CONVOLUTION THEOREM

If $L^{-1}\left[\overline{f}(s)\right]=f(t)$ and $L^{-1}\left[\overline{g}(s)\right]=g(t)$ then $L^{-1}\left[\overline{f}(s).\overline{g}(s)\right]=\int_{u=0}^{t} f(u)\,g(t-u)\,du$

By employing convolution theorem, evaluate $L^{-1}\left\{\dfrac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right\}$.

(VTU 2003, 2011, 2012)

Solution:

Let, $\overline{f}(s) = \dfrac{s}{\left(s^2+a^2\right)}$ and $\overline{g}(s)=\dfrac{s}{\left(s^2+b^2\right)}$

Taking Laplace inverse transform on both side

$$L^{-1}\left[\overline{f}(s)\right] = L^{-1}\left[\frac{s}{\left(s^2+a^2\right)}\right] \quad \text{and} \quad L^{-1}\left[\overline{g}(s)\right]=L^{-1}\left[\frac{s}{\left(s^2+b^2\right)}\right]$$

$$\Rightarrow \quad f(t) = \cos at \quad \text{and} \quad g(t)=\cos bt$$

By convolution theorem, we have $L^{-1}\left[\overline{f}(s).\overline{g}(s)\right] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$

$$\therefore \quad L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_{u=0}^{t} \cos au.\cos b(t-u)\,du$$

$$= \int_{u=0}^{t} \frac{1}{2}\left[\cos(au+bt-bu)+\cos(au-bt+bu)\right]du$$

$$= \frac{1}{2}\left[\frac{\sin(au+bt-bu)}{a-b}+\frac{\sin(au-bt+bu)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left\{\left[\frac{\sin(at+bt-bt)}{a-b}+\frac{\sin(at-bt+bt)}{a+b}\right]-\left[\frac{\sin(bt)}{a-b}+\frac{\sin(-bt)}{a+b}\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\sin(at)}{a-b}+\frac{\sin(at)}{a+b}\right]-\left[\frac{\sin(bt)}{a-b}-\frac{\sin(bt)}{a+b}\right]\right\}$$

$$= \frac{1}{2}\left[\frac{(a+b)\sin at+(a-b)\sin at-(a+b)\sin bt+(a-b)\sin bt}{(a-b)(a+b)}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at(a+b+a-b)-\sin bt(a+b-a+b)}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at(2a)-\sin bt(2b)}{a^2-b^2}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

By employing convolution theorem, evaluate $L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\}$.

(VTU 2004, 2005, 2013)

Solution:

Let, $\bar{f}(s) = \frac{1}{\left(s^2+a^2\right)}$ and $\bar{g}(s) = \frac{s}{\left(s^2+a^2\right)}$

Taking Laplace inverse transform on both side

$$L^{-1}\left[\overline{f}(s)\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\right] \text{ and } L^{-1}\left[\overline{g}(s)\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$$

$$f(t) = \frac{\sin at}{a} \quad \text{and} \quad g(t) = \cos at$$

By convolution theorem, we have $L^{-1}\left[\overline{f}(s).\overline{g}(s)\right] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a}.\cos a(t-u)\,du$$

$$= \frac{1}{a}\int_{u=0}^{t} \sin au.\cos a(t-u)\,du$$

$$= \frac{1}{a}\int_{u=0}^{t} \frac{1}{2}\left[\sin(au+at-au)+\sin(au-at+au)\right]du$$

$$= \frac{1}{2a}\int_{u=0}^{t}\left[\sin at + \sin(2au - at)\right]du \quad = \frac{1}{2a}\left[\sin at\int_{u=0}^{t} du + \int_{u=0}^{t}\sin(2au - at)\,du\right]$$

$$= \frac{1}{2a}\left[\sin at\,(u)_0^t + \left(\frac{-\cos(2au - at)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[\sin at\,(t - 0) - \frac{\cos at}{2a} + \frac{\cos at}{2a}\right] = \frac{1}{2a}\left[t\sin at\right] \quad = \frac{t\sin at}{2a}$$

By employing convolution theorem, evaluate $L^{-1}\left\{\frac{s}{(s-1)(s^2+4)}\right\}$.

(VTU 2004, 2011, 2013)

Solution:

Let, $\quad \bar{f}(s) = \frac{1}{s-1} \quad$ and $\quad \bar{g}(s) = \frac{s}{s^2+4}$

Taking Laplace inverse transform on both side

$$L^{-1}\left[\overline{f}(s)\right] = L^{-1}\left[\frac{1}{s-1}\right] \quad \text{and} \quad L^{-1}\left[\overline{g}(s)\right] = L^{-1}\left[\frac{s}{s^2+4}\right]$$

$$f(t) = e^t \quad \text{and} \quad g(t) = \cos 2t$$

By convolution theorem, we have $L^{-1}\left[\overline{f}(s).\overline{g}(s)\right] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$

$$L^{-1}\left[\frac{s}{(s-1)(s^2+4)}\right] = \int_{u=0}^{t} e^u.\cos 2(t-u)\,du \quad = \int_{u=0}^{t} e^u.\cos(2t-2u)\,du$$

$$= \left[\frac{e^u}{1^2+(-2)^2}\left[\cos(2t-2u) - 2\sin(2t-2u)\right]\right]_0^t$$

$$= \frac{1}{5}\left\{\left[e^t(\cos 0 - 2\sin 0)\right] - \left[e^0(\cos 2t - 2\sin 2t)\right]\right\}$$

$$= \frac{1}{5}\left\{e^t - \cos 2t + 2\sin 2t\right\}$$

By employing convolution theorem, evaluate $L^{-1}\left\{\dfrac{s}{(s+2)(s^2+9)}\right\}$.

(VTU 2005, 2008)

Solution:

Let, $\overline{f}(s) = \dfrac{1}{s+2}$ and $\overline{g}(s) = \dfrac{s}{s^2+9}$

Taking Laplace inverse transform on both side

$$L^{-1}\left[\overline{f}(s)\right] = L^{-1}\left[\frac{1}{s+2}\right] \text{ and } L^{-1}\left[\overline{g}(s)\right] = L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$f(t) = e^{-2t} \quad \text{and} \quad g(t) = \cos 3t$$

By convolution theorem, we have $L^{-1}\left[\overline{f}(s).\overline{g}(s)\right] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$

$$L^{-1}\left[\frac{s}{(s-1)(s^2+4)}\right] = \int_{u=0}^{t} e^{-2u}.\cos 3(t-u)\, du = \int_{u=0}^{t} e^{-2u}.\cos(3t-3u)\, du$$

$$= \left[\frac{e^{-2u}}{(-2)^2 + (-3)^2} \left[-2\cos(3t - 3u) - 3\sin(3t - 3u) \right] \right]_0^t$$

$$= \frac{1}{13} \left\{ \left[e^{-2t} \left(-2\cos 0 - 3\sin 0 \right) \right] - \left[e^{0} \left(-2\cos 3t - 3\sin 3t \right) \right] \right\}$$

$$= \frac{1}{13} \left\{ -2e^{-2t} + 2\cos 3t + 3\sin 3t \right\}$$

By employing convolution theorem, evaluate $L^{-1}\left\{\dfrac{1}{s\left(s^2+a^2\right)}\right\}$. **(VTU 2006)**

Solution:

Let, $\overline{f}(s) = \dfrac{1}{s}$ and $\overline{g}(s) = \dfrac{1}{s^2 + a^2}$

Taking Laplace inverse transform on both side

$$L^{-1}\left[\overline{f}(s)\right] = L^{-1}\left[\frac{1}{s}\right] \text{ and } L^{-1}\left[\overline{g}(s)\right] = L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$f(t) = 1 \quad \text{and} \quad g(t) = \frac{\sin at}{a}$$

By convolution theorem, we have $L^{-1}\left[\overline{f}(s).\overline{g}(s)\right] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$

$$L^{-1}\left[\frac{1}{s\left(s^2+a^2\right)}\right] = \int_{u=0}^{t} 1.\frac{\sin a(t-u)}{a}du = \frac{1}{a}\int_{u=0}^{t}\sin(at-au)\,du$$

$$= \frac{1}{a}\left[\frac{-\cos(at-au)}{-a}\right]_0^t = \frac{1}{a}\left[\frac{\cos 0-\cos at}{a}\right]$$

$$= \frac{1-\cos at}{a^2}$$

By employing convolution theorem, evaluate $L^{-1}\left\{\frac{1}{s\left(s^2+9\right)}\right\}$. **(VTU 2006)**

Solution:

Let, $\bar{f}(s) = \frac{1}{s}$ and $\bar{g}(s) = \frac{1}{s^2+9}$

Taking Laplace inverse transform on both side

$$L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left[\frac{1}{s}\right] \text{ and } L^{-1}\left[\bar{g}(s)\right] = L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$f(t) = 1 \quad \text{and} \quad g(t) = \frac{\sin 3t}{3}$$

By convolution theorem, we have $$L^{-1}\left[\bar{f}(s).\bar{g}(s)\right] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

$$L^{-1}\left[\frac{1}{s\left(s^2+9\right)}\right] = \int_{u=0}^{t} 1.\frac{\sin 3(t-u)}{3} du \quad = \frac{1}{3}\int_{u=0}^{t} \sin(3t-3u)\, du$$

$$= \frac{1}{3}\left[\frac{-\cos(3t-3u)}{-3}\right]_0^t = \frac{1}{3}\left[\frac{\cos 0 - \cos 3t}{3}\right] = \frac{1 - \cos 3t}{9}$$

By employing convolution theorem, evaluate $L^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\}$. (VTU 2014)

Solution: Let, $\overline{f}(s) = \frac{1}{s-1}$ and $\overline{g}(s) = \frac{1}{s^2+1}$

Taking Laplace inverse transform on both side

By convolution theorem, we have $L^{-1}\left[\overline{f}(s).\overline{g}(s)\right] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \int_{u=0}^{t} e^u \sin(t-u)\, du = \left(\frac{e^u}{1^2+1^2}\left[\sin(t-u) - (-1)\cos(t-u)\right]\right)_0^t$$

$$= \frac{1}{2}\left[e^t(\sin 0 + \cos 0) - e^0(\sin t + \cos t)\right] = \frac{1}{2}\left[e^t - \sin t - \cos t\right]$$

By employing convolution theorem, evaluate $L^{-1}\left\{\frac{1}{4s^2-9}\right\}$. **(VTU 2013)**

Solution:

$$L^{-1}\left\{\frac{1}{4s^2-9}\right\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s^2-9/4}\right\} = \frac{1}{4}L^{-1}\left\{\frac{1}{(s+3/2)(s-3/2)}\right\}$$

$$\overline{f}(s) = \frac{1}{s+3/2} \quad \text{and} \quad \overline{g}(s) = \frac{1}{s-3/2}$$

Taking Laplace inverse transform on both side

$$L^{-1}\left[\overline{f}(s)\right] = L^{-1}\left[\frac{1}{s+3/2}\right] \quad \text{and} \quad L^{-1}\left[\overline{g}(s)\right] = L^{-1}\left[\frac{1}{s-3/2}\right]$$

$$f(t) = e^{-(3/2)t} \quad \text{and} \quad g(t) = e^{(3/2)t}$$

By convolution theorem, we have $L^{-1}\left[\overline{f}(s).\overline{g}(s)\right] = \int_{u=0}^{t} f(u)g(t-u)\,du$

$$\therefore \quad \frac{1}{4} L^{-1}\left\{\frac{1}{(s+3/2)(s-3/2)}\right\} = \frac{1}{4}\int_{u=0}^{t} e^{-(3/2)u}.e^{(3/2)(t-u)}du$$

$$= \frac{1}{4}e^{(3/2)t}\int_{u=0}^{t} e^{-(3/2)u}.e^{-(3/2)u}du$$

$$= \frac{1}{4}e^{(3/2)t}\int_{u=0}^{t} e^{-3u}du$$

$$= \frac{1}{4}e^{(3/2)t}\left[\frac{e^{-3u}}{-3}\right]_0^t = \frac{1}{4}e^{(3/2)t}\left[\frac{e^{-3t} - e^0}{-3}\right]$$

$$= \frac{e^{(3/2)t}\left(1-e^{-3t}\right)}{12}$$