Module-3

RELATIONS AND FUNCTIONS

PART - 1

CARTESIAN PRODUCT OF TWO SETS

CARTESIAN PRODUCT OF TWO SETS

Let A and B be any two sets. Then the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$ is called the Cartesian product of A and B and is denoted by $A \times B$.

$$\text{i.e.,} A \times B = \{(x, y) \mid x \in A, y \in B\}.$$

Similarly, $B \times A = \{(x, y) \mid x \in B, y \in A\}$.

In general, $A \times B \neq B \times A$

Note: $A \times B = B \times A \Leftrightarrow A = B$

Example: If $A = \{4, 5\}$ and $B = \{1, 2, 3\}$ then

$A \times B = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$

$B \times A = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

$A^2 = A \times A = \{(4, 4), (4, 5), (5, 4), (5, 5)\}$

$B^2 = B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

Cardinality: If A and B are finite sets having m and n elements respectively, then $A \times B$ has mn elements.

i.e., if $|A| = m$ and $|B| = n$ then $|AB| = mn$.

Note: If $A = \phi$ or $B = \phi$ then $A \times B = \phi$ and $B \times A = \phi$.

NOTE: In set theory, for any two sets A and B we have the following:

1. If $x \in A \cup B$ then $x \in A$ or $x \in B$
2. If $x \in A \cap B$ then $x \in A$ and $x \in B$
3. If $x \in A - B$ then $x \in A$ and $x \notin B$
4. Two sets A and B are said to be **equal sets** if and only if $A \subseteq B$ and $B \subseteq A$.
5. To show that two sets A and B are equal, i.e., to show $A = B$ we have to show that

   (i) $A \subseteq B$

   To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

   (ii) $B \subseteq A$

   To show that $B \subseteq A$, show that if x belongs to B then x also belongs to A.

STANDARD RESULTS

For any three sets A, B, C, we have

(1) $A \times (B \cup C) = (A \times B) \cup (A \times C)$,

(2) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(3) $(A \cup B) \times C = (A \times C) \cup (B \times C)$,

(4) $(A \cap B) \times C = (A \times C) \cap (B \times C)$,

(5) $A \times (B - C) = (A \times B) - (A \times C)$.

Proof:

Let $(x, y) \in A \times (B \cup C)$

$\Leftrightarrow x \in A$ and $y \in (B \cup C)$

$\Leftrightarrow x \in A$ and ($y \in B$ or $y \in C$)

$\Leftrightarrow$ ($x \in A$ and $y \in B$) or ($x \in A$ and $y \in C$)

$\Leftrightarrow (x, y) \in A \times B$ or $(x, y) \in A \times C$

$\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C)$

$\therefore\ A \times (B \cup C) = (A \times B) \cup (A \times C)$

(2) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof:

Let $(x, y) \in A \times (B \cap C)$

$\Leftrightarrow x \in A$ and $y \in (B \cap C)$

$\Leftrightarrow x \in A$ and $(y \in B$ and $y \in C)$

$\Leftrightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Leftrightarrow (x, y) \in A \times B$ and $(x, y) \in A \times C$

$\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C)$

$\therefore\ A \times (B \cap C) = (A \times B) \cap (A \times C)$

(3) $(A \cup B) \times C = (A \times C) \cup (B \times C)$,

Proof:

Let $(x, y) \in (A \cup B) \times C$

$\Leftrightarrow x \in (A \cup B)$ and $y \in C$

$\Leftrightarrow (x \in A$ or $x \in B)$ and $y \in C$

$\Leftrightarrow (x \in A$ and $y \in C)$ or $(x \in B$ and $y \in C)$

$\Leftrightarrow (x, y) \in A \times C$ or $(x, y) \in B \times C$

$\Leftrightarrow (x, y) \in (A \times C) \cup (B \times C)$

$\therefore \ (A \cup B) \times C = (A \times C) \cup (B \times C)$

(4) $(A \cap B) \times C = (A \times C) \cap (B \times C)$,

Proof:

Let $(x, y) \in (A \cap B) \times C$

$\Leftrightarrow x \in (A \cap B)$ and $y \in C$

$\Leftrightarrow (x \in A$ and $x \in B)$ and $y \in C$

$\Leftrightarrow (x \in A$ and $y \in C)$ and $(x \in B$ and $y \in C)$

$\Leftrightarrow (x, y) \in A \times C$ and $(x, y) \in B \times C$

$\Leftrightarrow (x, y) \in (A \times C) \cap (B \times C)$

$\therefore \ (A \cap B) \times C = (A \times C) \cap (B \times C)$

(5) $A \times (B - C) = (A \times B) - (A \times C)$.

Proof:

Let $(x, y) \in A \times (B - C)$

$\Leftrightarrow x \in A$ and $y \in (B - C)$

$\Leftrightarrow x \in A$ and $(y \in B$ and $y \notin C)$

$\Leftrightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $y \notin C)$

$\Leftrightarrow (x, y) \in A \times B$ and $(x, y) \notin A \times C$

$\Leftrightarrow (x, y) \in (A \times B) - (A \times C)$

$\therefore\ A \times (B - C) = (A \times B) - (A \times C)$

Examples

If $P = \{a, b, c\}$ and $Q = \{r\}$, form the sets $P \times Q$ and $Q \times P$. Are these two products equal?

Solution:

By the definition of the Cartesian product,

$P \times Q = \{(a, r), (b, r), (c, r)\}$ and

$Q \times P = \{(r, a), (r, b), (r, c)\}$

Since, by the definition of equality of ordered pairs, the pair (a, r) is not equal to the pair (r, a), we conclude that

$P \times Q \neq Q \times P$.

However, the number of elements in each set will be the same.

Let $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$. Find the following:
(i) $A \times (B \cap C)$ (ii) $(A \times B) \cap (A \times C)$ (iii) $A \times (B \cup C)$
(iv) $(A \times B) \cup (A \times C)$

Solution:

(i) By the definition of the intersection of two sets, $(B \cap C) = \{4\}$.

Therefore, $A \times (B \cap C) = \{(1,4), (2,4), (3,4)\}$.

(ii) Now $(A \times B) = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$ and

$(A \times C) = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

Therefore, $(A \times B) \cap (A \times C) = \{(1, 4), (2, 4), (3, 4)\}$.

(iii) Since, $(B \cup C) = \{3, 4, 5, 6\}$, we have

$A \times (B \cup C) = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)\}$

(iv) Using the sets $A \times B$ and $A \times C$ from part (ii) above, we obtain

$(A \times B) \cup (A \times C) = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)\}$.

If $A = \{1, 2, 3, 4\}$, $B = \{2, 5\}$ and $C = \{3, 4, 7\}$ then determine the following:

(i) $A \times B$, (ii) $B \times A$, (iii) $A \cup (B \times C)$, (iv) $(A \cup B) \times C$, (v) $(A \times C) \cup (B \times C)$, (vi) $(A \times C) \cap (B \times C)$.

If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$ then verify the following

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$