TRANSFROM CALCULUS, FOURIER SERIES AND NUMERICAL TECHNIQUES

III Semester B E Common to all Branches

Subject Code: 18MAT31 Credits: 03

SEE Marks: 60 CIE Marks: 40

Learning Objectives:

- Introduction
- Definition
- Properties of Laplace Transform
- Laplace Transforms of some Elementary Functions
- Periodic Function
- Laplace Transform of Periodic Function
- Unit step Function
- Laplace Transform of Unit step Function

LAPLACE TRANSFORM:

Definition:

Let $f(t)$ be a real valued function defined for $0 \leq t \leq \infty$, suppose that for a real or complex parameters, the integral $\int_0^\infty e^{-st}\ f(t)dt$ is called the Laplace transform of $f(t)$ denoted by L{$f(t)$}. i.e., L{$f(t)$} $= \int_0^\infty e^{-st}\ f(t)dt = F(s)$ or $\bar{f}(s)$.

Properties of Laplace Transform:

1. **Linearity property:** For any two functions $f(t)$ and $g(t)$ and any two constants $C_1\ and\ C_2$,

$$L\{C_1 f(t) + C_2 g(t)\} = C_1 L\{f(t)\} + C_2 L\{g(t)\}$$

2. **Change of scale property:** If $\{f(t)\} = F(s)$, the $\boldsymbol{L\{f(at)\} = \frac{1}{a}F(s)}$ where a is a positive constant.

Laplace Transform of some Standard functions:

1. Laplace Transform of a constant function $\{k\} = \frac{k}{s}$, if s>0

Proof: Let f$(t) = k$, a constant. Then the definition of $L\{f(t)\}$ gives,

$$L\{k\} = \int_0^\infty e^{-st}\ k\,dt = k\int_0^\infty e^{-st}\ dt = k\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{s}, \quad s > 0$$

2. Laplace Transform of a function $(t) = e^{at}$, $\{e^{at}\} = \frac{1}{s-a}, if\ s > a.$

Proof: Let f$(t) = e^{at}$. Then the definition of $L\{f(t)\}$ gives,

$$L\{e^{at}\} = \int_0^\infty e^{-st}\ e^{at}\ dt \int_0^\infty e^{-t(s-a)}\ dt\ \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}, \quad s > 0$$

3. Laplace Transform of a function $(t) = cosh\,at$, $\{cosh\,at\} = \frac{s}{s^2-a^2}$, $if\ s > a$

Proof: Let f(t) $= coshat = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{e^{at}} + \boldsymbol{e^{-at}})$. Then by linearity property of $L\{f(t)\}$,

$$L\{coshat\} = \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\} = \frac{1}{2}\frac{1}{s+a} + \frac{1}{2}\frac{1}{s-a} = \frac{s}{s^2-a^2}\ ;\ s > a$$

4. Laplace Transform of a function $(\boldsymbol{t})$ = $\mathbf{sinh}\, \boldsymbol{at}$, $\{\sinh at\} = \frac{a}{s^2-a^2}$, $if\ s > a$

Proof: Let f(t) $= sinhat = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{e^{at}} - \boldsymbol{e^{-at}})$.

Then by linearity property of $L\{f(t)\}$,

$$L\{sinhat\} = \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\} = \frac{1}{2}\frac{1}{s+a} - \frac{1}{2}\frac{1}{s-a} = \frac{a}{s^2-a^2}\ ;\ s > a$$

5. Laplace Transform of a function $\boldsymbol{f(t)} = \mathbf{cos}\, \boldsymbol{at}$, $L\{\cos at\} = \frac{s}{s^2+a^2}$, $if\ s > 0$

Proof: Let f(t) $= cosat$. Then the definition of $L\{f(t)\}$ gives,

$$L\{cosat\} = \int_0^\infty e^{-st}\ cosat\ dt \left[\frac{e^{-st}}{s^2+a^2}(-s\ cosat + a\ sinat)\right]_0^\infty = \frac{s}{s^2+a^2},\ \text{if } s > 0$$

6. Laplace Transform of a function $\boldsymbol{f(t) = \sin at}$, $L\{\sin at\} = \frac{a}{s^2+a^2}$, $if\ s > 0$

Proof: Let $\mathrm{f}(t) = sinat$.

Then the definition of $L\{f(t)\}$ gives,

$$L\{sinat\} = \int_0^\infty e^{-st}\ sinat\ dt\ \left[\frac{e^{-st}}{s^2+a^2}(-s\ sinat - a\ cosat)\right]_0^\infty = \frac{a}{s^2+a^2},\ \text{if } s > 0$$

Note: $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$

7. Laplace Transform of a function $f(t) = t^n$, $L\{t^n\} = \begin{cases} \frac{\Gamma(n+1)}{s^{n+1}}, & if\ n\ is\ real \\ \frac{n!}{s^{n+1}}, & if\ n = 0, 1, 2, \cdots \end{cases}$

Proof: By definition, $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

Put $st = x$

Differentiating w.r.t 't'

$$s(1) = \frac{dx}{dt} \Rightarrow dt = \frac{dx}{s}$$

$$\therefore L[t^n] = \int_0^\infty e^{-st}[t^n]\,dt$$

$$\therefore L[t^n] = \int_0^\infty e^{-x}\left[\left(\frac{x}{s}\right)^n\right]\cdot\frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^n\,dx = \frac{1}{s^{n+1}}\,\Gamma(n+1)$$

If n is a positive integer, $\Gamma(n+1) = n!$

$$\therefore L[t^n] = \frac{n!}{s^{n+1}}.$$

SL N0.	$f(t)$	$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$
1	a	$a/s, \qquad s > 0$
2	e^{at}	$\frac{1}{s-a}, \qquad s > a$
3	$cosat$	$\frac{s}{s^2+a^2}, \qquad s > 0$
4	$\cosh at$	$\frac{s}{s^2-a^2}, \qquad s > a$
5	$\sin at$	$\frac{a}{s^2+a^2}, \qquad s > 0$
6	$\sinh at$	$\frac{a}{s^2-a^2}, \qquad s > a$
7	t^n	$\frac{\mathrm{n}!}{s^{n+1}}, \qquad s > 0, \qquad n = 0, 1, 2, 3, \cdots$
8	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}, \qquad s > 0, \qquad non - integer$

Problems:

1. Find the Laplace transform of the following :

(1) $\boldsymbol{t^3 - 4e^{3t} + 5e^{-t}}$

Solution: Let $f(t) = 1 + 2t^3 - 4e^{3t} + 5e^{-t}$

$$L\{f(t)\} = L\{1 + 2t^3 - 4e^{3t} + 5e^{-t}\} = L\{1\} + 2L\{t^3\} - 4L\{e^{3t}\} + 5L\{e^{-t}\}$$

$$= \frac{1}{s} + 2.\frac{3!}{s^4} - 4.\frac{1}{s-3} + 5.\frac{1}{s+1}$$

(2) $\boldsymbol{\cos 3t + 2^t}$

Solution: Let $f(t) = \cos 3t + 2^t$

$$L\{f(t)\} = L\{\cos 3t + 2^t\} = L\{\cos 3t\} + L\{2^t\} = L\{\cos 3t\} + L\{e^{t\,log2}\} = \frac{s}{s^2+9} + \frac{1}{s - log2}$$

(3) $\boldsymbol{\cosh 4t + 4 \sin 3t}$

Solution: Let $f(t) = \cosh 4t + 4 \sin 3t$

$$L\{f(t)\} = L\{3 \cosh 4t + 4 \sin 3t\} = 3\, L\{\cosh 4t\} + 4\, L\{\sin 3t\}$$

$$= 3.\frac{s}{s^2+16} + 4.\frac{3}{s^2+9} = \frac{3s}{s^2+16} + \frac{12}{s^2+9}$$

(4) $cosh^2\ at$

Solution: Let $f(t) = cosh^2\ at = \left(\frac{e^{at}+e^{-at}}{2}\right)^2 = \frac{1}{4}(e^{2at} + e^{-2at} + 2) = \frac{1}{2}\ cosh2at + \frac{1}{2}$

$$\therefore\ L\{f(t)\} = \frac{1}{2}L\{\cosh 2at\} + \frac{1}{2}L\{1\} = \frac{1}{2}\ \frac{s}{s^2-(2a)^2} + \frac{1}{2}\left(\frac{1}{s}\right)$$

(5) $sinh^3 2t$

Solution: Let $f(t) = sinh^3\ t = \left(\frac{e^{t}-e^{-t}}{2}\right)^3 = \frac{1}{8}\ (e^{3t} - 3e^{t} + 3e^{-t} + e^{-3t})$

$$\therefore\ L\{f(t)\} = \frac{1}{8}L\left\{e^{3t}\right\} - \frac{3}{8}\ L\{e^{t}\} + \frac{3}{8}L\{e^{-t}\} + \frac{1}{8}\ L\{\ e^{-3t}\}$$

$$= \frac{1}{8}\ \frac{1}{s-3} - -\frac{3}{8}\frac{1}{s-1} + \frac{3}{8}\ \frac{1}{s+1} + \frac{1}{8}\ \frac{1}{s+3}$$

(6) $cos^2\ at + sin^2\ bt$

Solution: Let $f(t) = cos^2\ at + sin^2\ bt = \frac{1+cos2at}{2} + \frac{1-cos2bt}{2} = 1 + \frac{1}{2}\ (cos2at - cos2bt)$

$$\therefore\ \{f(t)\} = L\{1\} + \frac{1}{2}\ (L\{cos2at\} - L\{cos2bt\}) = \frac{1}{s} + \frac{1}{2}\left(\frac{s}{s^2-(2a)^2} - \frac{s}{s^2-(2b)^2}\right)$$

(7) $\cos at \ \cos bt$

Solution: Let $f(t) = \cos at \ \cos bt$

We have $cosat\ cosbt = \frac{1}{2}(\cos(a+b)\,t + \cos(a-b)t)$

$$\therefore\ L\{cosat\ cosbt\} = \frac{1}{2}\ [L\{co\ s(a+b)t\} + L\{co\ s(a-b)t\}] = \frac{1}{2}\left[\frac{s}{s^2+(a+b)^2} + \frac{s}{s^2+(a-b)^2}\right]$$

(8) $\cos^3 3t$

Solution: Let $f(t) = cos^3 3t = \frac{1}{4}[cos9t + 3cos3t]$ $\quad [\because\ cos3\theta = 4\cos^3\theta - 3cos\theta]$

$$\therefore\ L\{f(t)\} = L\{cos^3 3t\} = \frac{1}{4}L\{cos9t\} + \frac{3}{4}L\{cos3t\} = \frac{1}{4}\left(\frac{s}{s^2+81}\right) + \frac{3}{4}\left(\frac{s}{s^2+9}\right)$$

(9) $\sin^3 2t$

Solution: Let $f(t) = sin^3 2t = \frac{1}{4}[3sin2t - sin6t]$ $\quad [\because\ sin3\theta = 3sin\theta - 4\sin^3\theta]$

$$\therefore\ L\{f(t)\} = L\{sin^3 3t\} = \frac{3}{4}L\{sin6t\} - \frac{1}{4}L\{sin3t\} = \frac{3}{4}\left(\frac{6}{s^2+36}\right) - \frac{1}{4}\left(\frac{3}{s^2+9}\right)$$

(10) $cos\,2t\,sin\,3t$

Solution: Let $f(t) = \sin 3t \cos 2t\, cost = \frac{1}{2}[\sin(5t) + \sin(t)]cost = \frac{1}{2} sin5t\ cost + \frac{1}{2} sint\ cost$

$$f(t) = \frac{1}{2}\,\frac{1}{2}\,[\sin(6t) + \sin(4t)] + \frac{1}{2}\,\frac{1}{2} sin2t = \frac{1}{4} sin(6t) + \frac{1}{4} sin(4t) + \frac{1}{4} sin(2t)$$

$$\left[\because\ sinA\ cosB = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}\right]$$

$$\therefore L\{f(t)\} = \frac{1}{4} L\{sin6t\} + \frac{1}{4} L\{sin4t\} + \frac{1}{4} L\{sin2t\} = \frac{1}{4}\left(\frac{6}{s^2+36}\right) + \frac{1}{4}\left(\frac{4}{s^2+16}\right) + \frac{1}{4}\left(\frac{2}{s^2+4}\right)$$

$$= \frac{1}{2}\left(\frac{3}{s^2+36}\right) + \left(\frac{1}{s^2+16}\right) + \frac{1}{2}\left(\frac{1}{s^2+4}\right)$$

(11) If $f(t) = \begin{cases} 2 & 0 < t < 3 \\ t & t > 3 \end{cases}$ then find $L\{f(t)\}$ (VTU 2014)

Solution: By definition, $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^3 e^{-st}(2)dt + \int_3^\infty e^{-st}(t)dt \;=\; 2\int_0^3 e^{-st}dt + \int_3^\infty te^{-st}dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^3 + \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_3^\infty$$

$$= 2\left[\frac{e^{-3s}-1}{-s}\right] + \left[(0-0) - \left(\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2}\right)\right]$$

$$= \frac{2}{s} + \frac{e^{-3s}}{s} + \frac{e^{-3s}}{s^2} = \frac{2}{s} + e^{-3s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$$

12. If $f(t)=\begin{cases}1 & 0<t\le 1\\ t & 1<t\le 2\\ 0 & t>2\end{cases}$ **then find** $L\{f(t)\}$

Solution: By definition, $L[f(t)] = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 e^{-st}(1)\,dt + \int_1^2 e^{-st}(t)\,dt + \int_2^\infty e^{-st}(0)\,dt$

$$= \int_0^1 e^{-st}dt + \int_1^2 t\,e^{-st}dt + 0 = \left(\frac{e^{-st}}{-s}\right)_0^1 + \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_1^2$$

$$= \int_0^1 e^{-st}dt + \int_1^2 t\,e^{-st}dt + 0 = \left(\frac{e^{-st}}{-s}\right)_0^1 + \left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_1^2$$

$$= \left(\frac{e^{-s}-1}{-s}\right) + \left\{\left(\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right\}$$

$$= \frac{1}{s} - \frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2}$$

1.2 .3 Transforms of $e^{at}f(t)$, $t^n f(t)$, $\frac{1}{t} f(t)$:-

Transform of $e^{at} f(t)$: If $L\{f(t)\} = F(s)$, then $L\{e^{at} f(t)\} = F(s-a)$

SL NO	$f(t)$	$L\{f(t)\} = F(s)$
1	$e^{at} cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
2	$e^{at} sin(bt)$	$\frac{a}{(s-a)^2+b^2}$
3	$e^{at} cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
4	$e^{at} sin(bt)$	$\frac{a}{(s-a)^2-b^2}$
5	$e^{at} t^n$	$\begin{cases} \frac{\Gamma(n+1)}{(s-a)^{n+1}}, & if\ n\ is\ real\ s-a>0 \\ \frac{n!}{(s-a)^{n+1}}, & if\ n=0,1,2,\cdots \end{cases}$

Problems:

1. **Find the Laplace transform of the following functions:**

(1) $e^{2t}\, t^3$

Solution: Let $f(t) = e^{2t}\, t^3$

$$\therefore\ L\{f(t)\} = L\{e^{2t}\, t^3\} = \frac{3!}{(s-2)^4}$$

(2) $e^{3t}\, sin^2 t$

Solution: Let $f(t) = e^{3t}\, sin^2 t$

$$L\{\sin^2 t\} = L\left\{\frac{1-cos2t}{2}\right\} = \frac{1}{2}L\{1\} - \frac{1}{2}L\{cos2t\} = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2+4}\right)$$

$$\therefore\ L\{e^{3t}\, sin^2 t\} = \left[\frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2+4}\right)\right]_{s\to s-3} = \frac{1}{2(s-3)} - \frac{s-3}{2[(s-3)^2+4]}$$

(3) e^{-3t} *sin* $5t$ *sin* $3t$

Solution: Let $f(t) = e^{-3t}\sin 5t \sin 3t$

$$L\{\sin 5t \sin 3t\} = L\left\{\frac{1}{2}(cos2t - cos8t)\right\} = \frac{1}{2}L\{cos2t\} - \frac{1}{2}L\{cos8t\} = \frac{1}{2}\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{s}{s^2+64}\right)$$

$$\therefore \quad L\{e^{-3t}\sin 5t \sin 3t\} = \left[\frac{1}{2}\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{s}{s^2+64}\right)\right]_{s\to s+3} = \frac{s+3}{2[(s+3)^2+4]} - \frac{s+3}{2[(s+3)^2+64]}$$

(4) e^{-4t} *cos* $5t$ *cos* $3t$.

Solution: Let $f(t) = e^{-4t}\cos 5t \cos 3t$

$$L\{\cos 5t \cos 3t\} = L\left\{\frac{1}{2}(cos8t + cos2t)\right\} = \frac{1}{2}L\{cos8t\} + \frac{1}{2}L\{cos2t\} = \frac{1}{2}\left(\frac{s}{s^2+64}\right) + \frac{1}{2}\left(\frac{s}{s^2+4}\right)$$

$$\therefore \quad L\{e^{-4t}\cos 5t \cos 3t\} = \left[\frac{1}{2}\left(\frac{s}{s^2+64}\right) + \frac{1}{2}\left(\frac{s}{s^2+4}\right)\right]_{s\to s+4}$$

$$L\{e^{-4t}\cos 5t \cos 3t\} = \frac{s+4}{2[(s+4)^2+64]} - \frac{s+4}{2[(s+4)^2+4]}$$

Laplace Transforms of $t^n f(t)$: If $L\{f(t)\} = F(s)$, then for a positive integer n

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

Find the Laplace transform of the following functions:

(1) $t \cos at$

Solution:

(1) $t \cos at$

Let $f(t) = t \cos at$

$L\{cosat\} = \frac{s}{s^2+a^2}$

$$\therefore \quad L\{t\ cosat\} = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{-1}{(s^2+a^2)^2}[(s^2+a^2)(1) - s(2s)] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$L\{t\ cosat\} = \frac{s^2-a^2}{(s^2+a^2)^2}$

(2) $t(\sin^3 t - \cos^3 t)$

Solution: Let $f(t) = t(\sin^3 t - \cos^3 t)$

$$L\{\sin^3 t - \cos^3 t\} = L\left\{\frac{1}{4}[3sint - sin3t] - \frac{1}{4}[cos3t + 3cost]\right\}$$

$$= \frac{3}{4}L\{sint\} - \frac{1}{4}L\{sin3t\} - \frac{1}{4}L\{cos3t\} - \frac{3}{4}L\{cost\}$$

$$= \frac{3}{4}\left(\frac{1}{s^2+1}\right) - \frac{1}{4}\left(\frac{3}{s^2+9}\right) - \frac{1}{4}\left(\frac{s}{s^2+9}\right) - \frac{3}{4}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{3}{4}\left(\frac{1-s}{s^2+1}\right) - \frac{1}{4}\left(\frac{3+s}{s^2+9}\right)$$

$$\therefore\ L\{t\,(\sin^3 t - \cos^3 t)\} = -\frac{d}{ds}\left(\frac{3}{4}\left(\frac{1-s}{s^2+1}\right) - \frac{1}{4}\left(\frac{3+s}{s^2+9}\right)\right)$$

$$= \frac{-3}{4(s^2+1)^2}[(s^2+1)(-1) - (1-s)(2s)] + \frac{1}{4(s^2+9)^2}[(s^2+9)(1) - (3+s)(2s)]$$

$$L\{t\,(\sin^3 t - \cos^3 t)\} = \frac{[6-6s-s^2]}{4(s^2+9)^2} - \frac{3[s^2-2s-1]}{4(s^2+1)^2}$$

(3) $t\, e^{-2t} \sin 4t$

Solution: Let $f(t) = t\, e^{-2t} \sin 4t$

$$L\{\sin 4t\} = \frac{4}{s^2+16}$$

$$L\{t\,(\sin 4t)\} = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right) = \frac{-4}{(s^2+16)^2}[(s^2+16)(0) - 2s] = \frac{8s}{(s^2+16)^2}$$

$$\therefore\ L\{e^{-2t}\, t\,(\sin 4t)\} = \left[\frac{8s}{(s^2+16)^2}\right]_{s\to s+2} = \frac{8(s+2)}{[(s+2)^2+16}$$

(4) $t^2\, e^{-2t} \cos t$

Solution: Let $f(t) = t^2\, e^{-2t} \cos t \ \Rightarrow\ L\{\cos t\} = \frac{s}{s^2+1}$

$$L\{t^2 \cos t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{d}{ds}\left(\frac{1}{(s^2+1)^2}[(s^2+1)(1) - (s)2s]\right)$$

$$= \frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right) = \frac{1}{(s^2+1)^4}[(s^2+1)^2(-2s) - (1-s^2)2s(s^2+1)] = \frac{-4s}{(s^2+1)^3}$$

$$\therefore\ L\{\, e^{-2t} t^2 \cos t\} = \left[\frac{-4s}{(s^2+1)^3}\right]_{s\to s+2} = \frac{-4(s+2)}{\left((s+2)^2+1\right)^3}$$

Laplace Transform of $\frac{1}{t}\,f(t)$: If $\mathrm{L}\{f(t)\} = F(s)$, then $\mathrm{L}\left\{\frac{f(t)}{t}\right\} = \int_0^\infty F(s)ds$.

3. Find the Laplace transform of the following functions:

(1) $\frac{\sin at}{t}$

Solution: We know that $L\{sinat\} = \frac{a}{s^2+a^2} = F(s)$

$$\therefore L\left\{\frac{sinat}{t}\right\} = \int_s^\infty F(s)ds = \int_s^\infty \frac{a}{s^2+a^2}\,ds = a\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\frac{s}{a} = \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

$$L\left\{\frac{sinat}{t}\right\} = \cot^{1}\frac{s}{a}$$

(2) $\frac{e^{-at}-e^{-bt}}{t}$

Solution: We know that $L\{e^{-at} - e^{-bt}\} = L\{e^{-at}\} - L\{e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b} = F(s)$

$$\therefore L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^{\infty} F(s)ds = \int_s^{\infty}\left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds = \ [log(s+a) - \log(s+b)]_s^{\infty}$$

$$L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \left[log\frac{(s+a)}{(s+b)}\right]_s^{\infty} = \left[log\frac{\left(1+\frac{a}{s}\right)}{\left(1+\frac{b}{s}\right)}\right]_s^{\infty} = 0 - log\frac{(s+a)}{(s+b)} = log\frac{(s+b)}{(s+a)}$$

(3) $\boldsymbol{\frac{\cos at - \cos bt}{t}}$

Solution: We know that $L\{\cos at - \cos bt\} = L\{\cos at\} - L\{\cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = F(s)$

$$\therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^{\infty} F(s)ds = \int_s^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}[log(s^2+a^2) - \log(s^2+b^2)]_s^{\infty}$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \left[log\frac{(s^2+a^2)}{(s^2+b^2)}\right]_s^{\infty} = \left[log\frac{\left(1+\frac{a^2}{s^2}\right)}{\left(1+\frac{b^2}{s^2}\right)}\right]_s^{\infty} = 0 - log\frac{(s^2+a^2)}{(s^2+b^2)} = log\frac{(s^2+b^2)}{(s^2+a^2)}$$

(4) $\frac{1-\cos at}{t}$

Solution: We know that $L\{1-\cos at\} = L\{1\} - L\{\cos at\} = \frac{1}{s} - \frac{s}{s^2+a^2} = F(s)$

$$\therefore L\left\{\frac{1-\cos at}{t}\right\} = \int_s^\infty F(s)ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right) ds$$

$$= \left[log(s) - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$$L\left\{\frac{1-\cos at}{t}\right\} = \left[log\frac{s}{\sqrt{s^2+a^2}}\right]_s^\infty = \left[log\frac{1}{\sqrt{1+\frac{a^2}{s^2}}}\right]_s^\infty = 0 - log\frac{s}{\sqrt{s^2+a^2}}$$

$$= log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$$

Laplace Transform of n^{th} derivative : If $L\{f(t)\} = F(s)$, then

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots f^{(n-1)}(0)$$

Note: (i) $L\{f'(t)\} = sL\{f(t)\} - f(0)$

(ii) $L\{f''(t)\} = s^2 L\{f(t)\} - s\,f(0) - f'(0)$

(iii) $L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - s\,f'(0) - f''(0)$

(iv) $L\{f^{1v}(t)\} = s^4 L\{f(t)\} - s^3 f(0) - s^2 f'(0) - s\,f''(0) - f'''(0)$

Laplace Transform of the integral : If $L\{f(t)\} = F(s)$, then $L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s} L\{f(t)\}$

Find $L\{\int_0^t e^{-t}\, sin2t\, sin3t\, dt\ \}$ **(VTU 2005)**

Solution: Let

$$f(t) = e^{-t}\sin 2t \sin 3t = e^{-t}\left\{-\frac{1}{2}[\cos 5t - \cos t]\right\}$$

$$\therefore \quad L[f(t)] = -\frac{1}{2}L\{e^{-t}\cos 5t - e^{-t}\cos t\}$$

$$\overline{f}(s) = -\frac{1}{2}\left\{\frac{s+1}{(s+1)^2+25} - \frac{s+1}{(s+1)^2+1}\right\} = -\frac{1}{2}\left\{\frac{s+1}{s^2+2s+26} - \frac{s+1}{s^2+2s+2}\right\}$$

$$= -\frac{1}{2}\left\{\frac{(s+1)(s^2+2s+2)-(s+1)(s^2+2s+26)}{(s^2+2s+26)(s^2+2s+2)}\right\}$$

$$= -\frac{1}{2}\left\{\frac{-24s-24}{(s^2+2s+26)(s^2+2s+2)}\right\} = \frac{12(s-1)}{(s^2+2s+26)(s^2+2s+2)}$$

$$= -\frac{1}{2}\left\{\frac{-24s-24}{(s^2+2s+26)(s^2+2s+2)}\right\} = \frac{12(s-1)}{(s^2+2s+26)(s^2+2s+2)}$$

We have, $L\left[\int_0^t f(t)\,dt\right] = \frac{\overline{f}(s)}{s}$

$$\therefore \qquad L\left[\int_0^t e^{-t}\sin 2t \sin 3t\,dt\right] = \frac{12(s-1)}{s(s^2+2s+26)(s^2+2s+2)}$$

Find the value of $\int_0^\infty t\,e^{-3t}\,sint\,dt$ using Laplace transform. **(VTU 2007)**

Solution: By definition of Laplace transform, we have $\quad L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$f(t) = \sin t$$

$$L[f(t)] = L[\sin t] = \frac{1}{s^2+1}$$

$$L[t\sin t] = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = \frac{2s}{(s^2+1)^2}$$

$$\therefore \quad \int_0^\infty e^{-st}\, t\sin t\,dt = L[t\sin t]$$

$$\int_0^\infty e^{-st}\, t\sin t\,dt = \frac{2s}{(s^2+1)^2}$$

Put $s = 3$

$$\int_0^\infty e^{-3t}\, t\sin t\, dt = \frac{2(3)}{\left(3^2+1\right)^2} = \frac{3}{50}$$

Find $L\left\{\frac{e^{-t}\, sint}{t}\right\}$ and hence find $\int_0^\infty \frac{e^{-t}\, sint}{t}\, dt$ (VTU 2009, 2013)

Solution: $$L\left[e^{-t}\sin t\right] = \frac{1}{(s+1)^2+1^2} = \bar{f}(s)$$

We have, $$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds$$

$$\therefore\ L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty \frac{1}{(s+1)^2+1^2}\, ds = \left[\tan^{-1}(s+1)\right]_s^\infty = \tan^{-1}(\infty) - \tan^{-1}(s+1) = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

$$\therefore\ \int_0^\infty e^{-st}\left(\frac{\sin t}{t}\right) dt = L\left(\frac{\sin t}{t}\right)$$

$$f(t) = \sin t$$

$$L[f(t)] = L[\sin t] = \frac{1}{s^2 + 1}$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1} ds = \left[\tan^{-1} s\right]_s^\infty = \tan^{-1}(\infty) - \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{-st} \left(\frac{\sin t}{t}\right) dt = \cot^{-1} s$$

Put $s = 1$

$$\int_0^\infty \frac{e^{-t} \sin t}{t} dt = \cot^{-1} 1 = \frac{\pi}{4}$$

Laplace Transform of a Periodic function:

A function $f(t)$ is said to be a periodic function of period $T > 0$,

if $f(t) = f(t+T) = f(t+2T) = f(t+3T) = \cdots$

i.,e $f(t) = f(t+nT),\ for\ \ n = 1, 2, 3, \cdots$

Example: $sint,\ \cos t$ are periodic functions of period 2π.

Transform of a periodic function:

Let $f(t)$ is a periodic function of period T, then $\mathrm{L}\{\boldsymbol{f(t)}\} = \frac{\mathbf{1}}{\mathbf{1}-\boldsymbol{e^{-sT}}} \int_{\mathbf{0}}^{\boldsymbol{T}} \boldsymbol{e^{-st} f(t) dt}$

If a periodic function of period $\frac{2\pi}{\omega}$ is defined by $f(t)=\begin{cases} E\sin\omega t, & \text{if } 0<t<\pi/\omega \\ 0, & \text{if } \pi/\omega<t<2\pi/\omega \end{cases}$

where E and ω are constants, then show that $L[f(t)]=\dfrac{E\omega}{\left(s^2+\omega^2\right)\left(1-e^{-\pi s/\omega}\right)}$

Solution: **Given** $T = \dfrac{2\pi}{\omega}$

We have $L[f(t)] = \dfrac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt = \dfrac{1}{1-e^{-s(2\pi/\omega)}}\int_0^{2\pi/\omega} e^{-st} f(t)\, dt$

$$= \frac{1}{1-e^{-s(2\pi/\omega)}}\left[\int_0^{\pi/\omega} e^{-st}\left(E\sin\omega t\right)dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\,(0)\, dt\right]$$

$$= \frac{1}{1-e^{-s(2\pi/\omega)}}\left[E\int_0^{\pi/\omega} e^{-st}\sin\omega t\, dt + 0\right]$$

$$= \frac{E}{1-\left(e^{-\pi s/\omega}\right)^2} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt = \frac{E}{1-\left(e^{-\pi s/\omega}\right)^2} \left[\frac{e^{-st}}{(-s)^2+\omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{E}{1-\left(e^{-\pi s/\omega}\right)^2} \cdot \frac{1}{s^2+\omega^2} \left[e^{-s\frac{\pi}{\omega}} \left(0-\omega(-1)\right) - 1\left(0-\omega(1)\right) \right]$$

$$= \frac{E}{1^2-\left(e^{-\pi s/\omega}\right)^2} \cdot \frac{1}{s^2+\omega^2} \left[e^{-s\frac{\pi}{\omega}} \omega + \omega \right]$$

$$= \frac{E\omega\left(e^{-\pi s/\omega}+1\right)}{\left(1-e^{-\pi s/\omega}\right)\left(1+e^{-\pi s/\omega}\right)\left(s^2+\omega^2\right)}$$

$$= \frac{E\omega}{\left(s^2+\omega^2\right)\left(1-e^{-\pi s/\omega}\right)}$$

If a periodic function of period $2a$ is defined by $f(t)=\begin{cases} t, & \text{if } 0 \le t \le a \\ 2a-t, & \text{if } a \le t \le 2a \end{cases}$ **then show that** $L[f(t)]=\dfrac{1}{s^2}\tanh\left(\dfrac{as}{2}\right)$. **(VTU 2003, 2008, 2011)**

Solution: Given $T = 2a$

$$\text{We have, } L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt = \frac{1}{1-e^{-s(2a)}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2sa}}\left[\int_0^a te^{-st}\,dt + \int_a^{2a}(2a-t)e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2sa}}\left[\left(t\frac{e^{-st}}{-s}-(1)\frac{e^{-st}}{s^2}\right)_0^a + \left((2a-t)\frac{e^{-st}}{-s}-(-1)\frac{e^{-st}}{s^2}\right)_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2sa}}\left[\left(\frac{ae^{-as}}{-s}-\frac{e^{-as}}{s^2}\right)-\left(0-\frac{1}{s^2}\right)+\left(0+\frac{e^{-2as}}{s^2}\right)-\left(\frac{ae^{-as}}{-s}+\frac{e^{-as}}{s^2}\right)\right]$$

$$= \frac{1}{1-e^{-2sa}}\left[\frac{e^{-2as}}{s^2}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1^2-\left(e^{-sa}\right)^2}\left[\frac{1^2+\left(e^{-as}\right)^2-2e^{-as}}{s^2}\right]$$

$$= \frac{1}{\left(1+e^{-sa}\right)\left(1-e^{-sa}\right)}\left[\frac{\left(1-e^{-as}\right)^2}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{\left(1-e^{-sa}\right)}{\left(1+e^{-sa}\right)}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right) \qquad \left[\because\ \tanh\left(\frac{\theta}{2}\right)=\frac{1-e^{-\theta}}{1+e^{-\theta}}\right]$$

If a periodic function of period a is defined by $f(t)=\begin{cases} E, & \text{if} \quad 0<t<a/2 \\ -E, & \text{if} \quad a/2<t<a \end{cases}$ then show that $L[f(t)]=\dfrac{E}{s}\tanh\left(\dfrac{as}{4}\right)$. (VTU 2006, 2011)

Solution: Given T = a We have, $L[f(t)] = \dfrac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$

$$= \frac{1}{1-e^{-s(a)}}\int_0^a e^{-st} f(t)\, dt \qquad = \frac{1}{1-e^{-sa}}\left[\int_0^{a/2} e^{-st}(E)dt + \int_{a/2}^{a} e^{-st}(-E)\, dt\right]$$

$$= \frac{E}{1-e^{-sa}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right] = \frac{E}{1-e^{-sa}}\left[\left(\frac{e^{-s(a/2)}-e^{0}}{-s}\right)-\left(\frac{e^{-as}-e^{-s(a/2)}}{-s}\right)\right]$$

$$= \frac{E}{1-e^{-sa}}\left[\frac{e^{-s(a/2)}-1-e^{-as}+e^{-s(a/2)}}{-s}\right] = \frac{E}{1^2-\left(e^{-sa/2}\right)^2}\left[\frac{1^2+\left(e^{-as/2}\right)^2-2e^{-s(a/2)}}{s}\right]$$

$$= \frac{E}{\left(1+e^{-sa/2}\right)\left(1-e^{-sa/2}\right)}\left[\frac{\left(1-e^{-as/2}\right)^2}{s}\right] \quad = \frac{E}{s}\frac{\left(1-e^{-sa/2}\right)}{\left(1+e^{-sa/2}\right)}$$

$$= \frac{E}{s}\tanh\left(\frac{as}{4}\right) \qquad \left[\because \tanh\left(\frac{\theta}{2}\right) = \frac{1-e^{-\theta}}{1+e^{-\theta}}\right]$$

Find the Laplace transform of the periodic function defined by $f(t) = \frac{kt}{T},\ 0 < t < T$ **given that** $f(t+T) = f(t)$. **(VTU 2007)**

Solution: Given that T = T, We have $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\left(\frac{kt}{T}\right) dt = \frac{k}{T\left(1-e^{-sT}\right)}\int_0^T t\, e^{-st}\ dt$$

$$= \frac{k}{T\left(1-e^{-sT}\right)}\left[t\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^T$$

$$= \frac{k}{T\left(1-e^{-sT}\right)}\left[\left(\frac{Te^{-sT}}{-s}-\frac{e^{-sT}}{s^2}\right)-\left(0-\frac{e^0}{s^2}\right)\right]$$

$$= \frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{-Te^{-sT}}{s}-\frac{e^{-sT}}{s^2}+\frac{1}{s^2}\right]$$

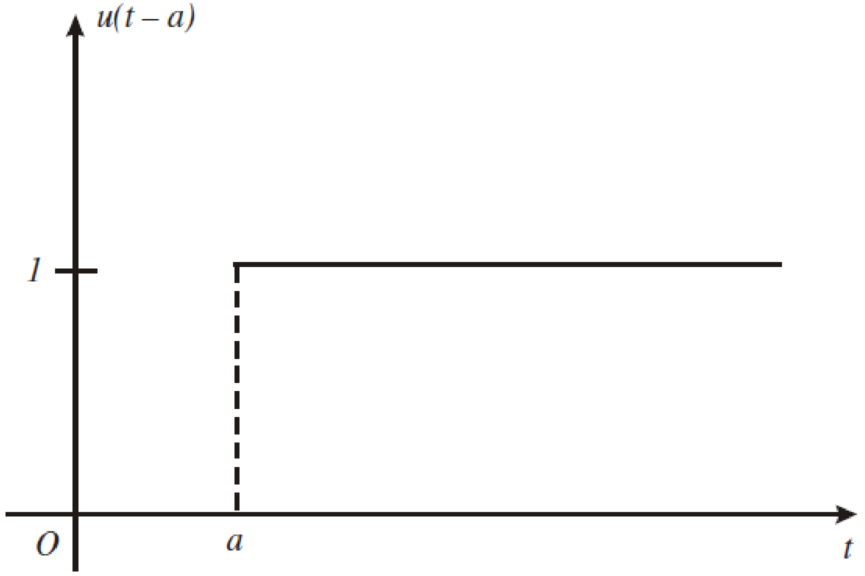
$$= \frac{k}{T\left(1-e^{-sT}\right)}\left[\frac{-Te^{-sT}}{s}+\frac{1-e^{-sT}}{s^2}\right]$$

$$= \frac{-ke^{-sT}}{s\left(1-e^{-sT}\right)}+\frac{k}{s^2T}$$

Laplace Transform of the step function: A discontinuous function $H(t-a)$ defined as $\mathrm{H}(t-a) = \begin{cases} 0, & for\ t \leq a \\ 1 & for\ t > a \end{cases}$ where $'a'$ is a non-negative constant.

This function is known as the unit step function or Heaviside function.

Graph:



In particular, $a = 0$ the function $H(t-a)$ becomes $f(t) = \begin{cases} 0, & for\ t \leq 0 \\ 1 & for\ t > 0 \end{cases}$

Transform of the step function $\boldsymbol{L\{u(t-a)\} = \frac{1}{s}e^{-as}}$ **and** $\mathbf{L}\{\boldsymbol{u(t)}\} = \frac{\mathbf{1}}{\boldsymbol{s}}$

Example:

(1) $L[u(t-1)] = \dfrac{e^{-s}}{s}$ (2) $L[u(t-2)] = \dfrac{e^{-2s}}{s}$ (3) $L[u(t-\pi)] = \dfrac{e^{-\pi s}}{s}$

Heaviside shift theorem: $\mathbf{L}\{\boldsymbol{f(t-a)u(t-a)}\} = \boldsymbol{e^{-as}L\{f(t)\}}$

__Remark:__ (i) $a = 0,\ L\{f(t)\,u(t)\} = L\{f(t)\}$

(ii) for $f(t) = 1$, $\mathrm{L}\{u(t-a)\} = e^{-as}\frac{1}{s}$ *and* $L\{u(t)\} = \frac{1}{s}$

Find the Laplace transforms of the following functions

$(i)\ (t-1)^2 u(t-1) \quad (ii) \sin t\, u(t-\pi) \quad (iii) e^{-3t} u(t-2)$

Solution: (i) Let $f(t-a)u(t-a) = (t-1)^2 u(t-1)$ Here, $a = 1$ and $f(t-a) = (t-1)^2$

$$f(t-1) = (t-1)^2$$

$$f(t) = (t+1-1)^2 = t^2$$

$$L[f(t)] = L[t^2]$$

$$\bar{f}(s) = \frac{2}{s^3}$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$$\therefore \quad L\{(t-1)^2 u(t-1)\} = e^{-s}\left(\frac{2}{s^3}\right) = \frac{2e^{-s}}{s^3}$$

(*ii*) Let $f(t-a)u(t-a) = \sin t\ u(t-\pi)$

Here, $a = \pi$ and $f(t-a) = \sin t$

$$f(t-\pi) = \sin t$$

$$f(t) = \sin(t+\pi) = -\sin t$$

$$L[f(t)] = L[-\sin t] = -\frac{1}{s^2+1}$$

We have, $$L\left[f(t-a)u(t-a)\right] = e^{-as}\,\overline{f}(s)$$

$$\therefore \quad L\{\sin t\; u(t-\pi)\} = e^{-\pi s}\left(-\frac{1}{s^2+1}\right) = -\frac{e^{-\pi s}}{s^2+1}$$

(*iii*) Let $f(t-a)u(t-a) = e^{-3t}u(t-2)$

Here, $a = 2$ and $f(t-a) = e^{-3t}$

$$f(t-2) = e^{-3t}$$

$$f(t) = e^{-3(t+2)} = e^{-6}e^{-3t}$$

$$L[f(t)] = L\left[e^{-6}e^{-3t}\right] = e^{-6}L\left[e^{-3t}\right]$$

$$= e^{-6}\left(\frac{1}{s+3}\right)$$

We have, $$L\left[f(t-a)u(t-a)\right] = e^{-as}\,\overline{f}(s)$$

$$\therefore \quad L\{e^{-3t}u(t-2)\} = e^{-2s}\left(e^{-6}\left(\frac{1}{s+3}\right)\right) = \frac{e^{-2s-6}}{s+3} = \frac{e^{-2(s+3)}}{s+3}$$

Results:

1. If a function $f(t)$ is defined by $(t) = \begin{cases} f_1(t), & for\ t \leq a \\ f_2(t) & for\ t > a \end{cases}$ Verify that

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}H(t-a).$$

2. If a function $f(t)$ is defined by $(t) = \begin{cases} f_1(t) & for\ \ 0 < t \leq a \\ f_2(t) & for\ \ a < t \leq b \\ f_3(t) & \quad for\ \ \ t > b \end{cases}$ Verify that

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}H(t-a) + \{f_3(t) - f_2(t)\}H(t-b).$$

Express the function $f(t)=\begin{cases}2t & \text{if } 0<t<\pi \\ 1 & \text{if } t>\pi\end{cases}$

in terms of unit step function and hence find its Laplace transform. **(VTU 2013)**

Solution: Let $f_1(t) = 2t, \quad f_2(t)=1$

We have, $$f(t) = f_1(t)+\left[f_2(t)-f_1(t)\right]u(t-a)$$

$$f(t) = 2t+[1-2t]u(t-\pi)$$

Its Laplace transform is

$$L\left[f(t)\right] = 2L(t)+L\left[(1-2t)u(t-\pi)\right] \quad \text{---- (1)}$$

Consider, $L\left[(1-2t)u(t-\pi)\right]$

It is in the form $L\left[f(t-a)u(t-a)\right]$

Here, $a = \pi$ and $f(t-a)=(1-2t)$

$\Rightarrow \quad f(t-\pi) = 1-2t \qquad f(t) = 1-2(t+\pi)=1-2t-2\pi$

$$L[f(t)] = L[1-2t-2\pi] = \frac{1}{s} - \frac{2}{s^2} - \frac{2\pi}{s} \qquad L(t) = \frac{1}{s^2} \quad \text{----} (2)$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$$\therefore \qquad L[(1-2t)u(t-\pi)] = e^{-\pi s}\left(\frac{1}{s} - \frac{2}{s^2} - \frac{2\pi}{s}\right) \quad \text{----} (3)$$

Substituting (2) and (3) in (1), we get

$$L[f(t)] = \frac{2}{s^2} + e^{-\pi s}\left(\frac{1}{s} - \frac{2}{s^2} - \frac{2\pi}{s}\right)$$

Express the function $f(t) = \begin{cases} \pi - t & \text{if } 0 < t \le \pi \\ \sin t & \text{if } t > \pi \end{cases}$ in terms of unit step function and hence find its Laplace transform. (VTU 2006)

Solution: Let $f_1(t) = \pi - t, \quad f_2(t) = \sin t$

We have, $$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$$

$$f(t) = (\pi - t) + [\sin t - (\pi - t)]u(t - \pi)$$

Its Laplace transform is

$$L[f(t)] = L(\pi - t) + L[(\sin t - \pi + t)u(t - \pi)] \quad \text{---- (1)}$$

$$L(\pi - t) = \frac{\pi}{s} - \frac{1}{s^2} \quad \text{---- (2)}$$

Consider, $L[(\sin t - \pi + t)u(t - \pi)]$

Here, $a = \pi$ and $f(t-a) = (\sin t - \pi + t) \Longrightarrow f(t - \pi) = \sin t - \pi + t$

$$f(t) = \sin(t + \pi) - \pi + (t + \pi) = -\sin t + t$$

$$L[f(t)] = L[-\sin t + t] = \left(-\frac{1}{s^2 + 1} + \frac{1}{s^2}\right)$$

We have, $\quad L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$$\therefore \qquad L[(4t-t^2)u(t-\pi)] = e^{-\pi s}\left(-\frac{1}{s^2+1}+\frac{1}{s^2}\right) \quad \text{---- (3)}$$

Substituting (2) and (3) in (1), we get

$$L[f(t)] = \left(\frac{\pi}{s}-\frac{1}{s^2}\right)+e^{-\pi s}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)$$

Express the function $f(t)=\begin{cases}\cos t & if \quad 0<t<\pi\\ \cos 2t & if \quad \pi<t<2\pi\\ \cos 3t & if \quad t>2\pi\end{cases}$ **in terms of unit step function and hence find its Laplace transform.** **(VTU 2003)**

Solution: Let $\quad f_1(t) = \cos t,\ \ f_2(t)=\cos 2t,\ \ f_3(t)=\cos 3t$

We have, $$f(t) = f_1(t)+[f_2(t)-f_1(t)]u(t-a)+[f_3(t)-f_2(t)]u(t-b)$$

$$f(t) = \cos t + [\cos 2t - \cos t]u(t-\pi) + [\cos 3t - \cos 2t]u(t-2\pi)$$

Its Laplace transform is

$$L[f(t)] = L[\cos t] + L\{[\cos 2t - \cos t]u(t-\pi)\} + L\{[\cos 3t - \cos 2t]u(t-2\pi)\} \quad \text{---- (1)}$$

$$L[\cos t] = \frac{s}{s^2+1} \quad \text{---- (2)}$$

Consider, $L\{[\cos 2t - \cos t]u(t-\pi)\}$

It is in the form $L[f(t-a)u(t-a)]$ Here, $a = \pi$ and $f(t-a) = [\cos 2t - \cos t]$

$$f(t-\pi) = [\cos 2t - \cos t]$$

$$f(t) = \cos 2(t+\pi) - \cos(t+\pi) = \cos(2t+2\pi) - \cos(t+\pi) = \cos 2t + \cos t$$

$$L[f(t)] = L[\cos 2t + \cos t] = \frac{s}{s^2+4} + \frac{s}{s^2+1}$$

We have, $L\left[f(t-a)u(t-a)\right] = e^{-as}\,\overline{f}(s)$

$$\therefore \quad L\{[\cos 2t - \cos t]u(t-\pi)\} = e^{-\pi s}\left(\frac{s}{s^2+4}+\frac{s}{s^2+1}\right) \qquad \text{---- (3)}$$

Consider, $L\{[\cos 3t - \cos 2t]u(t-2\pi)\}$

It is in the form $L\left[f(t-a)u(t-a)\right]$ Here, $a = 2\pi$ and $f(t-a) = [\cos 3t - \cos 2t]$

$$f(t-2\pi) = [\cos 3t - \cos 2t]$$

$$f(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi) = \cos(3t+3\times 2\pi) - \cos(2t+2\times 2\pi) = \cos 3t - \cos 2t$$

$$L\left[f(t)\right] = L[\cos 3t - \cos 2t] = \frac{s}{s^2+9} - \frac{s}{s^2+4}$$

We have, $L\left[f(t-a)u(t-a)\right] = e^{-as}\,\overline{f}(s)$

$$\therefore L\{[\cos 3t - \cos 2t]u(t-2\pi)\} = e^{-2\pi s}\left(\frac{s}{s^2+9}-\frac{s}{s^2+4}\right) \qquad \text{---- (4)}$$

Substituting (2), (3) and (4) in (1), we get

$$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Express the function $f(t) = \begin{cases} \sin t & \text{if} \quad 0 < t < \pi \\ \sin 2t & \text{if} \quad \pi < t < 2\pi \\ \sin 3t & \text{if} \quad t > 2\pi \end{cases}$ **in terms of unit step function and hence find its Laplace transform.** **(VTU 2004)**

Solution: Let $f_1(t) = \sin t$, $f_2(t) = \sin 2t$, $f_3(t) = \sin 3t$

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$$

$$f(t) = \sin t + [\sin 2t - \sin t]u(t-\pi) + [\sin 3t - \sin 2t]u(t-2\pi)$$

Its Laplace transform is

$$L[f(t)] = L[\sin t] + L\{[\sin 2t - \sin t]u(t-\pi)\} + L\{[\sin 3t - \sin 2t]u(t-2\pi)\} \quad \text{----} \ (1)$$

$$L[\sin t] = \frac{1}{s^2+1} \quad \text{----} \ (2)$$

Consider, $L\{[\sin 2t - \sin t]u(t-\pi)\}$ It is in the form $L[f(t-a)u(t-a)]$

Here, $a = \pi$ and $f(t-a) = [\sin 2t - \sin t] \Rightarrow f(t-\pi) = [\sin 2t - \sin t]$

$$f(t) = \sin 2(t+\pi) - \sin(t+\pi) = \sin(2t+2\pi) - \sin(t+\pi) = \sin 2t + \sin t$$

$$L[f(t)] = L[\sin 2t + \sin t] = \frac{2}{s^2+4} + \frac{1}{s^2+1}$$

We have, $L[f(t-a)u(t-a)] = e^{-as}\,\overline{f}(s)$

$$\therefore \quad L\{[\sin 2t - \sin t]u(t-\pi)\} = e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right)$$