Transform calculus , Fourier series and Numerical Techniques

Module 2

Fourier Series: Periodic functions, Dirichlet's condition. Fourier series of periodic functions period 2π and arbitrary period. Half range Fourier series. Practical harmonic analysis.

What is Fourier series? What are the applications?

Fourier series is a representation of a function using a series of sinusoidal functions of different frequencies. They are useful to represent functions that are periodic in nature.

- *Telecomms* - GSM/cellular phones,
- *Electronics/IT* - most DSP-based applications,
- *Entertainment* - music, audio, multimedia,
- *Accelerator control* (tune measurement for beam steering/control),
- *Imaging, image processing,*
- *Industry/research* - X-ray spectrometry, chemical analysis (FT spectrometry), PDE solution, radar design,
- *Medical* - (PET scanner, CAT scans & MRI interpretation for sleep disorder & heart malfunction diagnosis,
- *Speech analysis* (voice activated "devices", biometry, …).

- Signal Processing
- Image processing
- Heat distribution mapping
- Wave simplification
- Light Simplication(Interference , Deffraction etc.)
- Radiation measurements etc.

Prerequisites

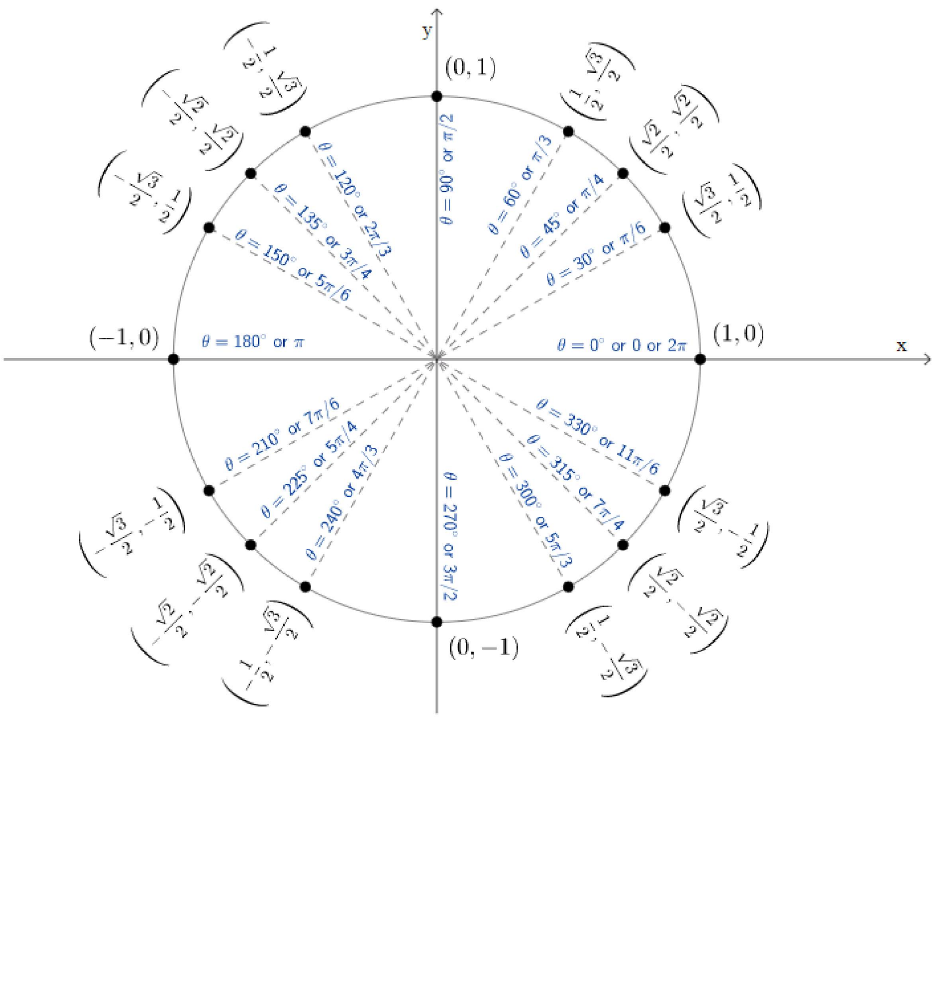
Sum of the terms S_n	n^{th} term t_n
$1+3+5+7+\cdots$	$2n-1$
$3-5+7-9+\cdots$	$(-1)^{n+1}(2n+1)$
$-2+4-6+\cdots$	$(-1)^n\,(2n)$

	0	$\frac{\pi}{2}$	π	$n\pi$
$\sin x$	0	1	0	0
$\cos x$	1	0	-1	$(-1)^n$

$\sin x\cos y=\frac{1}{2}[\sin(x+y)+\sin(x-y)]$	$\cos x\cos y=\frac{1}{2}[\cos(x+y)+\cos(x-y)]$
$\cos x\sin y=\frac{1}{2}[\sin(x+y)-\sin(x-y)]$	$\sin x\sin y=\frac{1}{2}[\cos(x-y)-\cos(x+y)]$
$\cos^2 x=\frac{1}{2}(1+\cos x),\sin^2 x=\frac{1}{2}(1-\cos x)$	

$\int\sin nx\,dx=-\frac{\cos nx}{n}+c$	$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$
$\int\cos nx\,dx=\frac{\sin nx}{n}+c$	$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$
$\int uv\,dx=uv_1-u'v_2+u''v_3-\cdots$	

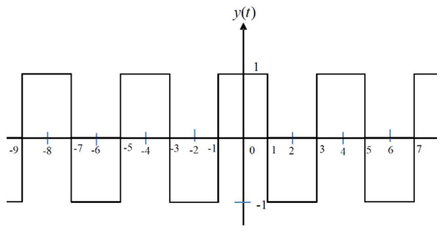
Trigonometric ratios of standard angles

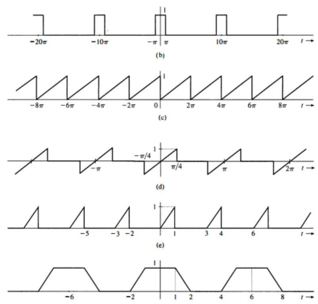


Periodic functions

$$y(t)=\begin{cases}1 & -1\le t\le 1\\ -1 & 1\le t\le 3\end{cases}$$

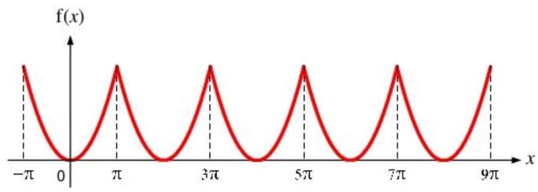
$$y(t+4)=y(t)\Rightarrow \text{period} \quad T=4$$

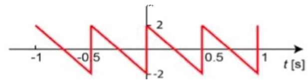


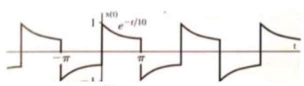


$$\boldsymbol{f(x)=x^2 \quad when \quad -\pi\le x\le \pi}$$

$$\boldsymbol{f(\theta+2\pi)=f(\theta)}$$

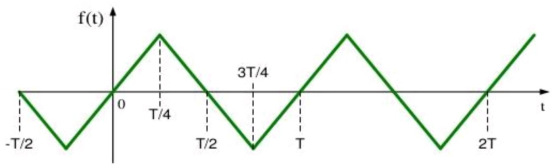


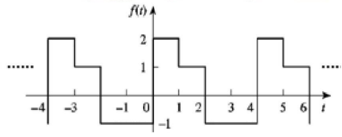




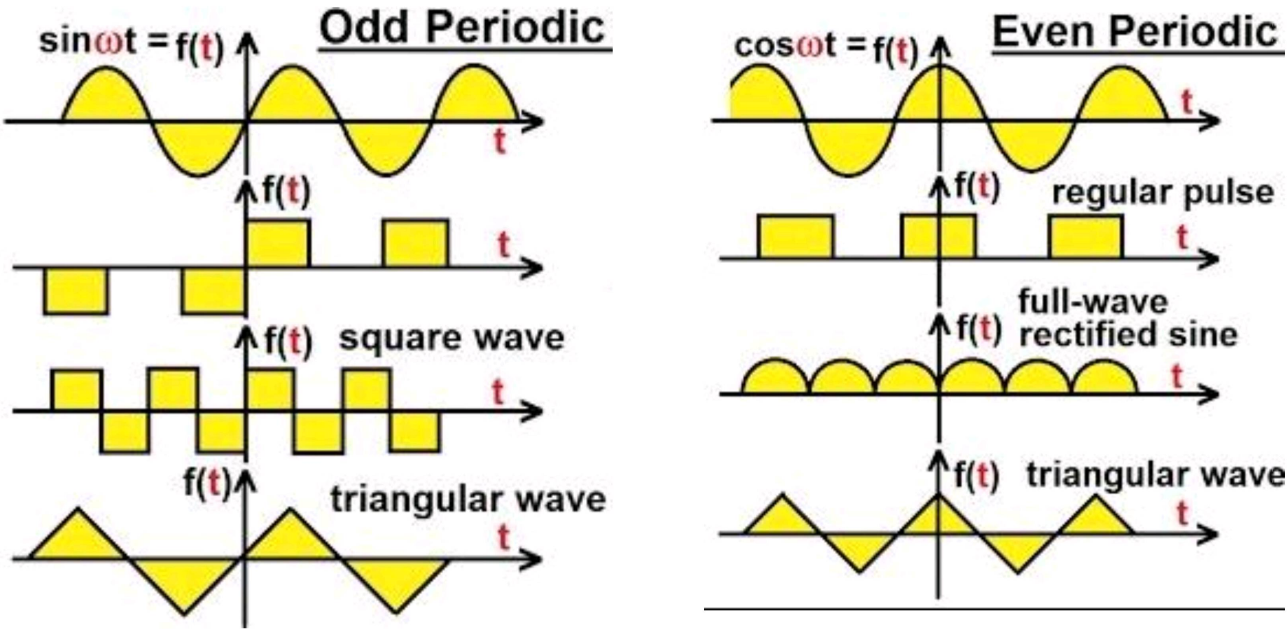
$$\boldsymbol{f(t)=t \quad when -\frac{T}{4}\le t\le \frac{T}{4}}$$

$$\boldsymbol{=-t+\frac{T}{2} \quad when \quad \frac{T}{4}\le t\le \frac{3T}{4}}$$



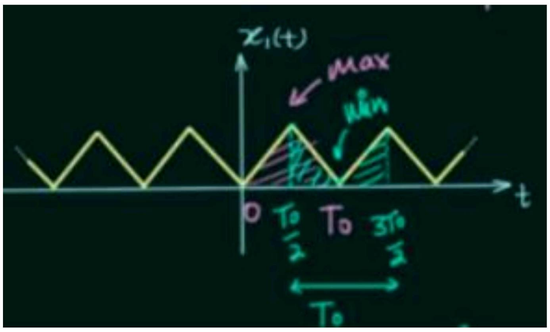


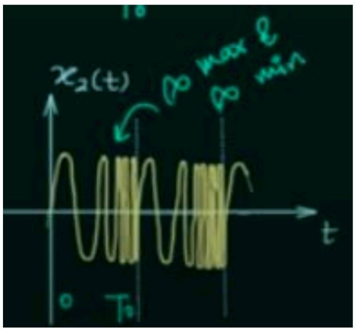
Odd and Even functions



Dirichlet’s condition 1

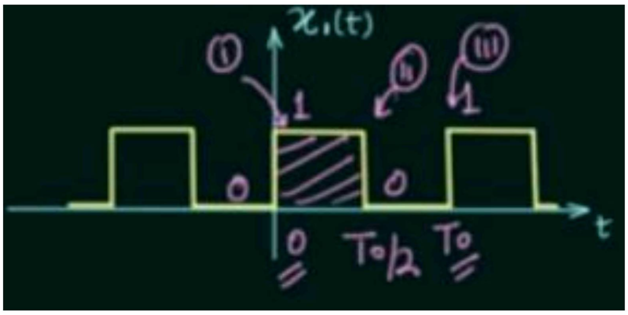
❖Signals should have a finite number of maxima and minima over the range of time period.

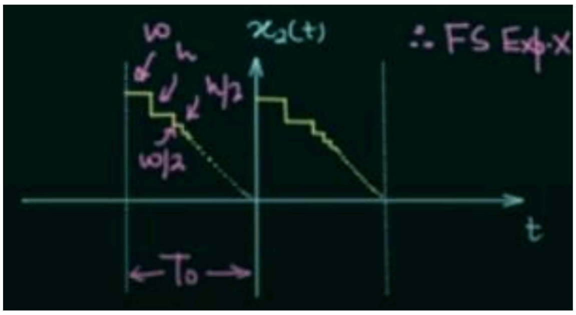




Dirichlet’s condition 2

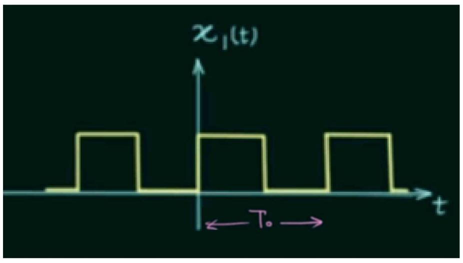
❖Signals should have a finite number of discontinuities over the range of time period.

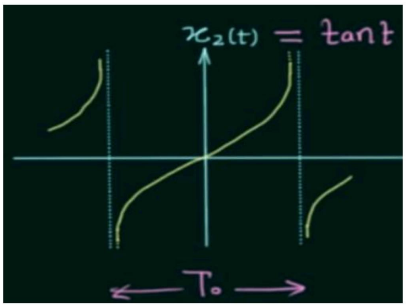




Dirichlet's condition 3

❖Signals should be absolutely integrable over the range of time period.





$$\int_{T_0} x_1(t)\,dt = (1)\left(\frac{T_0}{2}\right) < \infty$$

$$\int_{T_0} x_2(t)\,dt = \infty$$

Important points

1. <u>Integration of odd/even function:</u>

$$\int_{-l}^{l} f(x)dx = \begin{cases} 2\int_0^l f(x)dx, & if\ f(x)\ is\ even \\ 0, & if\ f(x)\ is\ odd \end{cases}$$

2. <u>Periodic function:</u>

$f(x)$is said to be periodic function with period T if $f(x + nT) = f(x)$ for all n.

3. <u>Point of discontinuity:</u>

If $x = c$ is a point of discontinuity of $f(x)$then $f(c) = \frac{1}{2}[f(c-0) + f(c+0)]$.

4. <u>Dirichlet's conditions:</u>

- $f(x)$ is periodic, single valued and finite
- $f(x)$ has a finite number of discontinuities in any one period
- $f(x)$ has atmost a finite number of maxima and minima

5. <u>Existance of Fourier series:</u>

Fourier expansion of $f(x)$ exists if $f(x)$ satisfies Dirichlet's conditions.

Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n cos\left(\frac{n\pi x}{l}\right) + b_n sin\left(\frac{n\pi x}{l}\right)\right], \qquad c < x < c + 2l$$

Where, $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$, $a_n = \frac{1}{l}\int_c^{c+2l} f(x)cos\left(\frac{n\pi x}{l}\right) dx$, $b_n = \frac{1}{l}\int_c^{c+2l} f(x)sin\left(\frac{n\pi x}{l}\right) dx$

	$c < x < c + 2l$	$l = \pi$
$c = 0$	$0 < x < 2l$	$0 < x < 2\pi$
$c = -l$	$-l < x < l$	$-\pi < x < \pi$

Odd / Even function

Consider the function	$f(x)$ is an even function if	$f(x)$ is an odd function if
$f(x), -l < x < l. (c = -l)$	$f(-x) = f(x)$	$f(-x) = -f(x)$
$f(x) = \begin{cases} \phi(x), -l < x < 0 \\ \chi(x), \ \ 0 < x < l \end{cases}$	$\chi(-x) = \phi(x)$	$\chi(-x) = -\phi(x)$
$f(x), 0 < x < 2l. (c = 0)$	$f(2l - x) = f(x)$	$f(2l - x) = -f(x)$
$f(x) = \begin{cases} \phi(x), 0 < x < l \\ \chi(x), \ \ l < x < 2l \end{cases}$	$\chi(2l - x) = \phi(x)$	$\chi(2l - x) = -\phi(x)$

Constants	If $f(x)$ is an even function	If $f(x)$ is an odd function
a_0	$\frac{2}{l}\int_0^l f(x)\,dx$	0
a_n	$\frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx$	0
b_n	0	$\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\,dx$

1. **Obtain the Fourier series for the function $f(x) = x$ in $(-\pi, \pi)$ and hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$.**

Solution:

Given, $f(x) = x$ in $(-\pi, \pi)$

$f(-x) = (-x) = -f(x)$

$\therefore$ $f(x)$ is an odd function

Since $f(x)$ is an odd function,

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\pi\left(-\frac{\cos n\pi}{n}\right) - 1\left(-\frac{\sin n\pi}{n^2}\right)\right] - [0 - 0]\right\}$$

$$= \frac{2}{\pi}\left\{\left[\pi\left(\frac{-(-1)^n}{n}\right) - 0\right] - [0 - 0]\right\}$$

$$= \frac{-2(-1)^n}{n}$$

1. **Obtain the Fourier series for the function $f(x) = x$ in $(-\pi, \pi)$ and hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$.**

Cont…

Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$x = 0+0+\sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

Put $x = \frac{\pi}{2}$, we get

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots = \frac{\pi}{4}$$

2. Expand $f(x)=\dfrac{\pi-x}{2}$ *as a Fourier series expansion in* $0<x<2\pi$. Hence deduce that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots=\frac{\pi}{4}$$

Solution:

$$f(2\pi-x)=\frac{x-\pi}{2}=-f(x).$$

$\therefore\ f(x)$ is an odd function

Since $f(x)$ is an odd function,

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi-x}{2}\right)\sin nx\ dx$$

$$=\frac{1}{\pi}\left[(\pi-x)\frac{1}{n}(-\cos nx)-\frac{(-1)1}{n^2}(-\sin nx)\right]_0^{\pi}$$

$$=\frac{1}{n}$$

$\therefore$ The required Fourier series expansion is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}\sin nx$$

$$\frac{\pi-x}{2}=0+0+\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$\frac{\pi-x}{2}=\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

2. **Expand** $f(x)=\dfrac{\pi-x}{2}$ *as a Fourier series expansion in* $0<x<2\pi$ **. Hence deduce that**

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots=\frac{\pi}{4}$$

$$\therefore \frac{\pi-x}{2}=\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots$$

Put $x=\frac{\pi}{2}$,

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\cdots$$

3. **If $f(x)$ given by $f(x)=\begin{cases}-k & \text{for} \quad -\pi<x<0\\ k & \text{for} \quad 0<x<\pi\end{cases}$ Where $k>0$ is a constant, Expand f(x) as a Fourier Series for $-\pi<x<\pi$. Deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\text{.........}=\frac{\pi}{4}$**

Solution: $f(x)=\begin{cases}-k & for \quad -\pi<-x<0 \ \ or \ \ 0<x<\pi\\ k & for \quad 0<-x<\pi \ \ or \ -\pi<x<0\end{cases}=-f(x)$ Therefore, it is an odd function.

$\therefore \ a_0 = 0 = a_n$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx=\frac{2}{\pi}\int_0^{\pi} k\sin nx\, dx$$

$$b_n=\frac{2k}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}=\frac{2k}{\pi n}\left[-\cos n\pi+1\right] \quad =\frac{2k}{\pi n}(1-\cos n\pi)$$

$\therefore$ The Fourier Series is given by $f(x)=\sum_{n=1}^{\infty} b_n \sin(nx)$

$$f(x)=\sum_{n=1}^{\infty}\frac{2k}{\pi n}\left(1-\cos n\pi\right)\cdot \sin nx$$

3. **If *f(x)* given by** $f(x)=\begin{cases} -k & \text{for} \quad -\pi<x<0 \\ k & \text{for} \quad 0<x<\pi \end{cases}$ **Where $k>0$ is a constant, Expand f(x) as a Fourier Series for $-\pi<x<\pi$. Deduce that** $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots\ldots=\frac{\pi}{4}$

$$f(x)=\frac{2k}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n}\right)\sin nx$$

$$f(x)=\frac{4k}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\frac{1}{7}\sin 7x+\ldots\right]$$

Put $x=\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=k$

$$k=\frac{4k}{\pi}\left[\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+\frac{1}{7}\sin\frac{7\pi}{2}+\cdots\right]$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

4. **Obtain the Fourier series for the function *f(x)* = x^2 over the interval $-\pi < x < \pi$. Hence deduce that**

(i) $\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\ldots\ldots = \dfrac{\pi^2}{6}$ **(ii)** $\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\ldots\ldots = \dfrac{\pi^2}{1^2}$ **(VTU 2004)**

Solution: $f(-x) = (-x)^2 = x^2 = f(x)$

$f(x) = x^2$ over the interval $(-\pi, \pi)$ is an even function

$\therefore\ b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3}-0\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0+\frac{2\pi}{n^2}\cos n\pi - 0\right] = \frac{4\pi}{\pi n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

4. **Obtain the Fourier series for the function *f(x)* = x^2 over the interval – π < *x* < π. Hence deduce that**

(i) $$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\,........=\frac{\pi^2}{6}$$ **(ii)** $$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\,........=\frac{\pi^2}{1^2}$$ **(VTU 2004)**

Therefore, $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \quad =\frac{1}{2}\frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$$x^2=\frac{\pi^2}{3}-4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+...\right)$$

Put $x=\pi$,

$$\pi^2=\frac{\pi^2}{3}-4\left(-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-...\right)$$

$$\Rightarrow \frac{2\pi^2}{3}=4\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...\right)$$

$$\therefore \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...=\frac{\pi^2}{6}$$

Put $x = 0$

$$0=\frac{\pi^2}{3}-4\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...\right)$$

$$4\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\right)=\frac{\pi^2}{3}$$

$$\therefore \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$$

5. **Express $f(x) = |x|\ -\pi < x < \pi$, as Fourier series. Hence deduce $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+........=\frac{\pi^2}{8}$**

(VTU 2017)

Solution: $f(-x) = |-x| = |x| = f(x)$

$f(x) = |x|$, over the interval $(-\pi, \pi)$ is an even function

$\therefore \quad b_n = 0. \qquad f(x)=|x|=\begin{cases} x & for \quad x\geq 0 \\ -x & for \quad x<0 \end{cases}$

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx=\frac{2}{\pi}\left[\int_0^{\pi} x\ dx\right]=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\frac{2}{\pi}\ \frac{\pi^2}{2}$$

$a_0 = \pi$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx=\frac{2}{\pi}\int_0^{\pi} x\ .\ \cos nx\ dx\ =\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi n^2}\left[\cos n\pi-1\right]$$

5. **Express *f(x)* = |*x*| – π < *x* < π, as Fourier series. Hence deduce** $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+........=\frac{\pi^2}{8}$

(VTU 2017)

Therefore $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$|x|=\frac{\pi}{2}+\sum_{n=1}^{\pi}\frac{2}{\pi n^2}(\cos n\pi-1)\cos nx$$

Put $x = 0$,

$$0=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(\cos n\pi-1)}{n^2}$$

$$\frac{-\pi}{2}=\frac{2}{\pi}\left[\frac{-2}{1^2}+0-\frac{2}{3^2}+0-\frac{2}{5^2}+\cdots\right]$$

$$\frac{-4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]=\frac{-\pi}{2}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$$

6. **If $f(x) = x(2\pi - x)$ over $[0, 2\pi]$. Find the Fourier series expansion of $f(x)$ and hence deduce that,**

(i) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ **(ii)** $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

Solution: $f(2\pi - x) = (2\pi - x)x = f(x)$ It is an even function.

$\therefore$ $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (2\pi x - x^2)dx = \frac{2}{\pi}\left[\pi x^2 - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{2\pi^3}{3}\right] = \frac{4\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (2\pi x - x^2)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[(2\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (2\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\left(\frac{2\pi}{n^2}\right)\right] = -\frac{4}{n^2}$$

6. **If $f(x) = x(2\pi - x)$ over $[0, 2\pi]$. Find the Fourier series expansion of $f(x)$ and hence deduce that,**

(i) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ **(ii)** $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

Therefore, $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

$$x(2\pi - x) = \frac{1}{2} \cdot \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{-4}{n^2} \right) \cos nx$$

$$x(2\pi - x) = \frac{2\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

Put $x = 0$ $\quad 0 = \dfrac{2\pi^2}{3} - 4 \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2} \Rightarrow \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6} \qquad (1)$$

Put $x = \pi$

$$\pi^2 = \frac{2\pi^2}{3} - 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots \right]$$

$$4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots \right] = \frac{2\pi^2}{3} - \pi^2$$

$$-4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \right] = \frac{-\pi^2}{3}$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \qquad (2)$$

Add (1) and (2) to get the value of $\frac{\pi^2}{8}$.

7. **Find the Fourier Series expansion of the function** $f(x)=\begin{cases} x, & 0\le x\le \pi \\ 2\pi - x, & \pi < x \le 2\pi \end{cases}$ **and hence deduce that** $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

Solution:

$$f(x)=\begin{cases} \phi(x)=x, & 0\le x\le \pi \\ \chi(x)=2\pi-x, & \pi\le x\le 2\pi \end{cases}$$

$\chi(2\pi-x)=2\pi-(2\pi-x)=x=\phi(x).$ $f(x)$ is an even function

Therefore, $b_n=0$

$$a_0=\frac{2}{\pi}\int_0^{\pi} x\,dx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\pi$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} x\cos nx\;dx=\frac{2}{\pi}\left[\frac{x\sin nx}{n}-\frac{(1)(-\cos nx)}{n^2}\right]_0^{\pi}$$

$$=\frac{2}{\pi n^2}\left[(-1)^n-1\right]$$

7. **Find the Fourier Series expansion of the function** $f(x)=\begin{cases} x, & 0\le x\le\pi \\ 2\pi-x, & \pi<x\le 2\pi \end{cases}$ **and hence deduce that** $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

Therefore, $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi n^2}\{(-1)^n-1\}\cos nx$$

Put $x=0$

$$0=\frac{\pi}{2}+\frac{2}{\pi}\left\{-\frac{2}{1^2}-\frac{2}{3^2}-\frac{2}{5^2}\dots\right\}=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$

8. **Obtain the Fourier series of $f(x) = 2x - x^2$ in $(0, 2)$**

Solution: Here the interval is $(0, 2)$

It is in the form of $(0, 2l)$

$\therefore \quad 2l = 2 \Rightarrow l = 1$

$$f(2l - x) = f(2 - x) = 2(2 - x)^2 - (2 - x)^2$$

$$= (2 - x)[2 - (2 - x)] = (2 - x)\,x = 2x - x^2 = f(x)$$

$\therefore \quad f(x) = 2x - x^2$ is even and hence $b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= 2\int_0^1 \left(2x - x^2\right) dx = 2\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left\{\left[1 - \frac{1}{3}\right] - [0]\right\} = \frac{4}{3}$$

8. **Obtain the Fourier series of $f(x) = 2x - x^2$ in $(0, 2)$**

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{1}\int_0^1 (2x - x^2)\cos nx\,dx$$

$$= 2\left[(2x - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (2 - 2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left\{[0 + 0 + 0] - \left[0 + \left(\frac{2}{n^2\pi^2}\right) + 0\right]\right\} = \frac{-4}{n^2\pi^2}$$

$\therefore$ Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2}\cos n\pi x$$

9. **Obtain the Fourier series expansion of** $f(x)=\begin{cases} 2 & \text{for} \quad -2\le x\le 0 \\ x & \text{for} \quad 0<x<2 \end{cases}$

Solution:

The given function *f(x)* defined in the interval (–2, 2) = (–*l*, *l*) ⇒ *l* = 2

$$a_0=\frac{1}{l}\int_{-l}^{l} f(x)dx=\frac{1}{2}\int_{-2}^{2} f(x)dx$$

$$=\frac{1}{2}\left[\int_{-2}^{0} 2dx+\int_{0}^{2} x\ dx\right]=\frac{1}{2}\left\{[2x]_{-2}^{0}+\left[\frac{x^2}{2}\right]_0^2\right\}$$

$$=\frac{1}{2}\left[0-2(-2)+\frac{4}{2}-0\right]=\frac{1}{2}(4+2)=\frac{1}{2}(6)=3$$

$$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)\ dx=\frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)\ dx$$

$$=\frac{1}{2}\left[\int_{-2}^{0} 2\cos\left(\frac{n\pi x}{2}\right)\ dx+\int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)\ dx\right]$$

$$a_n = \frac{1}{2}\left\{\left[2\cdot\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-2}^{0} + \left[x\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right) - (1)\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right)\right]_{0}^{2}\right\}$$

$$= \frac{1}{2}.\frac{4}{n^2\pi^2}(\cos n\pi - 1) = \frac{2}{n^2\pi^2}(\cos n\pi - 1)$$

$$b_n = \frac{1}{l}\int_{-l}^{1} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\sin\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{2} x\sin\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \frac{1}{2}\left\{2\left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_{-2}^{0} + \left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(\frac{-\sin\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right)\right]_{0}^{2}\right\}$$

$$b_n = \frac{1}{2}\left[\frac{-4}{n\pi}(1-\cos n\pi) - \frac{4}{n\pi}\cos n\pi + 0 - 0 - 0\right]$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\left\{-1+\cos n\pi - \cos n\pi\right\}\right] = \frac{-2}{n\pi}$$

The Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2(\cos n\pi - 1)}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty}\left(\frac{-2}{n\pi}\right)\sin\left(\frac{n\pi x}{2}\right)$$

$$= \frac{3}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos n\pi - 1}{n^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{2}\right)$$

10. **Find a Fourier Series to represent *f(x)* = *x* – *x*² from *x* = – π to *x* = π**

Solution:

For the given function *f(x)* = x – x^{2,} defined over the interval (−π, π).

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right] = \frac{-2\pi^3}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(\frac{\sin nx}{n}\right) - (1 - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[0+(1-2\pi)\frac{\cos n\pi}{n^2}+0-0-(1+2\pi)\frac{\cos n\pi}{n^2}-0\right]$$

$$= \frac{1}{\pi}\left[(1-2\pi-1-2\pi)\frac{\cos n\pi}{n^2}\right]$$

$$= \frac{-4}{n^2}\cos n\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right)-(1-2x)\left(-\frac{\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi-\pi^2+\pi+\pi^2)\frac{\cos n\pi}{n}\right]$$

$$= -\frac{2}{n}\cos n\pi$$

Therefore, the Fourier series is given by $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n \cos nx+\sum_{n=1}^{\infty}b_n \sin nx$

We get $$x-x^2=\frac{-\pi^3}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$