1.7 PERIODIC FUNCTION

A function $f(t)$ is said to be periodic function of period T if $f(t+nT)=f(t)$

1.7.1 LAPLACE TRANSFORM OF PERIODIC FUNCTION

If $f(t)$ is a periodic function with period T then $L\left[f(t)\right]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$

If a periodic function of period 2π is defined by $f(t) = \begin{cases} t, & \text{if } 0 < t < \pi \\ \pi - t, & \text{if } \pi < t < 2\pi \end{cases}$ then find its Laplace transform. **(VTU 2012)**

Solution:

Given, $T = 2\pi$

We have, $$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-s(2\pi)}}\int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s\pi}}\left[\int_0^{\pi} te^{-st}\,dt + \int_{\pi}^{2\pi}(\pi - t)e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2s\pi}}\left[\left(t\frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right)_0^{\pi} + \left((\pi - t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{1-e^{-2s\pi}}\left[\left(\frac{\pi e^{-\pi s}}{-s} - \frac{e^{-\pi s}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right) + \left(0 + \frac{e^{-2\pi s}}{s^2}\right) - \left(\frac{\pi e^{-\pi s}}{-s} + \frac{e^{-\pi s}}{s^2}\right)\right]$$

$$= \frac{1}{1-e^{-2s\pi}}\left[\frac{-\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2\pi s}}{s^2} + \frac{\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2s\pi}}\left[\frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} - \frac{e^{-\pi s}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2s\pi}}\left[\frac{e^{-2\pi s} + 1 - 2e^{-\pi s}}{s^2}\right]$$

$$= \frac{1}{1^2 - \left(e^{-s\pi}\right)^2}\left[\frac{1^2 + \left(e^{-\pi s}\right)^2 - 2e^{-\pi s}}{s^2}\right] = \frac{1}{\left(1+e^{-s\pi}\right)\left(1-e^{-s\pi}\right)}\left[\frac{\left(1-e^{-\pi s}\right)^2}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{\left(1-e^{-s\pi}\right)}{\left(1+e^{-s\pi}\right)} = \frac{1}{s^2}\tanh\left(\frac{\pi s}{2}\right) \qquad \left[\because \tanh\left(\frac{\theta}{2}\right) = \frac{1-e^{-\theta}}{1+e^{-\theta}}\right]$$

If a periodic function of period $\dfrac{2\pi}{\omega}$ is defined by $f(t) = \begin{cases} E\sin\omega t, & \text{if } 0 < t < \pi/\omega \\ 0, & \text{if } \pi/\omega < t < 2\pi/\omega \end{cases}$

where E and ω are constants, then show that $L[f(t)] = \dfrac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

(VTU 2004, 2005, 2013)

Solution:

Given, $$T = \frac{2\pi}{\omega}$$

We have,

$$\begin{aligned}
L[f(t)] &= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt \\
&= \frac{1}{1-e^{-s(2\pi/\omega)}}\int_0^{2\pi/\omega} e^{-st} f(t)\, dt \\
&= \frac{1}{1-e^{-s(2\pi/\omega)}}\left[\int_0^{\pi/\omega} e^{-st}(E\sin\omega t)\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}(0)\, dt\right] \\
&= \frac{1}{1-e^{-s(2\pi/\omega)}}\left[E\int_0^{\pi/\omega} e^{-st}\sin\omega t\, dt + 0\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{E}{1-\left(e^{-\pi s/\omega}\right)^2}\int_0^{\pi/\omega} e^{-st}\sin\omega t\, dt \\
&= \frac{E}{1-\left(e^{-\pi s/\omega}\right)^2}\left[\frac{e^{-st}}{(-s)^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega} \\
&= \frac{E}{1-\left(e^{-\pi s/\omega}\right)^2}\cdot\frac{1}{s^2+\omega^2}\left[e^{-s\frac{\pi}{\omega}}\left(-s\sin\omega\frac{\pi}{\omega} - \omega\cos\omega\frac{\pi}{\omega}\right) - e^0(-s\sin 0 - \omega\cos 0)\right] \\
&= \frac{E}{1-\left(e^{-\pi s/\omega}\right)^2}\cdot\frac{1}{s^2+\omega^2}\left[e^{-s\frac{\pi}{\omega}}(-s\sin\pi - \omega\cos\pi) - 1(-s\sin 0 - \omega\cos 0)\right] \\
&= \frac{E}{1-\left(e^{-\pi s/\omega}\right)^2}\cdot\frac{1}{s^2+\omega^2}\left[e^{-s\frac{\pi}{\omega}}(0-\omega(-1)) - 1(0-\omega(1))\right] \\
&= \frac{E}{1^2-\left(e^{-\pi s/\omega}\right)^2}\cdot\frac{1}{s^2+\omega^2}\left[e^{-s\frac{\pi}{\omega}}\omega + \omega\right] = \frac{E\omega\left(e^{-\pi s/\omega}+1\right)}{\left(1-e^{-\pi s/\omega}\right)\left(1+e^{-\pi s/\omega}\right)\left(s^2+\omega^2\right)} \\
&= \frac{E\omega}{(s^2+\omega^2)\left(1-e^{-\pi s/\omega}\right)}
\end{aligned}$$

If a periodic function of period a is defined by $f(t)=\begin{cases} E, & \text{if } 0<t<a/2 \\ -E, & \text{if } a/2<t<a \end{cases}$ then show that $L[f(t)]=\frac{E}{s}\tanh\left(\frac{as}{4}\right)$. (VTU 2006, 2011)

Solution:

Given, $T=a$

We have, $$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-s(a)}}\int_0^a e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-sa}}\left[\int_0^{a/2} e^{-st}(E)dt + \int_{a/2}^{a} e^{-st}(-E)\, dt\right]$$

$$= \frac{E}{1-e^{-sa}}\left[\int_0^{a/2} e^{-st}dt - \int_{a/2}^{a} e^{-st}\, dt\right]$$

$$= \frac{E}{1-e^{-sa}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$= \frac{E}{1-e^{-sa}}\left[\left(\frac{e^{-s(a/2)}-e^{0}}{-s}\right)-\left(\frac{e^{-as}-e^{-s(a/2)}}{-s}\right)\right]$$

$$= \frac{E}{1-e^{-sa}}\left[\frac{e^{-s(a/2)}-1-e^{-as}+e^{-s(a/2)}}{-s}\right]$$

$$= \frac{E}{1-e^{-sa}}\left[\frac{1+e^{-as}-2e^{-s(a/2)}}{s}\right]$$

$$= \frac{E}{1^2-\left(e^{-sa/2}\right)^2}\left[\frac{1^2+\left(e^{-as/2}\right)^2-2e^{-s(a/2)}}{s}\right]$$

$$= \frac{E}{\left(1+e^{-sa/2}\right)\left(1-e^{-sa/2}\right)}\left[\frac{\left(1-e^{-as/2}\right)^2}{s}\right]$$

$$= \frac{E\left(1-e^{-sa/2}\right)}{s\left(1+e^{-sa/2}\right)}$$

$$= \frac{E}{s}\tanh\left(\frac{as}{4}\right) \qquad \left[\because \tanh\left(\frac{\theta}{2}\right)=\frac{1-e^{-\theta}}{1+e^{-\theta}}\right]$$

If a periodic function of period $2a$ is defined by $f(t)=\begin{cases} t, & \text{if } 0\le t\le a \\ 2a-t, & \text{if } a\le t\le 2a \end{cases}$ then show that $L[f(t)]=\dfrac{1}{s^2}\tanh\left(\dfrac{as}{2}\right)$. **(VTU 2003, 2008, 2011)**

Solution:

Given, $\quad T = 2a$

We have,
$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-s(2a)}}\int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2sa}}\left[\int_0^a te^{-st}\, dt + \int_a^{2a} (2a-t)e^{-st}\, dt\right]$$

$$= \frac{1}{1-e^{-2sa}}\left[\left(t\frac{e^{-st}}{-s}-(1)\frac{e^{-st}}{s^2}\right)_0^a + \left((2a-t)\frac{e^{-st}}{-s}-(-1)\frac{e^{-st}}{s^2}\right)_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2sa}}\left[\left(\frac{ae^{-as}}{-s}-\frac{e^{-as}}{s^2}\right)-\left(0-\frac{1}{s^2}\right)+\left(0+\frac{e^{-2as}}{s^2}\right)-\left(\frac{ae^{-as}}{-s}+\frac{e^{-as}}{s^2}\right)\right]$$

$$= \frac{1}{1-e^{-2sa}}\left[\frac{-ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2sa}}\left[\frac{e^{-2as}}{s^2}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2sa}}\left[\frac{e^{-2as}+1-2e^{-as}}{s^2}\right]$$

$$= \frac{1}{1^2-\left(e^{-sa}\right)^2}\left[\frac{1^2+\left(e^{-as}\right)^2-2e^{-as}}{s^2}\right] = \frac{1}{\left(1+e^{-sa}\right)\left(1-e^{-sa}\right)}\left[\frac{\left(1-e^{-as}\right)^2}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{\left(1-e^{-sa}\right)}{\left(1+e^{-sa}\right)}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right) \qquad \left[\because \tanh\left(\frac{\theta}{2}\right)=\frac{1-e^{-\theta}}{1+e^{-\theta}}\right]$$