Combinations with repetitions

Introduction:

$\binom{n+r-1}{r}$ represents any one of the following:

- The number of combinations of n distinct objects, taken r at a time, with repetitions allowed.

- The number of ways in which r identical objects can be distributed among n distinct containers.

- The number of non-negative integer solutions of the equation $x_1 + x_2 + \cdots + x_n = r$.

1. In how many ways can we distribute 10 identical marbles among 6 distinct containers?

By data, $n = 6, r = 10.$

Therefore,required number $= \binom{n+r-1}{r} = \binom{15}{10} = 3003$

2. In how many ways can 10 identical pencils be distributed among 5 children so that
(a) Each child gets at least one pencil? (b) The youngest child gets at least two pencils?

(a) Distribute one pencil to each child. Now, 5 pencils remaining. $n = 5, r = 5.$

Therefore,required number $= \binom{n+r-1}{r} = \binom{9}{5} = 126$

(b) Give 2 pencils to the youngest child. Now,8 pencils remaining. $n = 5, r = 8.$

Therefore, required number $= \binom{n+r-1}{r} = \binom{12}{8} = 495$

3. In how many ways can one distribute 8 identical balls into 4 distinct containers so that (a) No container is left empty? (b) The 4^{th}container gets an odd no. of balls? (July 2014)

(a) Distribute one ball to each container. Now, 4 balls remaining. $n = 4, r = 4.$

Therefore, required number $= \binom{n+r-1}{r} = \binom{7}{4} = 35$

(b) • Put one ball into 4^{th} container. Now, 7 balls and 3 containers remaining.$n = 3, r = 7.$ No. of ways of distributing 7 balls into 3 containers is $\binom{n+r-1}{r} = \binom{9}{7} = 36.$

• Put 3 balls into 4^{th} container. Now, 5 balls and 3 containers remaining. $n = 3, r = 5.$ No. of ways of distributing 5 balls into 3 containers is $\binom{n+r-1}{r} = \binom{7}{5} = 21$

- Put 5 balls into 4^{th} container. Now, 3 balls and 3 containers remaining. $n = 3, r = 3$. No. of ways of distributing 3 balls into 3 containers is $\binom{n+r-1}{r} = \binom{5}{3} = 10$

- Put 7 balls into 4^{th} container. Now, 1 ball and 3 containers remaining. $n = 3, r = 1$. No. of ways of distributing 1 ball into 3 containers is $\binom{n+r-1}{r} = \binom{3}{1} = 3$

- By sum rule, the required number $= 36 + 21 + 10 + 3 = 70$.

4. In how many ways can we distribute 7 apples and 6 oranges among 4 children so that each child gets at least 1 apple? (Jan 2015)

- Distribute 1 apple to each child. Now, 3 apples and 6 oranges remaining.
- No. of ways of distributing 3 apples to 4 children $= \binom{n+r-1}{r} = \binom{6}{3} = 20$
- No. of ways of distributing 6 oranges to 4 children $= \binom{n+r-1}{r} = \binom{9}{6} = 84$
- By product rule, the required number$= 20 \times 84 = 1680$.

5. A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 blank spaces between the symbols, with at least 3 spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

- 12 symbols can be arranged in 12! ways. There are 11 positions between the symbols.
- Distribute 3 spaces to each one of 11 positions. There are 12 spaces remaining. (45-33)
- No. of ways of distributing 12 spaces to 11 positions = $\binom{n+r-1}{r} = \binom{22}{12} =$ 646646
- By product rule, the required number = $646646 \times 12!$

6. Find the number of ways of giving 10 identical gift boxes to 6 persons A, B, C, D, E, F in such a way that the total number of boxes given to A and B together does not exceed 4.

- Suppose r boxes out of 10 identical boxes given to A and B, $0 \leq r \leq 4$.
- No. of ways of distributing r boxes to 2 persons $\binom{n+r-1}{r.} = \binom{r+1}{r.} = r+1$
- Distribute remaining $10-r$ identical boxes to 4 persons C,D,E and F.
- No. of ways of distributing $10-r$ boxes to 4 persons

$$= \binom{n+r-1}{r} = \binom{4+10-r-1}{10-r} = \binom{13-r}{10-r} = \binom{13-r}{3}$$

- By sum rule, required number= $\sum_{i=1}^{4}(r+1) \times \binom{13-r}{3}$

Find the number of non-negative integer solutions of the inequality

$$x_1 + x_2 + x_3 + \cdots + x_6 < 10.$$

Solution: $x_1 + x_2 + x_3 + \cdots + x_6 < 10.$

$$x_1 + x_2 + x_3 + \cdots + x_6 \leq 9$$

$$x_1 + x_2 + x_3 + \cdots + x_6 = 9 - x_7$$

$$x_1 + x_2 + x_3 + \cdots + x_6 + x_7 = 9.$$

Here, $n = 7, r = 9.$

Therefore, the required number= $\binom{n+r-1}{r} = \binom{7+9-1}{9} = \binom{15}{9} = 5005.$

Find the number of integer solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 30$, where $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2,\ x_5 \geq 0$.

Let $y_1 = x_1 - 2, y_2 = x_2 - 3\ , y_3 = x_3 - 4, y_4 = x_4 - 2, y_5 = x_5$.

Then $x_1 + x_2 + x_3 + x_4 + x_5 = 30$

$$\Rightarrow y_1 + 2 + y_2 + 3 + y_3 + 4 + y_4 + 2 + y_5 = 30$$

$$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = 30 - 11 = 19.$$

Where $y_1,\ y_2,\ y_3,\ y_4,\ y_5 \geq 0$.

$$n = 5, r = 19.$$

Therefore, the required number$= \binom{n+r-1}{r} = \binom{5+19-1}{19} = \binom{23}{19} = 8855$.