH1 are on the list of computers under attack, we know that A(CS2) and A(MATH1) are true.

6. Let A(c, n) denote the statement "Computer c is connected to network n," where c is a variable representing a computer and n is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the truth values of

(i) A (MATH1, CAMPUS1) and

(ii) A (MATH1, CAMPUS2)?

Solution:

Because MATH1 is not connected to the CAMPUS1 network, we see that A(MATH1, CAMPUS1) is false.

However, because MATH1 is connected to the CAMPUS2 network, we see that A(MATH1, CAMPUS2) is true.

QUANTIFIERS

The expressions which convey the idea of quantity are called **Quantifiers**. In English, the words “all”, “some”, “many”, “none”, “for every”, “at least”, “there exists”, “at least one” and “few” are used in quantifications.

The Quantifiers are classified into two types namely

(i) Universal Quantifiers and

(ii) Existential Quantifiers.

UNIVERSAL QUANTIFIERS:

The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain". It is denoted by $\forall x\, P(x)$. Here $\forall$ is called the universal quantifier. We read $\forall x\, P(x)$ as "for all x $P(x)$" or "for every x $P(x)$."

The truth value of universal quantifier $\forall x\, P(x)$ is **true** if for all values of x in the domain, $P(x)$ is true and the truth value of universal quantifier $\forall x\, P(x)$ is **false** if for at least one value of x in the domain, $P(x)$ is not true.

Remark: When all the elements in the domain can be listed say $x_1, x_2, x_3 \ldots x_n$, it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots \wedge P(x_n),$$

because this conjunction is true if and only if $P(x_1)$, $P(x_2)$, $P(x_3)$, … $P(x_n)$ are all true.

EXISTENTIAL QUANTIFIERS:

The existential quantification of P(x) is the proposition "There exists an element x in the domain such that P(x)". It is denoted by $\exists x\ P(x)$. Here $\exists$ is called the existential quantifier.

The truth value of existential quantifier $\exists x\ P(x)$ is **true** if for at least one value of x in the domain, P(x) is true and the truth value of existential quantifier $\exists x\ P(x)$ is **false** if for all values of x in the domain, P(x) is not true.

The existential quantification $\exists x\ P(x)$ is read as
"There is an x such that P(x),"
"There is at least one x such that P(x),"
Or "For some x P(x)."

Remark: Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. If the domain is empty, then $\exists x\ Q(x)$ is false whenever $Q(x)$ is a propositional function because when the domain is empty, there can be no element x in the domain for which $Q(x)$ is true.

When all the elements in the domain can be listed say $x_1, x_2, x_3 \ldots x_n$, the existential quantification $\exists x\ P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots \vee P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2), P(x_3), \ldots P(x_n)$ is true.

Note:

Statement	When True?	When False?
∀x P(x)	P(x) is true for every x.	There is an x for which P(x) is false.
∃x P(x)	There is an x for which P(x) is true.	P(x) is false for every x.

EXAMPLE:

Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

Solution:

Because $P(x)$ is true for all real numbers x, the quantification $\forall x\, P(x)$ is true.

Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Solution:

$Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false.

That is, $x = 3$ is a counterexample for the statement $\forall x\, Q(x)$. Thus $\forall x\, Q(x)$ is false.

Suppose that $P(x)$ is "$x^2 > 0$." Show that the statement $\forall x\, P(x)$ is false where the universe of discourse consists of all integers.

Solution:

We give a counterexample. We see that $x = 0$ is a counterexample because $x = 0$ when $x = 0$, so that x is not greater than 0 when $x = 0$. Thus $\forall x\, P(x)$ is false.

What is the truth value of $\forall x\ P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution:

The statement $\forall x\ P(x)$ is the same as the conjunction $P(1) \land P(2) \land P(3) \land P(4)$, because the domain consists of the integers 1, 2, 3, and 4.

Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x\ P(x)$ is false.

What does the statement $\forall x\ N(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

Solution:

The statement $\forall x\ N(x)$ means that for every computer x on campus, that computer x is connected to the network.

This statement can be expressed in English as "Every computer on campus is connected to the network."

Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?

Solution:

Because "$x > 3$" is sometimes true, for instance, when $x = 4$, the existential quantification of $P(x)$, which is $\exists x\, P(x)$, is true. Observe that the statement $\exists x\, P(x)$ is false if and only if there is no element x in the domain for which $P(x)$ is true.

That is, $\exists x\, P(x)$ is false if and only if $P(x)$ is false for every element of the domain.

Let Q(x) denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

Solution:

Because Q(x) is false for every real number x, the existential quantification of Q(x), which is $\exists x\, Q(x)$, is false.

What is the truth value of $\exists x\, P(x)$, where P(x) is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution:

Because the domain is {1, 2, 3, 4}, the proposition $\exists x\, P(x)$ is the same as the disjunction $P(1) \vee P(2) \vee P(3) \vee P(4)$.

Because P(4), which is the statement "$4^2 > 10$," is true, it follows that $\exists x\, P(x)$ is true.