Inverse Laplace Transform

If $L[f(t)] = \overline{f}(s)$, then $f(t)$ is called the **inverse Laplace transform** of $\overline{f}(s)\}$ and is denoted by $L^{-1}\{\overline{f}(s)\} = f(t)$.

Here L^{-1} denotes the inverse Laplace transform.

List of Laplace Transform and Inverse Laplace Transform of Some Standard Functions

Sl. No	Laplace Transform	Inverse Laplace Transform
1	$L[a]=\frac{a}{s},\ s>0$ where 'a' is a constant	$L^{-1}\left[\frac{a}{s}\right]=a$
2	$L[1]=\frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right]=1$
3	$L\left[e^{at}\right]=\frac{1}{s-a},\ s>a$	$L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$
4	$L[\sinh at]=\frac{a}{s^2-a^2},\ s>\lvert a\rvert$	$(i)\,L^{-1}\left[\frac{a}{s^2-a^2}\right]=\sinh at,\ (ii)\,L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{\sinh at}{a}$
5	$L[\cosh at]=\frac{s}{s^2-a^2},\ s>\lvert a\rvert.$	$L^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh at$

6	$L[\sin at]=\dfrac{a}{s^2+a^2},\ s>0$	$(i)\,L^{-1}\left[\dfrac{a}{s^2+a^2}\right]=\sin at,\ (ii)\,L^{-1}\left[\dfrac{1}{s^2+a^2}\right]=\dfrac{\sin at}{a}$
7	$L[\cos at]=\dfrac{s}{s^2+a^2},\ s>0$	$L^{-1}\left[\dfrac{s}{s^2+a^2}\right]=\cos at$
8	$(i)\,L\left[t^n\right]=\dfrac{n!}{s^{n+1}},$ where n is a positive integer. $(ii)\,L\left[t^n\right]=\dfrac{\Gamma(n+1)}{s^{n+1}},$ when n is a fraction.	$(i)\,L^{-1}\left[\dfrac{1}{s^{n+1}}\right]=\dfrac{t^n}{n!},\ (ii)\,L^{-1}\left[\dfrac{1}{s^n}\right]=\dfrac{t^{n-1}}{(n-1)!}$

9	$L\left[e^{at}t^{n}\right]=\dfrac{n!}{(s-a)^{n+1}}$, where n is a positive integer.	$(i)\,L^{-1}\left[\dfrac{1}{(s-a)^{n+1}}\right]=\dfrac{e^{at}t^{n}}{n!}$, $(ii)\,L^{-1}\left[\dfrac{1}{(s-a)^{n}}\right]=\dfrac{e^{at}t^{n-1}}{(n-1)!}$
10	$L\left[e^{at}\sinh bt\right]=\dfrac{b}{(s-a)^{2}-b^{2}}$	$(i)\,L^{-1}\left[\dfrac{b}{(s-a)^{2}-b^{2}}\right]=e^{at}\sinh bt$, $(ii)\,L^{-1}\left[\dfrac{1}{(s-a)^{2}-b^{2}}\right]=\dfrac{e^{at}\sinh bt}{b}$
11	$L\left[e^{at}\cosh bt\right]=\dfrac{s-a}{(s-a)^{2}-b^{2}}$.	$L^{-1}\left[\dfrac{s-a}{(s-a)^{2}-b^{2}}\right]=e^{at}\cosh bt$

12	$L\left[e^{at}\,\mathbf{sin}\,bt\right]=\dfrac{b}{(s-a)^2+b^2}$	$(i)\,L^{-1}\left[\dfrac{b}{(s-a)^2+b^2}\right]=e^{at}\,\mathbf{sin}\,bt,$ $(ii)\,L^{-1}\left[\dfrac{1}{(s-a)^2+b^2}\right]=\dfrac{e^{at}\,\mathbf{sin}\,bt}{b}$
13	$L\left[e^{at}\,\mathbf{cos}\,bt\right]=\dfrac{s-a}{(s-a)^2+b^2}.$	$L^{-1}\left[\dfrac{s-a}{(s-a)^2+b^2}\right]=e^{at}\,\mathbf{cos}\,bt$
14	$L\left[t\,f(t)\right]=-\dfrac{d}{ds}\left[\overline{f}(s)\right]$	$L^{-1}\left\{\dfrac{d}{ds}\left[\overline{f}(s)\right]\right\}=-t\,f(t)$

15	$L\left[t^2 f(t)\right]=\frac{d^2}{ds^2}\left[\overline{f}(s)\right]$	$L^{-1}\left\{\frac{d^2}{ds^2}\left[\overline{f}(s)\right]\right\}=t^2 f(t)$
16	$L\left[t^3 f(t)\right]=-\frac{d^3}{ds^3}\left[\overline{f}(s)\right]$	$L^{-1}\left\{\frac{d^3}{ds^3}\left[\overline{f}(s)\right]\right\}=-t^3 f(t)$
17	$L\left[\frac{f(t)}{t}\right]=\int_s^\infty \overline{f}(s)ds$	$L^{-1}\left[\int_s^\infty \overline{f}(s)ds\right]=\frac{f(t)}{t}$
18	$L\left[\int_0^t f(u)du\right]=\frac{\overline{f}(s)}{s}$	$L^{-1}\left[\frac{\overline{f}(s)}{s}\right]=\int_0^t f(u)du$

In this section, we determine the inverse Laplace transform of the given function by applying any one of following suitable method :

1. Inverse Laplace transform by completing the square

To find the inverse Laplace transforms, we first express the given $\overline{f}(s)$ in completing the square form and then write its inverse by using standard results

2. Inverse Laplace transform by the method of Partial fractions

To find the inverse Laplace transforms, we first express the given $\overline{f}(s)$ into partial fractions and then write its inverse by using standard results

Inverse Laplace transform of logarithmic and Inverse functions

We know that

$$L[t f(t)] = -\frac{d}{ds}[\bar{f}(s)]$$

or $L[t f(t)] = -\bar{f}'(s)$

$$\Rightarrow \boxed{L^{-1}[-\bar{f}'(s)] = t f(t)} \longrightarrow (*)$$

Working procedure

1. Take given problem as $\bar{f}(s)$
2. Differentiate $\bar{f}(s)$ after using properties of logarithms to get $\bar{f}'(s)$
3. Multiply both sides by -1
4. Apply Inverse Laplace transform and use the above property $(*)$.

EXAMPLES:

Find the inverse Laplace transform of $\dfrac{s+1}{(s-1)^2(s+2)}$ **(VTU 2005)**

Solution:

$$L^{-1}\left[\frac{s+1}{(s-1)^2(s+2)}\right] = L^{-1}\left[\frac{A}{(s-1)}+\frac{B}{(s-1)^2}+\frac{C}{(s+2)}\right]$$

Now, $s+1 = A(s-1)(s+2)+B(s+2)+C(s-1)^2$

Put $s=1 \Rightarrow 2 = B(3) \quad \Rightarrow B = \dfrac{2}{3}$

Put $s=-2 \Rightarrow -1 = C(9) \quad \Rightarrow C = -\dfrac{1}{9}$

Put $s=0 \Rightarrow 1 = -2A+2B+C \Rightarrow A = \dfrac{1}{9}$

$$L^{-1}\left[\frac{s+1}{(s-1)^2(s+2)}\right] = L^{-1}\left[\frac{1/9}{(s-1)}+\frac{2/3}{(s-1)^2}+\frac{-1/9}{(s+2)}\right]$$

$$= \frac{1}{9}L^{-1}\left[\frac{1}{(s-1)}\right]+\frac{2}{3}L^{-1}\left[\frac{1}{(s-1)^2}\right]-\frac{1}{9}L^{-1}\left[\frac{1}{(s+2)}\right]$$

$$= \frac{1}{9}e^{t}+\frac{2}{3}t\,e^{t}-\frac{1}{9}e^{-2t}$$

Find the inverse Laplace transform of $\frac{1}{s^2(s+1)}$ **(VTU 2005)**

Solution:

$$L^{-1}\left[\frac{1}{s^2(s+1)}\right] = L^{-1}\left[\frac{A}{s}+\frac{B}{s^2}+\frac{C}{(s+1)}\right]$$

Now, $1 = As(s+1)+B(s+1)+Cs^2$

Put $s=0 \Rightarrow 1 = B(1) \quad \Rightarrow B=1$

Put $s=-1 \Rightarrow 1 = C(1) \quad \Rightarrow C=1$

Put $s=1 \Rightarrow 1 = 2A+2B+C \Rightarrow A=-1$

$$\therefore \quad L^{-1}\left[\frac{1}{s^2(s+1)}\right] = L^{-1}\left[\frac{-1}{s}+\frac{1}{s^2}+\frac{1}{(s+1)}\right]$$

$$= -L^{-1}\left[\frac{1}{s}\right]+L^{-1}\left[\frac{1}{s^2}\right]+L^{-1}\left[\frac{1}{(s+1)}\right]$$

$$= -1+t+e^{-t}$$

Find the inverse Laplace transform of $\dfrac{s-2}{s^2+7s+12}$

Solution: $$L^{-1}\left[\frac{s-2}{s^2+7s+12}\right] = L^{-1}\left[\frac{s-2}{(s+3)(s+4)}\right] = L^{-1}\left[\frac{A}{(s+3)}+\frac{B}{(s+4)}\right]$$

Now, $s-2 = A(s+4)+B(s+3)$

Put $s=-3 \Rightarrow -5 = A(1) \quad \Rightarrow A=-5$

Put $s=-4 \Rightarrow -6 = B(-1) \quad \Rightarrow B=6$

$$\therefore \quad L^{-1}\left[\frac{s-2}{s^2+7s+12}\right] = L^{-1}\left[\frac{-5}{(s+3)}+\frac{6}{(s+4)}\right]$$

$$= -5L^{-1}\left[\frac{1}{s+3}\right]+6L^{-1}\left[\frac{1}{s+4}\right]$$

$$= -5e^{-3t}+6e^{-4t}$$

Find the inverse Laplace transform of $\dfrac{3s+7}{s^2-2s-3}$ **(VTU 2004)**

Solution:

$$L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = L^{-1}\left[\frac{3s+7}{s^2-2s+1-1-3}\right]$$

$$= L^{-1}\left[\frac{3s+7}{(s-1)^2-4}\right] = L^{-1}\left[\frac{3(s-1+1)+7}{(s-1)^2-4}\right] = L^{-1}\left[\frac{3(s-1)+10}{(s-1)^2-4}\right]$$

$$= 3L^{-1}\left[\frac{(s-1)}{(s-1)^2-4}\right] + 10L^{-1}\left[\frac{1}{(s-1)^2-4}\right]$$

$$= 3e^t\cosh 2t + 10\left(\frac{e^t\sinh 2t}{2}\right)$$

$$= 3e^t\cosh 2t + 5e^t\sinh 2t$$

Find the inverse Laplace transform of $\dfrac{7s+4}{4s^2+4s+9}$ (VTU 2003)

Solution:

$$L^{-1}\left[\frac{7s+4}{4s^2+4s+9}\right] = L^{-1}\left[\frac{7s+4}{4\left(s^2+s+9/4\right)}\right] = \frac{1}{4}L^{-1}\left[\frac{7s+4}{s^2+s+9/4}\right]$$

$$= \frac{1}{4}L^{-1}\left[\frac{7s+4}{s^2+s+\frac{1}{4}-\frac{1}{4}+\frac{9}{4}}\right] = \frac{1}{4}L^{-1}\left[\frac{7s+4}{\left(s+\frac{1}{2}\right)^2+2}\right]$$

$$= \frac{1}{4}L^{-1}\left[\frac{7\left(s+\frac{1}{2}-\frac{1}{2}\right)+4}{\left(s+\frac{1}{2}\right)^2+\sqrt{2}^2}\right] = \frac{1}{4}L^{-1}\left[\frac{7\left(s+\frac{1}{2}\right)-\frac{7}{2}+4}{\left(s+\frac{1}{2}\right)^2+\sqrt{2}^2}\right]$$

$$= \frac{1}{4}L^{-1}\left[\frac{7\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\sqrt{2}^2}\right]$$

$$= \frac{1}{4}\left\{7L^{-1}\left[\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\sqrt{2}^2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\sqrt{2}^2}\right]\right\}$$

$$= \frac{1}{4}\left\{7e^{-(1/2)t}\cos\sqrt{2}t+\frac{1}{2}\left(\frac{e^{-(1/2)t}\sin\sqrt{2}t}{\sqrt{2}}\right)\right\}$$

$$= e^{-t/2}\left\{\frac{7}{4}\cos\sqrt{2}t+\frac{1}{8\sqrt{2}}\sin\sqrt{2}t\right\}$$

Find the inverse Laplace transform of $\dfrac{5s+3}{(s-1)\left(s^2+2s+5\right)}$ **(VTU 2005, 2011, 2012)**

Solution:

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{A}{s-1}+\frac{Bs+C}{(s^2+2s+5)}\right]$$

Now, $5s+3 = A(s^2+2s+5)+(Bs+C)(s-1)$

Put $s=1 \Rightarrow 8 = A(8) \quad \Rightarrow A=1$

Put $s=0 \Rightarrow 3 = 5A-C \Rightarrow C=2$

Put $s=2 \Rightarrow 13 = 13A+2B+C \Rightarrow B=-1$

$$\therefore \qquad L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{1}{s-1}+\frac{-s+2}{(s^2+2s+5)}\right]$$

$$= L^{-1}\left[\frac{1}{s-1}\right]-L^{-1}\left[\frac{s-2}{(s^2+2s+5)}\right]$$

$$= L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{s-2}{\left(s^2+2s+1-1+5\right)}\right] \quad = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{s-2}{(s+1)^2+4}\right]$$

$$= L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{s+1-3}{(s+1)^2+2^2}\right] = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^t - e^{-t}\cos 2t + 3\left(\frac{e^{-t}\cos 2t}{2}\right)$$

Find the inverse Laplace transform of $\dfrac{2s-1}{s^2+2s+17}$ (VTU 2006, 2013)

Solution: $$L^{-1}\left[\frac{2s-1}{s^2+2s+17}\right] = L^{-1}\left[\frac{2s-1}{s^2+2s+1-1+17}\right]$$

$$= L^{-1}\left[\frac{2s-1}{(s+1)^2+16}\right] = L^{-1}\left[\frac{2(s+1-1)-1}{(s+1)^2+4^2}\right]$$

$$= L^{-1}\left[\frac{2(s+1)-3}{(s+1)^2+4^2}\right] \quad = 2L^{-1}\left[\frac{(s+1)}{(s+1)^2+4^2}\right]-3L^{-1}\left[\frac{1}{(s+1)^2+4^2}\right] \quad = 2e^{-t}\cos 4t-3\left(\frac{e^{-t}\sin 4t}{4}\right)$$

Find the inverse Laplace transform of $\log\left(\frac{s^2+1}{s(s+1)}\right)$ (VTU 2004, 2008, 2009)

Solution: Let

$$\overline{f}(s) = \log\left(\frac{s^2+1}{s(s+1)}\right)$$

$$\overline{f}(s) = \log(s^2+1)-\log s-\log(s+1)$$

Differentiate w.r.t.'s'

$$\overline{f}'(s) = \frac{1}{(s^2+1)}(2s)-\frac{1}{s}-\frac{1}{(s+1)}$$

Multiply both side by '$-$'

$$-\overline{f}'(s) = -\frac{2s}{(s^2+1)}+\frac{1}{s}+\frac{1}{(s+1)}$$

Taking inverse Laplace transform on both sides

$$L^{-1}\left[-\bar{f}'(s)\right] = -2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s+1}\right)$$

$$t\,f(t) = -2\cos t + 1 + e^{-t}$$

$$f(t) = \frac{-2\cos t + 1 + e^{-t}}{t}$$

Find the inverse Laplace transform of $\cot^{-1}\left(\frac{s}{a}\right)$

Solution:

$$\bar{f}(s) = \cot^{-1}\left(\frac{s}{a}\right)$$

Differentiate w.r.t.'s'

$$\bar{f}'(s) = -\frac{1}{1+(s/a)^2}\left(\frac{1}{a}\right)$$

Multiply both side by '−'

$$-\overline{f}'(s) = \frac{a}{s^2 + a^2}$$

Taking inverse Laplace transform on both sides

$$L^{-1}\left[-\overline{f}'(s)\right] = L^{-1}\left[\frac{a}{s^2 + a^2}\right]$$

$$t\,f(t) = \sin at$$

$$f(t) = \frac{\sin at}{t}$$

Find the inverse Laplace transform of $\log\left(\frac{s+1}{s-1}\right)$ (VTU 2013)

Solution:

$$\overline{f}(s) = \log\left(\frac{s+1}{s-1}\right)$$

$$\overline{f}(s) = \log(s+1) - \log(s-1)$$ Differentiate w.r.t.'s'

$$\overline{f}\,'(s) = \frac{1}{s+1} - \frac{1}{s-1}$$ Multiply both side by '$-$'

$$-\overline{f}\,'(s) = -\frac{1}{s+1} + \frac{1}{s-1}$$

Taking inverse Laplace transform on both sides

$$L^{-1}\left[-\overline{f}\,'(s)\right] = -L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left(\frac{1}{s-1}\right)$$

$$t\,f(t) = -e^{-t} + e^{t}$$

$$f(t) = \frac{e^{t} - e^{-t}}{t}$$

Find the inverse Laplace transform of $\cot^{-1}(s)$ (VTU 2005)

Solution: $\bar{f}(s) = \cot^{-1}(s)$

Differentiate w.r.t.'s'

$$\bar{f}'(s) = -\frac{1}{1+s^2}$$

Multiply both side by '$-$' $-\bar{f}'(s) = \frac{1}{s^2+1}$

Taking inverse Laplace transform on both sides $L^{-1}\left[-\bar{f}'(s)\right] = L^{-1}\left[\frac{1}{s^2+1}\right]$

$$t\,f(t) = \sin t$$

$$f(t) = \frac{\sin t}{t}$$

Find the inverse Laplace transform of $\tan^{-1}\left(\frac{2}{s^2}\right)$

Solution:

$$\overline{f}(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$$

Differentiate w.r.t.'s'

$$\overline{f}'(s) = \frac{1}{1+\left(2/s^2\right)^2}\left(-\frac{4}{s^3}\right) = \frac{-4s}{s^4+4}$$

Multiply both side by '$-$'

$$-\overline{f}'(s) = \frac{4s}{s^4+4}$$

Taking inverse Laplace transform on both sides

$$L^{-1}\left[-\overline{f}'(s)\right] = L^{-1}\left[\frac{4s}{s^4+4}\right]$$

$$t\,f(t) = L^{-1}\left[\frac{4s}{\left(s^2+2\right)^2-4s^2}\right] = L^{-1}\left[\frac{4s}{\left(s^2+2\right)^2-(2s)^2}\right]$$

$$t\,f(t) = L^{-1}\left[\frac{4s}{(s^2+2+2s)(s^2+2-2s)}\right] = L^{-1}\left[\frac{(s^2+2+2s)-(s^2+2-2s)}{(s^2+2+2s)(s^2+2-2s)}\right]$$

$$t\,f(t) = L^{-1}\left[\frac{1}{(s^2+2-2s)} - \frac{1}{(s^2+2+2s)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2-2s+1-1+2)} - \frac{1}{(s^2+2s+1-1+2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$= e^t \sin t - e^{-t}\sin t = \sin t\left(e^t - e^{-t}\right)$$

Find the inverse Laplace transform of $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$ **(VTU 2013)**

Solution:

$$L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right]=L^{-1}\left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right]=L^{-1}\left[\frac{A}{(s-1)}+\frac{B}{(s-2)}+\frac{C}{(s-3)}\right]$$

$$2s^2-6s+5=A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)$$

$$\text{Put } s=1 \Rightarrow 1=A(-1)(-2) \Rightarrow A=\frac{1}{2}$$

$$\text{Put } s=2 \Rightarrow 1=B(1)(-1) \quad \Rightarrow B=-1$$

$$\text{Put } s=3 \Rightarrow 5=C(2)(1) \quad \Rightarrow C=\frac{5}{2}$$

$$\therefore \qquad L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] = L^{-1}\left[\frac{1/2}{(s-1)}+\frac{-1}{(s-2)}+\frac{5/2}{(s-3)}\right]$$

$$= \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right) \quad = \frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$$

Find the inverse Laplace transform of $\frac{s^2+4}{s(s+4)(s-4)}$ **(VTU 2014)**

Solution:

$$L^{-1}\left[\frac{s^2+4}{s(s+4)(s-4)}\right] = L^{-1}\left[\frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s-4)}\right]$$

$$s^2 + 4 = A(s+4)(s-4) + Bs(s-4) + Cs(s+4)$$

Put $s = 0 \Rightarrow 4 = A(4)(-4) \Rightarrow A = -\frac{1}{4}$

Put $s = -4 \Rightarrow 20 = B(-4)(-8) \quad \Rightarrow B = \frac{5}{8}$

Put $s = 4 \Rightarrow 20 = C(4)(8) \quad \Rightarrow C = \frac{5}{8}$

$$\therefore \qquad L^{-1}\left[\frac{s^2+4}{s(s+4)(s-4)}\right] = L^{-1}\left[\frac{-1/4}{s}+\frac{5/8}{(s+4)}+\frac{5/8}{(s-4)}\right]$$

$$= -\frac{1}{4}L^{-1}\left(\frac{1}{s}\right)+\frac{5}{8}L^{-1}\left(\frac{1}{s+4}\right)+\frac{5}{8}L^{-1}\left(\frac{1}{s-4}\right)$$

$$= -\frac{1}{4}(1)+\frac{5}{8}e^{-4t}+\frac{5}{8}e^{4t}$$

$$= -\frac{1}{4}+\frac{5}{8}e^{-4t}+\frac{5}{8}e^{4t}$$

Find the inverse Laplace transform of $\frac{s+2}{s^2(s+1)(s-2)}$ (VTU 2003)

Solution: