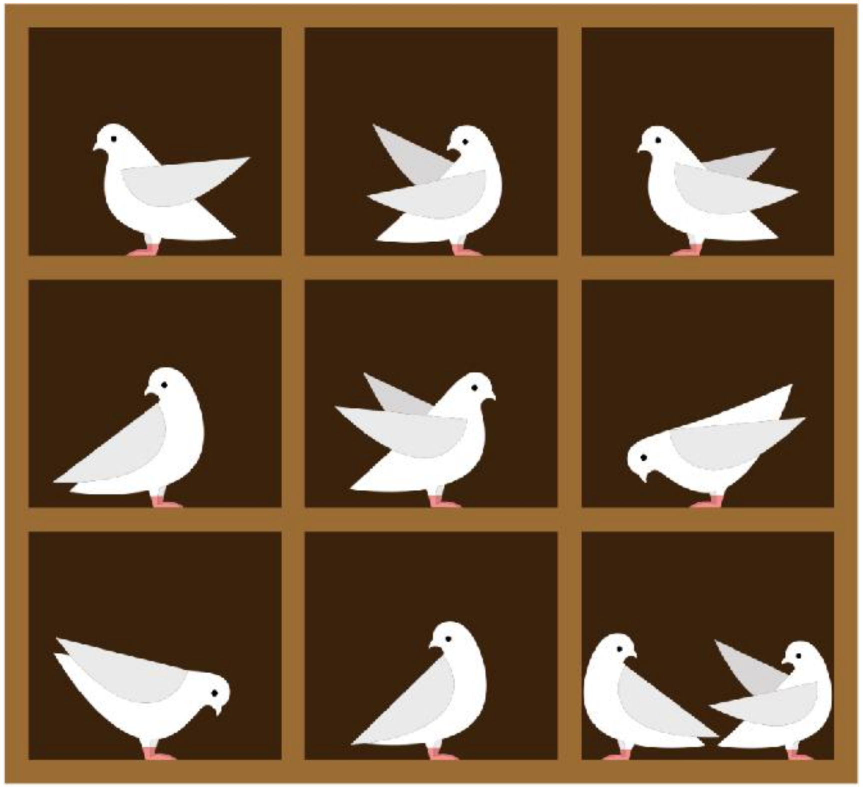
Module-3

RELATIONS AND FUNCTIONS



PART - 10

The Pigeonhole Principle

Pigeonhole - set of small open-fronted compartments in a workplace or other organization where letters or messages may be left for individuals.

THE PIGEONHOLE PRINCIPLE

If *m* pigeons occupy *n* pigeonholes and if $m > n$ then at least one pigeonhole must contain two or more pigeons in it.

The Generalized Pigeonhole Principle

If m pigeons occupy n pigeonholes and if $m > n$ then at least one pigeonhole must contain

$$\left\lfloor \left(\frac{m-1}{n} \right) \right\rfloor + 1$$ **or more pigeons.**

Floor and Ceiling Functions

Let x be any real number. Then

1. $\lfloor x \rfloor$ denote the greatest integer that is less than or equal to x and $\lfloor x \rfloor$ is called floor of x.
2. $\lceil x \rceil$ denote the least integer that is greater than or equal to x and $\lceil x \rceil$ is called ceiling of x.

$$\lfloor 3 \rfloor = 3 \qquad \lceil 3 \rceil = 3$$

$$\lfloor 4.2 \rfloor = 4 \qquad \lceil 4.2 \rceil = 5$$

$$\lfloor 5.9 \rfloor = 5 \qquad \lceil 5.9 \rceil = 6$$

$$\lfloor -6 \rfloor = -6 \qquad \lceil -6 \rceil = -6$$

$$\lfloor -7.3 \rfloor = -8 \qquad \lceil -7.3 \rceil = -7$$

$$\lfloor \sqrt{2} \rfloor = 1 \qquad \lceil \sqrt{2} \rceil = 2$$

$$\lfloor 0 \rfloor = 0 \qquad \lceil 0 \rceil = 0$$

EXAMPLES

Show that if 9 colors are used to paint 100 houses then at least 12 houses will be of the same color.

Let colors = Pigeonholes = n = 9

Houses = Pigeons = m = 100

By generalized Pigeonhole principle, we have

$$\left\lfloor \left(\frac{m-1}{n} \right) \right\rfloor + 1 \text{ or more}$$

$$\left\lfloor \left(\frac{100-1}{9} \right) \right\rfloor + 1 = 11 + 1 = 12$$

Thus, there are at least 12 houses of the same color.

Prove that if 30 dictionaries in a library contain a total of 61,327 pages, then at least one of the dictionaries must have at least 2045 pages.

Let dictionaries = Pigeonholes = n = 30

Pages = Pigeons = m = 61327

By generalized Pigeonhole principle, we have

$$\left\lfloor \left(\frac{m-1}{n}\right) \right\rfloor + 1 \text{ or more}$$

$$\left\lfloor \left(\frac{61327-1}{30}\right) \right\rfloor + 1 = \lfloor 2044.2 \rfloor + 1 = 2044 + 1 = 2045$$

Thus, there are at least 2045 pages in one of the dictionaries.

Prove that in any set of 29 persons at least five persons must have been born on the same day of the week.

Let days = Pigeonholes = n = 7

Persons = Pigeons = m = 29

By generalized Pigeonhole principle, we have

$$\left\lfloor \left(\frac{m-1}{n} \right) \right\rfloor + 1 \text{ or more}$$

$$\left\lfloor \left(\frac{29-1}{7} \right) \right\rfloor + 1 = 4 + 1 = 5$$

Thus, there are at least 5 persons must have born on the same day of the week.

Prove that in any set of 45 persons at least 4 persons must have been born on the same calendar month.

Let months = Pigeonholes = n = 12

Persons = Pigeons = m = 45

By generalized Pigeonhole principle, we have

$$\left\lfloor \left(\frac{m-1}{n} \right) \right\rfloor + 1 \text{ or more}$$

$$\left\lfloor \left(\frac{45-1}{12} \right) \right\rfloor + 1 = \lfloor 3.67 \rfloor + 1 = 3 + 1 = 4$$

Thus, there are at least 4 persons must have born on the same calendar month.

How many persons must be chosen in order that at least five of them will have birthdays in the same calendar month?

Let months = Pigeonholes = n = 12

Persons = Pigeons = m =?

Given that, there are at least 5 persons will have a birthday in the same calendar month.

By generalized Pigeonhole principle, we have

$$\left\lfloor \left(\frac{m-1}{n} \right) \right\rfloor + 1 = 5$$

$$\left\lfloor \left(\frac{m-1}{12} \right) \right\rfloor + 1 = 5$$

$$m - 1 = 48$$

$$m = 49$$

$\therefore$ 49 persons must be chosen in order that at least five of them will have birth days in the same calendar month.

How many persons must be chosen in order that at least nine of them will have birthdays in the same day of the week?

Let days = Pigeonholes = n = 7

Persons = Pigeons = m =?

Given that, there are at least 9 persons will have a birthday in the same day of the week.

By generalized Pigeonhole principle, we have

$$\left\lfloor \left(\frac{m-1}{n} \right) \right\rfloor + 1 = 9$$

$$\left\lfloor \left(\frac{m-1}{7} \right) \right\rfloor + 1 = 9$$

$$m - 1 = 56$$

$$m = 57$$

$\therefore$ 57 persons must be chosen in order that at least 9 of them will have birthdays in the same day of the week.

Seven members of a family have total Rs.2886 in their pockets. Show that at least one of them must have at least Rs.413 in his pocket.

Let members = Pigeonholes = n = 7

Amount = Pigeons = m = 2886

By generalized Pigeonhole principle, we have

$$\left\lfloor \left(\frac{m-1}{n} \right) \right\rfloor + 1 \text{ or more}$$

$$\left\lfloor \left(\frac{2886-1}{7} \right) \right\rfloor + 1 = \lfloor 412.143 \rfloor + 1 = 412 + 1 = 413$$

Thus, at least one of them must have at least Rs.413 in his pocket.

Find the least number of ways of choosing three different numbers from 1 to 10 so that all choices have the same sum.

Three numbers can be chosen from 1 to 10 in $\binom{10}{3} = 120$ ways.

The smallest possible sum is 1 + 2 + 3 = 6 and the largest sum is 8 + 9 + 10 = 27.

Thus, sums vary from 6 to 27 and there are 22 sums.

By generalized Pigeonhole principle, we have

$$\left\lfloor \left(\frac{m-1}{n}\right) \right\rfloor + 1 \text{ or more}$$

$$\left\lfloor \left(\frac{120-1}{22} \right) \right\rfloor + 1 = \lfloor 5.41 \rfloor + 1 = 5 + 1 = 6$$

Thus, 6 ways of choosing three different numbers from 1 to 10 in order to get all the choices have same sum.

Show that if any 5 numbers from 1 to 8 are chosen, then two of them will have their sum equal to 9.

The two numbers from 1 to 8 whose sum is 9 are as follows:

{1, 8}, {2,7}, {3, 6}, {4, 5}.

The 5 numbers chosen must belong to one of the above sets. Since there are only 4 sets, two of the 5 chosen numbers have to belong to the same set according to pigeonhole principle. These two numbers have their sum equal to 9.

A bag contains 12 pairs of socks (each pair in different color). If a person draws the socks one by one at random, determine at most how many draws are required to get at least one pair of matched socks.

Let n denote the number of the draw.

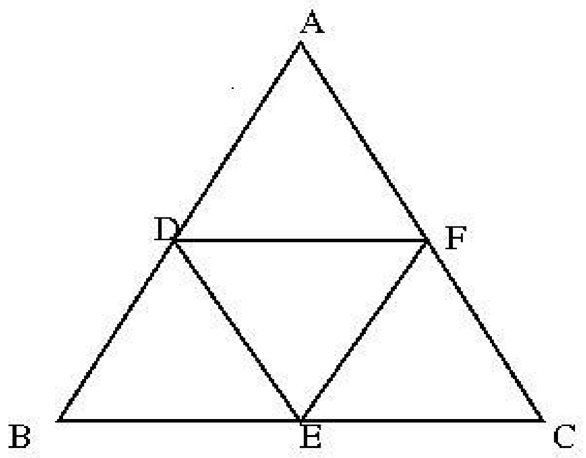
For $n \leq 12$, it is possible that the socks drawn are of different colors because there are 12 colors.

For $n = 13$, all socks cannot have different colors. At least two must have the same color.

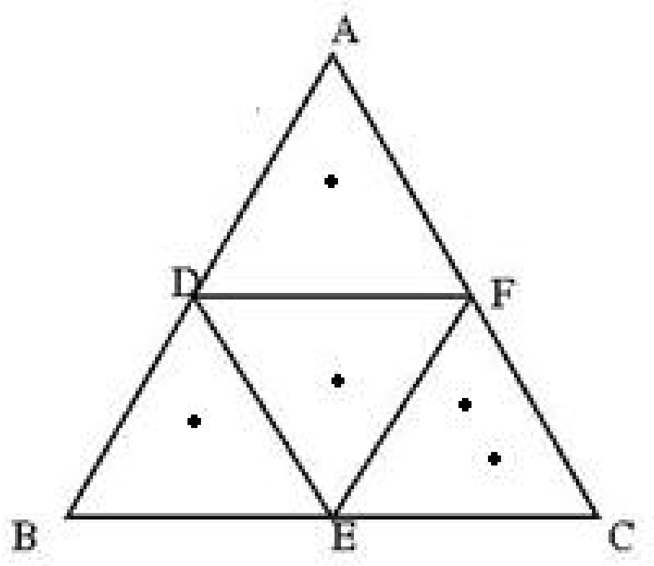
Treat $n = 13$ as pigeons and 12 colors as pigeonholes.

Thus, at most 13 draws are required to have at least one pair of socks of the same color.

ABC is an equilateral triangle whose sides are of length 1 cm each. If we select 5 points inside the triangle, prove that at least two of these points are such that the distance between them is less than ½ cm.



Consider the Δ DEF formed by the midpoints of the sides AB, BC and CA of the given Δ ABC. Then the Δ ABC is partitioned into four small equilateral triangles, each of which has sides equal to ½ cm.



Treating each of these four partitions as a pigeonhole and five points chosen inside the triangle as pigeons.

We find by using the pigeonhole principle that at least one portion must contain two or more points. Evidently, the distance between such points is less than ½ cm.