Module-I
Fundamentals of Logic

What is Logic ?

- Logic is the systematic study of the forms of inference, i.e. the relations that lead to the acceptance of one proposition on the basis of a set of other propositions.
- More broadly, logic is the analysis and appraisal of arguments.

What is the importance of logic ?

Logic is **important** because it influences every decision we make in our lives. **Logical** thinking allows us to learn and make decisions that will affect our lifestyle. If no one thought logically, we would all be running around like chickens with our heads cut off, and nothing would make any sense.

PROPOSITIONAL LOGIC

Introduction

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid arguments. A major goal of this topic is to read how to understand and how to construct correct mathematical arguments; we begin our study of discrete mathematics with an introduction to logic.

In addition to its importance in understanding mathematical reasoning, logic has numerous applications in computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways. Furthermore, software systems have been developed for constructing proofs automatically. We will discuss these applications of logic in the upcoming chapters.

Definition : PROPOSITIONS

A declarative sentence which is either true or false but not both at the same time is called a Proposition. Propositions are usually denoted by the lowercase letters $p, q, r, s, \ldots$

The truth value of a proposition is true, denoted by T or 1, if it is a true proposition and false, denoted by F or 0, if it is a false proposition.

Examples for proposition:

1. Bangalore is the capital city of India.
2. Three is a prime number.
3. $\sqrt{2}$ is an irrational number.
4. $\sqrt{2} + \sqrt{3} = \sqrt{5}$.
5. $3^2 + 4^2 = 7^2$.

The truth values of the above propositions are F, T, T, F and F.

Note: A sentences which involves with exclamation, question are not propositions.

Examples for not proposition:

1. Good Morning!
2. What time is it?
3. Read this carefully.
4. $x + 1 = 2$.

Definition : COMPOUND PROPOSITIONS

Mathematical statements are constructed by combining one or more simple propositions. New propositions, called compound propositions, are formed from existing propositions using logical operators. The words “and”, “or’, “If … then”, “iff”, “not” are used to form compound propositions which are called Logical connectives.

The following table gives name of the connectives and their symbols

S.No	Connectives	Name of the connectives	Symbols
1	and	Conjunction	$\wedge$
2	or	Disjunction	$\vee$
3	If … then	Conditional	$\rightarrow$
4	iff	Biconditional	$\boxed{\leftrightarrow}$
5	not	Negation	$\sim$ or $\neg$

Definition : NEGATION

Let p be a proposition. Then the proposition "not p" is called the negation of p, and is denoted by $\sim p$ or $\neg\, p$. The truth value of $\sim p$ is the opposite of the truth value of p.

Truth Table for the Negation

p	$\neg$ p
T	F
F	T

EXAMPLES:

- Let p: "Michael's PC runs Linux"

then its negation is

$\sim p$: "Michael's PC does not run Linux."

- Let q: "Vandana's smart phone has at least 32GB of memory"

then its negation is

$\sim q$: "Vandana's smart phone does not have at least 32GB of memory"

or even more simply as

"Vandana's smartphone has less than 32GB of memory."

Definition : CONJUNCTION

Let *p* and *q* be propositions. Then the compound proposition "***p* and *q***" is called a **conjunction** and is denoted by $\boldsymbol{p \wedge q}$. The conjunction $p \wedge q$ is true when both *p* and *q* are true and is false otherwise.

Truth Table for the Conjunction

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

EXAMPLE:

Let p: "Rebecca's PC has more than 16 GB free hard disk space" and

q: "The processor in Rebecca's PC runs faster than 1 GHz"

then its conjunction is

$p \wedge q$: "Rebecca's PC has more than 16 GB free hard disk space and the processor in Rebecca's PC runs faster than 1 GHz."

Definition : DISJUNCTION

Let *p* and *q* be propositions. Then the compound proposition "***p* or *q***" is called a **disjunction** and is denoted by ***p* ∨ *q***. The disjunction *p* ∨ *q* is false when both *p* and *q* are false and is true otherwise.

Truth Table for the Disjunction

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE:

Let p: "Rebecca's PC has more than 16 GB free hard disk space" and

q: "The processor in Rebecca's PC runs faster than 1 GHz"

then its disjunction is

$p \vee q$: "Rebecca's PC has at least 16 GB free hard disk space, or the processor in Rebecca's PC runs faster than 1 GHz."

Definition : Exclusive or

Let *p* and *q* be propositions. The exclusive or of p and q, denoted by ***p* $\veebar$ *q***, is the proposition that is true when exactly one of *p* and *q* is true and is false otherwise.

Truth Table for the Exclusive or

p	q	p $\veebar$ q
T	T	F
T	F	T
F	T	T
F	F	F

Example:

"Students, who have taken calculus or computer science, but not both, can enrol in this class."

Here, we mean that students who have taken both calculus and a computer science course cannot take the class. Only those who have taken exactly one of the two courses can take the class.

Similarly, when a menu at a restaurant states, "Soup or salad comes with an entrée," the restaurant almost always means that customers can have either soup or salad, but not both. Hence, this is an exclusive, or.

Definition : CONDITIONAL

Let p and q be propositions. Then the compound proposition "**If** $\boldsymbol{p}$ **then** $\boldsymbol{q}$" is called a **conditional** and is denoted by $\boldsymbol{p \rightarrow q}$. The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise.

Truth Table for the Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

EXAMPLE:

Let p:"Maria learns discrete mathematics" and

q:"Maria will find a good job"

then its conditional is

$p \rightarrow q$: "If Maria learns discrete mathematics, then she will find a good job."

Definition : BICONDITIONAL

Let p and q be propositions. Then the compound proposition "p **iff** $\boldsymbol{q}$" is called a **biconditional** and is denoted by $\boldsymbol{p} \boxed{\leftrightarrow} \boldsymbol{q}$. The biconditional statement $p \boxed{\leftrightarrow} q$ is true when p and q have the same truth values, and is false otherwise.

<u>Truth Table for the biconditional</u>

p	q	$p \boxed{\leftrightarrow} q$
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLE:

Let p:"You can take the flight," and

q:"You buy a ticket."

then its biconditional is

$p \boxed{\leftrightarrow} q$:"You can take the flight if and only if you buy a ticket."

Construction of Truth table

A truth table is a complete list of all the possible permutations of truth and falsity for a set of simple statements, showing the effect of each permutation on the truth value of a compound having those simple statements as components. Each permutation of truth values constitutes one row of a truth table and the number of rows in a truth table is 2^n where n equals the number of simple statements. In order to determine the truth value of a compound, examine the column under the dominant operator for that compound.

There is a completely mechanical procedure for constructing a truth table for a sentence. Three skills are necessary in order to do so:

- Producing an initial list of all permutations of truth and falsity assigned to the sentence letters in the sentence.
- Determining which columns in a truth table can be filled in at a given stage.
- Determining what truth values to fill into a given column.

EXAMPLE:

1. **How many rows are needed to construct a truth table for the compound proposition $(p \vee \neg q) \leftrightarrow [(\neg r \wedge s) \rightarrow t]$, where p, q, r, s and t are primitive statements?**

Solution:

Here, $n = 5$

Number of rows required to construct its truth table is

$2^n = 2^5 = 32$

2. **Construct the truth table of the compound proposition**

$$(p \lor \neg q) \rightarrow (p \land q).$$

Solution:

p	q	¬ q	p ∨ ¬ q	p ∧ q	(p ∨ ¬ q) → (p ∧ q)
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

TAUTOLOGY, CONTRADICTION AND CONTINGENCY

A compound proposition is said to be a **Tautology** if it is always **true** for all possible combinations of the truth values of its components. A compound proposition is said to be a **Contradiction** if it is always **false** for all possible combinations of the truth values of its components. A compound proposition is said to be a **Contingency** if it is neither Tautology nor Contradiction.

Example:

Verify the following compound propositions for Tautology, Contradiction or Contingency

$$[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$$

Solution:

Truth table for $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ is

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$ (a)	$(p \rightarrow q)$ x	$(p \rightarrow r)$ y	$x \rightarrow y$ (b)	$(a) \rightarrow (b)$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

From the last column, it is observed that all truth values are T.

Hence, the given compound proposition

$[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is a Tautology.

- Verify the following compound propositions for Tautology, Contradiction or Contingency

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

Solution:

Truth table for $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)]$ (x)	$p \rightarrow r$ (y)	$(x) \rightarrow (y)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the last column, it is observed that all truth values are T.

Hence, the given compound proposition

$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a Tautology.

- Verify the following compound propositions for Tautology, Contradiction or Contingency $q \leftrightarrow (\neg p \vee \neg q)$

Solution:

Truth table for $q \leftrightarrow (\neg p \vee \neg q)$ is

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$q \leftrightarrow (\neg p \vee \neg q)$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	T
F	F	T	T	T	F

From the last column, it is observed that truth values are T or F.

Hence, the compound proposition $q \leftrightarrow (\neg p \vee \neg q)$ is a contingency.

LOGICAL EQUIVALENCE

Two compound propositions A and B are said to be **logically equivalent** if their truth values are identical and in symbols this can be written as $A \equiv B$ or $A \Leftrightarrow B$.

EXAMPLE:

A demonstration that $\neg\,(p \vee q)$ and $\neg\, p \wedge \neg\, q$ are logically Equivalent.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the last two columns of the truth table, the truth values are identical.

Hence, $\neg (p \vee q) \equiv \neg p \wedge \neg q$

- **A demonstration that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are Logically Equivalent.**

p	q	r	q ∧ r	p ∨ (q ∧ r)	(p ∨ q)	(p ∨ r)	(p ∨ q) ∧ (p ∨ r)
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

From the 4th and 8th columns of truth table, the truth values are identical.

Hence, $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

LAWS OF LOGIC:

- **Idempotent laws**

  $p \vee p \Leftrightarrow p$

  $p \wedge p \Leftrightarrow p$

- **Law of Double negation** $\neg(\neg p) \Leftrightarrow p$

- **Commutative laws**

  $p \vee q \Leftrightarrow q \vee p$

  $p \wedge q \Leftrightarrow q \wedge p$

- **Associative laws**

  $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

  $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

- **Distributive laws**

  $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

  $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

- **De Morgan's laws**

  $\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$

  $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$

- **Inverse laws**

  $p \vee \neg p \Leftrightarrow \mathrm{T}$

  $p \wedge \neg p \Leftrightarrow \mathrm{F}$

- **Identity laws**

  $p \wedge \mathrm{T} \Leftrightarrow p$

  $p \vee \mathrm{F} \Leftrightarrow p$

- **Domination laws**

  $p \vee \mathrm{T} \Leftrightarrow \mathrm{T}$

  $p \wedge \mathrm{F} \Leftrightarrow \mathrm{F}$

- **Absorption laws**

$$p \vee (p \wedge q) \Leftrightarrow p$$
$$p \wedge (p \vee q) \Leftrightarrow p$$

- **Conditional laws**

$$(p \rightarrow r) \vee (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$$
$$p \rightarrow q \Leftrightarrow \neg p \vee q$$
$$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$$
$$p \vee q \Leftrightarrow \neg p \rightarrow q$$
$$p \wedge q \Leftrightarrow \neg (p \rightarrow \neg q)$$
$$\neg (p \rightarrow q) \Leftrightarrow p \wedge \neg q$$
$$(p \rightarrow q) \wedge (p \rightarrow r) \Leftrightarrow p \rightarrow (q \wedge r)$$
$$(p \rightarrow r) \wedge (q \rightarrow r) \Leftrightarrow (p \vee q) \rightarrow r$$
$$(p \rightarrow q) \vee (p \rightarrow r) \Leftrightarrow p \rightarrow (q \vee r)$$

- **Biconditional laws**

$$p \boxed{\leftrightarrow} q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$
$$p \boxed{\leftrightarrow} q \Leftrightarrow \neg p \boxed{\leftrightarrow} \neg q$$
$$p \boxed{\leftrightarrow} q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg (p \boxed{\leftrightarrow} q) \Leftrightarrow p \boxed{\leftrightarrow} \neg q$$

Note: The above laws can be verified by constructing truth tables.

Logically equivalent without constructing a truth table

By using the laws of logic, we can verify the given compound propositions are logically equivalent or not. We illustrate this by the following examples:

Show that $\neg (p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent without constructing a truth table.

Solution:

Consider, $\neg (p \vee (\neg p \wedge q))$

$\equiv \neg p \wedge \neg (\neg p \wedge q)$	by the second De Morgan law
$\equiv \neg p \wedge [\neg (\neg p) \vee \neg q]$	by the first De Morgan law
$\equiv \neg p \wedge (p \vee \neg q)$	by the double negation law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
$\equiv F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
$\equiv (\neg p \wedge \neg q) \vee F$	by the commutative law
$\equiv \neg p \wedge \neg q$	by the identity law

Consequently $\neg (p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Show that $\neg [\neg [(p \lor q) \land r] \lor \neg q]$ and $(q \land r)$ are logically equivalent without constructing a truth table.

Solution:

Consider, $\neg [\neg [(p \lor q) \land r] \lor \neg q]$

$\equiv \neg \neg [(p \lor q) \land r] \land \neg \neg q$ by De Morgan's law

$\equiv [(p \lor q) \land r] \land q$ by law of double negation

$\equiv (p \lor q) \land (r \land q)$ by associative law

$\equiv (p \lor q) \land (q \land r)$ by commutative law

$\equiv [(p \lor q) \land q] \land r$ by associative law

$\equiv (q \land r)$ by absorption law

Consequently $\neg (p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent.

Show that $(p \land q) \rightarrow (p \lor q)$ is a tautology without constructing a truth table.

Solution:

To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

$(p \land q) \rightarrow (p \lor q) \equiv \neg (p \land q) \lor (p \lor q)$	by conditional law
$\equiv (\neg p \lor \neg q) \lor (p \lor q)$	by De Morgan law
$\equiv (\neg p \lor p) \lor (\neg q \lor q)$	by the associative and commutative laws
$\equiv \mathrm{T} \lor \mathrm{T}$	by inverse laws
$\equiv \mathrm{T}$	by the domination law

Consequently, $(p \land q) \rightarrow (p \lor q)$ is a tautology.

CONVERSE, INVERSE, AND CONTRAPOSITIVE

If $p \to q$ is a conditional statement then

- The proposition $q \to p$ is called the **converse** of $p \to q$.
- The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$.
- The proposition $\neg q \to \neg p$ is called the **contrapositive** of $p \to q$.

Note: Only the contrapositive is equivalent to the original statement.

EXAMPLE:

"If oxygen is a gas then Gold is compound"

Let p: oxygen is a gas

q: Gold is compound

Given proposition is $p \to q$

- The converse is $q \to p$

i.e., "If Gold is compound then oxygen is a gas."

- The inverse is $\neg p \to \neg q$

i.e., "If oxygen is not a gas then Gold is not a compound"

- The contrapositive is $\neg q \to \neg p$

i.e., "If Gold is not compound then oxygen is not a gas."

"If it is raining, then the home team wins."

Let p: It is raining

q: Home team wins

Given proposition is $p \rightarrow q$

- The converse is $q \rightarrow p$

i.e., "If the home team wins, then it is raining."

- The inverse is $\neg p \rightarrow \neg q$

i.e., "If it is not raining, then the home team does not win."

- The contrapositive is $\neg q \rightarrow \neg p$

i.e., "If the home team does not win, then it is not raining."

Definition : DUAL

Let s be a statement. If s contains no logical connectives other than $\vee$, $\wedge$ and $\neg$ then the dual of s is denoted by s^d, is the statement obtained from s by replacing each occurrence of $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F, and each F by T.

EXAMPLE:

Let $s: \neg (p \wedge q) \vee (p \vee q)$ then its dual is

$s^d: \neg (p \vee q) \wedge (p \wedge q)$

Let $s: (p \wedge \neg q) \vee (r \wedge \mathrm{T})$ then its dual is

$s^d: (p \vee \neg q) \wedge (r \vee \mathrm{F})$

Note: $(s^d)^d \Leftrightarrow s$

The principle of Duality: Let s and t be statements that contain no logical connectives other than $\vee$, $\wedge$, and $\neg$. If $s \Leftrightarrow t$ then $s^d \Leftrightarrow t^d$.

Example:

Verify the principle of duality for the logical equivalence:

$[\neg(p \wedge q) \rightarrow \neg p \vee (\neg p \vee q)] \Leftrightarrow \neg p \vee q$

Solution:

Principle of duality: If $s \Leftrightarrow t$ then $s^d \Leftrightarrow t^d$.

If

If $s \Leftrightarrow t$ then

$s^d \Leftrightarrow [\neg(p \wedge q) \rightarrow \neg p \vee (\neg p \vee q)]^d$ by data

$\Leftrightarrow [(p \wedge q) \vee \neg p \vee (\neg p \vee q)]^d$ by the definition of conditional

$\Leftrightarrow [(p \wedge q) \vee (\neg p \vee \neg p) \vee q]^d$ by Associative law

$\Leftrightarrow [(p \wedge q) \vee \neg p \vee q]^d$ by idempotent law

$\Leftrightarrow [(p \wedge q) \vee q \vee \neg p]^d$ by commutative law

$\Leftrightarrow [q \vee \neg p]^d$ by Absorption law

$\Leftrightarrow [\neg p \vee q]^d$ by Commutative law

$\Leftrightarrow t^d$ by data

Hence, the principle of duality is verified.

OPERATORS NAND and NOR

The NAND operator denoted by ↑, also known as the Sheffer stroke, is a connective in logic equivalent to the composition NOT AND that yields true if any condition is false, and false if all conditions are true.

If p and q are two primitive statements then

$$(p \uparrow q) \Leftrightarrow \neg (p \wedge q).$$

Truth Table for the operator NAND

p	q	p ↑ q
T	T	F
T	F	T
F	T	T
F	F	T

The NOR operator denoted by $\downarrow$, also known as the **Peirce's arrow**, is a connective in logic equivalent to the composition NOT OR that yields true if both condition is false, and it is false otherwise.

If p and q are two primitive statements then

$$(p \downarrow q) \Leftrightarrow \neg (p \vee q).$$

Truth Table for he operator NOR

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Example: For any two statements p, q prove the following

(i) $\neg(p \downarrow q) \Leftrightarrow (\neg p \uparrow \neg q)$, (ii) $\neg(p \uparrow q) \Leftrightarrow (\neg p \downarrow \neg q)$.

Solution:

(i) Truth table for the given compound proposition is

p	q	$\neg p$	$\neg q$	$p \downarrow q$	$\neg(p \downarrow q)$	$(\neg p \uparrow \neg q)$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

From the last two columns of the truth table, we have

$\neg(p \downarrow q) \Leftrightarrow (\neg p \uparrow \neg q)$.

(ii) Truth table for the given compound proposition is

p	q	$\neg p$	$\neg q$	$p \uparrow q$	$\neg (p \uparrow q)$	$(\neg p \downarrow \neg q)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

From the last two columns of the truth table, we have
$\neg (p \uparrow q) \Leftrightarrow (\neg p \downarrow \neg q)$.