MODULE - 1

FUNDAMENTALS OF LOGIC

1.1 PROPOSITIONAL LOGIC

Introduction

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid arguments. A major goal of this topic is to read how to understand and how to construct correct mathematical arguments; we begin our study of discrete mathematics with an introduction to logic.

In addition to its importance in understanding mathematical reasoning, logic has numerous applications in computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways. Furthermore, software systems have been developed for constructing proofs automatically. We will discuss these applications of logic in the upcoming chapters.

Definition 1.1: PROPOSITIONS

A declarative sentence which is either true or false but not both at the same time is called a Proposition. Propositions are usually denoted by the lowercase letters $p, q, r, s,...$

The truth value of a proposition is true, denoted by T or 1, if it is a true proposition and false, denoted by F or 0, if it is a false proposition.

Examples for proposition:

1. Bangalore is the capital city of India.
2. Three is a prime number.
3. $\sqrt{2}$ is an irrational number.
4. $\sqrt{2} + \sqrt{3} = \sqrt{5}$.
5. $3^2 + 4^2 = 7^2$.

The truth values of the above propositions are F, T, T, F and F.

Note: A sentences which involves with exclamation, question are not propositions.

Examples for not proposition:

1. Good Morning!
2. What time is it?
3. Read this carefully.
4. x + 1 = 2.

Definition 1.2: COMPOUND PROPOSITIONS

Mathematical statements are constructed by combining one or more simple propositions. New propositions, called compound propositions, are formed from existing propositions using logical operators. The words "and", "or', "If … then", "iff", "not" are used to form compound propositions which are called Logical connectives.

The following table gives name of the connectives and their symbols

S.No	Connectives	Name of the connectives	Symbols
1.	and	Conjunction	$\wedge$
2.	or	Disjunction	$\vee$
3.	If … then	Conditional	$\rightarrow$
4.	iff	Biconditional	$\leftrightarrow$
5.	not	Negation	$\sim$ or $\neg$

Definition 1.3: NEGATION

Let p be a proposition. Then the proposition "not p" is called the negation of p, and is denoted by $\sim p$ or $\neg p$. The truth value of $\sim p$ is the opposite of the truth value of p.

<u>Truth Table for the Negation</u>

p	Ė ***p***
T	**F**
F	**T**

EXAMPLES:

1. Let p: "Michael's PC runs Linux" then its negation is
   $\sim p$: "Michael's PC does not run Linux."

2. Let q: "Vandana's smart phone has at least 32GB of memory" then its negation is
   $\sim q$: "Vandana's smart phone does not have at least 32GB of memory"
   or even more simply as
   "Vandana's smartphone has less than 32GB of memory."

Definition 1.4: CONJUNCTION

Let p and q be propositions. Then the compound proposition "***p* and *q***" is called a **conjunction** and is denoted by $\boldsymbol{p \wedge q}$. The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Truth Table for the Conjunction

p	q	$p \wedge q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

EXAMPLE: Let p: "Rebecca's PC has more than 16 GB free hard disk space" and q: "The processor in Rebecca's PC runs faster than 1 GHz" then its conjunction is

$p \wedge q$: "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz."

Definition 1.5: DISJUNCTION

Let p and q be propositions. Then the compound proposition "***p* or *q***" is called a **disjunction** and is denoted by $\boldsymbol{p \vee q}$. The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Truth Table for the Disjunction

p	q	$p \vee q$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

EXAMPLE: Let *p*: "Rebecca's PC has more than 16 GB free hard disk space" and *q*: "The processor in Rebecca's PC runs faster than 1 GHz" then its disjunction is

$p \vee q$:"Rebecca's PC has at least 16 GB free hard disk space, or the processor in Rebecca's PC runs faster than 1 GHz."

Definition 1.6: Exclusive or

Let *p* and *q* be propositions. The exclusive or of p and q, denoted by $\boldsymbol{p \veebar q}$, is the proposition that is true when exactly one of *p* and *q* is true and is false otherwise.

<u>Truth Table for the Exclusive or</u>

p	q	$p \veebar q$
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

<u>Example:</u>

"Students, who have taken calculus or computer science, but not both, can enrol in this class."

Here, we mean that students who have taken both calculus and a computer science course cannot take the class. Only those who have taken exactly one of the two courses can take the class.

Similarly, when a menu at a restaurant states, "Soup or salad comes with an entrée," the restaurant almost always means that customers can have either soup or salad, but not both. Hence, this is an exclusive, or.

Definition 1.7: CONDITIONAL

Let *p* and *q* be propositions. Then the compound proposition "**If *p* then *q***" is called a **conditional** and is denoted by $\boldsymbol{p \rightarrow q}$. The conditional statement $p \rightarrow q$ is false when *p* is true and *q* is false, and true otherwise.

Truth Table for the Conditional

p	q	$p \to q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

EXAMPLE: Let p: "Maria learns discrete mathematics" and q: "Maria will find a good job" then its conditional is

$p \to q$: "If Maria learns discrete mathematics, then she will find a good job."

Definition 1.8: BICONDITIONAL

Let p and q be propositions. Then the compound proposition "p **iff** $\boldsymbol{q}$" is called a **biconditional** and is denoted by $\boldsymbol{p \leftrightarrow q}$. The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.

Truth Table for the biconditional

p	q	$p \leftrightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

EXAMPLE: Let p: "You can take the flight," and let q: "You buy a ticket." then its biconditional is

$p \leftrightarrow q$:"You can take the flight if and only if you buy a ticket."

Construction of Truth table

A truth table is a complete list of all the possible permutations of truth and falsity for a set of simple statements, showing the effect of each permutation on the truth value of a compound having those simple statements as components. Each permutation of truth values constitutes one row of a truth table and the number of rows in a truth table is 2^n where n

equals the number of simple statements. In order to determine the truth value of a compound, examine the column under the dominant operator for that compound.

There is a completely mechanical procedure for constructing a truth table for a sentence. Three skills are necessary in order to do so:

- Producing an initial list of all permutations of truth and falsity assigned to the sentence letters in the sentence.
- Determining which columns in a truth table can be filled in at a given stage.
- Determining what truth values to fill into a given column.

EXAMPLE:

1. How many rows are needed to construct a truth table for the compound proposition $(p \vee \neg q) \leftrightarrow [(\neg r \wedge s) \rightarrow t]$, where p, q, r, s and t primitive statements?

Solution: Here, $n = 5$

Number of rows required to construct its truth table is $2^n = 2^5 = 32$

2. Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$.

Solution:

p	q	Ė q	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	**T**	**F**	**T**	**T**	**T**
T	**F**	**T**	**T**	**F**	**F**
F	**T**	**F**	**F**	**F**	**T**
F	**F**	**T**	**T**	**F**	**F**

1.2 TAUTOLOGY, CONTRADICTION AND CONTINGENCY

Definition 1.9: A compound proposition is said to be a **Tautology** if it is always **true** for all possible combinations of the truth values of its components. A compound proposition is said to be a **Contradiction** if it is always **false** for all possible combinations of the truth values of its components. A compound proposition is said to be a **Contingency** if it is neither Tautology nor Contradiction.

Example:

Verify the following compound propositions for Tautology, Contradiction or Contingency

(i) $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$

(ii) $[(p \to q) \land (q \to r)] \to (p \to r)$

(iii) $q \leftrightarrow (\neg p \lor \neg q)$

Solution:

(i) Truth table for $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is

p	q	r	$q \to r$	$p \to (q \to r)$ (a)	$(p \to q)$ x	$(p \to r)$ y	x → y (b)	(a) → (b)
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

From the last column, it is observed that all truth values are T.

Hence, the compound proposition $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is a Tautology.

(ii) Truth table for $[(p \to q) \land (q \to r)] \to (p \to r)$ is

p	q	r	$p \to q$	$q \to r$	$[(p \to q) \land (q \to r)]$ (x)	$p \to r$ (y)	$(x) \to (y)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the last column, it is observed that all truth values are T.

Hence, the compound proposition $[(p \to q) \land (q \to r)] \to (p \to r)$ is a Tautology.

(iii) Truth table for $q \leftrightarrow (\neg p \vee \neg q)$ is

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$q \leftrightarrow (\neg p \vee \neg q)$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	T
F	F	T	T	T	F

From the last column, it is observed that truth values are T or F.

Hence, the compound proposition $q \leftrightarrow (\neg p \vee \neg q)$ is a contingency.

1.3 LOGICAL EQUIVALENCE

Definition 1.10: Two compound propositions A and B are said to be **logically equivalent** if their truth values are identical and in symbols this can be written as $A \equiv B$ or $A \Leftrightarrow B$.

EXAMPLE:

1. **A demonstration that $\neg (p \vee q)$ and $\neg p \wedge \neg q$ are logically Equivalent.**

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg (p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the last two columns of the truth table, the truth values are identical.

Hence, $\neg (p \vee q) \equiv \neg p \wedge \neg q$

2. **A demonstration that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are Logically Equivalent.**

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$(p \vee q)$	$(p \vee r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

From the 4th and 8th columns of truth table, the truth values are identical.

Hence, $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

LAWS OF LOGIC:

(1) Idempotent laws	(i) $p \lor p \Leftrightarrow p$ (ii) $p \land p \Leftrightarrow p$
(2) Law of Double negation	$\neg(\neg p) \Leftrightarrow p$
(3) Commutative laws	(i) $p \lor q \Leftrightarrow q \lor p$ (ii) $p \land q \Leftrightarrow q \land p$
(4) Associative laws	(i) $(p \lor q) \lor r \Leftrightarrow p \lor (q \lor r)$ (ii) $(p \land q) \land r \Leftrightarrow p \land (q \land r)$
(5) Distributive laws	(i) $p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$ (ii) $p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$
(6) De Morgan's laws	(i) $\neg(p \land q) \Leftrightarrow \neg p \lor \neg q$ (ii) $\neg(p \lor q) \Leftrightarrow \neg p \land \neg q$
(7) Inverse laws	(i) $p \lor \neg p \Leftrightarrow \mathrm{T}$ (ii) $p \land \neg p \Leftrightarrow \mathrm{F}$
(8) Identity laws	(i) $p \land \mathrm{T} \Leftrightarrow p$ (ii) $p \lor \mathrm{F} \Leftrightarrow p$
(9) Domination laws	(i) $p \lor \mathrm{T} \Leftrightarrow \mathrm{T}$ (ii) $p \land \mathrm{F} \Leftrightarrow \mathrm{F}$
(10) Absorption laws	(i) $p \lor (p \land q) \Leftrightarrow p$ (ii) $p \land (p \lor q) \Leftrightarrow p$
(11) Conditional laws	(i) $p \to q \Leftrightarrow \neg p \lor q$ (ii) $p \to q \Leftrightarrow \neg q \to \neg p$ (iii) $p \lor q \Leftrightarrow \neg p \to q$ (iv) $p \land q \Leftrightarrow \neg(p \to \neg q)$

	(v) $\neg (p \to q) \Leftrightarrow p \wedge \neg q$ (vi) $(p \to q) \wedge (p \to r) \Leftrightarrow p \to (q \wedge r)$ (vii) $(p \to r) \wedge (q \to r) \Leftrightarrow (p \vee q) \to r$ (viii) $(p \to q) \vee (p \to r) \Leftrightarrow p \to (q \vee r)$ (ix) $(p \to r) \vee (q \to r) \Leftrightarrow (p \wedge q) \to r$
(12) Biconditional laws	(i) $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$ (ii) $p \leftrightarrow q \Leftrightarrow \neg p \leftrightarrow \neg q$ (iii) $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$ (iv) $\neg (p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \neg q$

Note: The above laws can be verified by constructing truth tables.

<u>Logically equivalent without constructing a truth table</u>

By using the laws of logic, we can verify the given compound propositions are logically equivalent or not. We illustrate this by the following examples:

EXAMPLES:

1. **Show that Ė (*p* ∨ (Ė *p* ∧ *q*)) and Ė *p* ∧ Ė *q* are logically equivalent without constructing a truth table.**

Solution:

Consider, $\neg (p \vee (\neg p \wedge q))$

$\equiv \neg p \wedge \neg (\neg p \wedge q)$	by the second De Morgan law
$\equiv \neg p \wedge [\neg (\neg p) \vee \neg q]$	by the first De Morgan law
$\equiv \neg p \wedge (p \vee \neg q)$	by the double negation law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
$\equiv F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
$\equiv (\neg p \wedge \neg q) \vee F$	by the commutative law for disjunction
$\equiv \neg p \wedge \neg q$	by the identity law for F

Consequently $\neg (p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

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2. **Show that Ė [Ė [($p \vee q$) $\wedge$ r] $\vee$ Ė q] and ($q \wedge r$) are logically equivalent without constructing a truth table.**

Solution: Consider, $\neg [\neg [(p \vee q) \wedge r] \vee \neg q]$

$\equiv \neg \neg [(p \vee q) \wedge r] \wedge \neg \neg q$ by De Morgan's law

$\equiv [(p \vee q) \wedge r] \wedge q$ by law of double negation

$\equiv (p \vee q) \wedge (r \wedge q)$ by associative law

$\equiv (p \vee q) \wedge (q \wedge r)$ by commutative law

$\equiv [(p \vee q) \wedge q] \wedge r$ by associative law

$\equiv (q \wedge r)$ by absorption law

Consequently $\neg (p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

3. **Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology without constructing a truth table.**

Solution:

To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

$(p \wedge q) \rightarrow (p \vee q) \equiv \neg (p \wedge q) \vee (p \vee q)$ by conditional law

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ by the first De Morgan law

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ by the associative and commutative laws

$\equiv T \vee T$ by inverse laws

$\equiv T$ by the domination law

Consequently, $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

1.4 CONVERSE, INVERSE, AND CONTRAPOSITIVE

If $p \rightarrow q$ is a conditional statement then

(i) The proposition $q \rightarrow p$ is called the **converse** of $p \rightarrow q$.

(ii) The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$.

(iii) The proposition $\neg q \rightarrow \neg p$ is called the **contrapositive** of $p \rightarrow q$.

Note: Only the contrapositive is equivalent to the original statement.

EXAMPLE:

1. "If oxygen is a gas then Gold is compound"

Let p: oxygen is a gas

q: Gold is compound

Given proposition is $p \to q$

The converse is $q \to p$

i.e., "If Gold is compound then oxygen is a gas."

The inverse is $\neg p \to \neg q$

i.e., "If oxygen is not a gas then Gold is not a compound"

The contrapositive is $\neg q \to \neg p$

i.e., "If Gold is not compound then oxygen is not a gas."

2. "If it is raining, then the home team wins."

Let p: It is raining

q: Home team wins

Given proposition is $p \to q$

The converse is $q \to p$

i.e., "If the home team wins, then it is raining."

The inverse is $\neg p \to \neg q$

i.e., "If it is not raining, then the home team does not win."

The contrapositive is $\neg q \to \neg p$

i.e., "If the home team does not win, then it is not raining."

Definition 1.11: DUAL

Let s be a statement. If s contains no logical connectives other than $\vee$, $\wedge$ and $\neg$ then the dual of s is denoted by s^d, is the statement obtained from s by replacing each occurrence of $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F, and each F by T.

EXAMPLE:

1. Let s: $\neg (p \wedge q) \vee (p \vee q)$ then its dual is s^d: $\neg (p \vee q) \wedge (p \wedge q)$
2. Let s: $(p \wedge \neg q) \vee (r \wedge T)$ then its dual is s^d: $(p \vee \neg q) \wedge (r \vee F)$

Note: $(s^d)^d \Leftrightarrow s$

The principle of Duality: Let s and t be statements that contain no logical connectives other than $\vee$, $\wedge$, and $\neg$. If $s \Leftrightarrow t$ then $s^d \Leftrightarrow t^d$.

Example:

Verify the principle of duality for the logical equivalence:

$$[\neg(p \wedge q) \rightarrow \neg p \vee (\neg p \vee q)] \Leftrightarrow \neg p \vee q$$

Solution:

Principle of duality: If $s \Leftrightarrow t$ then $s^d \Leftrightarrow t^d$.

If $s \Leftrightarrow t$ then

s^d	$\Leftrightarrow [\neg(p \wedge q) \rightarrow \neg p \vee (\neg p \vee q)]^d$	by data
	$\Leftrightarrow [(p \wedge q) \vee \neg p \vee (\neg p \vee q)]^d$	by the definition of conditional
	$\Leftrightarrow [(p \wedge q) \vee (\neg p \vee \neg p) \vee q]^d$	by Associative law
	$\Leftrightarrow [(p \wedge q) \vee \neg p \vee q]^d$	by idempotent law
	$\Leftrightarrow [(p \wedge q) \vee q \vee \neg p]^d$	by commutative law
	$\Leftrightarrow [q \vee \neg p]^d$	by Absorption law
	$\Leftrightarrow [\neg p \vee q]^d$	by Commutative law
	$\Leftrightarrow t^d$	by data

Hence, the principle of duality is verified.

1.5 OPERATORS NAND and NOR

The NAND operator denoted by $\uparrow$, also known as the Sheffer stroke, is a connective in logic equivalent to the composition NOT AND that yields true if any condition is false, and false if all conditions are true.

If p and q are two primitive statements then

$$(p \uparrow q) \Leftrightarrow \neg (p \wedge q).$$

<u>Truth Table for the operator NAND</u>

p	***q***	***p* ë *q***
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**T**

The NOR operator denoted by $\downarrow$, also known as the **Peirce's arrow**, is a connective in logic equivalent to the composition NOT OR that yields true if both condition is false, and it is false otherwise.

If p and q are two primitive statements then

$$(p \downarrow q) \Leftrightarrow \neg\,(p \vee q).$$

Truth Table for the operator NOR

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Example: For any two statements p, q prove the following

(i) $\neg\,(p \downarrow q) \Leftrightarrow (\neg\, p \uparrow \neg\, q)$, (ii) $\neg\,(p \uparrow q) \Leftrightarrow (\neg\, p \downarrow \neg\, q)$.

Solution:

(i) Truth table for the given compound proposition is

p	q	$\neg p$	$\neg q$	$p \downarrow q$	$\neg(p \downarrow q)$	$(\neg p \uparrow \neg q)$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

From the last two columns of the truth table, we have

$\neg\,(p \downarrow q) \Leftrightarrow (\neg\, p \uparrow \neg\, q)$.

(ii) Truth table for the given compound proposition is

p	q	$\neg p$	$\neg q$	$p \uparrow q$	$\neg(p \uparrow q)$	$(\neg p \downarrow \neg q)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

From the last two columns of the truth table, we have

$\neg\,(p \uparrow q) \Leftrightarrow (\neg\, p \downarrow \neg\, q)$.

PROBLEMS TO PRACTICE

1. Consider the following propositions concerned with a certain triangle

***p*: Δ ABC is isosceles, *q*: Δ ABC is equilateral, r : Δ ABC is equiangular.**

Write down the following propositions in words:

(a) $p \wedge \neg q$ **, (b)** $p \wedge \neg q$ **, (c)** $p \rightarrow q$**, (d)** $q \rightarrow p$**, (e)** $\neg r \rightarrow \neg q$**, (f)** $p \rightarrow \neg q$**.**

Solution:

(a) Δ ABC is isosceles and is not equilateral.

(b) Δ ABC is either not isosceles or equilateral.

(c) If Δ ABC is isosceles then it is equilateral.

(d) If Δ ABC is equilateral then it is isosceles.

(e) If Δ ABC is not equiangular then it is not equilateral.

(f) If Δ ABC is isosceles then it is not equilateral.

2. Given that p is true and q is false find the truth values of the following:

(a) $\neg p \wedge q$ **,** **(b)** $\neg(p \wedge q) \vee \neg(p \leftrightarrow q)$**,** **(c)** $\neg(p \rightarrow \neg q)$**,**

(d) $(p \rightarrow q) \vee \neg(p \leftrightarrow \neg q)$**,** **(e)** $(p \wedge q) \rightarrow (p \vee q)$**,** **(f)** $(p \rightarrow \neg q) \vee (q \rightarrow \neg p)$

Solution:

(a) $F \wedge F \Leftrightarrow F$

(b) $\neg(T \wedge F) \vee \neg(T \leftrightarrow F) \Leftrightarrow T \vee T \Leftrightarrow T$

(c) $\neg(T \vee T) \Leftrightarrow F$

(d) $(T \rightarrow F) \vee \neg(T \leftrightarrow T) \Leftrightarrow F \vee F \Leftrightarrow F$

(e) $(T \wedge F) \rightarrow (T \vee F) \Leftrightarrow F \rightarrow T \Leftrightarrow T$

(f) $(T \rightarrow T) \vee (F \rightarrow F) \Leftrightarrow (T \vee T) \Leftrightarrow T$

3. Determine the truth values of the following:

(a) $p \wedge q$ **is false and *q* is true. Find the truth value if *p*.**

(b) $p \vee q$ **is false and *q* is false. Find the truth value of *p*.**

(c) $p \rightarrow q$ **is true and *q* is false. Find *p*.**

(d) $p \leftrightarrow q$ **is true and *p* is false. Find *q*.**

(e) $p \rightarrow q$ **is false and *p* is false.**

Solution:

(a) $p \wedge T$ is false $\Leftrightarrow$ p is false.

(b) $p \vee F$ is false $\Leftrightarrow$ p is false.

(c) $p \to F$ is true $\Leftrightarrow$ p is false.

(d) $p \leftrightarrow F$ is true.

(e) $F \to q$ is false $\Leftrightarrow$ q may be true or false.

4. Find the possible truth values of *p, q* and *r* in the following cases:

(a) $p \to (q \vee r)$ **is false** **(b)** $p \wedge (q \to r)$ **is true.** **[July '09]**

Solution:

(a)

$p \to (q \vee r)$	p	$q \vee r$	q	r
0	1	0	0	0

Truth values of *p, q* and *r* are 1, 0, 0 respectively.

(b)

$p \wedge (q \to r)$	p	$q \to r$	q	r
1	1	1	0 or 1	1 or 0 1

All possible truth values are

p	q	r
1	1	1
1	0	1
1	0	0

5. Let *p, q* and *r* be propositions having truth values 0, 0, 1 respectively. Find the truth values of the following compound propositions:

(a) $p \to (q \wedge r)$ **(b)** $p \wedge (r \to q)$ **(c)** $(p \wedge q) \to r$ **(d)** $p \to (q \to \neg r)$ **[Jan '17**

Solution:

(a)

p	q	r	$q \wedge r$	$p \to (q \wedge r)$
0	0	1	0	1

(b)

p	q	r	$r \to q$	$p \wedge (r \to q)$
0	0	1	0	0

(c)

p	q	r	$p \wedge q$	$(p \wedge q) \to r$
0	0	1	0	1

(d)

p	q	r	$r \to q$	$p \to (q \to \neg r)$
0	0	1	1	1

6. Let *p* and *q* primitive statements for which the conditional $p \to q$ is false. Determine the truth values of the following compound propositions:

(a) $p \wedge q$ **(b)** $\neg p \vee q$ **(c)** $q \to p$ **(c)** $\neg q \to \neg p$ **[Jan'14**

Solution:

Since $p \to q$ is false, p is 1 and q is 0.

Therefore, (a) $p \wedge q$ is 0 (b) $\neg p \vee q$ is 0 (c) $q \to p$ is 1 (d) $\neg q \to \neg p$ is 0

7. Form the truth tables for the following:

(a) $(p \vee q) \wedge \neg p$ **(b)** $\neg(p \vee \neg q)$ **(c)** $p \to (q \to r)$ **(d)** $(p \to q) \to r$ **(e)** $[(p \wedge q) \vee \neg r] \leftrightarrow p$

Solution:

(a)

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$
0	0	0	1	0
0	1	1	1	1
1	0	1	0	0
1	1	1	0	0

(b)

p	q	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$
0	0	1	1	0
0	1	0	0	1
1	0	1	1	0
1	1	0	1	0

(c)

p	q	r	$q \to r$	$p \to (q \to r)$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

(d)

p	q	r	$p \to q$	$(p \to q) \to r$
0	0	0	1	0
0	0	1	1	1
0	1	0	1	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

(e)

p	q	r	$p \wedge q$	$\neg r$	$(p \wedge q) \vee \neg r$	$[(p \wedge q) \vee \neg r] \leftrightarrow p$
0	0	0	0	1	1	0
0	0	1	0	0	0	1
0	1	0	0	1	1	0
0	1	1	0	0	0	1
1	0	0	0	1	1	1
1	0	1	0	0	0	0
1	1	0	1	1	1	1
1	1	1	1	0	1	1

8. Show that for any propositions *p*, *q* and *r* the following are tautologies by constructing truth table:

(a) $[(p \to q) \wedge (q \to r)] \to (p \to r)$ **[Dec 2012]**

(b) $[p \wedge (p \to q) \wedge r] \to [(p \vee q) \to r]$

(c) $\{(p \vee q) \wedge [(p \to r) \wedge (q \to r)]\} \to r$ **[Dec '11, Jan '10, Jan '17]**

(d) $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ **[June 2012]**

(e) $[(p \to q) \wedge (q \to r)] \to [(p \vee q) \to r]$ **[Dec 2012]**

(f) $[\neg(p \vee q) \vee (\neg p \wedge q)] \vee p$

(g) $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$ **[July '13]**

Solution:

(a) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$ (1)	$p \to r$ (2)	(1) → (2)
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

(b) $[p \wedge (p \to q) \wedge r] \to [(p \vee q) \to r]$

p	q	r	$p \to q$	$p \wedge (p \to q) \wedge r$ (1)	$p \vee q$	$(p \vee q) \to r$ (2)	$(1) \to (2)$
0	0	0	1	0	0	1	1
0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	1
0	1	1	1	0	1	1	1
1	0	0	0	0	1	0	1
1	0	1	0	0	1	1	1
1	1	0	1	0	1	0	1
1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore, it is a ***Tautology***.

(c) $\{(p \vee q) \wedge [(p \to r) \wedge (q \to r)]\} \to r$

p	q	r	$p \vee q$ (1)	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$ (2)	$(1) \wedge (2)$	$[(1) \wedge (2)] \to r$
0	0	0	0	1	1	1	0	1
0	0	1	0	1	1	1	0	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology.***

(d) $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$

p	q	r	$q \to r$	$p \to (q \to r)$ (1)	$p \to q$	$p \to r$	$(p \to q) \to (p \to r)$ (2)	$(1) \to (2)$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	1	0	0	1	1
1	0	1	1	1	0	1	1	1
1	1	0	0	0	1	0	0	1
1	1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

(e) $[(p \to q) \wedge (q \to r)] \to [(p \vee q) \to r]$

p	q	r	$(p \to q)$	$(q \to r)$	$(p \to q) \wedge (q \to r)$ (1)	$p \vee q$	$(p \vee q) \to r$ (2)	$(1) \to (2)$
0	0	0	1	1	1	0	1	1
0	0	1	1	1	1	0	1	1
0	1	0	1	0	0	1	0	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	0	1
1	0	1	0	1	0	1	1	1
1	1	0	1	0	0	1	0	1
1	1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

(f) $[\neg(p \vee q) \vee (\neg p \wedge q)] \vee p$

p	q	$\neg p$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge q$	$\neg(p \vee q) \vee (\neg p \wedge q)$ (1)	$(1) \vee p$
0	0	1	0	1	0	1	1
0	1	1	1	0	1	1	1
1	0	0	1	0	0	0	1
1	1	0	1	0	0	0	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

(g) $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$

p	q	r	$p \to r$	$q \to r$	$(p \to q) \wedge (q \to r)$ (1)	$p \vee q$	$(p \vee q) \to r$ (2)	$(1) \to (2)$
0	0	0	1	1	1	0	1	1
0	0	1	1	1	1	0	1	1
0	1	0	1	0	0	1	0	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	0	1
1	0	1	0	1	0	1	1	1
1	1	0	1	0	0	1	0	1
1	1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

9. (a) S.T for any proposition *p* and *q*, $(p \veebar q) \vee (p \leftrightarrow q)$ is a tautology.

(b) S.T.for any proposition *p* and *q*, $(p \veebar q) \wedge (p \leftrightarrow q)$ is a contradiction.

(c) S.T. for any proposition *p* and *q*, $(p \veebar q) \wedge (p \to q)$ is a contingency.

Solution:

(a)

p	q	$p \veebar q$	$p \leftrightarrow q$	$(p \veebar q) \vee (p \leftrightarrow q)$
0	0	0	1	1
0	1	1	0	1
1	0	1	0	1
1	1	0	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

(b)

p	q	$p \veebar q$	$p \leftrightarrow q$	$(p \veebar q) \wedge (p \leftrightarrow q)$
0	0	0	1	0
0	1	1	0	0
1	0	1	0	0
1	1	0	1	0

Compound proposition is false for all possible combinations of truth values.

Therefore it is a ***contradiction.***

(c)

p	q	$p \veebar q$	$p \rightarrow q$	$(p \veebar q) \wedge (p \rightarrow q)$
0	0	0	1	0
0	1	1	1	1
1	0	1	0	0
1	1	0	1	0

Compound proposition is neither tautology nor contradiction.

Therefore it is a ***contingency***.

10. By constructing truth tables prove the following:

(a) $[(p \vee q) \rightarrow r] \Leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$ **[July '09]**

(b) $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)] \Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)]$

(c) $[(p \vee q) \rightarrow r] \Leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$ **[July '09, Dec '10]**

(d) $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$ **[July '13]**

(e) $p \rightarrow (q \vee r) \Leftrightarrow (p \wedge q) \rightarrow r$ **[July '16]**

Solution:

(a) $[(p \vee q) \rightarrow r] \Leftrightarrow [\neg r \rightarrow \neg(p \vee q)]]$

Denote $[(p \vee q) \rightarrow r]$ by u and $[\neg r \rightarrow \neg(p \vee q)]$ by v.

p	q	r	$p \vee q$	$[(p \vee q) \rightarrow r]$ (u)
0	0	0	0	1
0	0	1	0	1
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

p	q	r	$p \wedge q$	$\neg r$	$\neg(p \vee q)$	$[\neg r \to \neg(p \vee q)]$ (v)
0	0	0	0	1	1	1
0	0	1	0	0	1	1
0	1	0	1	1	0	0
0	1	1	1	0	0	1
1	0	0	1	1	0	0
1	0	1	1	0	0	1
1	1	0	1	1	0	0
1	1	1	1	0	0	1

Since, u and v have the same truth values for all possible combinations, $u \Leftrightarrow v$.

Therefore, $[(p \vee q) \to r] \Leftrightarrow [\neg r \to \neg(p \vee q)]]$

(b) $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)] \Leftrightarrow [(p \to q) \wedge (q \to r) \wedge (r \to p)]$

Denote $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$ by u and $[(p \to q) \wedge (q \to r) \wedge (r \to p)]$ by v.

p	q	r	$[(p \leftrightarrow q)$	$(q \leftrightarrow r)$	$(r \leftrightarrow p)$	$[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$ (u)
0	0	0	1	1	1	1
0	0	1	1	0	0	0
0	1	0	0	0	1	0
0	1	1	0	1	0	0
1	0	0	0	1	1	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

p	q	r	$(p \to q)$	$(q \to r)$	$(r \to p)$	$[(p \to q) \wedge (q \to r) \wedge (r \to p)]$ (v)
0	0	0	1	1	1	1
0	0	1	1	1	0	0
0	1	0	1	0	1	0
0	1	1	1	1	0	0
1	0	0	0	1	1	0
1	0	1	0	1	1	0
1	1	0	1	0	1	0
1	1	1	1	1	1	1

Since, u and v have the same truth values for all possible combinations, $u \Leftrightarrow v$.

Therefore, $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)] \Leftrightarrow [(p \to q) \wedge (q \to r) \wedge (r \to p)]$

(c) $[(p \vee q) \to r] \Leftrightarrow [(p \to r) \wedge (q \to r)]$

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$ (u)
0	0	0	0	1
0	0	1	0	1
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

p	q	r	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$ (v)
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Since, u and v have the same truth values for all possible combinations, $u \Leftrightarrow v$.

Therefore, $[(p \vee q) \rightarrow r] \Leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$

(d) $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

p	q	r	$(q \rightarrow r)$	$p \rightarrow (q \rightarrow r)$ (u)
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$ (v)
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

Since, u and v have the same truth values for all possible combinations, $u \Leftrightarrow v$.

Therefore, $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

(e) $p \rightarrow (q \vee r) \Leftrightarrow (p \wedge \neg q) \rightarrow r$

p	q	r	$q \vee r$	$p \to (q \vee r)$ (u)
0	0	0	0	1
0	0	1	1	1
0	1	0	1	1
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

p	q	r	$\neg q$	$p \wedge \neg q$	$(p \wedge \neg q) \to r$ (v)
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	0	0	1
0	1	1	0	0	1
1	0	0	1	1	0
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	0	0	1

Since, u and v have the same truth values for all possible combinations, $u \Leftrightarrow v$.

Therefore, $p \to (q \vee r) \Leftrightarrow (p \wedge \neg q) \to r$

11. **Prove the following using the laws of logic:**
    - **(a)** $(\neg p \vee \neg q) \wedge (F_0 \vee p) \wedge p \Leftrightarrow (p \wedge \neg q)$
    - **(b)** $p \to [q \to (p \wedge q)]$ **is a tautology.**
    - **(c)** $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$ **[Jan '17]**
    - **(d)** $[\neg p \wedge (\neg q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)] \Leftrightarrow r$ **[Jan'10]**
    - **(e)** $\{(p \vee q) \wedge \neg[\neg p \wedge (\neg q \vee \neg r)]\} \vee \{(\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)\}$ **is** T_0
    - **(f)** $[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow (p \vee q)$ **[Jan '17]**
    - **(g)** $[\neg (p \wedge q)] \to [\neg p \vee (\neg p \vee q)] \Leftrightarrow \neg p \vee q$
    - **(h)** $[(\neg p \vee \neg q) \to (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$ **[Jan '14]**
    - **(i)** $(p \to q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow \neg (q \vee p)$ **[Dec '12]**
    - **(j)** $[(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))] \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ **is a tautology.** **[Jan '09, Dec '10]**

Solution:

(a) $(\neg p \vee \neg q) \wedge (F_0 \vee p) \wedge p$

$\Leftrightarrow (\neg p \vee \neg q) \wedge p \wedge p$	by Identity law
$\Leftrightarrow (\neg p \vee \neg q) \wedge p$	by Idempotent law
$\Leftrightarrow (\neg p \wedge p) \vee (\neg q \wedge p)$	by distribution law
$\Leftrightarrow F_0 \vee (\neg q \wedge p)$	by inverse law
$\Leftrightarrow (\neg q \wedge p)$	by Identity law
$\Leftrightarrow (p \wedge \neg q)$	by commutative law

(b) $p \to [q \to (p \wedge q)]$

$\Leftrightarrow p \to [\neg q \vee (p \wedge q)]$ by the definition of conditional

$\Leftrightarrow \neg p \vee [\neg q \vee (p \wedge q)]$ by the definition of conditional

$\Leftrightarrow (\neg p \vee \neg q) \vee (p \wedge q)]$ by Associative law

$\Leftrightarrow \neg(p \wedge q) \vee (p \wedge q)]$ by Demorgan's law

$\Leftrightarrow T_0$ by inverse law

(c) $p \to (q \to r)$

$\Leftrightarrow p \to (\neg q \vee r)$ by the definition of conditional

$\Leftrightarrow \neg p \vee (\neg q \vee r)$ by the definition of conditional

$\Leftrightarrow (\neg p \vee \neg q) \vee r$ by Associative law

$\Leftrightarrow \neg(p \wedge q) \vee r$ by Demorgan's law

$\Leftrightarrow (p \wedge q) \to r$ by the definition of conditional

(d) $[\neg p \wedge (\neg q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)]$

$\Leftrightarrow [(\neg p \wedge \neg q) \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)]$ by Associative law

$\Leftrightarrow [(\neg p \wedge \neg q) \wedge r)] \vee [(q \vee p) \wedge r]$ by Distributive law

$\Leftrightarrow [(\neg(p \vee q) \wedge r)] \vee [(p \vee q) \wedge r)]$ by Demorgan's law

$\Leftrightarrow [\neg(p \vee q) \vee (p \vee q)] \wedge r$ by Distributive law

$\Leftrightarrow T_0 \wedge r$ by inverse law

$\Leftrightarrow r$ by Identity law

(e) $\{(p \vee q) \wedge \neg[\neg p \wedge (\neg q \vee \neg r)]\} \vee \{(\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)\}$

$\Leftrightarrow \{(p \vee q) \wedge \neg[\neg p \wedge \neg(q \wedge r)]\} \vee \{\neg(p \vee q) \vee \neg(p \vee r)\}$ by Demorgan's law

$\Leftrightarrow \{(p \vee q) \wedge [p \vee (q \wedge r)]\} \vee \neg\{(p \vee q) \wedge (p \vee r)\}$ by Demorgan's law

$\Leftrightarrow \{(p \vee q) \wedge (p \vee q) \wedge (p \vee r)\} \vee \neg\{(p \vee q) \wedge (p \vee r)\}$ by Distributive law

$\Leftrightarrow \{(p \vee q) \wedge (p \vee r)\} \vee \neg\{(p \vee q) \wedge (p \vee r)\}$ by Idempote

$\Leftrightarrow T_o$ by inverse law

(f) $[(p \vee q) \wedge (p \vee \neg q)] \vee q$

$\Leftrightarrow [(p \vee (q \wedge \neg q)] \vee q$ by Distributive law

$\Leftrightarrow [p \vee F_0] \vee q$ by inverse law

$\Leftrightarrow p \vee q$ by Identity law

(g) $[\neg(p \wedge q)] \rightarrow [\neg p \vee (\neg p \vee q)]$

$\Leftrightarrow (p \wedge q) \vee [\neg p \vee (\neg p \vee q)]$ by conditional la

$\Leftrightarrow (p \wedge q) \vee [(\neg p \vee \neg p) \vee q]$ by Associative law

$\Leftrightarrow (p \wedge q) \vee [\neg p \vee q]$ by Idempotent law

$\Leftrightarrow (p \wedge q) \vee [q \vee \neg p]$ by commutative law

$\Leftrightarrow [(p \wedge q) \vee q] \vee \neg p$ by Associative la

$\Leftrightarrow q \vee \neg p$ by Absorption law

$\Leftrightarrow \neg p \vee q$ by commutative law

(h) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)]$

$\Leftrightarrow [\neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)]$ by the definition of conditional

$\Leftrightarrow [(p \wedge q) \vee ((p \wedge q) \wedge r)]$ by Demorgan's law and associative law

$\Leftrightarrow (p \wedge q)$ by Absorbtion law

(i) $(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)]$

$\Leftrightarrow (p \rightarrow q) \wedge [\neg q \wedge (\neg q \vee r)]$ by commutative law

$\Leftrightarrow (p \rightarrow q) \wedge \neg q$ by Absorbtion law

$\Leftrightarrow (\neg p \vee q) \wedge \neg q$ by the definition of conditional

$\Leftrightarrow \neg(p \wedge \neg q) \wedge \neg q$ by Demorgan's law

$\Leftrightarrow \neg[(p \wedge \neg q) \vee q]$ by Demorgan's law

$\Leftrightarrow \neg[(p \wedge q) \vee (\neg q \wedge q)]$ by distributive law

$\Leftrightarrow \neg[(p \wedge q) \vee F_0]$ by inverse law

$\Leftrightarrow \neg(p \wedge q)$ by identity law

$\Leftrightarrow \neg(q \wedge p)$ by commutative law

(j)

$[(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))] \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

$\Leftrightarrow [(P \vee Q) \wedge (P \vee (Q \wedge R))] \vee \neg(P \vee Q) \vee \neg(P \vee R)$ by Demorgan's law

$\Leftrightarrow [(P \vee Q) \wedge \{(P \vee Q) \wedge (P \vee R)\}] \vee \neg(P \vee Q) \vee \neg(P \vee R)$ by distributive law

$\Leftrightarrow [\{(P \vee Q) \wedge (P \vee Q)\} \wedge (P \vee R)\}] \vee \neg(P \vee Q) \vee \neg(P \vee R)$ by Associative law

$\Leftrightarrow [(P \vee Q) \wedge (P \vee R)] \vee \neg[(P \vee Q) \wedge (P \vee R)]$ by idempotent law

$\Leftrightarrow T_0$

It is a Tautology.

12. Show that the compound propositions $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are not logically equivalent. [Jan '14]

Solution:

Truth tables for $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$

p	q	r	$\neg q$	$\neg q \vee r$	$p \wedge (\neg q \vee r)$ (u)
0	0	0	1	1	0
0	0	1	1	1	0
0	1	0	0	0	0
0	1	1	0	1	0
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	0	1	1

p	q	r	$\neg q$	$p \wedge \neg q$	$(p \wedge \neg q) \to r$ (v)
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	0	0	1
0	1	1	0	0	1
1	0	0	1	1	0
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	0	0	1

u and v do not have the same truth values in all possible situations.

Therefore, $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are not logically equivalent.

13. Write the duals of the following propositions:

(a) $\boldsymbol{p \to q}$ (b) $\boldsymbol{p \leftrightarrow r}$ (c) $\boldsymbol{p \veebar q}$

(d) $(\boldsymbol{p} \vee \boldsymbol{T_0}) \wedge (\boldsymbol{q} \vee \boldsymbol{F_0}) \vee [(\boldsymbol{r} \wedge \boldsymbol{s}) \vee \boldsymbol{F_0}]$ [Dec '10, Dec '12, Jan '09]

(e) $(\boldsymbol{p} \wedge \boldsymbol{q}) \wedge [(\neg \boldsymbol{p} \vee \boldsymbol{q}) \wedge (\neg \boldsymbol{r} \vee \boldsymbol{s})] \vee (\boldsymbol{r} \wedge \boldsymbol{s})$

(f) $\neg(\boldsymbol{p} \vee \boldsymbol{q}) \wedge [\boldsymbol{p} \vee \neg(\boldsymbol{q} \wedge \neg \boldsymbol{s})]$

(g) $\boldsymbol{p} \to (\boldsymbol{q} \to \boldsymbol{r})$

Solution:

(a) $u \equiv p \to q \equiv \neg p \vee q$

$u^d \equiv \neg p \wedge q$

(b) $u \equiv p \leftrightarrow r \equiv (p \to r) \wedge (r \to p) \equiv (\neg p \vee q) \wedge (\neg r \vee p)$
$u^d \equiv (\neg p \wedge q) \vee (\neg r \wedge p)$

(c) $u \equiv p \underline{\vee} q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$
$u^d \equiv (\neg p \vee q) \wedge (p \vee \neg q)$

(d) $u \equiv (p \vee T_0) \wedge (q \vee F_0) \vee [(r \wedge s) \vee F_0]$
$u^d \equiv (p \wedge F_0) \vee (q \wedge T_0) \wedge [(r \vee s) \wedge T_0]$

(e) $u \equiv (p \wedge q) \wedge [(\neg p \vee q) \wedge (\neg r \vee s)] \vee (r \wedge s)$
$u^d \equiv (p \vee q) \vee [(\neg p \wedge q) \vee (\neg r \wedge s)] \wedge (r \wedge s)$

(f) $u \equiv \neg(p \vee q) \wedge [p \vee \neg(q \wedge \neg s)]$
$u^d \equiv \neg(p \wedge q) \vee [p \wedge \neg(q \vee \neg s)]$

(g) $u \equiv p \to (q \to r) \equiv \neg p \vee (q \to r) \equiv \neg p \vee (\neg q \vee r)$
$u^d \equiv \neg p \wedge (\neg q \wedge r)$

14. Verify the principle of duality for the logical equivalence:

$[\neg(\boldsymbol{p} \wedge \boldsymbol{q}) \to \neg \boldsymbol{p} \vee (\neg \boldsymbol{p} \vee \boldsymbol{q})] \Leftrightarrow \neg \boldsymbol{p} \vee \boldsymbol{q}$ **[Jan '10, Jan '17]**

Solution:

Principle of duality: If $u \Leftrightarrow v$ then $u^d \Leftrightarrow v^d$

If $u \Leftrightarrow v$ then

u^d	$\Leftrightarrow [\neg(p \wedge q) \to \neg p \vee (\neg p \vee q)]^d$	by data
	$\Leftrightarrow [(p \wedge q) \vee \neg p \vee (\neg p \vee q)]^d$	by the definition of conditional
	$\Leftrightarrow [(p \wedge q) \vee (\neg p \vee \neg p) \vee q]^d$	by Associative law
	$\Leftrightarrow [(p \wedge q) \vee \neg p \vee q]^d$	by idempotent law
	$\Leftrightarrow [(p \wedge q) \vee q \vee \neg p]^d$	by commutative law
	$\Leftrightarrow [q \vee \neg p]^d$	by Absorption law
	$\Leftrightarrow [\neg p \vee q]^d$	by Commutative law
	$\Leftrightarrow v^d$	by data

15. For any proposition $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{r}$ prove the following:

(a) $\boldsymbol{p} \uparrow \boldsymbol{q} \Leftrightarrow \boldsymbol{q} \uparrow \boldsymbol{p}$ (b) $\boldsymbol{p} \downarrow \boldsymbol{q} \Leftrightarrow \boldsymbol{q} \downarrow \boldsymbol{p}$ (c) $\neg(\boldsymbol{p} \uparrow \boldsymbol{q}) \Leftrightarrow \neg \boldsymbol{q} \downarrow \neg \boldsymbol{p}$

(d) $\neg(\boldsymbol{p} \downarrow \boldsymbol{q}) \Leftrightarrow \neg \boldsymbol{q} \uparrow \neg \boldsymbol{p}$ (e) $\boldsymbol{p} \uparrow (\boldsymbol{q} \uparrow \boldsymbol{r}) \Leftrightarrow \neg \boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r})$ [Jan '17]

(f) $\neg(\boldsymbol{p} \uparrow \boldsymbol{q}) \Leftrightarrow \neg \boldsymbol{p} \downarrow \neg \boldsymbol{q}$ (g) $\boldsymbol{p} \downarrow (\boldsymbol{q} \downarrow \boldsymbol{r}) \Leftrightarrow \neg \boldsymbol{p} \wedge (\boldsymbol{q} \vee \boldsymbol{r})$ [Jan '17]

Solution:

(a) $p \uparrow q$	$\Leftrightarrow \neg(p \wedge q)$	by the definition of NAND
	$\Leftrightarrow \neg(q \wedge p)$	by commutative law
	$\Leftrightarrow q \uparrow p$	by the definition of NAND

(b) $p \downarrow q \quad \Leftrightarrow \neg(p \vee q)$ — by the definition of NOR

$\Leftrightarrow \neg(q \vee p)$ — by commutative law

$\Leftrightarrow q \downarrow p$ — by the definition of NOR

(c) $\neg(p \uparrow q) \Leftrightarrow \neg\neg(p \wedge q)$ — by the definition of NAND

$\Leftrightarrow \neg(\neg p \vee \neg q)$ — by Demorgan's law

$\Leftrightarrow \neg(\neg q \vee \neg p)$ — by commutative law

$\Leftrightarrow \neg q \downarrow \neg p$ — by the definition of NOR

(d) $\neg(p \downarrow q) \quad \Leftrightarrow \neg\neg(p \vee q)$ — by the definition of NOR

$\Leftrightarrow \neg(\neg p \wedge \neg q)$ — by Demorgan's law

$\Leftrightarrow \neg q \uparrow \neg p$ — by the definition of NAND

(e) $p \uparrow (q \uparrow r) \Leftrightarrow \neg(p \wedge (q \uparrow r))$ — by the definition of NAND

$\Leftrightarrow \neg(p \wedge \neg(q \wedge r))$ — by the definition of NAND

$\Leftrightarrow \neg p \vee (q \wedge r)$ — by Demorgan's law

(f) $\neg(p \uparrow q) \quad \Leftrightarrow \neg\neg(p \wedge q)$ — by the definition of NAND

$\Leftrightarrow \neg(\neg p \vee \neg q)$ — by Demorgan's law

$\Leftrightarrow \neg p \downarrow \neg q$ — by the definition of NOR

(g) $p \downarrow (q \downarrow r) \Leftrightarrow \neg(p \vee (q \uparrow r))$ — by the definition of NAND

$\Leftrightarrow \neg(p \vee \neg(q \vee r))$ — by the definition of NAND

$\Leftrightarrow \neg p \wedge (q \vee r)$ — by Demorgan's law

16. Represent $p \vee q, p \wedge q$ and $p \rightarrow q$ using only $\uparrow$ and using only $\downarrow$. **[June '12]**

Solution:

(a) $p \vee q \quad \Leftrightarrow \neg\neg(p \vee q)$

$\Leftrightarrow \neg(\neg p \wedge \neg q)$

$\Leftrightarrow \neg(p \downarrow q)$

$\Leftrightarrow \neg(p \downarrow q) \wedge \neg(p \downarrow q)$

$\Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q)$

This is the representation of $p \vee q$ using only $\downarrow$.

$$
\begin{aligned}
p \vee q &\Leftrightarrow \neg\neg(p \vee q) \\
&\Leftrightarrow \neg(\neg p \wedge \neg q) \\
&\Leftrightarrow \neg(\neg p) \vee \neg(\neg q) \\
&\Leftrightarrow (\neg p) \uparrow (\neg q) \\
&\Leftrightarrow (\neg p \vee \neg p) \uparrow (\neg q \vee \neg q) \\
&\Leftrightarrow (p \uparrow p) \uparrow (q \uparrow q)
\end{aligned}
$$

This is the representation of $p \vee q$ using only $\uparrow$.

(b)
$$
\begin{aligned}
p \wedge q &\Leftrightarrow \neg\neg(p \wedge q) \\
&\Leftrightarrow \neg(\neg p \vee \neg q) \\
&\Leftrightarrow \neg(\neg p) \wedge \neg(\neg q) \\
&\Leftrightarrow (\neg p) \downarrow (\neg q) \\
&\Leftrightarrow (\neg p \wedge \neg p) \downarrow (\neg q \wedge \neg q) \\
&\Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q)
\end{aligned}
$$

This is the representation of $p \vee q$ using only$\downarrow$.

$$
\begin{aligned}
p \wedge q &\Leftrightarrow \neg\neg(p \wedge q) \\
&\Leftrightarrow \neg(\neg p \vee \neg q) \\
&\Leftrightarrow \neg(p \uparrow q) \\
&\Leftrightarrow \neg(p \uparrow q) \vee \neg(p \uparrow q) \\
&\Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q)
\end{aligned}
$$

This is the representation of $p \vee q$ using only $\uparrow$.

(c)
$$
\begin{aligned}
p \to q &\Leftrightarrow \neg p \vee q \\
&\Leftrightarrow \neg p \vee \neg\neg q \\
&\Leftrightarrow p \uparrow (\neg q) \\
&\Leftrightarrow p \uparrow \neg(q \wedge q) \\
&\Leftrightarrow p \uparrow (q \uparrow q)
\end{aligned}
$$

This is the representation of $p \to q$ using only $\uparrow$.

$$
\begin{aligned}
p \to q &\Leftrightarrow \neg p \vee q \\
&\Leftrightarrow \neg\neg(\neg p \vee q) \\
&\Leftrightarrow \neg(p \wedge \neg q)
\end{aligned}
$$

$$\Leftrightarrow \neg(\neg p \downarrow q)$$

$$\Leftrightarrow \neg(\neg p \downarrow q) \wedge \neg(\neg p \downarrow q)$$

$$\Leftrightarrow (\neg p \downarrow q) \downarrow (\neg p \downarrow q)$$

$$\Leftrightarrow ((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$$

This is the representation of $p \rightarrow q$ using only $\downarrow$.

17. Write the converse, inverse and contra positive of the following conditionals:

(a) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

(b) If a real number x^2 is greater than zero, then x is not equal to zero.

(c) If a triangle is not isosceles, then it is not equilateral. [July '13]

(d) If Ram can solve the puzzle, then Ram can solve the problem. [July '16]

Solution:

(a) Let p: Quadrilateral is a parallelogram.

q: Diagonals of the quadrilateral bisect each other.

The given conditional is $p \rightarrow q$.

<u>Converse</u>: $(q \rightarrow p)$ If the diagonals of the quadrilateral bisect each other, then it is a parallelogram.

<u>Inverse</u>: $(\neg p \rightarrow \neg q)$ If a quadrilateral is not a parallelogram, then its diagonals do not bisect each other.

<u>Contra positive</u>: $(\neg q \rightarrow \neg p)$ If the diagonals of the quadrilateral do not bisect each other, then it is not a parallelogram.

(b) Let p: A real number x^2 is greater than zero.

q: x is not equal to zero

The given conditional is $p \rightarrow q$.

<u>Converse</u>: $(q \rightarrow p)$ If a real number$x$ is not equal to zero, then x^2 is greater than zero.

<u>Inverse</u>: $(\neg p \rightarrow \neg q)$ If a real number x^2 is not greater than zero, then x is equal to zero.

<u>Contrapositive</u>: $(\neg q \rightarrow \neg p)$ If a real number$x$ is equal to zero, then x^2 is not greater than zero.

(c) Let p: A triangle is not isosceles

q: Triangle is not equilateral

The given conditional is $p \to q$.

<u>Converse</u>: $(q \to p)$ If a triangle is not equilateral, then it is not isosceles.

<u>Inverse</u>: $(\neg p \to \neg q)$ If a triangle is isosceles, then it is equilateral.

<u>Contra positive</u>: $(\neg q \to \neg p)$ If a triangle is equilateral, then it is isosceles.

(d) Let p : Ram can solve the puzzle.
q : Ram can solve the problem.

The given conditional is $p \to q$.

<u>Converse</u>: $(q \to p)$ If Ram can solve the problem, then he can solve the puzzle.

<u>Inverse</u>: $(\neg p \to \neg q)$ If Ram cannot solve the puzzle, then he cannot solve the problem.

<u>Contrapositive</u>: $(\neg q \to \neg p)$ If Ram cannot solve the problem, then he cannot solve the puzzle.

EXERCISE – 1.1

1. Define: (i) Proposition, (ii) Conjunction, (iii) Disjunction, (iv) Conditional, (v) Biconditioanl, (vi) Negation.
2. How many rows appear in a truth table for each of the following compound propositions?

   (i) $(q \to \neg p) \vee (\neg p \to \neg q)$, **(ii)** $(p \vee \neg t) \wedge (p \vee \neg s)$,

   (iii) $(p \to r) \vee (\neg s \to \neg t) \vee (\neg u \to v)$ **(iv)** $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$,

   (v) $q \vee p \vee \neg s \vee \neg r \vee \neg t \vee u$, **(vi)** $(p \wedge r \wedge t) \leftrightarrow (q \wedge t)$.
3. Construct a truth table for $(p \to q) \leftrightarrow (\neg q \to \neg p)$.
4. Using truth tables verify the **(i)** commutative laws, **(ii)** associative laws, **(iii)** distributive laws, **(iv)** De Morgan's laws, **(v)** absorption laws.
5. Define tautology, contradiction or contingency. Verify the following compound propositions for tautology, contradiction or contingency:

   (i) $(p \to q) \wedge (q \to r) \to (p \to r)$, **(ii)** $(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$,

   (iii) $[\neg p \wedge (p \vee q)] \to q$, **(iv)** $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$,

   (v) $\neg(p \to q) \to \neg q$, **(vi)** $(\neg p \wedge (p \to q)) \to \neg q$,

   (vii) $(\neg q \wedge (p \to q)) \to \neg p$.

6. Verify the following compound propositions are logically equivalent or not

(i) $\neg p \to (q \to r)$ and $q \to (p \vee r)$, (ii) $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$,

(iii) $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$, (iv) $p \to q$ and $\neg q \to \neg p$,

(v) $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$, (vi) $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$,

(vii) $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$, (viii) $(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$,

(ix) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$, (x) $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$,

(xi) $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$, (xii) $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$,

(xiii) $(p \to q) \to r$ and $p \to (q \to r)$, (xiv) $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$,

(xv) $(p \to q) \to (r \to s)$ and $(p \to r) \to (q \to s)$.

7. Find the dual of the following compound propositions.

(i) $p \vee \neg q$, (ii) $p \wedge (q \vee (r \wedge T))$, (iii) $(p \wedge \neg q) \vee (q \wedge F)$,

(iv) $p \wedge \neg q \wedge \neg r$, (v) $(p \wedge q \wedge r) \vee s$, (vi) $(p \vee F) \wedge (q \vee T)$

1.6 PREDICATES

Definition 1.12: An expression P(x) in the variable x is called an **open sentence or a predicate** on a set S, if P (a), $\forall$ a $\in$ S is a proposition. The set S is called a replacement set or domain of the open sentence.

i.e., P(x) is called open sentence if p(a) is true or p(a) is false ? a Č S.

A Predicate in two variables is denoted by P(x, y) and in three variables is denoted by P (x, y, z). In general, a predicate involving the *n* variables $x_1, x_2, x_3 \ldots x_n$ can be denoted by P ($x_1, x_2, x_3, \ldots, x_n$).

EXAMPLE:

1. **Let P(x) denote the statement "x > 3." What are the truth values of P(4) and P(2)?**

Solution:

We obtain the statement P (4) by setting x = 4 in the statement "x > 3."

Hence, P (4), which is the statement "4 > 3," is true.

However, P (2), which is the statement "2 > 3," is false.

---

2. **Let P(x) denotes the statement "x ≤ 4." What are the truth values of P(0), P(4) and P(6) ?**

   Solution:

   We obtain the statement P (0) by setting x = 0 in the statement "x ≤ 4"

   Hence, P(0), which is the statement "0 ≤ 4", is true.

   P(4) is the statement "4 ≤ 4", is true.

   However, P (6), which is the statement "6 ≤ 4", is false.

3. **Let Q(x, y) denote the statement "x = y + 3." What is the truth values of the propositions Q (1, 2) and Q (3, 0)?**

   Solution:

   To obtain Q(1, 2), set x = 1 and y = 2 in the statement Q(x, y).

   Hence, Q(1, 2) is the statement "1 = 2 + 3," which is false.

   The statement Q(3, 0) is the proposition "3 = 0 + 3," which is true.

4. **Let R(x, y, z) denote the statement ''x + y = z." Find the truth values of the propositions R (1, 2, 3) and R (0, 0, 1)?**

   Solution:

   The proposition R (1, 2, 3) is obtained by seeting x = 1, y = 2 and z = 3 in the statement R(x, y, z). We see that R(1, 2, 3) is the statement "1 + 2 = 3," which is true.

   Also note that R(0, 0, 1), which is the statement "0 + 0 = 1," is false.

5. **Let A(x) denote the statement "Computer x is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of A(CS1), A(CS2), and A(MATH1)?**

   Solution:

   We obtain the statement A(CS1) by setting x = CS1 in the statement "Computer x is under attack by an intruder." Because CS1 is not on the list of computers currently under attack, we conclude that A(CS1) is false.

   Similarly, because CS2 and MATH1 are on the list of computers under attack, we know that A(CS2) and A(MATH1) are true.

6. **Let A(c, n) denote the statement "Computer c is connected to network n," where c is a variable representing a computer and n is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of A (MATH1, CAMPUS1) and A (MATH1, CAMPUS2)?**

Solution:

Because MATH1 is not connected to the CAMPUS1 network, we see that A(MATH1, CAMPUS1) is false.

However, because MATH1 is connected to the CAMPUS2 network, we see that A(MATH1, CAMPUS2) is true.

1.7 QUANTIFIERS

The expressions which convey the idea of quantity are called **Quantifiers**. In English, the words “all”, “some”, “many”, “none”, “for every”, “at least”, “there exists”, “at least one” and “few” are used in quantifications.

The Quantifiers are classified into two types namely (i) Universal Quantifiers and (ii) Existential Quantifiers.

UNIVERSAL QUANTIFIERS:

The universal quantification of P (x) is the statement “P (x) for all values of x in the domain”. It is denoted by ∀x P (x). Here ∀ is called the universal quantifier. We read ∀x P (x) as “for all x P (x)” or “for every x P (x).”

The truth value of universal quantifier ∀x P (x) is **true** if for all values of x in the domain, P (x) is true and the truth value of universal quantifier ∀x P (x) is **false** if for at least one value of x in the domain, P (x) is not true.

Remark: When all the elements in the domain can be listed say x_1, x_2, x_3… x_n, it follows that the universal quantification ∀x P (x) is the same as the conjunction

$$P(x_1) \land P(x_2) \land P(x_3) \land \dots \land P(x_n),$$

because this conjunction is true if and only if $P(x_1)$, $P(x_2)$, $P(x_3)$, … $P(x_n)$ are all true.

EXISTENTIAL QUANTIFIERS:

The existential quantification of P (x) is the proposition “There exists an element x in the domain such that P (x)”. It is denoted by ∃x P (x). Here ∃ is called the existential quantifier.

The truth value of existential quantifier ∃x P (x) is **true** if for at least one value of x in the domain, P (x) is true and the truth value of existential quantifier ∃x P (x) is **false** if for all values of x in the domain, P (x) is not true.

The existential quantification ∃x P (x) is read as

"There is an x such that P (x),"

"There is at least one x such that P (x),"

Or "For some x P (x)."

Remark: Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. If the domain is empty, then $\exists x\ Q(x)$ is false whenever Q(x) is a propositional function because when the domain is empty, there can be no element x in the domain for which Q(x) is true.

When all the elements in the domain can be listed say $x_1, x_2, x_3 \ldots x_n$, the existential quantification $\exists x\ P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots \vee P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2), P(x_3), \ldots P(x_n)$ is true.

Note:

Statement	When True?	When False?
$\forall x\ P(x)$	P (x) is true for every x.	There is an x for which P (x) is false.
$\exists x\ P(x)$	There is an x for which P (x) is true.	P (x) is false for every x.

EXAMPLE:

1. **Let P (x) be the statement "x + 1 > x." What is the truth value of the quantification $\forall x\ P(x)$, where the domain consists of all real numbers?**

   Solution:

   Because P (x) is true for all real numbers x, the quantification $\forall x\ P(x)$ is true.

2. **Let Q(x) be the statement "x < 2." What is the truth value of the quantification $\forall x\ Q(x)$, where the domain consists of all real numbers?**

   Solution:

   Q(x) is not true for every real number x, because, for instance, Q(3) is false.

   That is, x = 3 is a counterexample for the statement $\forall x\ Q(x)$. Thus $\forall x\ Q(x)$ is false.

3. **Suppose that P(x) is "$x^2 > 0$." Show that the statement $\forall x\ P(x)$ is false where the universe of discourse consists of all integers.**

   Solution:

   We give a counterexample. We see that x = 0 is a counterexample because x = 0 when x = 0, so that x is not greater than 0 when x = 0. Thus $\forall x\ P(x)$ is false.

4. **What is the truth value of ∀x P(x), where P(x) is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?**

   Solution:

   The statement ∀x P(x) is the same as the conjunction P(1) ∧ P(2) ∧ P(3) ∧ P(4), because the domain consists of the integers 1, 2, 3, and 4. Because P(4), which is the statement "$4^2 < 10$," is false, it follows that ∀x P (x) is false.

5. **What does the statement ∀x N(x) mean if N(x) is "Computer x is connected to the network" and the domain consists of all computers on campus?**

   Solution:

   The statement ∀x N(x) means that for every computer x on campus, that computer x is connected to the network. This statement can be expressed in English as "Every computer on campus is connected to the network."

6. **What is the truth value of $\forall x\,(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?**

   Solution:

   The universal quantification $\forall x\,(x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $\left(\frac{1}{2}\right)^2 < \frac{1}{2}$. Note that $x^2 \geq x$ if and only if $x^2 - x = x(x - 1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x\,(x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$). However, if the domain consists of the integers, $\forall x\,(x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$.

7. **Let P (x) denote the statement "x > 3." What is the truth value of the quantification ∃x P (x), where the domain consists of all real numbers?**

   Solution:

   Because "x > 3" is sometimes true, for instance, when x = 4, the existential quantification of P(x), which is ∃x P(x), is true.

   Observe that the statement ∃x P(x) is false if and only if there is no element x in the domain for which P (x) is true. That is, ∃x P(x) is false if and only if P(x) is false for every element of the domain.

8. **Let Q(x) denote the statement "x = x + 1." What is the truth value of the quantification ∃x Q(x), where the domain consists of all real numbers?**

   Solution:

Because Q(x) is false for every real number x, the existential quantification of Q(x), which is ∃x Q(x), is false.

9. What is the truth value of ∃x P(x), where P(x) is the statement "x^2 > 10" and the universe of discourse consists of the positive integers not exceeding 4?

Solution:

Because the domain is {1, 2, 3, 4}, the proposition ∃x P (x) is the same as the disjunction P (1) ∨ P (2) ∨ P (3) ∨ P (4). Because P (4), which is the statement "4^2 > 10," is true, it follows that ∃x P (x) is true.

1.7.1 NESTED QUANTFIERS

Nested quantifiers are quantifiers that occur within the scope of other quantifiers.

Example:

1. ∀x ∃y P(x, y)
2. ∀x ∃y (x + y = 0)

Problems

1. Let L(x, y) be the statement "x loves y," where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

- **a) Everybody loves Jerry.**
- **b) Everybody loves somebody.**
- **c) There is somebody whom everybody loves.**
- **d) Nobody loves everybody.**
- **e) Everyone loves himself or herself**

Solution:

- a) ∀x L(x, Jerry)
- b) ∀x ∃y L(x, y)
- c) ∃y ∀x L(x, y)
- d) ∀x ∃y ¬ L(x, y) or ¬ ∃x ∀y L(x, y)
- e) ∀x L(x, x)

2. Let W(x, y) mean that student x has visited website y, where the domain for x consists of all students in your school and the domain for y consists of all websites. Express each of these statements by a simple English sentence.

- **a) ∃y (W(Ashok, y) ∧ W(Cindy, y))**
- **b) ∃y ∀z (y ÷ (David) ∧ (W(David, z) ì W (y, z)))**
- **c) ∃x ∃y ∀z (((x ÷ y) ∧ (W(x, z) é W (y, z))))**

Solution:

a) There is a website that both Ashok and Cindy have visited.

b) There is a person besides David who has visited all the websites that David has visited.

c) There are two distinct people who have visited exactly the same sites.

3. Let M (x, y) be "x has sent y an e-mail message" and T (x, y) be "x has telephoned y," where the domain consists for all students in your class. Use quantifiers to express each of these statements.

a) There is a student in your class who has not received an e-mail message from anyone else in the class and who has not been called by any other student in the class.

b) Every student in the class has either received an e-mail message or received a telephone call from another student in the class.

c) There are at least two students in your class such that one student has sent the other e-mail and the second student has telephoned the first student

Solution:

a) $\exists x\ \forall y\ ((x \neq y) \rightarrow (\neg\ M\ (y, x) \wedge \neg\ T\ (y, x)))$

b) $\forall x\ \exists y\ ((x \neq y) \wedge (M\ (y, x) \vee T\ (y, x)))$

c) $\exists x\ \exists y\ ((x \neq y) \wedge (M\ (x, y) \wedge T\ (y, x)))$

FREE AND BOUND VARIABLES

Definition 1.13: If a Quantifier (Universal or Existential) is involving the quantified variable x, we say that this occurrence of the variable is **bound variable**. Let P(x) is an open sentence or predicate involving a variable x, then the variable x is called **free variable.**

EXAMPLE:

1. Identify the bound and free variables in the statement $\exists x\ (x + y = 1)$.

Solution:

In the statement $\exists x\ (x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable.

This illustrates that in the statement $\exists x\ (x + y = 1)$, x is bound, but y is free.

2. Identify the bound and free variables in the following statements:

(i) ?**y $\exists$z sin (x + y) = cos (z – x)** **(ii) $\exists x\ \exists y\ [(x^2 + y^2) = z^2]$**

Solution:

(i) x is free variable and y, z are bound variables.

(ii) z is free variable and x, y are bound variables.

1.7.3 LOGICAL EQUIVALENCES INVOLVING QUANTIFIERS

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example: Show that the following are logically equivalent (where the same domain is used throughout).

1. **$\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$.**
2. **$\forall x\,(P(x) \vee Q(x)) \equiv \forall x\, P(x) \vee \forall x\, Q(x)$.**

Solution:

To show that these statements are logically equivalent, we must show that they always take the same truth value, no matter what the predicates P and Q are, and no matter which domain of discourse is used. Suppose we have particular predicates P and Q, with a common domain. We can show that $\forall x\,(P(x) \wedge Q(x))$ and $\forall x\, P(x) \wedge \forall x\, Q(x)$ are logically equivalent by doing two things.

First, we show that if $\forall x\,(P(x) \wedge Q(x))$ is true, then $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true. Second, we show that if $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true, then $\forall x\,(P(x) \wedge Q(x))$ is true. So, suppose that $\forall x\,(P(x) \wedge Q(x))$ is true. This means that if a is in the domain, then $P(a) \wedge Q(a)$ is true. Hence, P(a) is true and Q(a) is true. Because P(a) is true and Q(a) is true for every element in the domain, we can conclude that $\forall x\, P(x)$ and $\forall x\, Q(x)$ are both true.

This means that $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true.

Next, suppose that $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true. It follows that $\forall x\, P(x)$ is true and $\forall x\, Q(x)$ is true. Hence, if a is in the domain, then P(a) is true and Q(a) is true [because P(x) and Q(x) are both true for all elements in the domain, there is no conflict using the same value of a here].

It follows that for all a, $P(a) \wedge Q(a)$ is true. It follows that $\forall x\,(P(x) \wedge Q(x))$ is true. We can now conclude that $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$.

Furthermore, we can also distribute an existential quantifier over a disjunction.

1.7.4 NEGATION OF QUANTIFIERS

The following are the rules for negations for quantified statements:

1. $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$
2. $\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x)$.

The rules for negations for quantifiers are called **De Morgan's laws** for quantifiers.

EXAMPLE:

1. "Every student in your class has taken a course in calculus."

Let P(x) denotes "x is in your class has taken a course in calculus".

Then the given statement is represented in symbolic form by $\forall x\ P(x)$, where the domain consists of all students.

The negation of this statement is $\neg \forall x\ P(x)$, which is equivalent to $\exists x \neg P(x)$

This negation can be expressed as "There is a student in your class who has not taken a course in calculus."

2. "There is an honest politician"

Let P(x) denotes "x is honest."

Then the given statement is represented in symbolic form by $\exists x\ P(x)$, where the domain consists of all politicians.

The negation of this statement is $\neg \exists x\ P(x)$, which is equivalent to $\forall x \neg P(x)$.

This negation can be expressed as "Every politician is dishonest." or "All politicians are not honest".

3. "All Americans eat cheeseburgers"

Let P(x) denote "x eats cheeseburgers."

Then the given statement is represented in symbolic form by $\forall x\ P(x)$, where the domain consists of all Americans.

The negation of this statement is $\neg \forall x\ P(x)$, which is equivalent to $\exists x \neg P(x)$.

This negation can be expressed as "Some American does not eat cheeseburgers" or "There is an American who does not eat cheeseburgers."

1.7.5 Converse, Inverse and Contrapositive for Quantified Statements:

For open statements $p(x)$, $q(x)$ — defined for a prescribed universe — and the universally quantified statement $\forall x\, [p(x) \rightarrow q(x)]$, we define:

1) The *contrapositive* of $\forall x\, [p(x) \rightarrow q(x)]$ to be $\forall x\, [\neg q(x) \rightarrow \neg p(x)]$.
2) The *converse* of $\forall x\, [p(x) \rightarrow q(x)]$ to be $\forall x\, [q(x) \rightarrow p(x)]$.
3) The *inverse* of $\forall x\, [p(x) \rightarrow q(x)]$ to be $\forall x\, [\neg p(x) \rightarrow \neg q(x)]$.

PROBLEMS TO PRACTICE

1. For the universe of all integers let $p(x)$: $x > 0$, $q(x)$: x is even, $r(x)$: x is a perfect square, $s(x)$: x is divisible by 3, $t(x)$: x is divisible by 7.Write the following statements in symbolic form:

(a)At least one integer is even, (b) There exists a positive integer that is even.

(c) Some even integers are divisible by 3. (d) Every integer is either even or odd.

(e) If x is even and a perfect square then x is not divisible by 3.

(f) If x is odd or is not divisible by 7 then x is divisible by 3. [Jan'10]

Solution:

(a) $\exists x, q(x)$

(b) $\forall x,\ [p(x) \wedge q(x)]$

(c) $\exists x, [q(x) \wedge s(x)]$

(d) $\forall x, [q(x) \vee \neg q(x)]$

(e) $\forall x, [q(x) \wedge r(x)] \rightarrow \neg s(x)$

(f) $\forall x, [q(x) \wedge t(x)] \rightarrow s(x)$

2. Consider the universe of all polygons with 3 or 4 sides and define the following open statements for this universe. a(x): All interior angles of x are equal, e(x): x is an equilateral triangle, i(x): x is an isosceles triangle, p(x): x has an interior angle that exceeds 180^0, q(x): x is a quadrilateral, r(x): x is a rectangle, s(x): x is a square, t(x): x is a triangle.

(a) $\forall x, [q(x) \vee t(x)]$ **(b)** $\forall x, [i(x) \vee e(x)]$ **(c)** $\exists x, [t(x) \wedge p(x)]$

(d) $\exists x, [q(x) \wedge \neg r(x)]$ **(e)** $\forall x, \{[a(x) \wedge t(x)] \leftrightarrow e(x)\}$ **(f)** $\forall x, t(x) \rightarrow \neg p(x)$

Solution:

(a) For any x, x is a quadrilateral or a triangle.

(b) For any x, x is an isosceles triangle or an equilateral triangle.

(c) For some x, x is a triangle and a quadrilateral.

(d) For some x, x is a quadrilateral and not a rectangle.

(e) For any x, all interior angles of x are equal and is a triangle if and only if x is an equilateral triangle.

(f) For any x, if x is a triangle then x does not have an interior angle that exceeds180^0.

3. Consider the following statements with a set of all real numbers as the universe.

$p(x): x \geq 0, q(x): x^2 \geq 0, r(x): x^2 - 3x - 4 = 0, s(x): x^2 - 3 > 0.$

Determine the truth values of the following:

(a) $\exists x, p(x) \wedge q(x)$ (b) $\forall x, p(x) \to q(x)$ (c) $\forall x, q(x) \to s(x)$

(d) $\forall x, r(x) \vee s(x)$ (e) $\exists x, p(x) \wedge r(x)$.

[July ’09, Dec’10, Dec’12, Jan’17]

Solution:

(a) For $x = 1,\ p(x): x \geq 0\ and\ q(x): x^2 \geq 0$ are true.

Therefore, $\exists x, p(x) \wedge q(x)$ is true.

(b) $q(x): x^2 \geq 0$ cannot be false for any real number x.

Therefore, $p(x) \to q(x)$ cannot be false for any real number x.

Therefore, $\forall x, p(x) \to q(x)$ is true.

(c) For $x = 1, q(x): x^2 \geq 0$ is true but $s(x): x^2 - 3 > 0$ is false.

Therefore, $\forall x, q(x) \to s(x)$ is false.

(d) For $x = 1, r(x): x^2 - 3x - 4 = 0\ and\ s(x): x^2 - 3 > 0$ are false.

Therefore, $\forall x, r(x) \vee s(x)$is false.

(e) For $x = 4, p(x): x \geq 0$ is true and$r(x): x^2 - 3x - 4 = 0$ is true.

Therefore, $\exists x, p(x) \wedge r(x)$ is true.

4. Negate and simplify each of the following:

(a) $\exists x, p(x) \vee q(x)$ (b) $\forall x,\ p(x) \to q(x)$

(c)$\forall x, p(x) \to \neg q(x)$ (d) $\exists x,\ \left(p(x) \vee q(x)\right) \to r(x)]$ **[Dec ’11, Dec ‘12]**

Solution:

(a) $\forall x, \neg p(x) \wedge \neg q(x)$

(b) $u \equiv \forall x,\ p(x) \to q(x) \equiv \forall x,\ \neg p(x) \vee q(x)$

Its negation is $\neg u \equiv \exists x, p(x) \wedge \neg q(x)$

(c) $u \equiv \forall x, p(x) \to \neg q(x) \equiv \forall x, \neg p(x) \vee \neg q(x)$

Its negation is $\neg u \equiv \exists x, p(x) \wedge q(x)$

(d) $u \equiv \exists x, \left[\left(p(x) \vee q(x)\right) \to r(x)\right] \equiv \exists x, [\neg\left(p(x) \vee q(x)\right) \vee r(x)]$

Its negation is $\neg u \equiv \forall x, [\left(p(x) \vee q(x)\right) \wedge \neg r(x)]$

5. Write down the following propositions in symbolic form and find their negation:

(a) For all integers, if n is not divisible by 2, then n is odd. [Jan’10, Jan’06]

(b) If l, n , n are any integers where $l - n$ and $n\ - n$ are odd then $l - n$ is even

[Jan’10]

(c) If x is a real number where x^2 is greater than 16 then x is less than J 4 or x is more than 4.

(d) All rational numbers are real and some real numbers are not rational.

(e) No real number is greater than its square.

(f) All integers are rational numbers and some rational numbers are not integers

[Dec'10, July'13]

(g) Some straight lines are parallel or all straight lines intersect. **[July'16]**

Solution:

(a) Let $p(n)$: n is divisible by 2 and $q(n)$: n is odd

$u \equiv$ For all integers, if n is not divisible by 2, then n is odd.

$\equiv \forall\, n, \neg p(n) \rightarrow q(n)$

$\equiv \forall n, p(n) \vee q(n)$

$\neg u \equiv \exists n, \neg p(n) \wedge \neg q(n)$

$\equiv$ Some integers are neither divisible by 2 nor odd.

(b) Let $p(x)$: $l - m$ is odd, $q(x)$: $m - n$ is odd and $r(x)$: $l - n$ is odd.

$u \equiv$ For any integers, if $l - m$ and $m - n$ are odd then $l - n$ is even

$\equiv \forall x, [p(x) \wedge q(x)] \rightarrow r(x)$

$\equiv \forall x, \neg[p(x) \wedge q(x)] \vee r(x)$

$\neg u \equiv \exists x,\ [p(x) \wedge q(x)] \wedge \neg r(x)$

$\equiv$ For some $l, m, n, l - m$ and $m - n$ are odd and $l - n$ is also odd.

(c) Let $p(x): x^2 > 16,\ q(x): x < -4, r(x): x > 4$

$u \equiv$ If x is a real number where $x^2 > 16$ then $x < -4$ or $x > 4$.

$\equiv \forall x \in R, p(x) \rightarrow [q(x) \vee r(x)]$

$\equiv \forall x \in R, \neg p(x) \vee [q(x) \vee r(x)]$

$\neg u \equiv \exists x \in R, p(x) \wedge \neg[q(x) \vee r(x)]$

$\equiv$ For some real number x, $x^2 > 16,\ x \geq -4$ *and* $x \leq 4$.

(d) Let $p(x)$: x is real and $q(x)$: x is rational

$u \equiv$ All rational numbers are real and some real numbers are not rational

$\equiv \forall x \in Q, p(x) \wedge \exists x \in R, \neg q(x)$

$\neg u \equiv \exists x \in Q, \neg\, p(x) \wedge \forall\, x \in R, q(x)$

$\equiv$ Some rational numbers are not real or all real numbers are rational.

(e) Let $p(x): x > x^2$

$u \equiv$ No real number is greater than its square.

$\equiv \forall x \in R, \neg p(x)$

$\neg u \equiv \exists x \in R, p(x)$

$\equiv$ Some real numbers are greater than its square.

(f) Let $p(x)$: x is rational number and $q(x)$: x is integer.

$u \equiv$ All integers are rational numbers and some rational numbers are not integers.

$\equiv \forall x \in Z, p(x) \wedge \exists x \in Q, \neg q(x)$

$\neg u \equiv \exists x \in Z, \neg p(x) \vee \forall x \in Q, q(x)$

$\equiv$ Some integers are not rational numbers or every rational number is an integer.

(g) Let $p(x)$: x is parallel and $q(x)$: x intersect.

$u \equiv$ Some straight lines are parallel or all straight lines intersect.

$\equiv \exists x \in L, p(x) \vee \forall x \in L, q(x)$

$\neg u \equiv \forall x \in L, \neg p(x) \wedge \exists x \in L, \neg q(x)$

$\equiv$ All straight lines are not parallel and some straight lines do not intersect.

6. Prove the following logical equivalences:

(a) $\exists \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x}) \rightarrow \boldsymbol{q}(\boldsymbol{x}) \equiv \forall \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x}) \rightarrow \exists \boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x})$

(b) $\exists \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x}) \rightarrow \forall \boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x}) \equiv \forall \boldsymbol{x}, [\boldsymbol{p}(\boldsymbol{x}) \rightarrow \boldsymbol{q}(\boldsymbol{x})]$

(c) $\neg[\exists \boldsymbol{x}, \neg \boldsymbol{p}(\boldsymbol{x})] \equiv \forall \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x})$

(d) $\forall \boldsymbol{x}, \{\boldsymbol{p}(\boldsymbol{x}) \wedge [\boldsymbol{q}(\boldsymbol{x}) \wedge \boldsymbol{r}(\boldsymbol{x})]\} \equiv \forall \boldsymbol{x}, [\{\boldsymbol{p}(\boldsymbol{x}) \wedge \boldsymbol{q}(\boldsymbol{x})\} \wedge \boldsymbol{r}(\boldsymbol{x})]$

Solution:

(a) $\exists x, p(x) \rightarrow q(x) \equiv \exists x, \neg p(x) \vee q(x) \equiv \exists x, \neg p(x) \vee \exists x, q(x)$

$\equiv \neg \forall x, p(x) \vee \exists x, q(x) \equiv \forall x, p(x) \rightarrow \exists x, q(x)$

(b) $\exists x, p(x) \rightarrow \forall x, q(x) \equiv \neg \exists x, p(x) \vee \forall x, q(x)$

$\equiv \forall x, \neg p(x) \vee \forall x, q(x)$

$\equiv \forall x, [\neg p(x) \vee q(x)] \equiv \forall x, [p(x) \rightarrow q(x)]$

(c) $\neg[\exists x, \neg p(x)] \equiv \forall x, \neg\neg p(x) \equiv \forall x, p(x)$

(d) $\forall x, \{p(x) \wedge [q(x) \wedge r(x)]\} \equiv \forall x, p(x) \wedge \forall x, [q(x) \wedge r(x)]$

$\equiv \forall x, p(x) \wedge [\forall x, q(x) \wedge \forall x, r(x)]$

$\equiv [\forall x, p(x) \wedge \forall x, q(x)] \wedge \forall x, r(x)$

$\equiv \forall x, [p(x) \wedge q(x)] \wedge \forall x, r(x)$

$\equiv \forall x, [\{p(x) \wedge q(x)\} \wedge r(x)]$

EXERCISE – 1.2

1. Let Q(x) be the statement "$x + 1 > 2x$." If the domain consists of all integers, then find the truth values of the following?

   (i) Q(0), (ii) Q(−1), (iii) Q(1), (iv) $\exists x\, Q(x)$, (v) $\forall x\, Q(x)$, (vi) $\exists x \neg Q(x)$ (vii) $\forall x \neg Q(x)$

2. Let P (x) be the statement "$x + 1 > x$". Find the truth values of the following quantifications, where the domain consists of all real numbers?

   (i) $\exists x\, P(x)$, (ii) $\forall x\, P(x)$, (iii) $\exists x \neg P(x)$ (iv) $\forall x \neg P(x)$

3. Let Q(x) be the statement "$x < 2$". Find the truth values of the following quantifications, where the domain consists of all real numbers?

   (i) $\exists x\, P(x)$, (ii) $\forall x\, P(x)$, (iii) $\exists x \neg P(x)$ (iv) $\forall x \neg P(x)$

4. Let P (x) be the statement "$x < 10$". Find the truth values of the following quantifications, where the domain consists of the positive integers not exceeding 4?

   (i) $\exists x\, P(x)$, (ii) $\forall x\, P(x)$, (iii) $\exists x \neg P(x)$ (iv) $\forall x \neg P(x)$

5. What is the truth value of $\forall x\, (x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

6. Let P (x) denote the statement "$x > 3$." Find the truth values of the following quantifications, where the domain consists of all real numbers?

   (i) $\exists x\, P(x)$, (ii) $\forall x\, P(x)$, (iii) $\exists x \neg P(x)$ (iv) $\forall x \neg P(x)$

7. Let P (x) be the statement "$x = x^2$". Find the truth values of the following quantifications, where the domain consists of all integers?

   (i) $\exists x\, P(x)$, (ii) $\forall x\, P(x)$, (iii) $\exists x \neg P(x)$ (iv) $\forall x \neg P(x)$

8. Determine the truth value of each of the following statements if the domain consists of all integers.

   (i) $\forall n\, (n + 1 > n)$, (ii) $\exists n\, (2n = 3n)$, (iii) $\exists n\, (n = -n)$, (iv) $\forall n\, (3n \leq 4n)$

9. Determine the truth value of each of these statements if the domain consists of all real numbers.

   a) $\exists x(x^3 = -1)$ **b)** $\exists x(x^4 < x^2)$
   c) $\forall x((-x)^2 = x^2)$ **d)** $\forall x(2x > x)$

10. Write the negation of the following propositions

   a) Some drivers do not obey the speed limit.
   b) All Swedish movies are serious.
   c) No one can keep a secret.
   d) There is someone in this class who does not have a good attitude.
   e) $\forall x\, (x^2 > x)$ and $\exists x\, (x^2 = 2)$

f) Some old dogs can learn new tricks.
g) No rabbit knows calculus.
h) Every bird can fly.
i) There is no dog that can talk.
j) There is no one in this class who knows French and Russian.
k) All dogs have fleas.
l) There is a horse that can add.
m) Every koala can climb.
n) No monkey can speak French.
o) There exists a pig that can swim and catch fish

1.8 RULES OF INFERENCE

Later in this chapter we will study proofs. Proofs in mathematics are valid arguments that establish the truth of mathematical statements. By an **argument**, we mean a sequence of statements that end with a conclusion. By **valid**, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or **premises**, of the argument. That is, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false. To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments. Rules of inference are our basic tools for establishing the truth of statements.

Before we study mathematical proofs, we will look at arguments that involve only compound propositions. We will define what it means for an argument involving compound propositions to be valid. Then we will introduce a collection of rules of inference in propositional logic. These rules of inference are among the most important ingredients in producing valid arguments. After we illustrate how rules of inference are used to produce valid arguments, we will describe some common forms of incorrect reasoning, called fallacies, which lead to invalid arguments.

After studying rules of inference in propositional logic, we will introduce rules of inference for quantified statements. We will describe how these rules of inference can be used to produce valid arguments. These rules of inference for statements involving existential and universal quantifiers play an important role in proofs in computer science and mathematics, although they are often used without being explicitly mentioned.

Finally, we will show how rules of inference for propositions and for quantified statements can be combined. These combinations of rule of inference are often used together in complicated arguments.

Definition 1.14: An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called **premises** and the final proposition is called the **conclusion**. An argument is valid if the truth of all its premises implies that the conclusion is true.

An argument form in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is valid no matter which particular propositions are substituted for the propositional variables in its premises; the conclusion is true if the premises are all true.

Consider a valid argument of the form

$$(p_1 \wedge p_2 \wedge p_3 \wedge \dots \wedge p_n) \rightarrow q.$$

Here, n is a positive integer, the statements $p_1, p_2, p_3, \dots, p_n$ are called the **premises** of the argument and the statement q is the **conclusion** for the argument.

The above argument can also be written as

$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \cdot \\ \cdot \\ \cdot \\ p_n \\ \hline \therefore\ q \end{array}$$

From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, p_3, \dots, p_n$ and conclusion q is valid, when $(p_1 \wedge p_2 \wedge p_3 \wedge \dots \wedge p_n) \rightarrow q$ is a tautology.

The key to showing that an argument in propositional logic is valid is to show that its argument form is valid. Consequently, we would like techniques to show that argument forms are valid. We will now develop methods for accomplishing this task.

1.8.1 Rules of Inference for Propositional Logic

We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the conclusion must also be true. However, this can be a tedious approach. For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid require 2^{10} = 1024 different rows. Fortunately, we do not have to resort to truth tables. Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference**. These rules of inference can be used as building blocks to construct more complicated valid argument forms. We will now introduce the most important rules of inference in propositional logic in the following table:

S.No	Rules of Inference	Related Logical Implication or Tautology	Name of Rule
1.	p $p \rightarrow q$ ——— $\therefore q$	$(p \land (p \rightarrow q)) \rightarrow q$	Rule of detachment or Modus ponens.
2.	$p \rightarrow q$ $q \rightarrow r$ ——— $\therefore p \rightarrow r$	$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Rule of syllogism
3.	$p \rightarrow q$ $\neg q$ ——— $\therefore \neg p$	$((p \rightarrow q) \land \neg q) \rightarrow \neg p$	Modus Tollens
4.	$p \lor q$ $\neg p$ ——— $\therefore q$	$((p \lor q) \land \neg p) \rightarrow q$	Rule of Disjunctive Syllogism
5.	p ——— $\therefore p \lor q$	$p \rightarrow (p \lor q)$	Rule of Disjunctive Amplification
6.	$p \land q$ ——— $\therefore p$	$(p \land q) \rightarrow p$	Rule of Conjunctive Simplification
7.	p q ——— $\therefore p \land q$	$((p) \land (q)) \rightarrow (p \land q)$	Rule of Conjunction
8.	$p \lor q$ $\neg p \lor r$ ——— $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$	Rule of Resolution

Remark: There are many rules of inference. In the above table we have considered only most important rules of inference in propositional logic.

Note: We could have used a truth table to show that whenever each of the hypotheses is true, the conclusion is also true. However, if we are working with more than three propositional variables, such a truth table would have more than 8 rows. It is very tedious to construct a truth table, so we can apply rules of inference to verify such arguments.

Examples:

1. Establish the validity of the following argument:

p ì r
Ė **p ì q**
q ì s

---

y Ė **r ì s**

Solution: We construct an argument to show that our premises lead to the desired conclusion as follows:

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\neg$ p $\to$ q	Given Premise
2	---	q $\to$ s	Given Premise
3	1 and 2	$\neg$ p $\to$ s	Rule of syllogism
4	3	$\neg$ s $\to$ p	By Contrapositive
5	---	p $\to$ r	Given Premise
6	4 and 5	$\neg$ s $\to$ r	Rule of syllogism
7	---	$\neg$ r $\to$ s	By Contrapositive

Hence, the given argument is valid argument.

2. Test the validity of the following argument

$$\boldsymbol{p \to q}$$
$$\boldsymbol{\neg r \vee s}$$
$$\boldsymbol{p \vee r}$$

$$\boldsymbol{\therefore \neg q \to s}$$

Solution: We construct an argument to show that our premises lead to the desired conclusion as follows:

Step Number	Steps Used	Main Steps	Rules Used
1	---	$p \to q$	Given Premise
2	---	$p \vee r$	Given Premise
3	---	$\neg$ $p \to q$ $p \vee r$ $r \vee s$	Given Premise
4	1	$\neg q \to \neg p$	By Contrapositive
5	2	$\neg p \to r$	By Conditional law
6	3	$r \to s$	By Conditional law
7	4 and 5	$\neg q \to r$	Rule of syllogism
8	7 and 6	$\neg q \to s$	Rule of syllogism

Hence, the given argument is valid argument.

3. Show that $R \lor S$ follows logically from the premises $C \lor D$, $(C \lor D) \to \neg H$, $\neg H \to (A \land \neg B)$ and $(A \land \neg B) \to (R \lor S)$.

Solution: We construct an argument to show that our premises lead to the desired conclusion as follows:

Step Number	Steps Used	Main Steps	Rules Used
1	---	$(C \lor D) \to \neg H$	Given Premise
2	---	$\neg H \to (A \land \neg B)$	Given Premise
3	1 and 2	$(C \lor D) \to (A \land \neg B)$	Rule of syllogism
4	---	$(A \land \neg B) \to (R \lor S)$	Given Premise
5	3 and 4	$(C \lor D) \to (R \lor S)$	Rule of syllogism
6	---	$C \lor D$	Given Premise
7	5 and 6	$(R \lor S)$	Modus ponens

A second way of showing our premises leads to the desired conclusion as follows:

Step Number	Steps Used	Main Steps	Rules Used
1	---	$C \lor D$	Given Premise
2	---	$(C \lor D) \to \neg H$	Given Premise
3	1 and 2	$\neg H$	Modus ponens
4	---	$\neg H \to (A \land \neg B)$	Given Premise
5	3 and 4	$(A \land \neg B)$	Modus ponens
6	---	$(A \land \neg B) \to (R \lor S)$	Given Premise
7	5 and 6	$(R \lor S)$	Modus ponens

This proves the required result.

4. Verify the validity of the following argument:

Rita is baking a cake.
If Rita is baking a cake then she is not practicing her flute.
If Rita is not practicing her flute, then her father will not buy her a car.
Therefore, Rita's father will not buy her a car.

Solution:

Let p: Rita is baking a cake.

q: she is practicing her flute.

r: Father will buy her a car.

The given argument in symbolic form is

$$\begin{array}{l} p \\ p \rightarrow \neg q \\ \neg q \rightarrow \neg r \\ \hline \therefore \neg r \end{array}$$

We establish the validity of the argument as follows:

Step Number	Steps Used	Main Steps	Rules Used
1	---	$p \rightarrow \neg q$	Given Premise
2	---	$\neg q \rightarrow \neg r$	Given Premise
3	1 and 2	$p \rightarrow \neg r$	Rule of syllogism
4	---	p	Given Premise
5	3 and 4	$\neg r$	Modus ponens

Hence, the given argument is valid argument.

A second way to validate the argument as follows:

Step Number	Steps Used	Main Steps	Rules Used
1	---	p	Given Premise
2	---	$p \rightarrow \neg q$	Given Premise
3	1 and 2	$\neg q$	Modus ponens
4	---	$\neg q \rightarrow \neg r$	Given Premise
5	3 and 4	$\neg r$	Modus ponens

Hence, the given argument is valid argument.

5. Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution:

Let p: "It is sunny this afternoon,"
q: "It is colder than yesterday,"
r: "We will go swimming,"
s: "We will take a canoe trip,"
t: "We will be home by sunset."

Then the given premises are $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$ and $s \rightarrow t$ and the conclusion is t.

We construct an argument to show that our premises lead to the desired conclusion as follows:

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\neg p \wedge q$	Given Premise
2	1	$\neg p$	Rule of Conjunctive Simplification
3	---	$r \rightarrow p$	Given Premise
4	2 and 3	$\neg r$	Modus tollens
5	---	$\neg r \rightarrow s$	Given Premise
6	4 and 5	s	Modus ponens
7	---	$s \rightarrow t$	Given Premise
8	6 and 7	t	Modus ponens

This proves the required result.

6. **Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."**

Solution:

Let p: "You send me an e-mail message,"
q: "I will finish writing the program,"
r: "I will go to sleep early,"
s: "I will wake up feeling refreshed."

Then the given premises are $p \rightarrow q$, $\neg p \rightarrow r$, $r \rightarrow s$ and the conclusion is $\neg q \rightarrow s$.

We construct an argument to show that our premises lead to the desired conclusion as follows:

Step Number	Steps Used	Main Steps	Rules Used
1		$p \rightarrow q$	Given Premise
2	1	$\neg q \rightarrow \neg p$	Contrapositive
3		$\neg p \rightarrow r$	Given Premise
4	2 and 3	$\neg q \rightarrow r$	Rule of syllogism
5		$r \rightarrow s$	Given Premise
6	4 and 5	$\neg q \rightarrow s$	Rule of syllogism

This proves the required result.

7. Verify the validity of the following argument:

$$\mathbf{p \rightarrow q}$$
$$\mathbf{q \rightarrow (r \wedge s)}$$
$$\mathbf{\neg r \vee (\neg t \vee u)}$$
$$\mathbf{p \wedge t}$$

$$\therefore \mathbf{u}$$

Solution:

We construct an argument to show that our premises lead to the desired conclusion as follows:

Step Number	Steps Used	Main Steps	Rules Used
1	---	$p \rightarrow q$	Given Premise
2	---	$q \rightarrow (r \wedge s)$	Given Premise
3	1 and 2	$p \rightarrow (r \wedge s)$	Rule of syllogism
4	---	$p \wedge t$	Given Premise
5	4	p	Rule of Conjunctive Simplification
6	5 and 3	$(r \wedge s)$	Modus ponens
7	6	r	Rule of Conjunctive Simplification
8	---	$\neg r \vee (\neg t \vee u)$	Given Premise
9	8	$\neg (r \wedge t) \vee u$	By associative and De Morgan's Laws
10	4	t	Rule of Conjunctive Simplification
11	7 and 10	$r \wedge t$	Rule of Conjunction
12	9 and 11	u	Rule of Disjunctive Syllogism

Hence the given argument is valid argument.

8. Show that $\mathbf{R \wedge (P \vee Q)}$ is a valid conclusion from the premises $\mathbf{P \vee Q}$, $\mathbf{Q \rightarrow R}$, $\mathbf{P \rightarrow M}$ and $\mathbf{\neg M}$.

Solution:

We construct an argument to show that our premises lead to the desired conclusion as follows:

Step Number	Steps Used	Main Steps	Rules Used
1	---	$P \vee Q$	Given Premise
2	1	$\neg P \rightarrow Q$	By Conditional law
3	---	$Q \rightarrow R$	Given Premise
4	2 and 3	$\neg P \rightarrow R$	Rule of syllogism
5	---	$P \rightarrow M$	Given Premise
6	---	$\neg M$	Given Premise
7	5 and 6	$\neg p$	Modus Tollens
8	7 and 4	R	Modus ponens
9	8 and 1	$R \wedge (P \vee Q)$	By Conjunction

This proves the required result.

1.8.2 Rules of Inference for Quantified Statements

We have discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

Universal Instantiation or Specification is the rule of inference used to conclude that P(c) is true, where c is a particular member of the domain, given the premise $\forall x\ P(x)$. Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the domain of all women.

Universal Generalization is the rule of inference that states that $\forall x\ P(x)$ is true, given the premise that P(c) is true for all elements c in the domain. Universal generalization is used when we show that $\forall x\ P(x)$ is true by taking an arbitrary element c from the domain and showing that P(c) is true. The element c that we select must be an arbitrary, and not a specific, element of the domain. That is, when we assert from $\forall x\ P(x)$ the existence of an element c in the domain, we have no control over c and cannot make any other assumptions about c other than it comes from the domain. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly. However, the error of adding unwarranted assumptions about the arbitrary element c when universal generalization is used is all too common in incorrect reasoning.

Existential Instantiation or Specification is the rule that allows us to conclude that there is an element c in the domain for which P(c) is true if we know that ∃x P(x) is true. We cannot select an arbitrary value of c here, but rather it must be a c for which P(c) is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

Existential Generalization is the rule of inference that is used to conclude that ∃x P(x) is true when a particular element c with P(c) true is known. That is, if we know one element c in the domain for which P(c) is true, then we know that ∃x P(x) is true.

We summarize these rules of inference in the following Table. We will illustrate how some of these rules of inference for quantified statements are used in Examples.

S.No	Rule of Inference	Name
1	$\dfrac{\forall x\, P(x)}{\therefore P(c)}$	Universal Instantiation
2	$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x\, P(x)}$	Universal Generalization
3	$\dfrac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c}$	Existential Instantiation
4	$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x\, P(x)}$	Existential Generalization

EXAMPLES:

1. **Establish the validity of the following argument:**

?x **[p(x) ì q(x)]**
?x **[q(x) ì r(x)]**

---

y ?x **[p(x) → r(x)]**

Solution:

The following steps can be used to establish the conclusion from the premises.

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\forall x\ [p(x) \rightarrow q(x)]$	Given Premise
2	1	$p(a) \rightarrow q(a)$	Universal Instantiation
3	---	$\forall x\ [q(x) \rightarrow r(x)]$	Given Premise
4	3	$q(a) \rightarrow r(a)$	Universal Instantiation
5	2 and 4	$p(a) \rightarrow r(a)$	Rule of Syllogism
6	5	$\forall x\ [p(x) \rightarrow r(x)]$	Universal Generalization

Hence, the given argument is valid argument.

2. **Establish the validity of the following argument:**

?x [p(x) ì q(x)]
?x [q(x) ì r(x)]
Ė r(c)
∴ ¬ p(c)

Solution:

The following steps can be used to establish the conclusion from the premises.

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\forall x\ [p(x) \rightarrow q(x)]$	Given Premise
2	1	$p(c) \rightarrow q(c)$	Universal Instantiation
3	---	$\forall x\ [q(x) \rightarrow r(x)]$	Given Premise
4	3	$q(c) \rightarrow r(c)$	Universal Instantiation
5	2 and 4	$p(c) \rightarrow r(c)$	Rule of Syllogism
6	5	$\neg r(c)$	Given Premise
7	6 and 5	$\neg p(c)$	Modus Tollens

Hence, the given argument is valid argument.

3. **Establish the validity of the following argument:**

?x [p(x) Ę q(x)]
Ax Ė p(x)
?x [Ė q(x) Ę r(x)]
?x [s(x) → Ė r(x)]
y Ax Ė s(x)

Solution:

The following steps can be used to establish the conclusion from the premises.

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\forall x\ [p(x) \lor q(x)]$	Given Premise
2	1	$p(a) \lor q(a)$	Universal Instantiation
3	---	$\exists x \neg p(x)$	Given Premise
4	3	$\neg p(a)$	Existential Instantiation
5	2 and 4	$q(a)$	Rule of Disjunctive Syllogism
6	---	$\forall x\ [\neg q(x) \lor r(x)]$	Given Premise
7	6	$\neg q(a) \lor r(a)$	Universal Instantiation
8	5 and 7	$r(a)$	Rule of Disjunctive Syllogism
9	---	$\forall x\ [s(x) \rightarrow \neg r(x)]$	Given Premise
10	9	$s(a) \rightarrow \neg r(a)$	Universal Instantiation
11	8 and 10	$\neg s(a)$	Modus Tollens
12	11	$\exists x \neg s(x)$	Existential Generalization

Hence, the given argument is valid argument.

4. Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution:

Let D(x) denote "x is in this discrete mathematics class,"

let C(x) denote "x has taken a course in computer science."

Then the premises are $\forall x\ (D(x) \rightarrow C(x))$ and D(Marla) and the conclusion is C(Marla).

The following steps can be used to establish the conclusion from the premises.

Step Number	Steps Used	Main Steps	Rules Used
1	---	$\forall x(D(x) \rightarrow C(x))$	Given Premise
2	1	$D(Marla) \rightarrow C(Marla)$	Universal instantiation
3	---	D(Marla)	Given Premise
4	2 and 3	C(Marla)	Modus ponens

This proves the required result.

5. **Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."**

Solution:

Let C(x) be "x is in this class,"

Let B(x) be "x has read the book," and P(x) be "x passed the first exam."

The premises are $\exists x\,(C(x) \wedge \neg B(x))$ and $\forall x\,(C(x) \rightarrow P(x))$ and the conclusion is $\exists x\,(P(x) \wedge \neg B(x))$.

The following steps can be used to establish the conclusion from the premises.

Step Number	Steps Used	Main Steps	Rules Used
1		$\exists x\,(C(x) \wedge \neg B(x))$	Given Premise
2	1	$C(a) \wedge \neg B(a)$	Existential instantiation
3	2	$C(a)$	Rule of Conjunctive Simplification
4		$\forall x\,(C(x) \rightarrow P(x))$	Given Premise
5	4	$C(a) \rightarrow P(a)$	Universal instantiation
6	3 and 5	$P(a)$	Modus ponens
7	2	$\neg B(a)$	Rule of Conjunctive Simplification
8	6 and 7	$P(a) \wedge \neg B(a)$	Conjunction
9	8	$\exists x\,(P(x) \wedge \neg B(x))$	Existential generalization

This proves the required result.

1.8.3 Combining Rules of Inference for Propositions and Quantified Statements

We have developed rules of inference both for propositions and for quantified statements. Note that in our arguments in the above Examples 4 and 5 we used both universal instantiation, a rule of inference for quantified statements, and modus ponens, a rule of inference for propositional logic. We will often need to use this combination of rules of inference. Because universal instantiation and modus ponens are used so often together, this combination of rules is sometimes called universal modus ponens. This rule tells us that if $\forall x\,(P(x) \rightarrow Q(x))$ is true, and if P(a) is true for a particular element a in the domain of the universal quantifier, then Q(a) must also be true. To see this, note that by universal instantiation, $P(a) \rightarrow Q(a)$ is true. Then, by modus ponens, Q(a) must also be true. We can describe **Universal Modus Ponens** as follows:

$\forall x\ (P(x) \rightarrow Q(x))$

P(a), where a is a particular element in the domain

$\therefore Q(a)$

Another useful combination of a rule of inference from propositional logic and a rule of inference for quantified statements is **Universal Modus Tollens**. Universal Modus Tollens combines Universal Instantiation and Modus Tollens and can be expressed in the following way:

$\forall x\ (P(x) \rightarrow Q(x))$

$\neg$ Q(a), where a is a particular element in the domain

$\therefore \neg P(a)$

EXAMPLE:

1. **Assume that "For all positive integers *n*, if *n* is greater than 4, then $n^2 < 2^n$ " is true. Use universal modus ponens to show that $100^2 < 2^{100}$.**

   Solution:

   Let $P(n)$: $n > 4$ and $Q(n)$: $n^2 < 2^n$

   Then the given statement can be represented by $\forall n\ (P(n) \rightarrow Q(n))$,where the domain consists of all positive integers. We are assuming that $\forall n\ (P(n) \rightarrow Q(n))$ is true. Note that P (100) is true because 100 > 4. It follows by universal modus ponens that Q(100) is true, namely that $100^2 < 2^{100}$

2. **Verify the validity of the following argument:**

   No Engineering student of First or Second Semester studies Logic.

   Anil is an Engineering student who studies Logic.

   y **Anil is not in Second Semester.**

   Solution:

   Let us take the universe to be the set of all Engineering students.

   Let P(x): x is in First Semester

   Q(x): x is in Second Semester

   R(x): x studies Logic

   a: Anil

   Then the given statement can be represented by

   $\forall x\ [\{P(x) \vee Q(x)\} \rightarrow \neg R(x)]$

   R(a)

   $\therefore \neg Q(a)$

By Universal Modus Tollens, it follows that $\neg\{P(a) \vee Q(a)\}$

$$\Leftrightarrow \neg P(a) \wedge \neg Q(a) \quad \text{(by De Morgan's Law)}$$

$$\Leftrightarrow \neg Q(a) \quad \text{(by Rule of Conjunctive Simplification)}$$

This proves that the given argument is valid argument.

3. Verify the validity of the following argument:

No Engineering student is bad in studies.

Anil is not bad in studies.

---

y Anil is an Engineering student.

Solution:

Let us take the universe to be the set of all students.

Let P(x): x is an Engineering student

Q(x): x is bad in studies

a: Anil

Then the given statement can be represented by

$$\forall x\,[P(x) \rightarrow \neg Q(x)]$$

$$\neg Q(a)$$

$$\therefore \; P(a)$$

By Universal Modus Tollens, it follows that $\neg P(a)$

This proves that the given argument is not valid argument.

PROBLEMS TO PRACTICE

1. Test the validity of the following arguments:

(a) If there is a strike by students then exam will be postponed. Exam was not postponed. Therefore there were no strikes by students. [Jan'09]

(b) If Sachin hits a century then he gets a free car. Sachin gets a free car. Therefore Sachin has hit a century.

(c) If I drive to work, then I will arrive tired. I am not tired. Therefore I do not drive to work.

(d) If interest rate falls then stock market will rise. The stock market will not rise. Therefore the interest rates will not fall. [Jan '06]

(e) If Ravi studies, then he will pass in DMS. If Ravi does not play cricket, then he will study. Ravi failed in DMS. Therefore, Ravi played cricket. [Jan '09]

(f) If I study then I will not fail in exam. If I do not watch TV in the evening then I will study. I failed in exam. Therefore I must have watched TV in the evening. [Jan '10, Dec '10, Jan' 17]

(g) If Rochelle gets the supervisor's position and work hard, then she will get a rise in her payment. If she gets a rise, then she will buy a car. She has not purchased the car. Therefore either Rochelle did not get the supervisor's position or she did not work hard.

(h) Let p, q and r be the primitive statements, p: Ram studies, q: Ram plays tennis, r: Ram passes in DMS. Let H_1 , H_2 , H_3 be the premises. H_1: If Ram studies then he will pass in DMS. H_2: If Ram does not play Tennis then he will study. H_3: Ram did not pass in DMS. Show that q follows from H_1 , H_2 , H_3 .

(i) I will get grade A in this course or I will not graduate. If I do not graduate, I will join army. I got grade A. Therefore, I will not join the army. [Jan 2008]

(j) **I will become famous or I will not become a musician. I will become a musician. Therefore, I will become famous. [July 2007]**

Solution:

(a) Let p: There is a strike by students and q: Exam is postponed

Given premises are $p \to q$, $\neg q$

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \to q$	Premise 1
2	---	$\neg q$	Premise 2
3	1, 2	$\neg p$	Modus Tollens

Therefore, there were no strikes by students.

Therefore, the given argument is valid.

(b) Let p: Sachin hits a century and q: Sachin gets a free car

Given premises are $p \to q$, q

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \to q$	Premise 1
2	---	q	Premise 2
3	1, 2	p need not be true	

$\therefore p$ need not be true.

Therefore Sachin may not hit a century.

Therefore, the given argument is invalid.

(c) Let p: I drive to work and q: I will arrive tired

Given premises are $p \to q,\ \neg q$

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \to q$	Premise 1
2	---	$\neg q$	Premise 2
3	1, 2	$\neg p$	Modus Tollens

Therefore I do not drive to work.

Therefore, the given argument is valid.

(d) Let p: interest rate falls and q: stock market will rise.

Given premises are $p \to q,\ \neg q.$

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \to q$	Premise 1
2	---	$\neg q$	Premise 2
3	1, 2	$\neg p$	Modus Tollens

Therefore the interest rates will not fall.

Therefore, the given argument is valid.

(e) Let p: Ravi studies, q: He will pass in DMS and r: Ravi plays cricket

Given premises are $p \to q,\ \neg r \to p,\ \neg q.$

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \to q$	Premise 1
2	---	$\neg q$	Premise 3
3	1, 2	$\neg p$	Modus Tollens
4	---	$\neg r \to p$	Premise 2
5	3, 4	r	Modus Tollens

Therefore Ravi played cricket. Therefore, the argument is valid.

(f) Let p: I study, q: I fail in exam. and r: I watch TV in the evening.

Given premises are $p \to \neg q,\ \neg r \to p,\ q$

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \to \neg q$	Premise 1
2	---	q	Premise 2
3	1, 2	$\neg p$	Modus Tollens
4	---	$\neg r \to p$	Premise 3
5	3, 4	r	Modus Tollens

Therefore I watch TV in the evening. Therefore, the given argument is valid.

(g) Let p: Rochelle gets the supervisor's position,

q: Rochelle works hard

r: Rochelle gets a rise in her payment, s: Rochelle purchased the car.

Given premises are $(p \land q) \to r,\ r \to s,\ \neg s$

Step Number	Steps used	Main Steps	Rules Used
1	---	$r \to s$	Premise 2
2	---	$\neg s$	Premise 3
3	1, 2	$\neg r$	Modus Tollens
4	---	$(p \wedge q) \to r$	Premise 1
5	3, 4	$\neg(p \wedge q)$	Modus Tollens
6	5	$\neg p \vee \neg q$	By De Morgan's law

$\therefore$ Either Rochelle did not get the supervisor's position or she did not work hard.

Therefore, the given argument is valid.

(h) Let p: Ram studies q: Ram plays tennis, r: Ram passes in DMS

Given premises are $H_1: p \to r,\ H_2: \neg q \to p, H_3: \neg r$

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \to r$	Premise H_1
2	---	$\neg r$	Premise H_3
3	1, 2	$\neg p$	Modus Tollens
4	---	$\neg q \to p$	Premise H_2
5	4,3	q	Modus Tollens

Therefore, q follows from H_1 , H_2 *and* H_3.

(i) Let p: I get grade A in this course. q: I am not graduate. r : I will join army.

Given premises are $p \vee q$, $q \to r$, p

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \vee q$	Premise 1
2	---	$\neg p \to q$	Conditional law
3	1	$q \to r$	Premise 2
4	1, 2	$\neg p \to r$	Rule of syllogism
5	---	p	Premise 3

$\therefore \neg r$ need not be true.

Therefore, this argument is not valid.

(j) Let p: I will become famous q: I will become a musician.

Given premises are $p \vee \neg q$, q

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \vee \neg q$	Premise 1
2	---	q	Premise 2
3	1, 2	p	Disjunctive syllogism

Therefore, I will become a musician. Therefore, this argument is valid.

2. Test the validity of the following arguments:

(a) $p \to q$, $\neg r \vee s$, $p \vee r$ $\therefore \neg q \to s$

(b) $p \vee q$, $\neg p \vee r$, $\neg r$ $\therefore q$ **[June 2012]**

(c) $p \wedge \neg q$, $p \to (q \to r)$ $\therefore \neg r$ **[Dec'11, July'11]**

(d) $p, p \to q$, $s \vee r$, $r \to \neg q$ $\therefore s \vee t$ **[July'11, Jan'17]**

(e) $p \wedge q$, $p \to (q \to r)$ $\therefore r$

(f) $p \to r$, $q \to r$ $\therefore (p \vee q) \to r$

(g) p, $p \to r$, $p \to (q \vee \neg r)$, $\neg q \vee \neg s$ $\therefore s$ **[June 2012]**

(h) $p \leftrightarrow q$, $q \to r$, $r \vee \neg s$, $\neg s \to q$ $\therefore s$ **[July 2011]**

Solution:

(a) Given premises are $p \to q$, $\neg r \lor s$, $p \lor r$

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \to q$	Premise 1
2	---	$p \lor r$	Premise 2
3	---	$\neg r \lor s$	Premise 3
4	1	$\neg q \to \neg p$	Contra positive
5	2	$\neg p \to r$	Conditional law
6	3	$r \to s$	Conditional law
7	1,2, 3	$\neg q \to s$	Rule of syllogism

Therefore, this argument is valid.

(b)

Step Number	Steps used	Main Steps	Rules Used
1	---	$\neg p \lor r$	Premise 2
2	---	$\neg r$	Premise 3
3	1, 2	$\neg p$	Disjunctive syllogism
4	---	$p \lor q$	Premise 1
5	3, 4	q	Disjunctive syllogism

Therefore, this argument is valid.

(c)

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \wedge \neg q$	Premise 1
2	---	$p \rightarrow (q \rightarrow r)$	Premise 2
3	1	p	Conjunctive simplification
4	2, 3	$q \rightarrow r$	Modus pones
5	1	$\neg q$	Conjunctive simplification
6	4, 5	r or $\neg r$	---

Therefore, this argument is invalid.

(d)

Step Number	Steps used	Main Steps	Rule Used
1	---	p	Premise p_1
2	---	$p \rightarrow q$	Premise p_2
3	1, 2	q	Contra positive
4	---	$r \rightarrow \neg q$	Premise p_3
5	3, 4	$\neg r$	Conditional law
6	---	$s \vee r$	Disjunctive syllogism
7	5, 6	s	

Therefore, this argument is valid.

(e)

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \wedge q$	Premise p_1
2	1	p	Conjunctive simplification
3	---	$p \to (q \to r)$	Premise p_2
4	2, 3	$q \to r$	Modus pones
5	1	q	Conjunctive simplification
6	4, 5	r	Modus pones

(f)

Step Number	Steps used	Main Steps	Rules Used
1	---	$p \to r$	Premise p_1
2	1	$\neg p \vee r$	Conditional law
3	---	$q \to r$	Premise p_2
4	3	$\neg q \vee r$	Conditional law
5	2, 4	$(\neg p \wedge \neg q) \vee r$	Distributive law
6	5	$\neg(p \vee q) \vee r$	Demorgen’s law
7	6	$(p \vee q) \to r$	Conditional law

Therefore this argument is valid.

(g)

Step Number	Steps used	Main Steps	Rules Used
1	---	p	Premise p_1
2	---	$p \to r$	Premise p_2
3	---	$p \to (q \vee \neg r)$	Premise p_3
4	1, 2	r	Modus pones
5	1, 3	$q \vee \neg r$	Modus pones
6	4, 5	q	Disjunctive syllogism
7	---	$\neg q \vee \neg s$	Conditional law
8	6, 7	$\neg s$	Disjunctive syllogism

Therefore, this argument is invalid.

(h)

Step Number	Steps used	Main Steps	Rules Used
1	---	$r \vee \neg s$	Premise p_3
2	1	$\neg r \to \neg s$	Conditional law
3	---	$\neg s \to q$	Premise p_4
4	---	$q \to r$	Premise p_2
5	2, 3, 4	$\neg r \to r$	Rule of syllogism
6	5	$r \vee r$	Conditional law
7	5	r	Disjunctive simplification
8	1, 7	s need not be true	Contra positive

Therefore, this argument is invalid.

3. Find whether the following arguments valid:

(a) If a triangle has two equal sides, then it is isosceles.

If a triangle is isosceles, then it has two equal angles.

A certain triangle ABC does not have two equal angles.

∴ The triangle ABC does not have two equal sides. **[Jan'09]**

(b) No Engineering student of first or second semester studies logic.

Anil is an Engineering student who studies Logic.

∴ Anil is not in second semester. **[Jan'07]**

(c) All Mathematics professors have studied calculus.

Ramanujan is a mathematics professor.

∴ Ramanujan have studied calculus. **[Jan'06]**

(d) All employers pay their employees.

Anil is an employer.

∴ Anil pays his employees. **[July'16]**

Solution:

(a) Let p(x): x has equal sides, q(x): x is isosceles, r(x): x has two equal angles.

Let c denote the triangle ABC.

By data,

$$\begin{array}{l} \forall x,[p(x) \to q(x)] \\ \forall x,[q(x) \to r(x)] \\ \neg r(c) \\ \hline \therefore \neg p(c) \end{array}$$

Step Number	Steps used	Main Steps	Rules Used
1	---	$\forall x,[p(x) \to q(x)]$	Premise 1
2	---	$\forall x,[q(x) \to r(x)]$	Premise 2
3	1	$p(c) \to q(c)$	Universal specification
4	2	$q(c) \to r(c)$	Universal specification
5	1, 2	$p(c) \to r(c)$	Rule of syllogism
6	---	$\neg r(c)$	Premise 3
7	5, 6	$\neg p(c)$	Modus Tollens

∴ The triangle ABC does not have two equal sides.

(b) Let $p(x): x$ is in first semester, $q(x):$ x is in second semester, $r(x):$ x studies logic and c: Anil

By data, $\forall x, [p(x) \vee q(x)] \rightarrow \neg r(x)$

$r(c)$

$\therefore \neg q(c)$

Step Number	Steps used	Main Steps	Rules Used
1	---	$\forall x, [p(x) \vee q(x)] \rightarrow \neg r(x)$	Premise 1
2	1	$[p(c) \vee q(c)] \rightarrow \neg r(c)$	Universal specification
3	---	$r(c)$	Premise 2
4	2, 3	$\neg[p(c) \vee q(c)]$	Modus Tollens
5	4	$\neg p(c) \wedge \neg q(c)$	Demorgen's law
6	5	$\neg q(c)$	Conjunctive simplification

$\therefore$ Anil is not in second semester.

(c) Let $p(x): x$ is a Mathematics professor, $q(x):$ x is studies calculus, c: Ramanujan.

By data, $\forall x, [p(x) \rightarrow q(x)]$

$p(c)$

$\therefore q(c)$

Step Number	Steps used	Main Steps	Rules Used
1	---	$\forall x, [p(x) \rightarrow q(x)]$	Premise 1
2	1	$p(c) \rightarrow q(c)$	Universal specification
3	---	$p(c)$	Premise 2
4	2, 3	$q(c)$	Modus pones

$\therefore$ Ramanujan have studied calculus.

(d) Let $p(x): x$ is an employer, $q(x):$ x pays his employee, c: Anil

By data, $\forall x, [p(x) \rightarrow q(x)]$

$p(a)$

$\therefore \neg q(c)$

Step Number	Steps used	Main Steps	Rules Used
1	---	$\forall x, [p(x) \rightarrow q(x)]$	Premise 1
2	---	$\forall x, [q(x) \rightarrow r(x)]$	Premise 2
3	1	$p(c) \rightarrow q(c)$	Universal specification
4	2	$q(c) \rightarrow r(c)$	Universal specification
5	4 and 5	$p(c) \rightarrow r(c)$	Rule of syllogism
6	---	$\neg r(c)$	Premise 3
7	5 and 6	$\neg p(c)$	Modus Tollens

4. Prove that the following arguments are valid:

(a) $\forall x, [p(x) \rightarrow q(x)]$

$\forall x, [q(x) \rightarrow r(x)]$

$\therefore \forall x, [p(x) \rightarrow r(x)]$ **[Jan'14]**

(b) $\forall x, \{p(x) \rightarrow [q(x) \wedge r(x)]\}$

$\forall x, [p(x) \wedge s(x)]$

$\therefore \forall x, [r(x) \wedge s(x)]$ **[Jan'17]**

(c) $\forall x, [p(x) \vee q(x)]$

$\forall x, [\{\neg p(x) \wedge q(x)\} \rightarrow r(x)]$

$\therefore \forall x, [\neg r(x) \rightarrow p(x)]$ **[July'11, Dec'12]**

(d) $\forall x, [p(x) \vee q(x)]$

$\exists x, \neg p(x)$

$\forall x, [\neg q(x) \vee r(x)]$

$\forall x, [s(x) \rightarrow \neg r(x)]$

$\therefore \exists x, \neg s(x)$ **[Dec'11, Jan'17]**

Solution:

(a)

Step Number	Steps used	Main Steps	Rules Used
1	---	$\forall x,[p(x) \to q(x)]$	Premise 1
2	---	$\forall x,[q(x) \to r(x)]$	Premise 2
3	1	$p(c) \to q(c)$	Universal specification
4	2	$q(c) \to r(c)$	Universal specification
5	1, 2	$p(c) \to r(c)$	Rule of syllogism
6	5	$\forall x,[p(x) \to r(x)]$	Universal generalisation

(b)

Step Number	Steps used	Main Steps	Rules Used
1	---	$\forall x,\{p(x) \to [q(x) \wedge r(x)]\}$	Premise 1
2	---	$\forall x,[p(x) \wedge s(x)]$	Premise 2
3	1	$p(c) \to [q(c) \wedge r(c)]$	Universal specification
4	2	$p(c) \wedge s(c)$	Universal specification
5	4	$p(c)$	Conjunctive simplification
6	3, 5	$q(c) \wedge r(c)$	Modus pones
7	6	$r(c)$	Conjunctive simplification
8	4	$s(c)$	Conjunctive simplification
9	7, 8	$r(c) \wedge s(c)$	Conjunction
10	9	$\therefore \forall x,[r(x) \wedge s(x)]$	Universal Generalisation

(c)

Step Number	Steps used	Main Steps	Rules Used
1	---	$\forall x, [\{\neg p(x) \wedge q(x)\} \rightarrow r(x)]$	Premise 1
2	1	$\{\neg p(c) \wedge q(c)\} \rightarrow r(c)$	Universal specification
3	2	$\neg\{\neg p(c) \wedge q(c)\} \vee r(c)$	Conditional law
4	3	$\{p(c) \vee \neg q(c)\} \vee r(c)$	De Morgan's law
5	4	$\{p(c) \vee \{\neg q(c) \vee r(c)\}$	Associative law
6	---	$\{p(c) \vee \{q(c) \rightarrow r(c)\}$	Conditional law
7	6	$\forall x, [p(x) \vee q(x)]$	Premise 2
8	5, 7	$p(c) \vee q(c)$	Universal specification
9	6, 8	$p(c) \vee \{[q(c) \rightarrow r(c)] \wedge q(c)\}$	Distributive law
10	9	$p(c) \vee r(c)$	Modus pones
11	10	$r(c) \vee p(c)$	Commutative law
12	11	$\neg r(c) \rightarrow p(c)$	Conditional law

(d)

Step Number	Steps used	Main Steps	Rules Used
1	---	$\forall x,[p(x) \vee q(x)]$	Premise 1
2	1	$p(c) \vee q(c)$	Universal specification
3	---	$\exists x, \neg p(x)$	Premise 2
4	3	$\neg p(c)$	Universal specification
5	2, 4	$q(c)$	Rule of syllogism
6	---	$\forall x,[\neg q(x) \vee r(x)]$	Premise 3
7	6	$\neg q(c) \vee r(c)$	Universal specification
8	5, 7	$r(c)$	Disjunctive syllogism
9	---	$\forall x,[s(x) \to \neg r(x)]$	Premise 3
10	9	$s(c) \to \neg r(c)$	Universal specification
11	8, 10	$\neg s(c)$	Rule of syllogism
12	11	$\forall x, \neg s(x)$	Universal Generalisation

EXERCISE – 1.3

1. Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."
2. Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."
3. Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."
4. Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

5. Show that the premises “It is not sunny this afternoon and it is colder than yesterday,” “We will go swimming only if it is sunny,” “If we do not go swimming, then we will take a canoe trip,” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset.”
6. Show that the following argument is valid. If today is Tuesday, I have a test in Mathematics or Economics. If my Economics Professor is sick, I will not have a test in Economics. Today is Tuesday and my Economics Professor is sick. Therefore, I have a test in Mathematics.
7. Show that the premises “Everyone in this discrete mathematics class has taken a course in computer science” and “Marla is a student in this class” imply the conclusion “Marla has taken a course in computer science.”
8. Show that the premises “A student in this class has not read the book,” and “Everyone in this class passed the first exam” imply the conclusion “Someone who passed the first exam has not read the book.”
9. Use rules of inference to show that the hypotheses “All men are mortal. Socrates is a man. Therefore, Socrates is mortal.”

1.9 INTRODUCTION TO PROOFS

In this section we introduce the notion of a proof and describe methods for constructing proofs. A proof is a valid argument that establishes the truth of a mathematical statement. A proof can use the hypotheses of the theorem, if any, axioms assumed to be true and previously proven theorems. Using these ingredients and rules of inference, the final step of the proof establishes the truth of the statement being proved.

The methods of proof discussed in this chapter are important not only because they are used to prove mathematical theorems, but also for their many applications to computer science. These applications include verifying that computer programs are correct, establishing that operating systems are secured, making inferences in artificial intelligence, and showing that system specifications are consistent, and so on. Consequently, understanding the techniques used in proofs is essential both in mathematics and in computer science.

Remark: Before we introduce methods for proving theorems, we need to understand how many mathematical theorems are stated. Many theorems assert that a property holds for all elements in a domain, such as the integers or the real numbers. Although the precise statement of such theorems needs to include a universal quantifier, the standard convention in mathematics is to omit it.

For example, the statement

“If $x > y$, where x and y are positive real numbers then $x^2 > y^2$.”

really means
"For all positive real numbers x and y, if $x > y$ then $x^2 > y^2$."

Furthermore, when theorems of this type are proved, the first step of the proof usually involves selecting a general element of the domain. Subsequent steps show that this element has the property in question. Finally, universal generalization implies that the theorem holds for all members of the domain.

METHODS OF PROOFS

Proving mathematical theorems can be difficult. To construct proofs we need all available ammunition, including a powerful battery of different proof methods. These methods provide the overall approach and strategy of proofs. Understanding these methods is a key component of learning how to read and construct mathematical proofs. One we have chosen a proof method, we use axioms, definitions of terms, previously proved results, and rules of inference to complete the proof. Note that in this book we will always assume the axioms for real numbers. We will also assume the usual axioms whenever we prove a result about geometry. When you construct your own proofs, be careful not to use anything but these axioms, definitions, and previously proved results as facts!

1. **Direct Proofs:** A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true.

   Working Procedure:

   Given a conditional statement $p \to q$,

   First step: Assume that p is true;

   Second step: By using rules of inference/ laws of logic and other known facts, infer that q is true.

   Third step: It follows that the given conditional $p \to q$ is true.

EXAMPLE: Give a direct proof of the theorem "If *n* is an odd integer, then n^2 is odd."

Solution:

Given, $P \to Q$, where P is "n is an odd integer" and Q "n^2 is odd."

Assume that P is true.

i.e., n is an odd integer

$\Rightarrow n = 2k + 1$ where $k \in Z$

Now, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$ where $m = 2k^2 + 2k$

$\Rightarrow n^2 = 2m + 1$ which is odd

$\therefore$ Q is true.

It follows that P $\rightarrow$ Q is true.

2. **Proof by Contraposition (Indirect Proofs):**

Conditional statement p $\rightarrow$ q is equivalent to its contrapositive, $\neg$ q $\rightarrow$ $\neg$ p.

An indirect proof or Proofs by contraposition shows that a conditional statement p $\rightarrow$ q is true if its contrapositive, $\neg$ q $\rightarrow$ $\neg$ p, is true.

Working Procedure:

Given a conditional statement p $\rightarrow$ q,

First step: Assume that $\neg$ q is true;

Second step: By using rules of inference/ laws of logic and other known facts, infer that $\neg$ p is true.

Third step: It follows that the contrapositive, $\neg$ q $\rightarrow$ $\neg$ p, is true, hence the given conditional p $\rightarrow$ q is true.

EXAMPLE:

Prove by Contraposition "If *n* is an integer and 3*n* + 2 is odd, then *n* is odd."

Solution:

Given, P $\rightarrow$ Q, where P is "*n* is an integer" and Q "3*n* + 2 is odd."

To prove P $\rightarrow$ Q is true by Contraposition, we need to prove $\neg$ Q $\rightarrow$ $\neg$ P is true.

Assume that $\neg$ Q is true.

i.e., Q is false

$\Rightarrow$ $3n + 2$ is not odd

$\Rightarrow$ $3n + 2$ is even

$\Rightarrow$ $3n + 2 = 2k$ where $k \in Z$

$\Rightarrow$ $3n = 2k - 2$

$\Rightarrow$ $n = (2k - 2) / 3$ which is not integer.

$\therefore$ P is false.

$\Rightarrow$ $\neg$ P is true.

It follows that $\neg$ Q $\rightarrow$ $\neg$ P is true which is equivalent to P $\rightarrow$ Q is true.

3. Proof by Contradiction:

In this method, we prove a conditional statement $p \rightarrow q$ is true by the following steps:

Working Procedure:

Given a conditional statement $p \rightarrow q$,

First step: Assume that $p \rightarrow q$ is false;

i.e., p is true and q is false

Second step: Start with the hypothesis q is false, by using rules of inference/ laws of logic and other known facts, infer that p is false.

Third step: This contradicts our assumption that p is true. Hence our assumption is wrong.

It follows that the given conditional $p \rightarrow q$ is true.

Example:

Give a proof by contradiction of the theorem "If 3n + 2 is odd, then n is odd."

Solution:

Given, $P \rightarrow Q$, where P is "n is an integer" and Q "$3n + 2$ is odd."

To prove $P \rightarrow Q$ is true by Contradiction.

Assume that $P \rightarrow Q$ is false;

i.e., P is true and Q is false

Start with Q is false

$\Rightarrow$ $3n + 2$ is not odd

$\Rightarrow$ $3n + 2$ is even

$\Rightarrow$ $3n + 2 = 2k$ where $k \in Z$

$\Rightarrow$ $3n = 2k - 2$

$\Rightarrow$ $n = (2k - 2) / 3$ which is not integer.

$\therefore$ P is false.

This contradicts our assumption that P is true.

Hence our assumption is wrong.

$\therefore$ $P \rightarrow Q$ is true by Contradiction.

PROBLEMS TO PRACTICE

1. Give a direct proof for the following statements:

(a) The sum of two odd integers is an even integer.

(b) The square of an odd integer is an odd integer **[Jan'10, Jan'14]**

(c) If an integer a is such that $a - 2$ is divisible by 3, then $a^2 - 1$ is divisible by 3.

(d) For all positive integers m and n, if m and n are perfect squares, then mn is also a perfect square.

(e) For all integers k and l, if k and l are both even, then $k + l$ is even. **[Jan'09]**

Solution:

(a) Let p: x is a sum of two odd integers and q: x is an even integer.

Assume p is true.

$\Rightarrow x$ is a sum of two odd integers

$\Rightarrow x = (2m + 1) + (2n + 1), where\ m, n \in I$

$\Rightarrow x = 2m + 1 + 2n + 1$

$\Rightarrow x = 2m + 2n + 2$

$\Rightarrow x = 2(m + n + 1)$

$\Rightarrow x$ is an even integer

$\Rightarrow q$ is true.

$\therefore\ p \to q$ is true.

$\therefore$ The given statement is true by direct proof.

(b) Let p: x is a square of an odd integer and q: x is an odd integer.

Assume p is true.

$\Rightarrow x$ is a square of an odd integer

$\Rightarrow x = (2n + 1)^2, where\ n \in I$

$\Rightarrow x = 4n^2 + 4n + 1$

$\Rightarrow x$ is an odd integer

$\Rightarrow q$ is true.

$\therefore\ p \to q$ is true.

$\therefore$ The given statement is true by direct proof.

(c) Let p: $a - 2$ is divisible by 3 and q: $a^2 - 1$ is divisible by 3

Assume p is true.

$\Rightarrow a - 2$ is divisible by 3

$\Rightarrow a - 2 = 3n, n \in I$

$\Rightarrow a = 3n + 2$

$\Rightarrow a^2 - 1 = (3n + 2)^2 - 1$

$= (9n^2 + 12n + 4) - 1$

$= 9n^2 + 12n + 3$

$= 3(3n^2 + 4n + 1)$

$\Rightarrow a^2 - 1$ is divisible by 3.

$\Rightarrow q$ is true.

$\therefore\ p \to q$ is true.

$\therefore$ The given statement is true by direct proof.

(d) Let p: m and n are perfect squares and q: mn is a perfect square.

Assume p is true.

$\Rightarrow m$ and n are perfect squares

$\Rightarrow m = a^2, n = b^2, where\ a, b \in I$

$\Rightarrow mn = a^2b^2, where\ a, b \in I$

$\Rightarrow m = (ab)^2, where\ ab \in I$

$\Rightarrow mn$ is a perfect square.

$\Rightarrow q$ is true.

$\therefore\ p \to q$ is true.

$\therefore$ The given statement is true by direct proof.

(e) For all integers k and l, if $k$and l are both even, then $k + l$ is even.

Assume p is true.

$\Rightarrow k$ and l are both even

$\Rightarrow k = 2m, l = 2n, where\ m, n \in I$

$\Rightarrow k = 2m + 2n$

$\Rightarrow k = 2(m + n)$

$\Rightarrow k + l$ is even

$\Rightarrow q$ is true.

$\therefore\ p \to q$ is true.

$\therefore$ The given statement is true by direct proof.

2. Give an indirect proof for the following statements:

(a) For any real number x, if $x^2 > 0$ then $x \neq 0$.

(b) If n is an even integer then $n + 7$ is an odd integer.

(c) If x and y are integers such that xy is odd, then x and y are both odd.

(d) If an integer n is such that n^2 is odd then n is odd. **[July'16]**

(e) If n is a product of two positive integers a and b, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution:

(a) Let for any real number x, $p: x^2 > 0$ and $q: x \neq 0$.

Assume $\neg q$ is true.

□ $x = 0, x \in R$

$\Rightarrow$ $x^2 = 0, x \in R$

$\Rightarrow$ $\neg p$ is true.

∴ $\neg q \to \neg p$ is true and hence $p \to q$ is true.

∴ The given statement is true by indirect proof.

(b) Let p: m is an even integer and $q: m + 7$ is an odd integer.

Assume $\neg q$ is true.

□ $m + 7$ is an even integer.

□ $m + 7 = 2k, k \in I$.

□ $m = 2k - 7, k \in I$

$= 2k - 8 + 1, k \in I$

$= 2(k - 4) + 1, k \in I$

□ m is an odd integer

□ $\neg p$ is true.

□ $\neg q \to \neg p$ is true and hence $p \to q$ is true.

□ The given statement is true by indirect proof.

(c) Let if x and y are integers p: xy is odd and q: x and y are both odd.

Assume $\neg q$ is true.

□ x is even or y is even

□ $x = 2m$ or $y = 2n$, $m, n \in I$

□ $xy = 2my$ or $xy = x2n$.

□ $xy = 2(my)$ or $2(xn)$

□ xy is even

□ $\neg p$ is true.

□ $\neg q \to \neg p$ is true and hence $p \to q$ is true.

□ The given statement is true by indirect proof.

(d) Let p: n^2 is odd integer and q: n is odd integer.

Assume $\neg q$ is true.

□ n is an even integer

□ $n = 2k$, $k \in I$

□ $n^2 = (2k)^2 = 4k^2$

□ n^2 is even integer

□ $\neg p$ is true.

∴ $\neg q \rightarrow \neg p$ is true and hence $p \rightarrow q$ is true.

∴ The given statement is true by indirect proof.

(e) Let p: n is a product of two positive integers a and b, q: $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Assume $\neg q$ is true.

□ $a > \sqrt{n}$ and $b > \sqrt{n}$.

□ $ab > n$

□ n is not a product of a and b.

□ $\neg p$ is true.

□ $\neg q \rightarrow \neg p$ is true and hence $p \rightarrow q$ is true.

□ The given statement is true by indirect proof.

3. Prove the following statements by the method of contradiction:

(a) For all positive real numbers x and y, if the product $xy > 25$, then $x > 5$ or $y > 5$.

(b) The sum of two prime numbers, each larger than 2, is not a prime number.

(c) If n and n are positive integers which are perfect squares, then $n + n$ is also a perfect square.

(d) The sum of two odd integers is an even integer.

Solution:

(a) Let Let $p: xy > 25$ and $q: x > 5$ or $y > 5$.

Assume $p \rightarrow q$ is false.

That is, p is true and q is false.

Start with q is false

□ $x \leq 5$ and $y \leq 5$.

□ $xy \leq 25$

□ p is false.

Which is a contradiction to our assumption that p is true.

Therefore, the given statement is true by the method of contradiction.

(b) Let $p: a\ and\ b$ are prime numbers, greater than 2

$q: a + b$ is not a prime number.

Assume $p \to q$ is false.

That is, p is true and q is false.

Start with q is false

□ $a + b$ is a prime number.

□ $a + b = odd\ number$

□ $a = odd - b = odd - odd = even$

□ $a = even, a > 2$

□ a is not a prime number

□ p is false.

Which is a contradiction to our assumption that p is true.

Therefore, the given statement is true by the method of contradiction.

(c) Let $p: a\ and\ b$ are prime numbers, greater than 2

$q: a + b$ is not a prime number.

Assume $p \to q$ is false. That is, p is true and q is false.

Start with q is false

□ $a + b$ is a prime number.

□ $a + b = odd\ number$

□ $a = odd - b = odd - odd = even$

□ $a = even, a > 2$

□ a is not a prime number

□ p is false.

Which is a contradiction to our assumption that p is true.

Therefore, the given statement is true by the method of contradiction.

(d) Let $p: a\ and\ b$ are odd integers.

$q: a + b$ is an even integers.

Assume $p \to q$ is false.

That is, p is true and q is false.

Start with q is false

□ $a + b$ is an odd integer.

□ $a = odd - b = odd - odd = even$

□ $a = even$

□ p is false.

Which is a contradiction to our assumption that p is true.

Therefore, the given statement is true by the method of contradiction.

4. Give (i) a direct proof, (ii) an indirect proof (iii) proof by contradiction for the following statement: If n is an odd integer, then $n + 9$ is an even integer.

[Dec'10, July'13, Jan'17]

Solution:

Let p: n is an odd integer
q: $n + 9$ is an even integer.

(i) Direct proof:

Assume p is true

□n is an odd integer.

$\Rightarrow n = 2k + 1, k \in I$

$\Rightarrow n + 9 = 2k + 1 + 9 = 2k + 10 = 2(k + 5)$

$\Rightarrow n + 9$ is an even integer.

$\Rightarrow q$ is true. $\therefore p \to q$ is true.

Therefore, the given statement is true by a direct proof.

(ii) Indirect proof:

Assume $\neg q$ is true

$\Rightarrow n + 9$ is an odd integer.

□ $n + 9 = 2k + 1, for\ some\ k \in I$

□ $n = 2k + 1 - 9 = 2k - 8 = 2(k - 4), for\ some\ k \in I$

$\Rightarrow n$ is an even integer.

$\Rightarrow \neg p$ is true. $\therefore \neg q \to \neg p$ is true.

Therefore, the given statement is true by an indirect proof.

(iii) **Method of contradiction:**

Assume $p \to q$ is false. That is, p is true and q is false.

Start with q is false.

□ $n + 9$ is an odd integer.

□ $n + 9 = 2k + 1, for\ some\ k \in I$

□ $n = 2k - 8 = 2(k - 4)$

□ n is an even integer.

$\therefore$ p is false.

Which is a contradiction to our assumption that p is true.

Therefore, the given statement is true by the method of contradiction.

EXERCISE – 1.4

1. Give a direct proof that "If n is an odd integer, then n^2 is odd."
2. Give a direct proof that "If m and n are both perfect squares, then nm is also a perfect square."
3. Give an indirect proof that "If n is an integer and $3n + 2$ is odd, then n is odd."
4. Give a proof by contradiction of "If $3n + 2$ is odd, then n is odd."
5. Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd."
6. Use a direct proof to show that the sum of two odd integers is even.
7. Use a direct proof to show that the sum of two even integers is even.
8. Show that the square of an even number is an even number using a direct proof.
9. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.
10. Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even.
11. Use a direct proof to show that the product of two odd numbers is odd.
12. Use a direct proof to show that every odd integer is the difference of two squares.
13. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.
14. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.
15. Prove that if m and n are integers and mn is even, then m is even or n is even.
16. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using (i) a proof by contraposition, (ii) a proof by contradiction.
17. Prove that if n is an integer and $3n + 2$ is even, then n is even using (i) a proof by contraposition, (ii) a proof by contradiction.
18. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.
19. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.