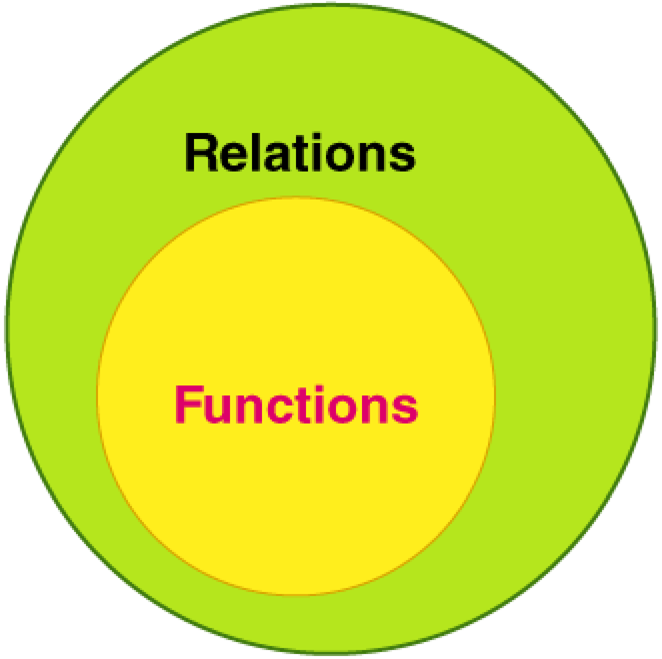
Module-3

RELATIONS AND FUNCTIONS

PART - 6

FUNCTIONS



Note:

All functions are relations, but not all relations are functions.

FUNCTIONS

Definition: Let A and B be any two non-empty sets. A relation f from set A to set B is denoted by $f : A \to B$ and is said to be a function if

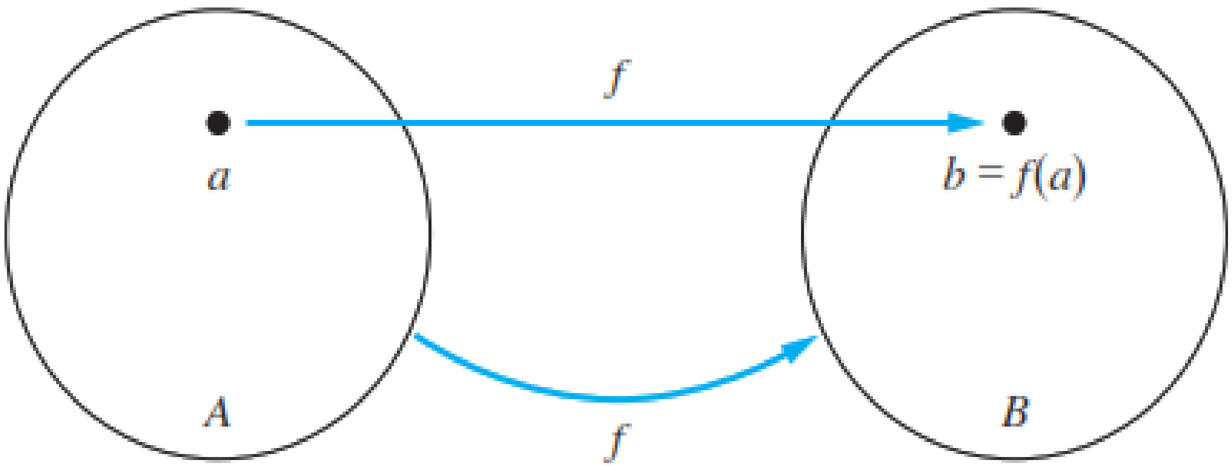
(i) every element of set A is related to some element in set B.

(ii) no element of set A related to two or more elements in set B.

i.e., each element of set A is related to unique element in set B

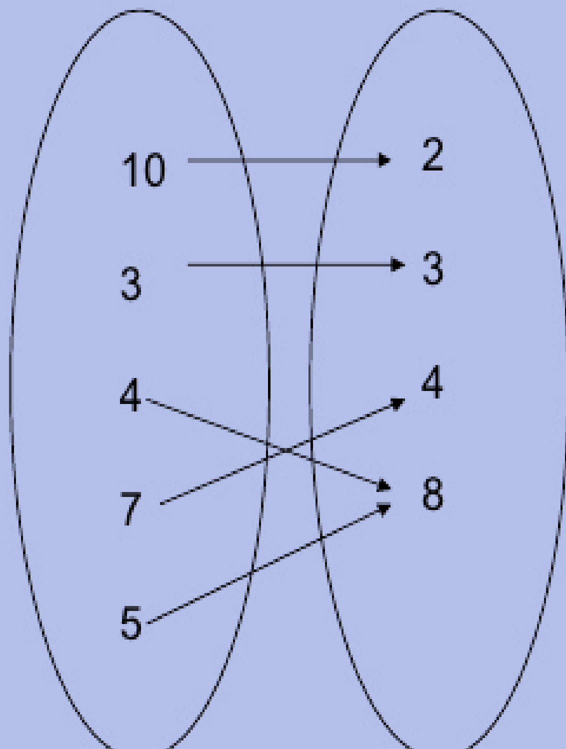
- Here the elements of set A are called **domain** & the elements of set B are called **co-domain**.

If f is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where b is called the **image** of a under f and a is called the **preimage** of b under f.

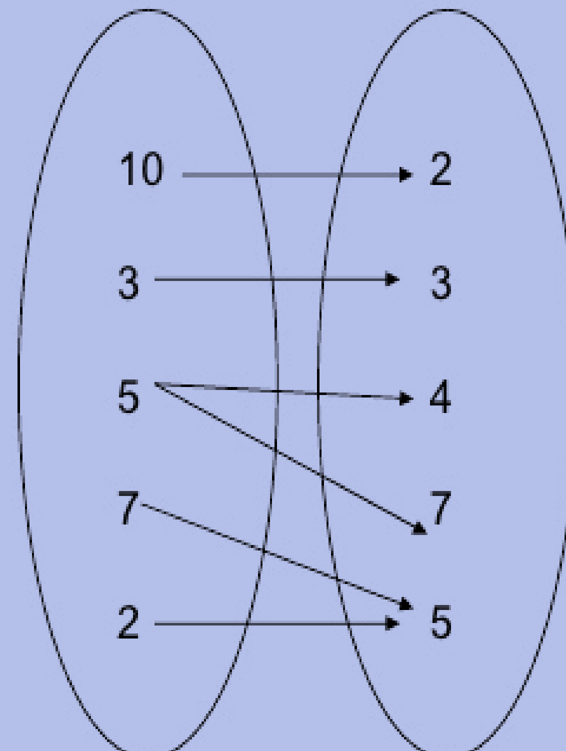


The set B consisting of the images of all elements of A under f is called the **range** of f and is denoted by $f(A)$.

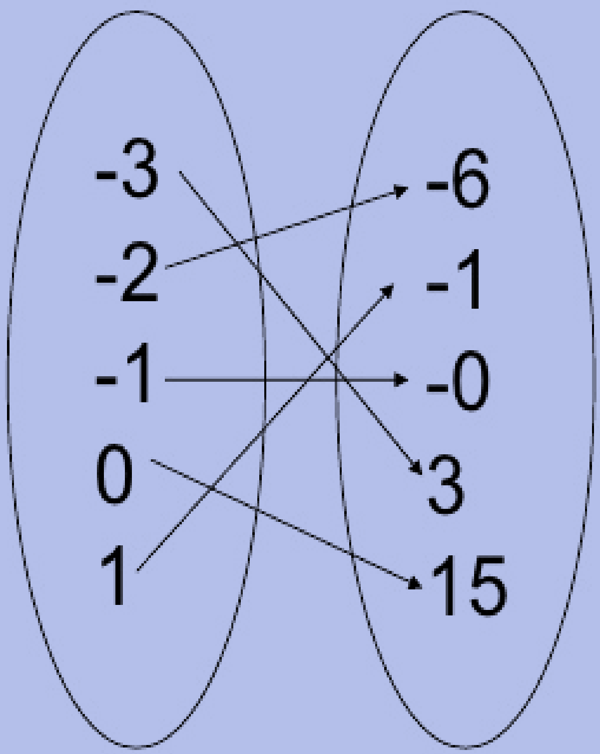
Function



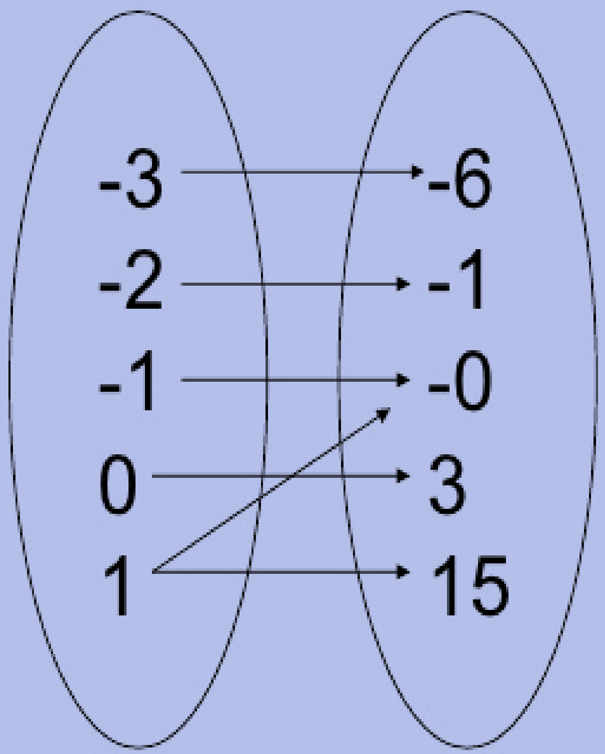
Not a Function



Function



Not a Function



Definition:

Let $f : X \to Y$ be a function and let $A \subseteq X$ and $B \subseteq Y$ then the **image of A** under f is defined as

$$f(A) = \{f(x) : x \in A\}$$

and the **pre-image of B** under f is defined as

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Note:

If $|A| = m$ and $|B| = n$ then

(i) the number of functions from A to B = n^m and

(ii) the number of functions from B to A = m^n.

Let A = {1, 2, 3} and B = {a, b, c, d}. Which of the following relations are functions? Give reasons. If it is a function, determine its domain, co-domain and range.

(i) $f : A \rightarrow B$ **defined by {(1, c), (2, b), (3, d), (2, a)}**

(ii) $g : A \rightarrow B$ **defined by {(1, c), (2, c), (3, d)}**

(iii) $h : A \rightarrow B$ **defined by {(1, a), (2, d), (3, c)}**

(iv) $k : A \rightarrow B$ **defined by {(1, a), (3, d)}**

(v) $l : B \rightarrow A$ **defined by {(a, 3), (b, 2), (c, 1)}**

(vi) $m : B \rightarrow A$ **defined by {(a, 2), (b, 1), (c, 3), (d, 1)}**

(i) *f* is not a function because element 2 in set A is related to two elements in set B.

(ii) *g* is a function because every element of set A is related to some element in set B and each element of set A is related to unique element in set B.

Domain = {1, 2, 3}, co-domain = {a, b, c, d}

and range = {c, d}

(iii) *h* is a function because every element of set A is related to some element in set B and each element of set A is related to unique element in set B.

Domain = {1, 2, 3}, co-domain = {a, b, c, d} and range = {a, c, d} *(iv)* *k* is not a function because every element of set A is not related to some element in set B.

(v) *l* is not a function because every element of set B is not related to some element in set A.

(vi) *m* is a function because every element of set B is related to some element in set A and each element of set B is related to unique element in set A.

Domain = {a, b, c, d}, co-domain = {1, 2, 3}

and range = {1, 2, 3}

Let A = {1, 3, 5, 6, 8}, B = {2, 4, 7, 8}, C = {1, 2, 6, 7} and D = {3, 5, 6, 7}. Which of the following relations are functions? Give reasons. If it is a function, determine its domain, co-domain and range.

(i) **$f : A \rightarrow B$ defined by {(1, 2), (3, 8), (5, 8), (6, 4), (8, 7)}**

(ii) **$g : B \rightarrow C$ defined by {(2, 1), (4, 1), (4, 6), (7, 7), (8, 6)}**

(iii) **$h : C \rightarrow D$ defined by {(1, 6), (6, 6), (7, 7)}**

(iv) **$k : A \rightarrow C$ defined by {(1, 2), (3, 6), (5, 1), (6, 1), (8, 2)}**

(i) f is a function because every element of set A is related to some element in set B and each element of set A is related to unique element in set B.

Domain = {1, 3, 5, 6, 8}, co-domain = {2, 4, 7, 8} and range = {2, 4, 7, 8} *(ii)* *g* is not a function because element 4 in set B is related to two elements in set C.

(iii) *h* is not a function because every element of set C is not related to some element in set D.

(iv) *k* is a function because every element of set A is related to some element in set C and each element of set A is related to unique element in set C.

Domain = {1, 3, 5, 6, 8}, co-domain = {1, 2, 6, 7}

and range = {1, 2, 6}

Let A = {0, ±1, ±2, 3} and f : A → R be a function defined by f (x) = $x^3 - 2x^2 + 3x + 1$ for x ∈ A, find the range of f.

Given, $f(x) = x^3 - 2x^2 + 3x + 1$

$f(0) = 0 - 0 + 0 + 1 = 1$

$f(1) = 1^3 - 2(1)^2 + 3(1) + 1 = 1 - 2 + 3 + 1 = 3$

$f(-1) = (-1)^3 - 2(-1)^2 + 3(-1) + 1 = -1 - 2 - 3 + 1 = -5$

$f(2) = 2^3 - 2(2)^2 + 3(2) + 1 = 8 - 8 + 6 + 1 = 7$

$f(-2) = (-2)^3 - 2(-2)^2 + 3(-2) + 1 = -8 - 8 - 6 + 1 = -21$

$f(3) = 3^3 - 2(3)^2 + 3(3) + 1 = 27 - 18 + 9 + 1 = 19$

Range of f = {1, 3, –5, 7, –21, 19}.

Let A = {1, 2, 3, 4, 5, 6}, B = {6, 7, 8, 9, 10} and $f : A \to B$ be a function defined by $f = \{(1, 7), (2, 7), (3, 8), (4, 6), (5, 9), (6, 9)\}$, determine $f^{-1}(6)$ and $f^{-1}(9)$. Also, if $B_1 = \{7, 8\}$ and $B_2 = \{8, 9, 10\}$ find $f^{-1}(B_1)$ and $f^{-1}(B_2)$.

By definition, we have

$f^{-1}(x) = \{x \in A: f(x) \in B\}$

$\therefore f^{-1}(6) = \{x \in A: f(x) = 6\} = \{4\}$ and

$f^{-1}(9) = \{x \in A: f(x) = 9\} = \{5, 6\}$

By definition, we have

$f^{-1}(B_1) = \{x \in A: f(x) \in B_1\}$

$f^{-1}(B_1) = f^{-1}(\{7, 8\}) = \{1, 2, 3\}$ and

$f^{-1}(B_2) = f^{-1}(\{8, 9, 10\}) = \{3, 5, 6\}$.

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 2x - 3, & \text{for} \quad x \leq 9 \\ x^2 - 4x + 7, & \text{for} \quad 9 < x < 100 \\ \cos \pi x, & \text{for} \quad 100 \leq x \end{cases}$$

Determine $f(7)$, $f(8)$, $f(9)$, $f(10)$ and $f(100)$.

By using the given definition of f, we find that

$f(7) = 2(7) - 3 = 11,$ $\qquad$ $f(8) = 2(8) - 3 = 13,$

$f(9) = 2(9) - 3 = 15,$ $\qquad$ $f(10) = 10^2 - 4(10) + 7 = 67,$

$f(100) = \cos \pi(100) = 1.$

Let f : R → R be a function defined by

$$f(x) = \begin{cases} 3x - 5, & \text{for } x > 0 \\ -3x + 1, & \text{for } x \leq 0 \end{cases}$$

(i) Determine $f(0)$, $f(-1)$, $f(5/3)$, $f(-5/3)$.

(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$ and $f^{-1}(-6)$.

(iii) What are $f^{-1}([-5, 5])$ and $f^{-1}([-6, 5])$?

(i) By using the given definition of f, we find that

$$f(0) = 0 + 1 = 1,$$

$$f(-1) = -3(-1) + 1 = 3 + 1 = 4,$$

$$f(5/3) = 3(5/3) - 5 = 0,$$

$$f(-5/3) = -3(-5/3) + 1 = 6.$$

Let f : R → R be a function defined by

$$f(x)=\begin{cases}3x-5, & \text{for } x>0\\ -3x+1, & \text{for } x\le 0\end{cases}$$

(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$ and $f^{-1}(-6)$.

(ii) By definition, we have

$f^{-1}(B) = \{x \in A: f(x) \in B\}$

$f^{-1}(0) = \{x \in R: f(x) = 0\}$

$= \{x \in R: 3x - 5 = 0 \text{ and } -3x + 1 = 0\}$

$= \{x \in R: x = 5/3 \text{ and } x = 1/3\}$

$x = 5/3$ which is > 0 and $x = 1/3$ which is ≤ 0

$\therefore f^{-1}(0) = 5/3$

Let f : R → R be a function defined by

$$f(x) = \begin{cases} 3x - 5, & \text{for } x > 0 \\ -3x + 1, & \text{for } x \leq 0 \end{cases}$$

(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$ and $f^{-1}(-6)$.

(ii) By definition, we have

$$f^{-1}(B) = \{x \in A: f(x) \in B\}$$

$$f^{-1}(1) = \{x \in R: f(x) = 1\}$$

$$= \{x \in R: 3x - 5 = 1 \text{ and } -3x + 1 = 1\}$$

$$= \{x \in R: x = 2 \text{ and } x = 0\}$$

$x = 2$ which is > 0 and $x = 0$ which is ≤ 0

$$\therefore f^{-1}(1) = \{2, 0\}$$

Let f : R → R be a function defined by

$$f(x) = \begin{cases} 3x - 5, & \text{for } x > 0 \\ -3x + 1, & \text{for } x \le 0 \end{cases}$$

(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$ and $f^{-1}(-6)$.

By definition, we have

$f^{-1}(B) = \{x \in A: f(x) \in B\}$

$f^{-1}(-1) = \{x \in R: f(x) = -1\}$

$= \{x \in R: 3x - 5 = -1 \text{ and } -3x + 1 = -1\}$

$= \{x \in R: x = 4/3 \text{ and } x = 2/3\}$

$x = 4/3$ which is > 0 and $x = 2/3$ which is ≤ 0

$\therefore f^{-1}(-1) = 4/3$

Let f : R → R be a function defined by

$$f(x)=\begin{cases} 3x-5, & \text{for } x>0 \\ -3x+1, & \text{for } x\le 0 \end{cases}$$

(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$ and $f^{-1}(-6)$.

By definition, we have

$f^{-1}(B) = \{x \in A: f(x) \in B\}$

$f^{-1}(3) = \{x \in R: f(x) = 3\}$

$= \{x \in R: 3x - 5 = 3 \text{ and } -3x + 1 = 3\}$

$= \{x \in R: x = 8/3 \text{ and } x = -2/3\}$

$x = 8/3$ which is > 0 and $x = -2/3$ which is ≤ 0

$\therefore f^{-1}(3) = \{8/3, -2/3\}$

Let f: R → R be a function defined by

$$f(x)=\begin{cases}3x-5, & \text{for } x>0\\ -3x+1, & \text{for } x\le 0\end{cases}$$

(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$ and $f^{-1}(-6)$.

By definition, we have

$f^{-1}(B) = \{x \in A: f(x) \in B\}$

$f^{-1}(-3) = \{x \in R: f(x) = -3\}$

$= \{x \in R: 3x - 5 = -3 \text{ and } -3x + 1 = -3\}$

$= \{x \in R: x = 2/3 \text{ and } x = 4/3\}$

$x = 2/3$ which is > 0 and $x = 4/3$ which is ≤ 0

$\therefore f^{-1}(-3) = 2/3$

Let f : R → R be a function defined by

$$f(x) = \begin{cases} 3x - 5, & \text{for } x > 0 \\ -3x + 1, & \text{for } x \leq 0 \end{cases}$$

(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$ and $f^{-1}(-6)$.

By definition, we have

$f^{-1}(\text{B}) = \{\text{x} \in \text{A}: f(\text{x}) \in \text{B}\}$

$f^{-1}(-6) = \{\text{x} \in \text{R}: f(\text{x}) = -6\}$

$= \{\text{x} \in \text{R}: 3x - 5 = -6 \text{ and } -3x + 1 = -6\}$

$= \{\text{x} \in \text{R}: x = -1/3 \text{ and } x = 7/3\}$

$x = -1/3$ which is $\not> 0$ and $x = 7/3$ which is $\not\leq 0$

$\therefore f^{-1}(-6) = \{\}$

Let $f : R \to R$ be a function defined by

$$f(x) = \begin{cases} 3x - 5, & \text{for } x > 0 \\ -3x + 1, & \text{for } x \le 0 \end{cases}$$

(iii) What are $f^{-1}([-5, 5])$ and $f^{-1}([-6, 5])$?

(iii) By definition, we have

$$f^{-1}(B) = \{x \in A : f(x) \in B\}$$

$$f^{-1}([-5, 5]) = \{x \in R : f(x) \in [-5, 5]\}$$

$$= \{x \in R : 3x - 5 \in [-5, 5] \text{ and } -3x + 1 \in [-5, 5]\}$$

$$= \{x \in R : -5 \le 3x - 5 \le 5 \text{ and } -5 \le -3x + 1 \le 5\}$$

$$= \{x \in R : 0 \le 3x \le 10 \text{ and } -6 \le -3x \le 4\}$$

$$= \{x \in R : 0 \le x \le 10/3 \text{ and } 2 \ge x \ge -4/3\}$$

$$= \{x \in R : -4/3 \le x \le 10/3\}$$

$$= [-4/3, 10/3]$$

Let f : R → R be a function defined by

$$f(x)=\begin{cases} 3x-5, & \text{for } x>0 \\ -3x+1, & \text{for } x\le 0 \end{cases}$$

(iii) What are $f^{-1}([-5, 5])$ and $f^{-1}([-6, 5])$?

By definition, we have

$$f^{-1}(B) = \{x \in A : f(x) \in B\}$$

$$f^{-1}([-6, 5]) = \{x \in R : f(x) \in [-6, 5]\}$$

$$= \{x \in R : 3x - 5 \in [-6, 5] \text{ and } -3x + 1 \in [-6, 5]\}$$

$$= \{x \in R : -6 \le 3x - 5 \le 5 \text{ and } -6 \le -3x + 1 \le 5\}$$

$$= \{x \in R : -1 \le 3x \le 10 \text{ and } -7 \le -3x \le 4\}$$

$$= \{x \in R : -1/3 \le x \le 10/3 \text{ and } 7/3 \ge x \ge -4/3\}$$

$$= \{x \in R : -4/3 \le x \le 10/3\}$$

$$= [-4/3, 10/3]$$

If $f : R \to R$ is a function defined by $f(x) = x^2$ then find the image of [–3, 7].

By definition, we have

$f(A) = \{f(x) : x \in A\}$

$f([-3, 7]) = \{x^2 : x \in [-3, 7]\}$

$= \{x^2 : -3 \le x \le 7\}$

$= \{x^2 : -3 \le x \le 0\} \cup \{x^2 : 0 \le x \le 7\}$

$= [0, 49].$

If $f : R \to R$ is a function defined by $f(x) = 2x - 3$ then find the preimage of $[-3, 8]$.

By definition, we have

$$f^{-1}(B) = \{x \in A : f(x) \in B\}$$

$$f^{-1}([-3, 8]) = \{x \in R : f(x) \in [-3, 8]\}$$

$$= \{x \in R : 2x - 3 \in [-3, 8]\}$$

$$= \{x \in R : -3 \leq 2x - 3 \leq 8\}$$

$$= \{x \in R : 0 \leq 2x \leq 11\}$$

$$= \{x \in R : 0 \leq x \leq 11/2\}$$

$$= [0, 11/2]$$

If A = {w, x, y, z} and B = {1, 2, 3} then find the number of functions from A to B and from B to A.

Here, $|A| = m = 4$ and $|B| = n = 3$

Number of functions from A to B $= n^m = 3^4 = 81$

Number of functions from B to A $= m^n = 4^3 = 64$.

Let $f : A \rightarrow B$ be a function and C and D are non-empty subsets of B then prove the following:

(i) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

(ii) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

(iii) $f^{-1}(C') = [f^{-1}(C)]'$

(i) Let $x \in f^{-1}(C \cup D)$

$\Leftrightarrow f(x) \in (C \cup D)$

$\Leftrightarrow f(x) \in C$ or $f(x) \in D$

$\Leftrightarrow x \in f^{-1}(C)$ or $x \in f^{-1}(D)$

$\Leftrightarrow x \in f^{-1}(C) \cup f^{-1}(D)$

$\therefore\ f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

(ii) Let $x \in f^{-1}(C \cap D)$

$\Leftrightarrow f(x) \in (C \cap D)$

$\Leftrightarrow f(x) \in C$ and $f(x) \in D$

$\Leftrightarrow x \in f^{-1}(C)$ and $x \in f^{-1}(D)$

$\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D)$

$\therefore f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

(iii) Let $x \in f^{-1}(C')$

$\Leftrightarrow f(x) \in C'$

$\Leftrightarrow f(x) \notin C$

$\Leftrightarrow x \notin f^{-1}(C)$

$\Leftrightarrow x \in [f^{-1}(C)]'$

$\therefore \; f^{-1}(C') = [f^{-1}(C)]'$