# Module-3

# RELATIONS AND FUNCTIONS

## PART - 2

## RELATIONS AND THEIR PROPERTIES

# RELATIONS

Let A and B be any two sets. Any subset of the Cartesian product A $\times$ B is called a **(binary) relation** from A to B.

Any subset of the Cartesian product A $\times$ A is called a relation from A to A or a relation on set A.

We use the notation aRb to denote that (a, b) $\in$ R and a$\not R$b to denote that (a, b) $\notin$ R. Moreover, when (a, b) belongs to R, a is said to be related to b by R.

Let A and B be finite sets with $|A| = m$ and $|B| = n$ then the number of relations from A to B $2^{mn}$.

The number of relations on set A = $2^{m^2}$.

The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the **domain** of the relation R. The set of all second elements in a relation R from a set A to a set B is called the **range** of the relation R. The whole set B is called the codomain of the relation R.

## Examples:

**Let A = {1, 2} and B = {3, 4} then find the number of relations from A to B.**

**Solution:**

Here, $|A| = m = 2$ and $|B| = n = 2$.

$\therefore$ Number of relations from A to B = $2^{mn} = 2^{2 \times 2} = 2^4 = 16$

**Let A = {x, y, z} and B = {1, 2}. Find the number of relations from A to B.**

**Solution:**

Here, $|A| = m = 3$ and $|B| = n = 2$.

$\therefore$ Number of relations from A to B = $2^{mn} = 2^{3 \times 2} = 2^6 = 64$

**If there are 4096 relations from the finite sets A and B with $|B| = 3$ then find $|A|$?**

**Solution:**

Let $|A| = m$ and given $|B| = n = 3$.

Number of relations from A to B $= 2^{mn} = 4096$

$\Rightarrow \quad 2^{m \times 3} = 4096$

$\Rightarrow \quad 2^{3m} = 2^{12}$

$\Rightarrow \quad 3m = 12$

$\Rightarrow \quad m = 4$

$\therefore \; |A| = 4$

**Let A = {1, 2, 3, 5} and B = {4, 6, 9}. Define a relation R from A to B by R = {(x, y): the difference between x and y is odd; x ∈ A, y ∈ B}. Write the elements R and write down the domain, codomain and range of R.**

**Solution:**

The Cartesian product of A and B is

$A \times B = \{(1, 4), (1, 6), (1, 9), (2, 4), (2, 6), (2, 9), (3, 4), (3, 6), (3, 9), (5, 4), (5, 6), (5, 9)\}$

By the definition of the relation, we have

$R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

Domain = {1, 2, 3, 5},

codomain = {4, 6, 9} and

range = {4, 6, 9}.

**Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by R = {(x, y): y = x + 1} Write the elements of R and write down the domain, codomain and range of R.**

**Solution:**

By the definition of the relation,

R = {(1,2), (2,3), (3,4), (4,5), (5,6)}.

Domain = {1, 2, 3, 4, 5,}, codomain = {1, 2, 3, 4, 5, 6} and

range = {2, 3, 4, 5, 6}

**Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?**

**Solution:**

Because (a, b) is in R if and only if a and b are positive integers

not exceeding 4 such that a divides b, we see that

R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.

**Define a relation R on the set N of natural numbers by R = {(x, y): y = x + 5, x is a natural number less than 4; x, y ∈ N}. Write the elements of R.**

**Solution:**

$x \in N$ and $x < 4 \Rightarrow x = \{1, 2, 3\}$

By definition of R, we have

$R = \{(1, 6), (2, 7), (3, 8)\}$

**Let A = {1, 2, 3} and B = {2, 4, 5}. Determine the following:**

- **$|A \times B|$**
- **Number of relations from A to B**
- **Number of relations on A**
- **Number of relations from A to B that contain (1, 2) and (1, 5).**
- **Number of relations from A to B that contain exactly 5 ordered pairs.**
- **Number of relations on A that contain at least 7 ordered pairs.**

(i) $|A \times B| = 3 \times 3 = 9$

(ii) Number of relations from A to B $= 2^{mn} = 2^{3 \times 3} = 2^9 = 512$

(iii) Number of relations on A = $2^{m^2} = 2^9 = 512$

(iv) Let $R_1 = \{(1, 2), (1, 5)\}$, we note that every relation from A to B that contains the elements of (1, 2), (1, 5) is of the form $R_1 \cup R_2$ where $R_2 \subseteq R_1'$ in $A \times B$

$|R_1'| = |A \times B| - |R_1| = 9 - 2 = 7.$

Number of subsets of $R_1'$ is $2^7 = 128$.

Thus there are 128 number of relations from A to B that contain the elements (1, 2) and (1, 5).

(v) Number of relations from A to B that contain exactly 5 ordered pairs is $\binom{9}{5} = 126.$

(vi) Number of relations on A that contain at least 7 ordered pairs is $\binom{9}{7} + \binom{9}{8} + \binom{9}{9} = 46.$

# PROPERTIES OF RELATIONS

- A relation R on a set A is called **empty relation**, if no element of A is related to any element of A,

  i.e., $R = \phi \subset A \times A$.

- A relation R in a set A is called **universal relation**, if each element of A is related to every element of A,

  i.e., $R = A \times A$.

**Note:** Both the empty relation and the universal relation are sometimes called trivial relations.

A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

$$\text{i.e., } \forall\, a \in A \Rightarrow (a, a) \in R.$$

**Example:**

**Which of the following relations on set $A = \{1, 2, 3, 4\}$ are reflexive?**

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

**Solution:**

The relations $R_3$ and $R_5$ are reflexive because they both contain all pairs of the form (a, a), namely, (1, 1), (2, 2), (3, 3), and (4, 4). The other relations are not reflexive because they do not contain all of these ordered pairs.

A relation R on a set A is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.

i.e., $\forall\, a \in A \Rightarrow (a, a) \notin R$

**Example:**

**Which of the following relations on set $A = \{1, 2, 3, 4\}$ are irreflexive?**

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

**Solution:**

The relations $R_4$ and $R_6$ are irreflexive because they both do not contain all pairs of the form (a, a), namely, (1, 1), (2, 2), (3, 3), and (4, 4). The other relations are not irreflexive because they contain at least one of these ordered pairs.

A relation R on a set A is called **symmetric** if (b, a) ∈ R whenever (a, b) ∈ R for all a, b ∈ A.

i.e., $\forall\ (a, b) \in R \Rightarrow (b, a) \in R$

**Example:**

**Which of the following relations on set A = {1, 2, 3, 4} are symmetric?**

**$R_1$ = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)},**

**$R_2$ = {(1, 1), (1, 2), (2, 1)},**

**$R_3$ = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)},**

**$R_4$ = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)},**

**$R_5$ = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)},**

**$R_6$ = {(3, 4)}.**

**Solution:**

The relations $R_2$ and $R_3$ are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. The other relations are not symmetric. This is done by finding a pair (a, b) such that it is in the relation but (b, a) is not.

A relation R on a set A is called **asymmetric** if $(b, a) \notin R$ whenever $(a, b) \in R$ for all $a, b \in A$.

i.e., $\forall\ (a, b) \in R \Rightarrow (b, a) \notin R$

**Example:**

**Which of the following relations on set $A = \{1, 2, 3, 4\}$ are asymmetric?**

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

**Solution:**

The relations $R_4$ and $R_6$ are asymmetric, because in each case (b, a) does not belongs to the relation whenever (a, b) does. The other relations are not asymmetric. This is done by finding at least one pair (a, b) such that it is in the relation implies (b, a) is also present.

A relation R on a set A is called **antisymmetric** if for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

i.e., $\forall\ (a, b), (b, a) \in R \Rightarrow a = b$ or

$\forall\ (a, b) \in R, (b, a) \notin R \Rightarrow a \neq b$

**Example:**

**Which of the following relations on set $A = \{1, 2, 3, 4\}$ are asymmetric?**

**$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,**

**$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,**

**$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,**

**$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,**

**$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,**

**$R_6 = \{(3, 4)\}$.**

**Solution:**

The relations $R_4$, $R_5$ and $R_6$ are antisymmetric, because for all (a, b), (b, a) $\in$ R implies that a = b or for all (a, b) $\in$ R, (b, a) $\notin$ R implies that a $\neq$ b. The other relations are not antisymmetric. This is done by finding pairs (a, b), (b, a) $\in$ R which implies that a $\neq$ b.

A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

$$\text{i.e., } \forall\ (a, b), (b, c) \in R \Rightarrow (a, c) \in R$$

**Example:**

**Which of the following relations on set $A = \{1, 2, 3, 4\}$ are transitive?**

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

**Solution:**

The relations $R_4, R_5$ and $R_6$ are transitive, because for all (a, b), (b, c) belongs to the relation imply that (a, c) belongs the relation. The other relations are not transitive. This is done by finding a pairs (a, b) and (b, c) belongs to the relation implies that $(a, c) \notin R$.

A relation R on a set A is called an **Equivalence relation** if it is reflexive, symmetric and transitive.

**Example:**

**Which of the following relations on set $A = \{1, 2, 3, 4\}$ are Equivalence?**

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

**Solution:**

All the above relations are not equivalence.

**Example:**

**Which of the following relations on set A = {1, 2, 3, 4} are Equivalence?**

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (3, 3), (4, 1), (4, 2), (4, 4)\}$,

$R_4 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (3, 4), (4, 3), (4, 4)\}$,

The relations $R_2$ and $R_3$ are equivalence relations.