***Example 1.4* :**

Find the Laplace transform of $\cos^2 2t$.

Solution:

$$L\left(\cos^2 2t\right)=L\left(\frac{1+\cos 4t}{2}\right)=\frac{1}{2}L\left(1+\cos 4t\right)=\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+16}\right]$$

***Example 1.5* :**

Find the Laplace transform of $\sin^3 2t$. **(VTU 2014)**

Solution:

$$L\left(\sin^3 2t\right) = L\left(\frac{3\sin 2t-\sin 6t}{4}\right) \qquad \left[\because \sin^3\theta=\frac{3\sin\theta-\sin 3\theta}{4}\right]$$

$$= \frac{1}{4}\left\{3L\left[\sin 2t\right]-L\left[\sin 6t\right]\right\}$$

$$= \frac{1}{4}\left\{3.\frac{2}{s^2+2^2}-\frac{6}{s^2+6^2}\right\}=\frac{3}{2}\left\{\frac{1}{s^2+4}-\frac{1}{s^2+36}\right\}$$

Example 1.7 :

Evaluate $L[\sin t \sin 2t \sin 3t]$. **(VTU 2013)**

Solution:

$$
\begin{aligned}
L[\sin t \sin 2t \sin 3t] &= L\left[\sin t\left\{-\frac{1}{2}(\cos 5t - \cos t)\right\}\right] \\
&= -\frac{1}{2}L[\cos 5t \sin t - \sin t \cos t] \\
&= -\frac{1}{2}L\left[\frac{1}{2}\{\sin 6t - \sin 4t\} - \frac{1}{2}\{\sin 2t\}\right] \\
&= -\frac{1}{4}L[\sin 6t - \sin 4t - \sin 2t] \\
&= -\frac{1}{4}\left[\frac{6}{s^2+36} - \frac{4}{s^2+16} - \frac{2}{s^2+4}\right]
\end{aligned}
$$

Example 1.8 :

Evaluate $L[\cos t \cos 2t \cos 3t]$.

Solution:

$$
\begin{aligned}
L[\cos t \cos 2t \cos 3t] &= L\left[\cos t\left\{\frac{1}{2}(\cos 5t + \cos t)\right\}\right] \\
&= \frac{1}{2}L\left[\cos 5t \cos t + \cos^2 t\right] \\
&= \frac{1}{2}L\left[\frac{1}{2}\{\cos 6t + \cos 4t\} + \frac{1+\cos 2t}{2}\right] \\
&= \frac{1}{4}L[\cos 6t + \cos 4t + 1 + \cos 2t] \\
&= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{1}{s} + \frac{s}{s^2+4}\right]
\end{aligned}
$$

Find $L\left[\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^{3}\right]$

Solution:

$$
\begin{aligned}
L\left[\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^{3}\right] &= L\left[t^{3/2}-t^{-3/2}-3t^{1/2}+3t^{-1/2}\right] \\
&= L\left[t^{3/2}\right]-L\left[t^{-3/2}\right]-3L\left[t^{1/2}\right]+3L\left[t^{-1/2}\right] \\
&= \frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}}-\frac{\Gamma\left(-\frac{3}{2}+1\right)}{s^{-\frac{3}{2}+1}}-3\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}+3\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} \\
&= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}}-\frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-\frac{1}{2}}}-3\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}+3\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}=\frac{\frac{3\sqrt{\pi}}{4}}{s^{\frac{5}{2}}}-\frac{-2\sqrt{\pi}}{s^{-\frac{1}{2}}}-3\frac{\frac{\sqrt{\pi}}{2}}{s^{\frac{3}{2}}}+3\frac{\sqrt{\pi}}{s^{\frac{1}{2}}} \\
&= \sqrt{\pi}\left[\frac{3}{4s^{2}\sqrt{s}}+2\sqrt{s}-\frac{3}{2s\sqrt{s}}+\frac{3}{\sqrt{s}}\right]
\end{aligned}
$$

Properties of Laplace transforms

① Shifting property

(i) $\boxed{\text{If } L[f(t)] = F(s)}$ then

$$L[e^{at} f(t)] = F(s-a),$$

$$L[e^{-at} f(t)] = F(s+a)$$

(ii) $$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

(iii) $$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)\,ds$$

Examples

① Find $L[e^{t}\cos^2 2t]$ & $L[e^{-t}\cos^2 2t]$

Solu: We know $L[\cos^2 2t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right] = F(s)$

$\therefore$ $$L[e^{t}\cos^2 2t] = F(s-1)$$

$$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{(s-1)}{(s-1)^2+16}\right]$$

Similarly $L[e^{-t}\cos^2 2t] = F(s+1)$