Module-3
RELATIONS AND FUNCTIONS

PART - 5

PARTIAL ORDERINGS

PARTIAL ORDERINGS

A relation R on a set A is called a **partial ordering or partial order** if it is reflexive, antisymmetric and transitive. A set A together with a partial order R is called a partially ordered set or **POSET** and is denoted by (A, R). Members of A are called elements of the POSET.

EXAMPLES

$(Z, \leq)$ is a POSET.

Solution:

For any $a \in Z$, we have $a \leq a$. Hence R is reflexive.

If $a \leq b$ and $b \leq a$ then $a = b$. Hence R is antisymmetric.

If $a \leq b$ and $b \leq c$ then $a \leq c$. Hence R is transitive.

Thus R is a partial order.

$\therefore$ $(Z, \leq)$ is a POSET.

$(Z, \geq)$ is a POSET.

Solution:

For any $a \in Z$, we have $a \geq a$. Hence R is reflexive.

If $a \geq b$ and $b \geq a$ then $a = b$. Hence R is antisymmetric.

If $a \geq b$ and $b \geq c$ then $a \geq c$. Hence R is transitive.

Thus R is a partial order.

$\therefore$ $(Z, \geq)$ is a POSET.

(Z, <) and (Z, >) are not POSET's.

Solution:

(i) For any $a \in Z$, $a < a$ not possible. Hence R is not reflexive.

Thus R is not a partial order.

$\therefore$ (Z, <) is not a POSET.

(ii) For any $a \in Z$, $a > a$ not possible. Hence R is not reflexive.

Thus R is not a partial order.

$\therefore$ (Z, >) is not a POSET.

COMPARABLE

The elements a and b of a POSET (A, R) are called **comparable** if either a R b or b R a. When a and b are elements of A such that neither a R b nor b R a, a and b are called **incomparable**.

EXAMPLES

In the POSET $(Z, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution:

The integers 3 and 9 are comparable, because $3 \mid 9$.

The integers 5 and 7 are incomparable, because 5 does not divides 7 and 7 does not divides 5.

In the POSET $(Z, \leq)$, are the integers 6 and 5 comparable? Are 8 and 10 comparable?

Solution:

The integers 6 and 5 are comparable, because $5 \leq 6$.

The integers 8 and 10 are comparable, because $8 \leq 10$.

TOTAL ORDERED SET

If (A, R) is a POSET and every two elements of A are comparable then A is called a **totally ordered or linearly ordered set** and R is called a **total order or a linear order.** A totally ordered set is also called a chain.

EXAMPLES

1. The POSET $(Z, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.
2. The POSET $(Z, \geq)$ is totally ordered, because $a \geq b$ or $b \geq a$ whenever a and b are integers.
3. The POSET $(Z^+, \mid)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7.

HASSE DIAGRAMS

The pictorial representation of a POSET is called a **POSET diagram or Hasse diagram**.

- In other words, a Hasse diagram is a graphical rendering of a partially ordered set displayed via the cover relation of the partially ordered set with an implied upward orientation. A point is drawn for each element of the POSET which is called a vertex in the plane and line segments or curves are drawn between these points according to the following two rules:

- If $x < y$ in the POSET, then the point corresponding to x appears lower in the drawing than the point corresponding to y.
- The line segment between the points corresponding to any two elements x and y of the POSET is included in the drawing iff x covers y or y covers x.

Note:

- For a POSET (A, ≤), one represents each element of A as a vertex in the plane and draws a line segment or curve that goes upward from x to y whenever y covers x (that is, whenever $x < y$ and there is no z such that $x < z < y$). These curves may cross each other but must not touch any vertices other than their endpoints.
- In POSET, every element is related to itself, we eliminate self-loops in Hasse diagram.

EXAMPLES

Show that the set $A = \{1, 2, 3, 4, 6, 8, 12\}$ is a POSET with respect to the relation R defined as $\{(a, b) : a \text{ divides } b\}$and draw its Hasse diagram.

Solution:

Given, $R = \{(a, b) : a \text{ divides } b\}$ on set $A = \{1, 2, 3, 4, 6, 8, 12\}$.

(i) Every integer divides itself.

Hence, $\forall\ a \in A \Rightarrow a$ divides a

$$\Rightarrow (a, a) \in R.$$

Hence R is reflexive.

(ii) Let $a \, R \, b$ and $b \, R \, a \ \forall \ a, b \in A$

i.e., a divides b and b divides a

$\Rightarrow a = b.$

Hence R is antisymmetric.

(iii) Let $a \, R \, b$ and $b \, R \, c$

i.e., a divides b and b divides c

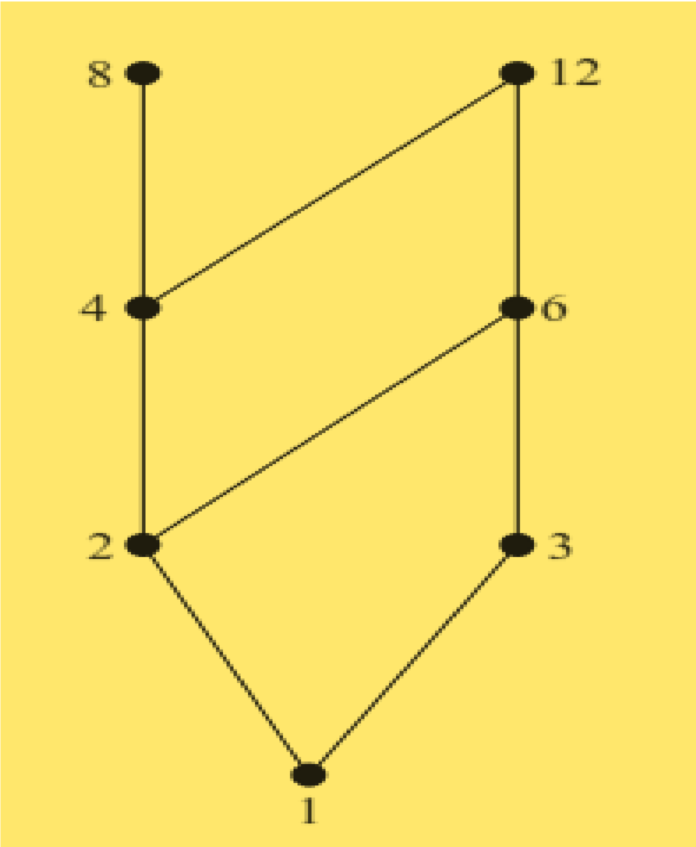
$\Rightarrow$ a divides c.

Hence R is transitive.

Thus R is a partial order.

$\therefore$ (A, R) is a POSET.

Hasse diagram



Let S = {a, b, c} and P(S) be the power set of S. On P(S), define the relation R by $\{(x, y) \mid x \subseteq y\}$. Show that this relation is a partial ordering on P(S). Draw its Hasse diagram.

Solution:

The power set of S = {a, b, c} is given by

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

(i) We know that every set is a subset of itself.

i.e., $\forall\, x \in P(S) \Rightarrow x \subseteq x$

$\Rightarrow (x, x) \in R.$

Hence R is reflexive.

(ii) Let x R y and y R x $\forall$ x, y $\in$ P(S)

i.e., $x \subseteq y$ and $y \subseteq x$

$\Rightarrow x = y$.

Hence R is antisymmetric.

(iii) Let x R y and y R z

i.e., $x \subseteq y$ and $y \subseteq z$

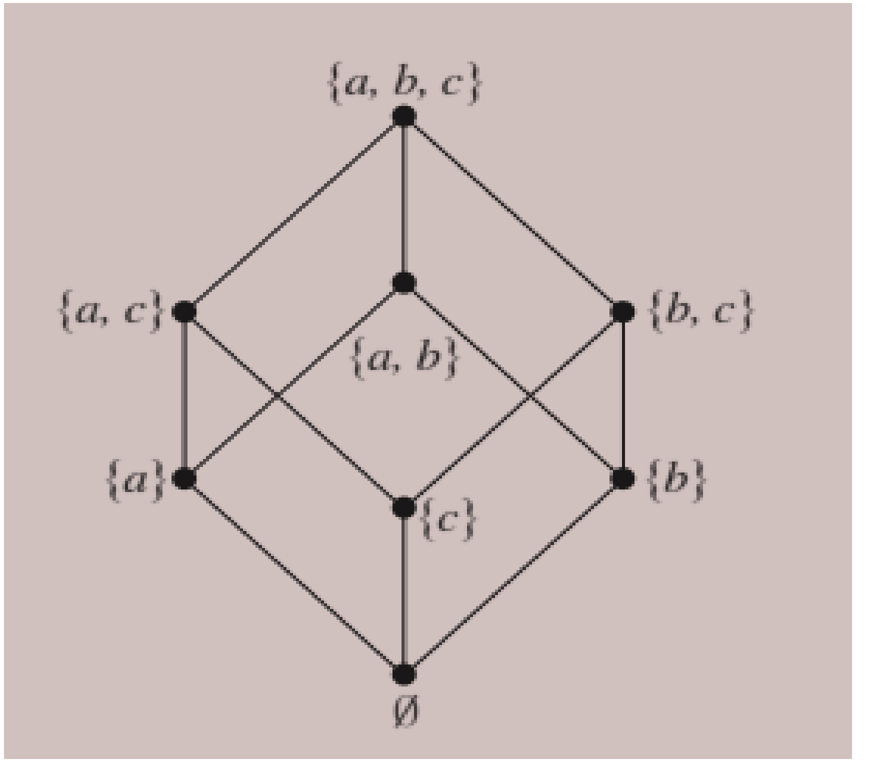
$\Rightarrow x \subseteq z$

Hence R is transitive.

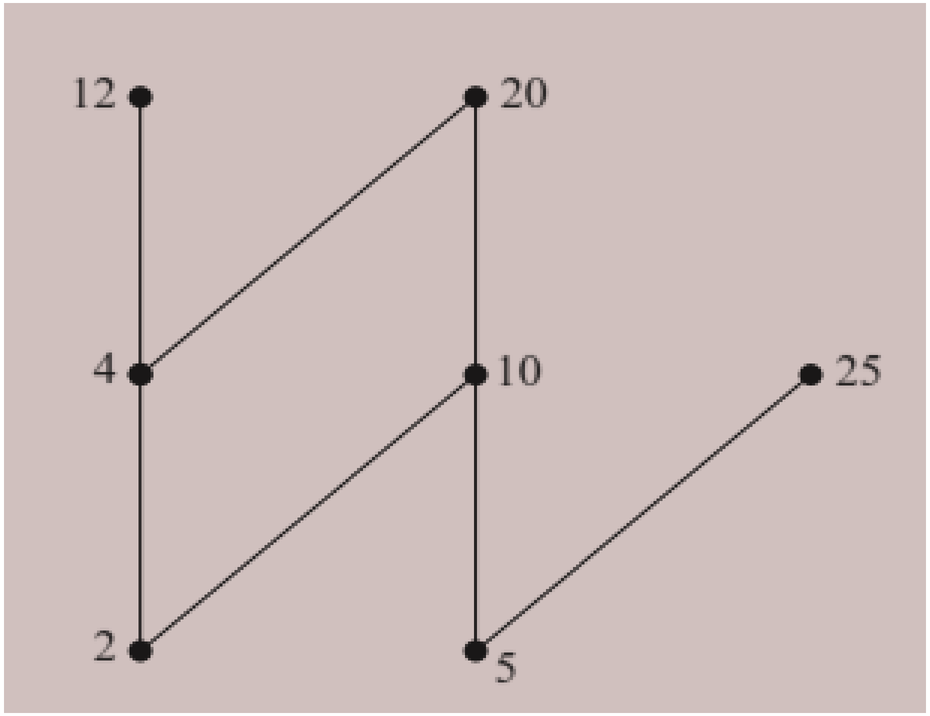
Thus R is a partial order.

$\therefore$ (P(S), R) is a POSET.

Hasse diagram



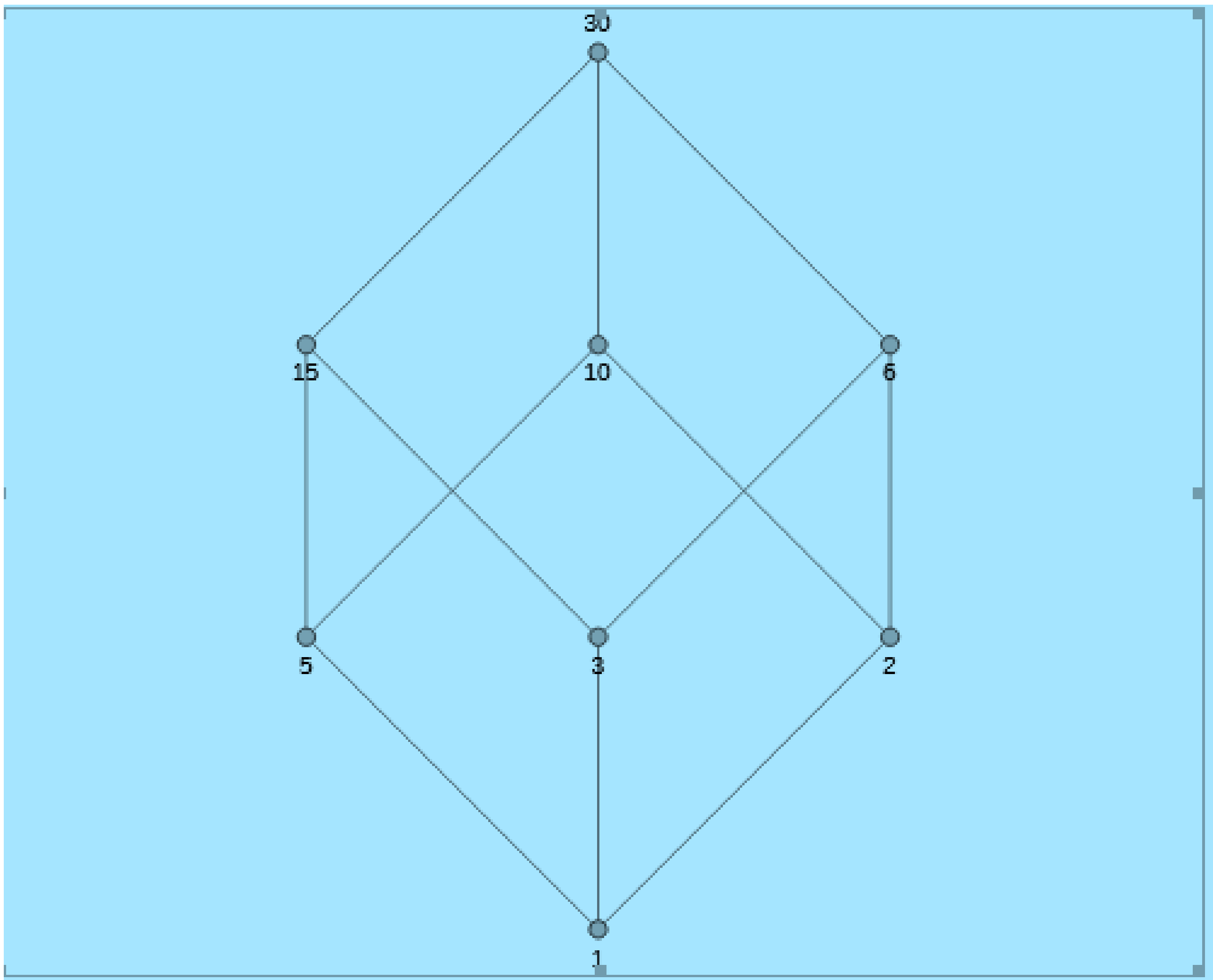
Draw the Hasse diagram of the POSET ({2, 4, 5, 10, 12, 20, 25}, |).



Draw the Hasse diagram for the positive divisors of 30 under the divisibility relation.

Let A = Positive divisors of 30 = {1, 2, 3, 5, 6, 10, 15, 30}

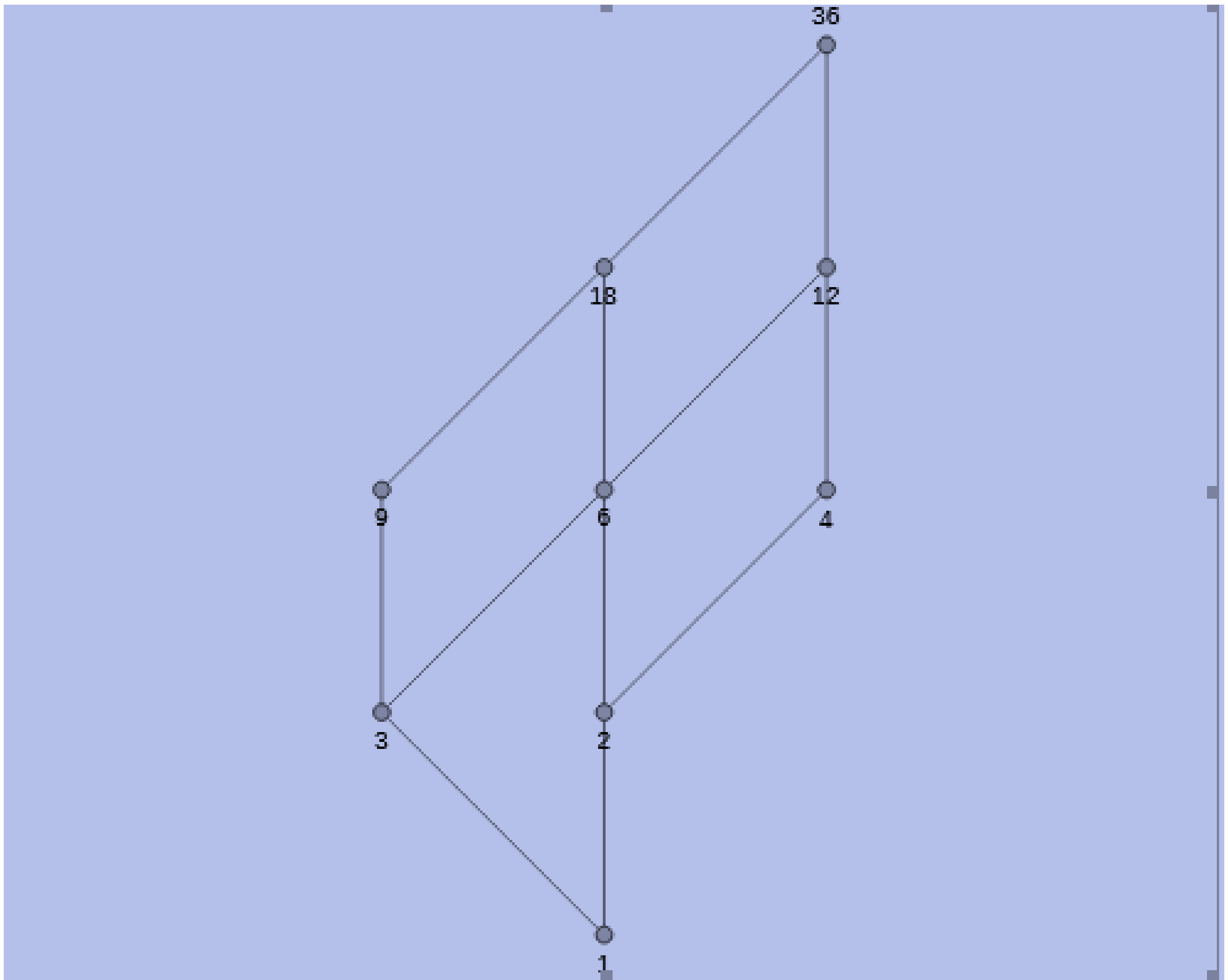
Hasse Diagram



Draw the Hasse diagram for the positive divisors of 36 under the divisibility relation.

Let A = Positive divisors of 36 = {1, 2, 3, 4, 6, 9, 12, 18, 36}

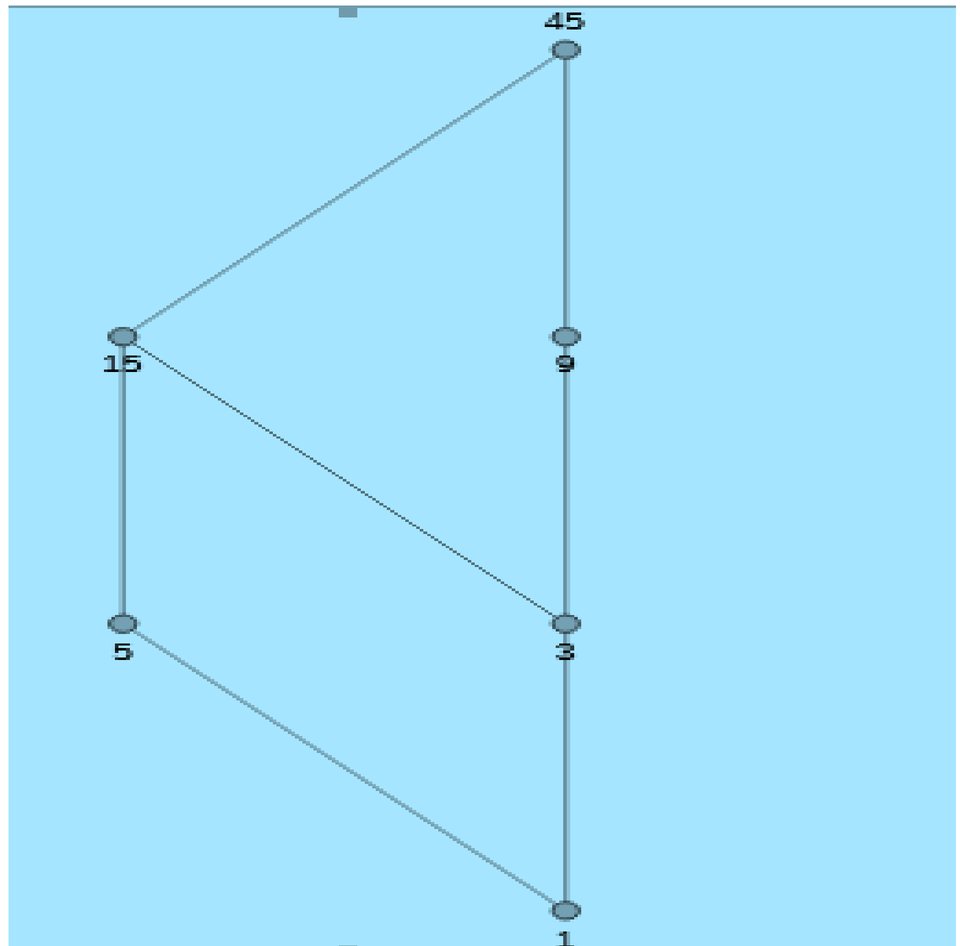
Hasse Diagram



Draw the Hasse diagram for the positive divisors of 45 under the divisibility relation.

Let A = Positive divisors of 45 = {1, 3, 5, 9, 15, 45}

Hasse Diagram



Extremal Elements in POSET

Let (A, R) be a POSET. Then we have the following special elements called the extremal elements.

- An element $a \in A$ is called a **maximal element** of A if $\exists$ no element x in A such that a R x.
- An element $a \in A$ is called a **minimal element** of A if $\exists$ no element x in A such that x R a.
- An element $a \in A$ is called a **greatest element** of A if x R a for all $x \in A$. The greatest element is unique if it exists.
- An element $a \in A$ is called a **lest element** of A if a R x for all $x \in A$. The lest element is unique if it exists.

- An element $a \in A$ is called an **upper bound (UB)** of x if x R a.
- An element $a \in A$ is called a **lower bound (LB)** of x if a R x.
- An element $a \in A$ is called the **least upper bound (LUB) or Supremum** of a subset B of A if

  (i) a is an UB of B.

  (ii) if d is an UB of B then a R d.
- An element $a \in A$ is called the **greatest lower bound (GLB) or Infimum** of a subset B of A if

  (i) a is an LB of B.

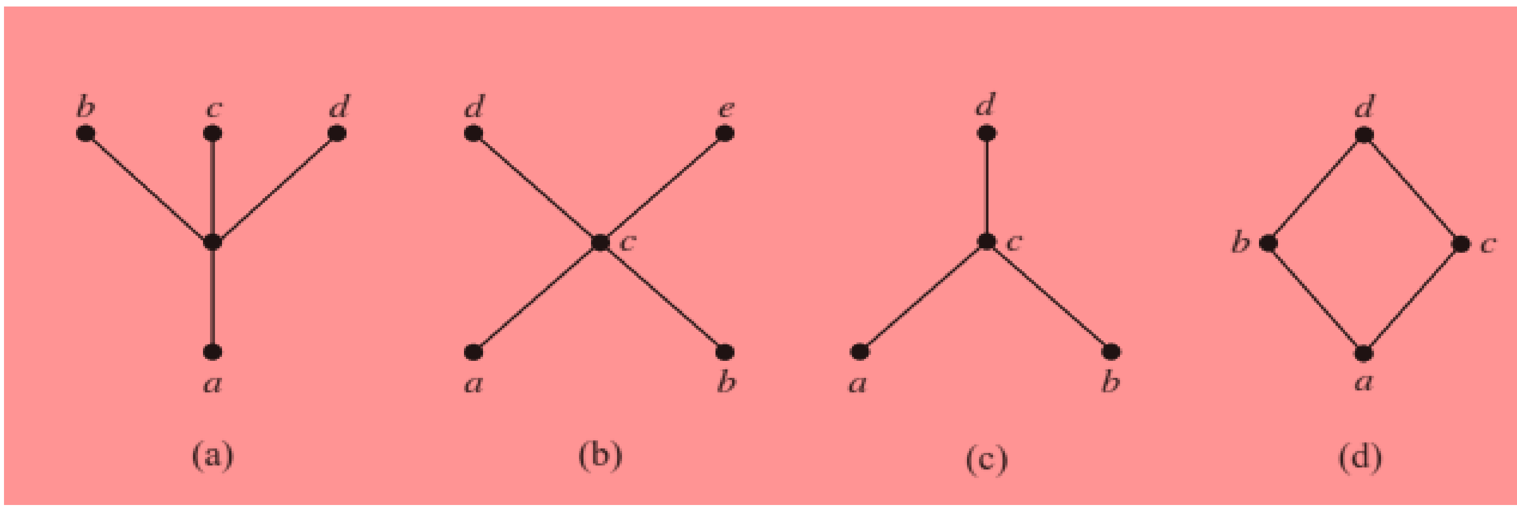
  (ii) if d is an LB of B then d R a.

Note: In the Hasse diagram of a POSET (A, R),

1. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram.
2. The elements which do not have downward are called minimal elements and the elements which do not have upward are called maximal elements.
3. If the element a is related to all the elements of A then a is called least element. If all the elements of A are related to any element a then a is called greatest element.
4. The least and minimal elements may be same.
5. The greatest and maximal elements may be same.

EXAMPLES

Determine whether the POSETS represented by each of the following Hasse diagrams have a greatest element and a least element. Also find minimal and maximal elements



Solution:

The least element of the POSET with Hasse diagram (a) is a. This POSET has no greatest element. The POSET with Hasse diagram (b) has neither a least nor a greatest element. The POSET with Hasse diagram (c) has no least element. Its greatest element is d. The POSET with Hasse diagram (d) has least element a and greatest element d.

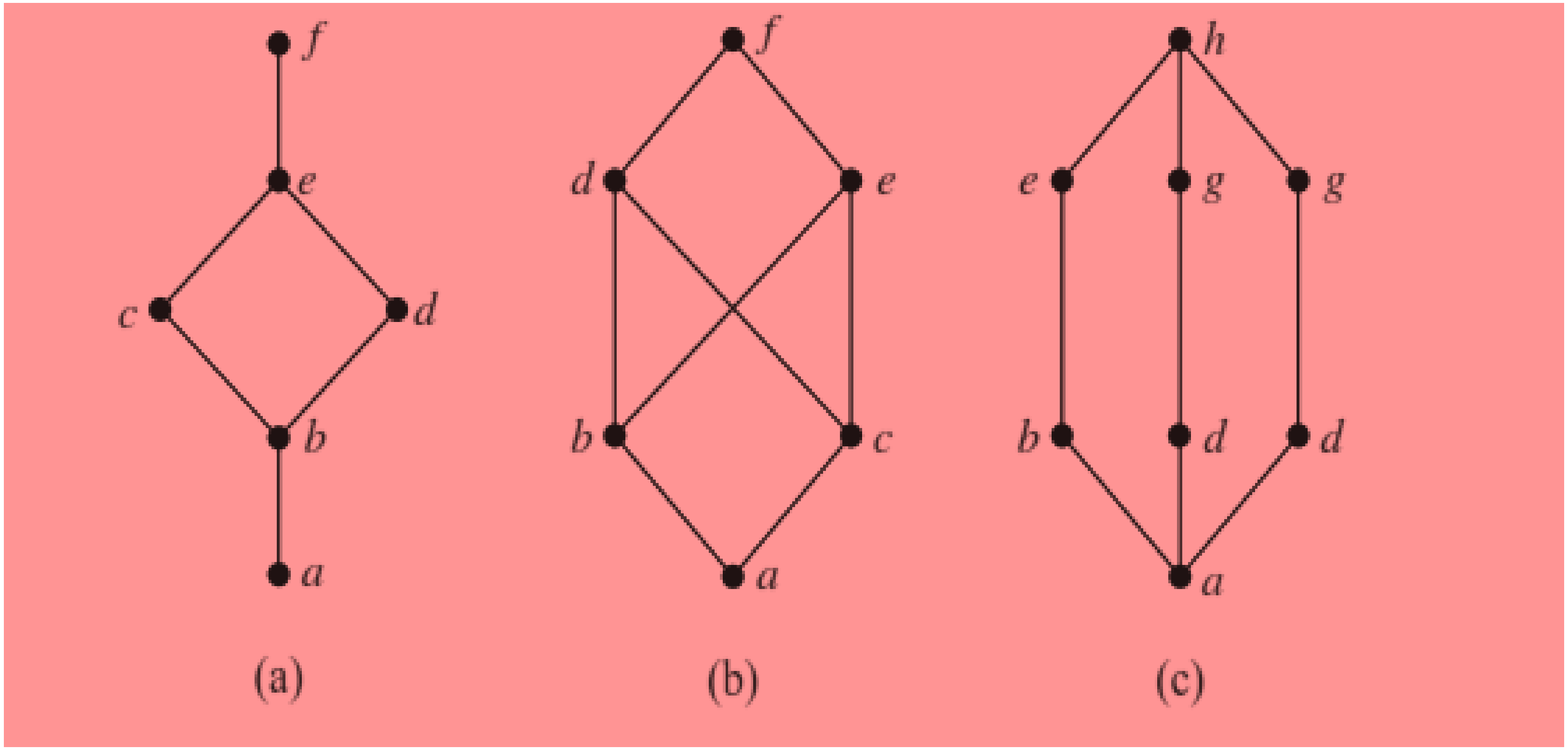
(a) Minimal elements = {a}, Maximal elements = {b, c, d}

(b) Minimal elements = {a, b}, Maximal elements = {d, e}

(c) Minimal elements = {a, b}, Maximal elements = {d}

(d) Minimal elements = {a}, Maximal elements = {d}

For the POSETS represented by each of the following Hasse diagrams: find (i) minimal and maximal elements (ii) least and greatest elements.



Solution:

(a) Minimal elements = {a}, Maximal elements = {f}

Least element = {a}, Greatest element = {f}

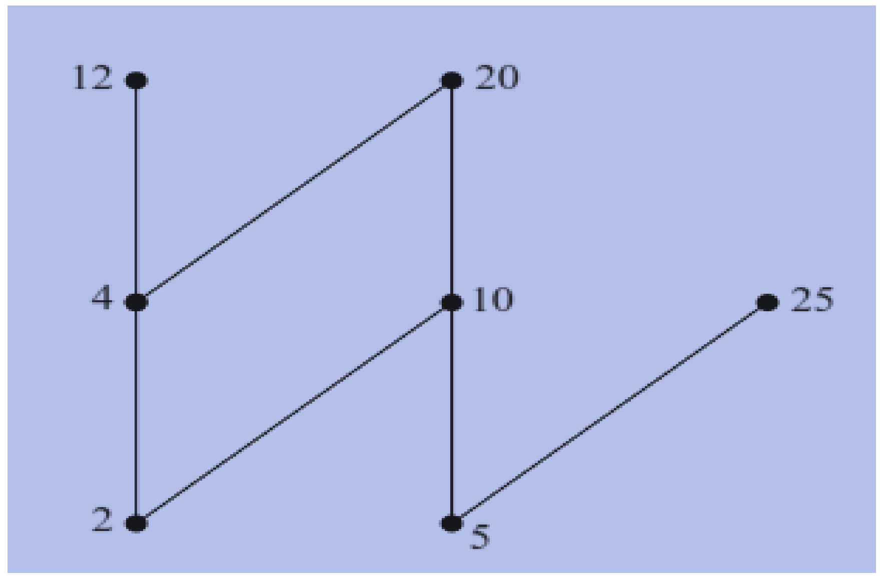
(b) Minimal elements = {a}, Maximal elements = {f}

Least element = {a}, Greatest element = {f}

(c) Minimal elements = {a}, Maximal elements = {h}

Least element = {a}, Greatest element = {h}

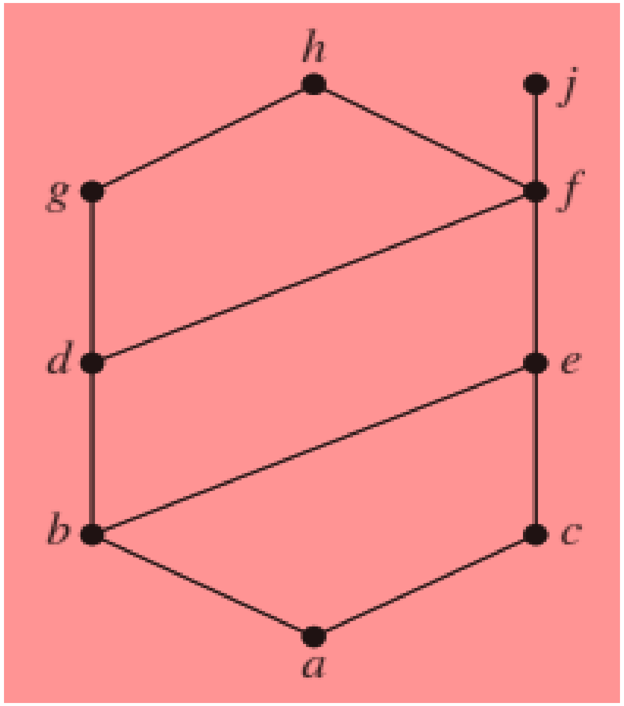
Which elements of the POSET ({2, 4, 5, 10, 12, 20, 25}, |) are maximal, and which are minimal? Also find least and greatest elements.



Minimal elements = {2, 5}, Maximal elements = {12, 20, 25}

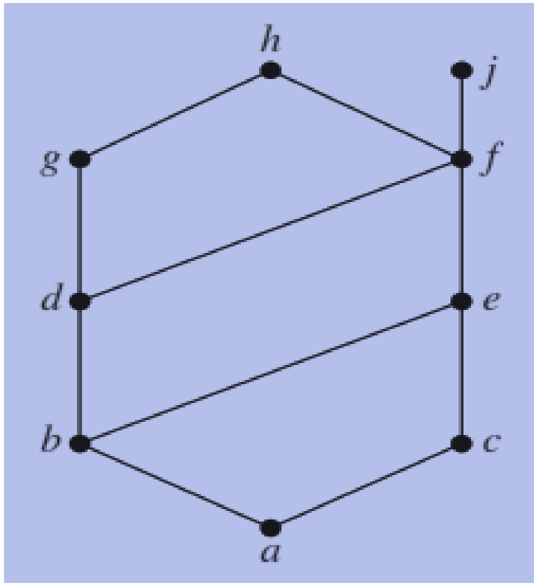
Least element = No, Greatest element = No

For the following Hasse diagram, find (i) minimal and maximal elements (ii) least and greatest elements. And, if B_1 = {a, b, c}, B_2 = {j, h} and B_3 = {a, c, d, f} find LUB and GLB of B_1, B_2 and B_3.



(i) Minimal elements = {a}, Maximal elements = {h, j}

(ii) Least element = {a}, Greatest element = No



Given, $B_1 = \{a, b, c\}$

UB of a = {a, b, c, d, e, f, g, h, j},

UB of b = {b, d, e, f, g, h, j} and

UB of c = {c, e, f, h, j}

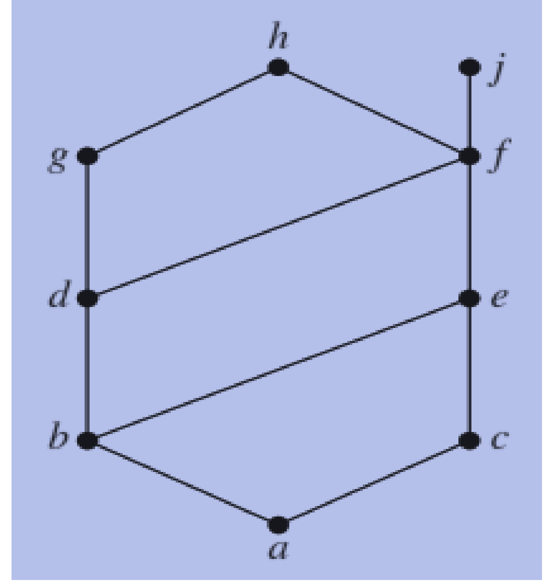
$\therefore$ UB of $B_1 = \{e, f, h, j\}$

Hence, LUB of $B_1 = \{e\}$

LB of a = {a}, LB of b = {b, a}, LB of c = {c, a}

$\therefore$ LB of $B_1 = \{a\}$

Hence, GLB of $B_1 = \{a\}$



Given, $B_2 = \{j, h\}$

UB of j = {j} and UB of h = {h}

$\therefore$ UB of $B_2 = \{\}$

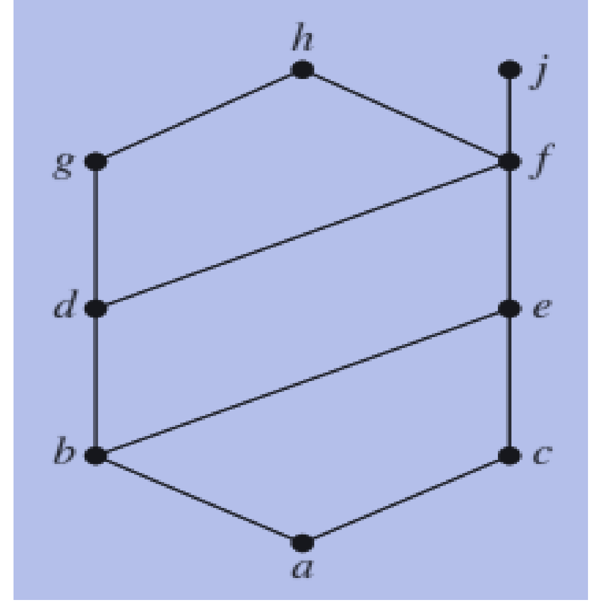
Hence, LUB of B_2 = No

LB of j = {j, f, e, c, a, d, b} and

LB of h = {h, g, f, d, e, b, c, a}

$\therefore$ LB of $B_2 = \{a, b, c, d, e, f\}$

Hence, GLB of $B_1 = \{f\}$



Given, $B_3 = \{a, c, d, f\}$

UB of a = {a, b, c, d, e, f, g, h, j},

UB of c = {c, e, f, h, j},

UB of d = {d, f, g, h, j} and UB of f = {f, h, j}

$\therefore$ UB of $B_3 = \{f, h, j\}$

Hence, LUB of $B_3 = \{f\}$

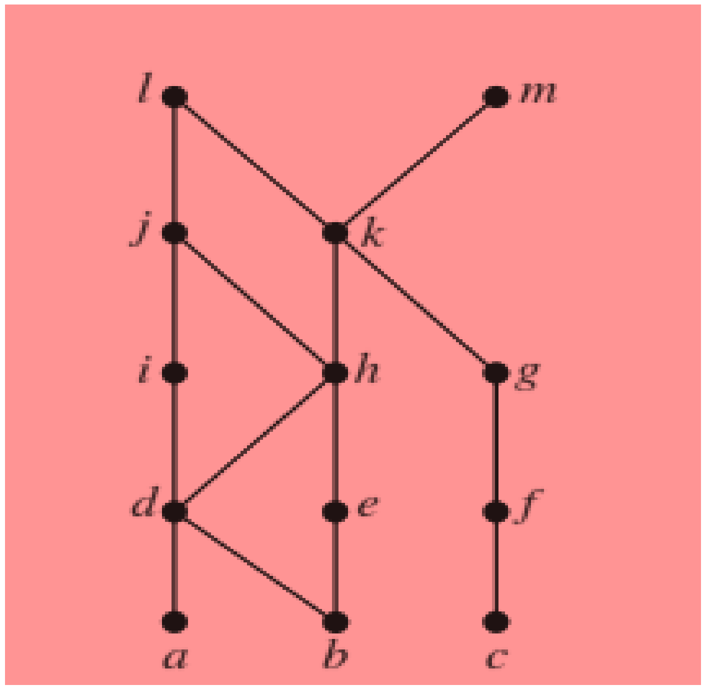
LB of a = {a}, LB of c = {c, a}, LB of d = {d, b, a} and

LB of f = {f, d, e, b, c, a}

$\therefore$ LB of $B_3 = \{a\}$

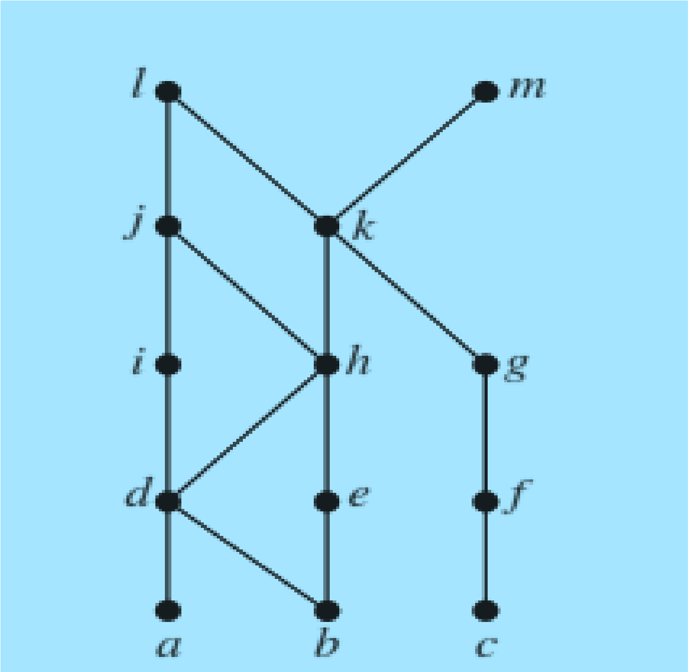
Hence, GLB of $B_3 = \{a\}$

For the following Hasse diagram, find (i) minimal and maximal elements (ii) least and greatest elements. And, if $B_1 = \{a, b, c\}$, $B_2 = \{f, g, h\}$ and $B_3 = \{i, j, k\}$ find LUB and GLB of B_1, B_2 and B_3



(i) Minimal elements = {a, b, c}, Maximal elements = {l, m}

(ii) Least element = No, Greatest element = No



Given, B_1 = {a, b, c}

UB of a = {a, d, i, h, j, k, l, m},

UB of b = {b, d, e, h, i, j, k, l, m}

and UB of c = {c, f, g, k, l, m}

$\therefore$ UB of B_1 = {k, l, m}

Hence, LUB of B_1 ={k}

LB of a = {a}, LB of b = {b},

LB of c = {c}

$\therefore$ LB of B_1 = {}

Hence, GLB of B_1 = No

Given, $B_2 = \{f, g, h\}$

UB of f = $\{f, g, k, l, m\}$, UB of g = $\{g, k, l, m\}$and

UB of h = $\{h, k, l, m, j\}$

$\therefore$ UB of $B_2 = \{k, l, m\}$

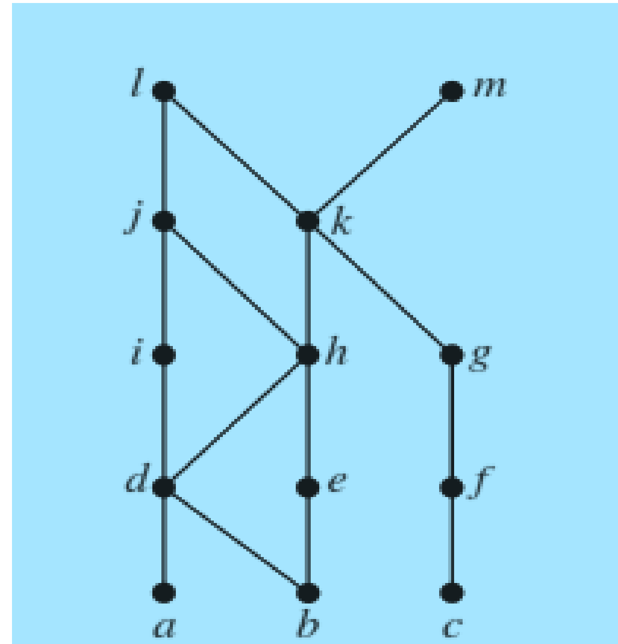
Hence, LUB of $B_2 = \{k\}$

LB of f = $\{f, c\}$, LB of g = $\{g, f, c\}$

and LB of h = $\{h, d, e, a, b\}$

$\therefore$ LB of $B_2 = \{\}$

Hence, GLB of B_2 = No



Given, $B_3 = \{i, j, k\}$

UB of $i = \{i, j, l\}$, UB of $j = \{j, l\}$

and UB of $k = \{k, l, m\}$

$\therefore$ UB of $B_3 = \{l\}$

Hence, LUB of $B_3 = \{l\}$

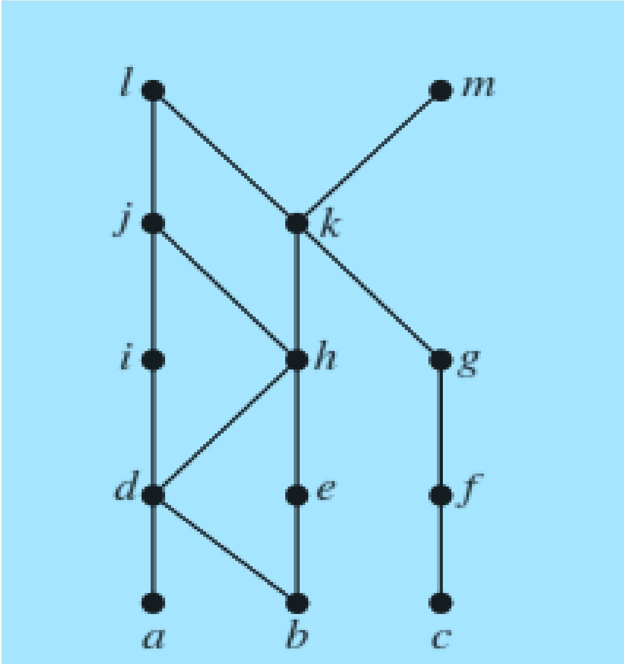
LB of $i = \{i, d, a, b\}$,

LB of $j = \{j, h, i, d, e, a, b\}$ and

LB of $k = \{k, h, g, d, e, f, a, b, c\}$

$\therefore$ LB of $B_3 = \{d, a, b\}$

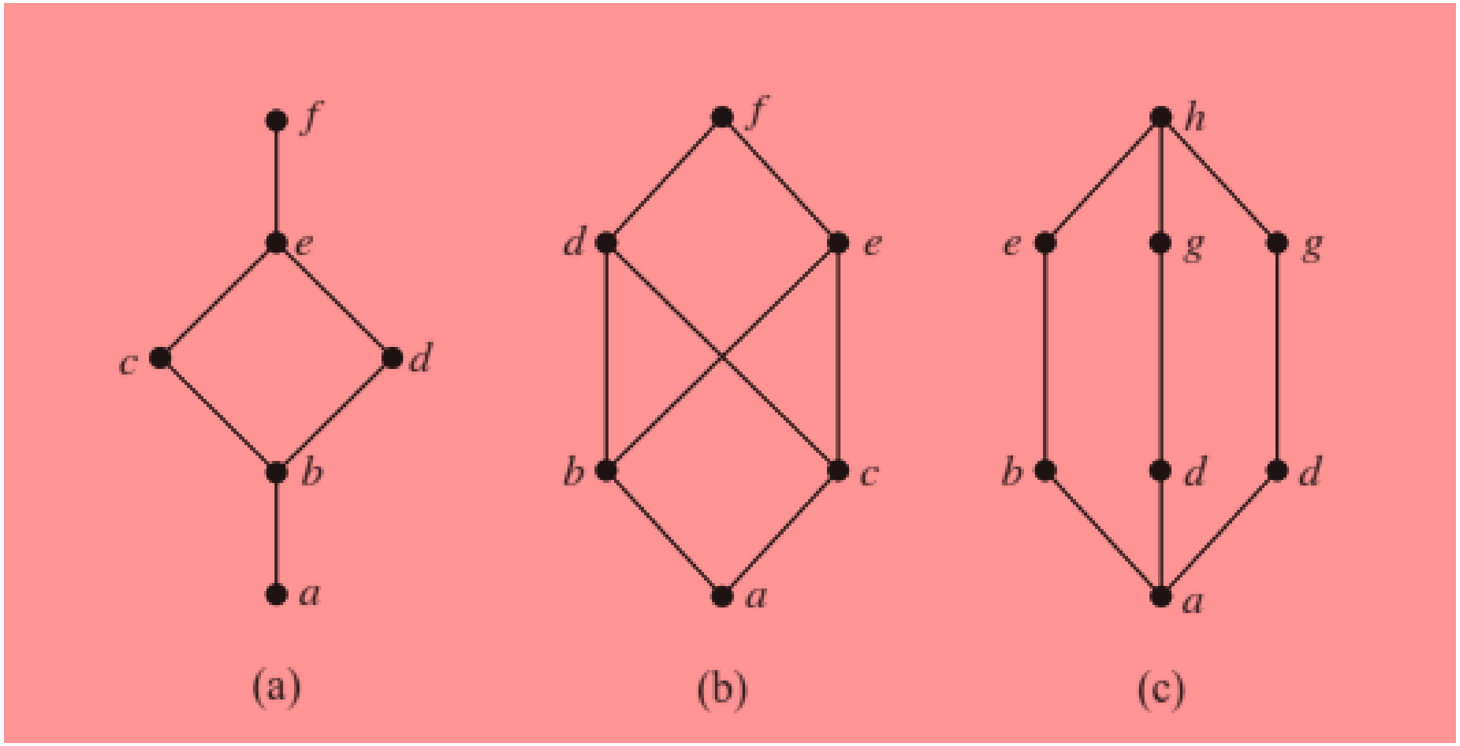
Hence, GLB of $B_3 = \{d\}$



LATTICES

- A partially ordered set (POSET) in which every pair of elements has both a least upper bound (LUB) and a greatest lower bound (GLB) is called a **lattice**.
- Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

Determine whether the POSET's represented by each of the following Hasse diagrams are lattices.



The POSET's represented by the Hasse diagrams in (a) and (c) are both lattices because in each POSET every pair of elements has both a least upper bound and a greatest lower bound, by verification.

On the other hand, the POSET with the Hasse diagram shown in (b) is not a lattice, because the elements b and c have no least upper bound. To see this, note that each of the elements d, e, and f is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this POSET.

Determine whether the POSET's ({1, 2, 3, 4, 5}, |) and ({1, 2, 4, 8, 16}, |) are lattices.

Because 2 and 3 have no upper bounds in ({1, 2, 3, 4, 5}, |), they certainly do not have a least upper bound. Hence, the first POSET is not a lattice.

Every two elements of the second POSET have both a least upper bound and a greatest lower bound. The least upper bound of two elements in this POSET is the larger of the elements and the greatest lower bound of two elements is the smaller of the elements, as the reader should verify. Hence, this second POSET is a lattice.

Is the POSET (Z^+, |) a lattice?

Let a and b be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively, by verification. It follows that this POSET is a lattice.

Note: Every total order is a partial order but every partial order is not a total order.

TOPOLOGICAL SORTING

Let R be a partial order on a set A, which is not a total order. To find a total order T on A, we have the following technique known as **Topological sorting**:

Step 1: Draw the Hasse diagram for the given POSET and denote it by H.

Step 2: Choose the minimal element (say $v_{1)}$. If it has more than one element choose anyone arbitrarily.

Step 3: Delete the minimal element v_1 and all edges begins at v_1 from H. Denote the resulting diagram by H_1.

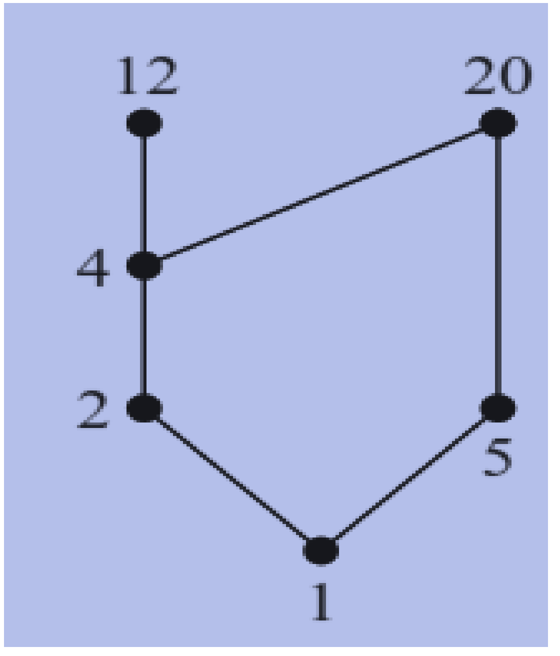
- Repeat steps 2 and 3 with H replaced by H_1 and v_1 replaced by v_2 and the diagram at the end of step 3 is denoted by H_2.
- Repeat the process until a final diagram is got. If $|A| = n$ then the final diagram H_n contains only one vertex v_n, with the vertices $v_1, v_2, v_3, \ldots$ removed one by one in that order.
- Now a relation T on A is defined by $(v_i, v_j) \in T$ if $i \leq j$, for $i, j = 1, 2, 3, \ldots$ then T is the required total order.
- i.e., $v_1 \prec v_2 \prec v_3 \prec \ldots \prec v_n$ is the required total order and it is not unique.

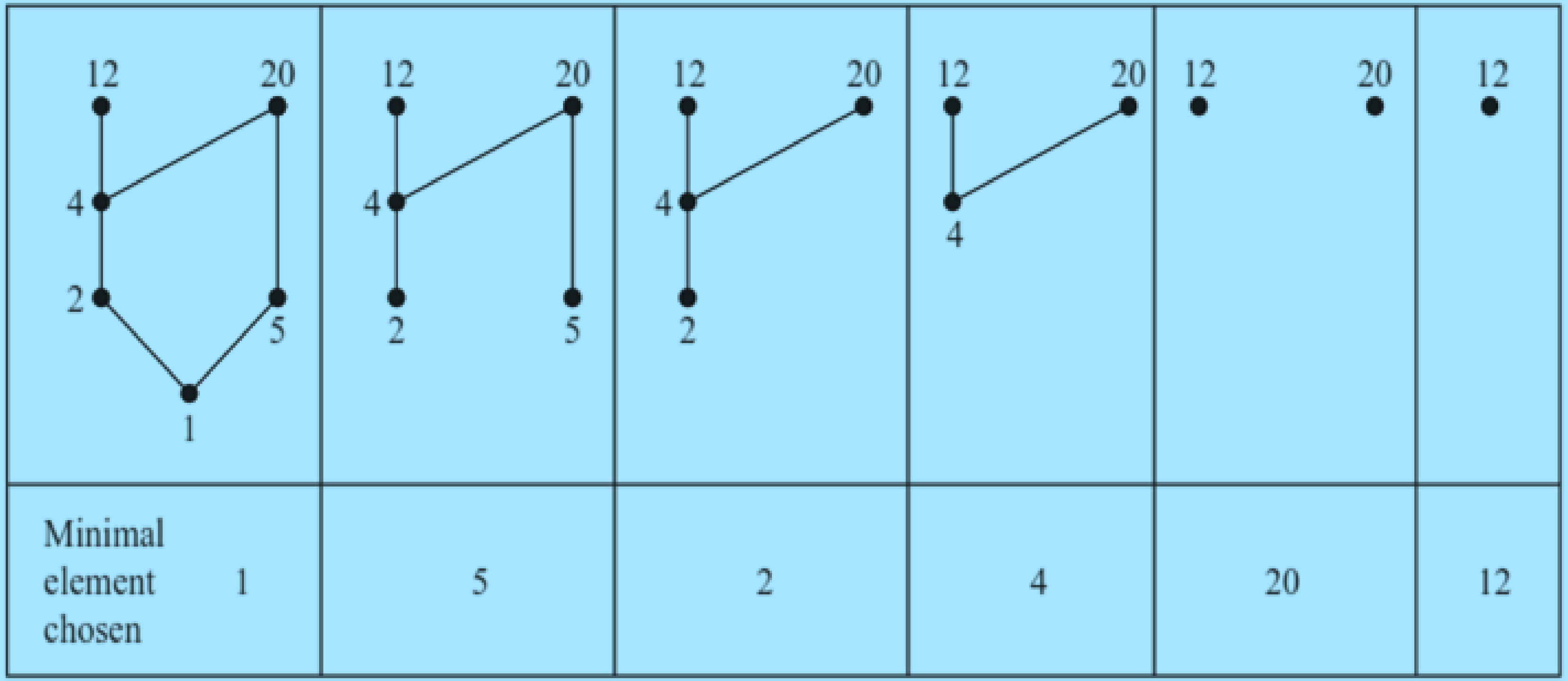
Topological sorting has an application to the scheduling of projects.

Remark:

- “Topological sorting” is terminology used by computer scientists; mathematicians use the terminology “linearization of a partial ordering” for the same thing.
- In mathematics, topology is the branch of geometry dealing with properties of geometric figures that hold for all figures that can be transformed into one another by continuous bijections.
- In computer science, a topology is any arrangement of objects that can be connected with edges.

Find a compatible total ordering for the POSET ({1, 2, 4, 5, 12, 20}, |).





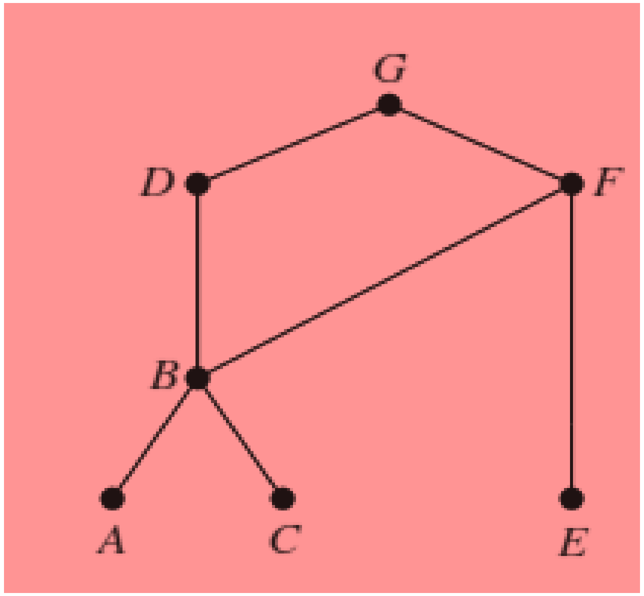
The first step is to choose a minimal element. This must be 1, because it is the only minimal element. Next, select a minimal element of ({2, 4, 5, 12, 20}, |). There are two minimal elements in this POSET, namely, 2 and 5. We select 5. The remaining elements are {2, 4, 12, 20}. The only minimal element at this stage is 2. Next, 4 is chosen because it is the only minimal element of ({4, 12, 20}, |). Because both 12 and 20 are minimal elements of ({12, 20}, |), either can be chosen next. We select 20, which leaves 12 as the last element left.

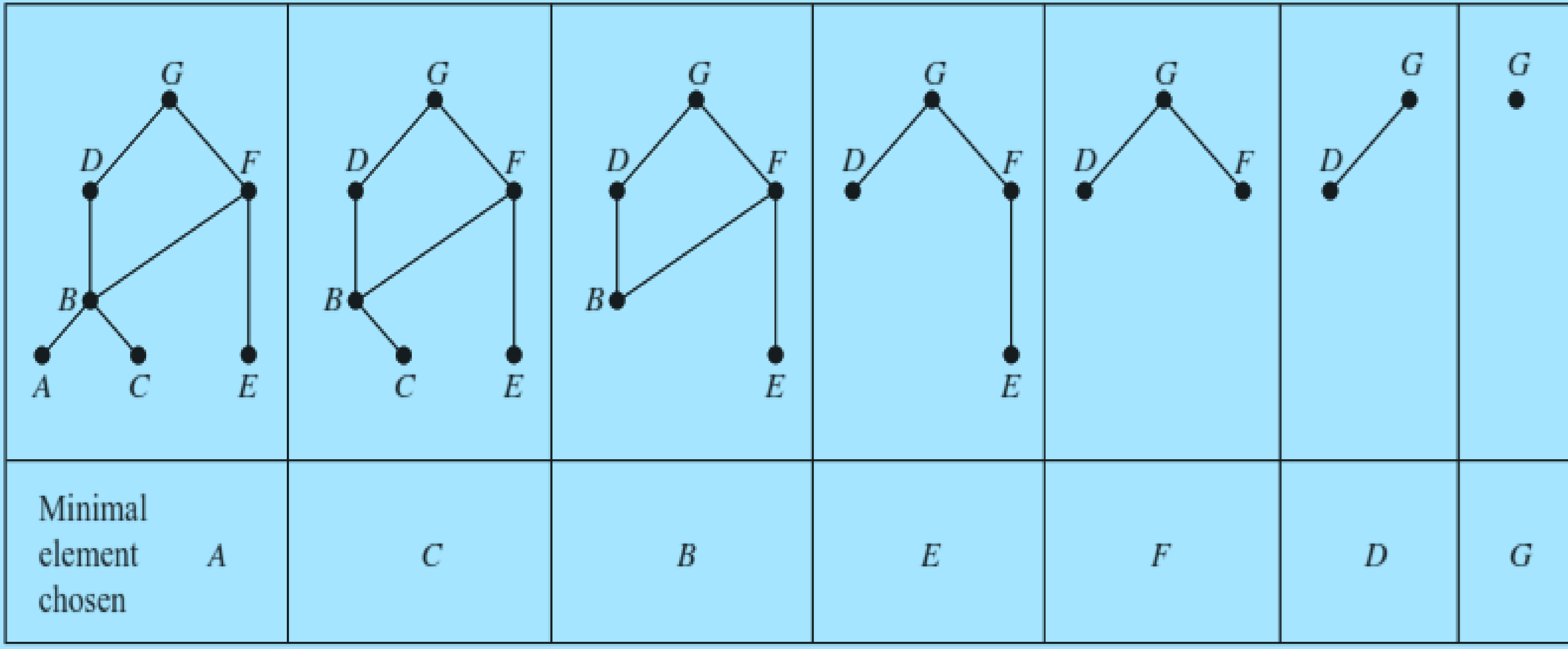
This produces the total ordering

$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12.$$

This total order is not unique.

A development project at a computer company requires the completion of seven tasks. Some of these tasks can be started only after other tasks are finished. A partial ordering on tasks is set up by considering task X ≺ task Y if task Y cannot be started until task X has been completed. The Hasse diagram for the seven tasks, with respect to this partial ordering, is shown in the following Figure. Find an order in which these tasks can be carried out to complete the project.





An ordering of the seven tasks can be obtained by performing a topological sort. The steps of a sort are illustrated in the above Figure.

The result of this sort, A $\prec$ C $\prec$ B $\prec$ E $\prec$ F $\prec$ D $\prec$ G, gives one possible order for the tasks.