Module-I
III Part

RULES OF INFERENCE

An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called **premises** and the final proposition is called the **conclusion**. An argument is valid if the truth of all its premises implies that the conclusion is true.

Consider a valid argument of the form

$$(p_1 \land p_2 \land p_3 \land \dots \land p_n) \to q.$$

Here, n is a positive integer, the statements $p_1, p_2, p_3, \dots, p_n$ are called the **premises** of the argument and the statement q is the **conclusion** for the argument.

The above argument can also be written as

$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \cdot \\ \cdot \\ \cdot \\ p_n \\ \hline \therefore\ q \end{array}$$

From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, p_3, \ldots, p_n$ and conclusion q is valid, when $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology.

Note: We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the conclusion must also be true. However, this can be a tedious approach. For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid require $2^{10} = 1024$ different rows. Fortunately, we do not have to resort to truth tables. Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference**. These rules of inference can be used as building blocks to construct more complicated valid argument forms.

Rules of Inference for Propositional Logic

We will now introduce the most important rules of inference in propositional logic:

1. **Rule of detachment or Modus ponens**

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array} \qquad (p \land (p \rightarrow q)) \rightarrow q$$

2. **Rule of syllogism**

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array} \qquad ((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

3. Modus Tollens

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

$$((p \to q) \land \neg q) \to \neg p$$

4. Rule of Disjunctive Syllogism

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

$$((p \lor q) \land \neg p) \to q$$

5. Rule of Disjunctive Amplification

$$\frac{p}{\therefore p \lor q}$$

$$p \rightarrow (p \lor q)$$

6. Rule of Conjunctive Simplification

$$\frac{p \land q}{\therefore p}$$

$$(p \land q) \rightarrow p$$

7. Rule of Conjunction

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore p \land q}$$

$$((p) \land (q)) \rightarrow (p \land q)$$

8. Rule of Resolution

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

$$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$$

Remark: There are many rules of inference. In the above we have considered only most important rules of inference in propositional logic.

Examples:

Establish the validity of the following argument:

$$\begin{array}{c} \mathbf{p \to r} \\ \mathbf{\neg p \to q} \\ \mathbf{q \to s} \\ \hline \therefore \mathbf{\neg r \to s} \end{array}$$

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\neg p \to q$	Given Premise
2	---	$q \to s$	Given Premise
3	1 and 2	$\neg p \to s$	Rule of syllogism
4	3	$\neg s \to p$	By Contrapositive
5	---	$p \to r$	Given Premise
6	4 and 5	$\neg s \to r$	Rule of syllogism
7	---	$\neg r \to s$	By Contrapositive

Hence, the given argument is valid argument.

Establish the validity of the following argument:

$$\begin{array}{c} p \to q \\ \neg r \vee s \\ p \vee r \\ \hline \therefore \neg q \to s \end{array}$$

Step Number	Steps Used	Main Steps	Rules Used
1	---	$p \to q$	Given Premise
2	---	$p \vee r$	Given Premise
3	---	$\neg r \vee s$	Given Premise
4	1	$\neg q \to \neg p$	By Contrapositive
5	2	$\neg p \to r$	By Conditional law
6	3	$r \to s$	By Conditional law
7	4 and 5	$\neg q \to r$	Rule of syllogism
8	7 and 6	$\neg q \to s$	Rule of syllogism

Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow (R \vee S)$.

Step Number	Steps Used	Main Steps	Rules Used
1	---	$(C \vee D) \rightarrow \neg H$	Given Premise
2	---	$\neg H \rightarrow (A \wedge \neg B)$	Given Premise
3	1 and 2	$(C \vee D) \rightarrow (A \wedge \neg B)$	Rule of syllogism
4	---	$(A \wedge \neg B) \rightarrow (R \vee S)$	Given Premise
5	3 and 4	$(C \vee D) \rightarrow (R \vee S)$	Rule of syllogism
6	---	$C \vee D$	Given Premise
7	5 and 6	$(R \vee S)$	Modus ponens

A second way of showing our premises leads to the desired conclusion as follows:

Step Number	Steps Used	Main Steps	Rules Used
1	---	$C \vee D$	Given Premise
2	---	$(C \vee D) \rightarrow \neg H$	Given Premise
3	1 and 2	$\neg H$	Modus ponens
4	---	$\neg H \rightarrow (A \wedge \neg B)$	Given Premise
5	3 and 4	$(A \wedge \neg B)$	Modus ponens
6	---	$(A \wedge \neg B) \rightarrow (R \vee S)$	Given Premise
7	5 and 6	$(R \vee S)$	Modus ponens

This proves the required result.

Verify the validity of the following argument:
Rita is baking a cake.
If Rita is baking a cake then she is not practicing her flute.
If Rita is not practicing her flute, then her father will not buy her a car. Therefore, Rita's father will not buy her a car.

Solution:

Let p: Rita is baking a cake.

q: she is practicing her flute.

r: Father will buy her a car.

The given argument in symbolic form is

$$\begin{array}{l} p \\ p \to \neg q \\ \neg q \to \neg r \\ \hline \therefore \neg r \end{array}$$

We establish the validity of the argument as follows:

$$\begin{array}{l} p \\ p \to \neg q \\ \neg q \to \neg r \\ \hline \therefore \neg r \end{array}$$

Step Number	Steps Used	Main Steps	Rules Used
1	---	$p \to \neg q$	Given Premise
2	---	$\neg q \to \neg r$	Given Premise
3	1 and 2	$p \to \neg r$	Rule of syllogism
4	---	p	Given Premise
5	3 and 4	$\neg r$	Modus ponens

Hence, the given argument is valid argument.

Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution:

Let p: "It is sunny this afternoon,"

q: "It is colder than yesterday,"

r: "We will go swimming,"

s: "We will take a canoe trip,"

t: "We will be home by sunset."

Then the given premises are $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$ and $s \rightarrow t$ and the conclusion is t.

We construct an argument to show that our premises lead to the desired conclusion as follows:

$$\begin{array}{c} \neg p \wedge q \\ r \rightarrow p \\ \neg r \rightarrow s \\ s \rightarrow t \\ \hline \therefore t \end{array}$$

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\neg p \wedge q$	Given Premise
2	1	$\neg p$	Rule of Conjunctive Simplification
3	---	$r \rightarrow p$	Given Premise
4	2 and 3	$\neg r$	Modus Tollens
5	---	$\neg r \rightarrow s$	Given Premise
6	4 and 5	s	Modus ponens
7	---	$s \rightarrow t$	Given Premise
8	6 and 7	t	Modus ponens

This proves the required result.

Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \rightarrow R$, $P \rightarrow M$ and $\neg M$.

Solution:

Step Number	Steps Used	Main Steps	Rules Used
1	---	$P \vee Q$	Given Premise
2	1	$\neg P \rightarrow Q$	By Conditional law
3	---	$Q \rightarrow R$	Given Premise
4	2 and 3	$\neg P \rightarrow R$	Rule of syllogism
5	---	$P \rightarrow M$	Given Premise
6	---	$\neg M$	Given Premise
7	5 and 6	$\neg p$	Modus Tollens
8	7 and 4	R	Modus ponens
9	8 and 1	$R \wedge (P \vee Q)$	By Conjunction

This proves the required result.

Rules of Inference for Quantified Statements

1. Universal Instantiation or Specification

$$\frac{\forall x\, P(x)}{\therefore P(c)}$$

2. Universal Generalization

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x\, P(x)}$$

3. Existential Instantiation or Specification

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c}$$

4. Existential Generalization

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x\, P(x)}$$

1. Establish the validity of the following argument:

$$\forall x\,[p(x) \rightarrow q(x)]$$
$$\forall x\,[q(x) \rightarrow r(x)]$$
$$\overline{\qquad\qquad\qquad\qquad}$$
$$\therefore \forall x\,[p(x) \rightarrow r(x)]$$

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\forall x\,[p(x) \rightarrow q(x)]$	Given Premise
2	1	$p(a) \rightarrow q(a)$	Universal Instantiation
3	---	$\forall x\,[q(x) \rightarrow r(x)]$	Given Premise
4	3	$q(a) \rightarrow r(a)$	Universal Instantiation
5	2 and 4	$p(a) \rightarrow r(a)$	Rule of Syllogism
6	5	$\forall x\,[p(x) \rightarrow r(x)]$	Universal Generalization

2. **Establish the validity of the following argument:**

$$\begin{array}{l} \forall x\, [p(x) \rightarrow q(x)] \\ \forall x\, [q(x) \rightarrow r(x)] \\ \neg\, r(c) \\ \hline \therefore\ \neg\, p(c) \end{array}$$

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\forall x\, [p(x) \rightarrow q(x)]$	Given Premise
2	1	$p(c) \rightarrow q(c)$	Universal Instantiation
3	---	$\forall x\, [q(x) \rightarrow r(x)]$	Given Premise
4	3	$q(c) \rightarrow r(c)$	Universal Instantiation
5	2 and 4	$p(c) \rightarrow r(c)$	Rule of Syllogism
6	5	$\neg\, r(c)$	Given Premise
7	6 and 5	$\neg\, p(c)$	Modus Tollens

Establish the validity of the following argument:

$\forall x\, [p(x) \vee q(x)]$

$\exists x\, \neg\, p(x)$

$\forall x\, [\neg\, q(x) \vee r(x)]$

$\forall x\, [s(x) \rightarrow \neg\, r(x)]$

$\therefore \quad \exists x\, \neg\, s(x)$

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\forall x\, [p(x) \vee q(x)]$	Given Premise
2	1	$p(a) \vee q(a)$	Universal Instantiation
3	---	$\exists x\, \neg\, p(x)$	Given Premise
4	3	$\neg\, p(a)$	Existential Instantiation
5	2 and 4	$q(a)$	Rule of Disjunctive Syllogism
6	---	$\forall x\, [\neg\, q(x) \vee r(x)]$	Given Premise
7	6	$\neg\, q(a) \vee r(a)$	Universal Instantiation
8	5 and 7	$r(a)$	Rule of Disjunctive Syllogism
9	---	$\forall x\, [s(x) \rightarrow \neg\, r(x)]$	Given Premise
10	9	$s(a) \rightarrow \neg\, r(a)$	Universal Instantiation
11	8 and 10	$\neg\, s(a)$	Modus Tollens
12	11	$\exists x\, \neg\, s(x)$	Existential Generalization

Hence, the given argument is valid argument.

Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution:

Let $D(x)$ denote "x is in this discrete mathematics class,"

let $C(x)$ denote "x has taken a course in computer science."

Then the premises are $\forall x\, (D(x) \rightarrow C(x))$ and $D(Marla)$ and the conclusion is $C(Marla)$.

$\forall x\,(D(x) \to C(x))$

$D(\text{Marla})$

$\therefore C(\text{Marla})$.

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\forall x(D(x) \to C(x))$	Given Premise
2	1	$D(\text{Marla}) \to C(\text{Marla})$	Universal instantiation
3	---	$D(\text{Marla})$	Given Premise
4	2 and 3	$C(\text{Marla})$	Modus ponens

This proves the required result.

Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution:

Let $C(x)$ be "x is in this class,"

Let $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam."

The premises are $\exists x\,(C(x) \wedge \neg B(x))$ and $\forall x\,(C(x) \to P(x))$ and the conclusion is $\exists x\,(P(x) \wedge \neg B(x))$.

Step Number	Steps Used	Main Steps	Rules Used
1		$\exists x\,(C(x) \wedge \neg B(x))$	Given Premise
2	1	$C(a) \wedge \neg B(a)$	Existential instantiation
3	2	$C(a)$	Rule of Conjunctive Simplification
4		$\forall x\,(C(x) \to P(x))$	Given Premise
5	4	$C(a) \to P(a)$	Universal instantiation
6	3 and 5	$P(a)$	Modus ponens
7	2	$\neg B(a)$	Rule of Conjunctive Simplification
8	6 and 7	$P(a) \wedge \neg B(a)$	Conjunction
9	8	$\exists x\,(P(x) \wedge \neg B(x))$	Existential generalization

This proves the required result.