LOGICAL EQUIVALENCES INVOLVING QUANTIFIERS

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

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Example:

Show that the following are logically equivalent (where the same domain is used throughout).

(1) $\forall x\,(P(x) \land Q(x)) \equiv \forall x\,P(x) \land \forall x\,Q(x)$.

(2) $\forall x\,(P(x) \lor Q(x)) \equiv \forall x\,P(x) \lor \forall x\,Q(x)$.

Solution:

To show that these statements are logically equivalent, we must show that they always take the same truth value, no matter what the predicates P and Q are, and no matter which domain of discourse is used. Suppose we have particular predicates P and Q, with a common domain. We can show that $\forall x\,(P(x) \land Q(x))$ and $\forall x\,P(x) \land \forall x\,Q(x)$ are logically equivalent by doing two things.

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First, we show that if $\forall x\ (P(x) \wedge Q(x))$ is true, then $\forall x\ P(x) \wedge \forall x\ Q(x)$ is true. Second, we show that if $\forall x\ P(x) \wedge \forall x\ Q(x)$ is true, then $\forall x\ (P(x) \wedge Q(x))$ is true.

So, suppose that $\forall x\ (P(x) \wedge Q(x))$ is true. This means that if a is in the domain, then $P(a) \wedge Q(a)$ is true. Hence, $P(a)$ is true and $Q(a)$ is true. Hence, $P(a)$ is true and $Q(a)$ is true. Because $P(a)$ is true and $Q(a)$ is true for every element in the domain, we can conclude that $\forall x\ P(x)$ and $\forall x\ Q(x)$ are both true.

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Next, suppose that $\forall x\ P(x) \land \forall x\ Q(x)$ is true. It follows that $\forall x\ P(x)$ is true and $\forall x\ Q(x)$ is true. Hence, if a is in the domain, then $P(a)$ is true and $Q(a)$ is true [because $P(x)$ and $Q(x)$ are both true for all elements in the domain, there is no conflict using the same value of a here].

It follows that for all a, $P(a) \land Q(a)$ is true. It follows that $\forall x\ (P(x) \land Q(x))$ is true.

We can now conclude that $\forall x\ (P(x) \land Q(x)) \equiv \forall x\ P(x) \land \forall x\ Q(x)$.

Furthermore, we can also distribute an existential quantifier over a disjunction.

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NEGATION OF QUANTIFIERS

The following are the rules for negations for quantified statements:

(i) $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

(ii) $\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x)$.

The rules for negations for quantifiers are called **De Morgan's laws** for quantifiers.

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EXAMPLE:

“Every student in your class has taken a course in calculus.”

Let $P(x)$ denotes “x is in your class has taken a course in calculus”.

Then the given statement is represented in symbolic form by $\forall x\, P(x)$, where the domain consists of all students.

The negation of this statement is $\neg \forall x\, P(x)$, which is equivalent to $\exists x \neg P(x)$

This negation can be expressed as “There is a student in your class who has not taken a course in calculus.”

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“There is an honest politician”

Let $P(x)$ denotes “x is honest.”

Then the given statement is represented in symbolic form by $\exists x\, P(x)$, where the domain consists of all politicians.

The negation of this statement is $\neg \exists x\, P(x)$, which is equivalent to $\forall x \neg P(x)$.

This negation can be expressed as “Every politician is dishonest.” or “All politicians are not honest”.

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“All Americans eat cheeseburgers”

Let $P(x)$ denote “x eats cheeseburgers.”

Then the given statement is represented in symbolic form by $\forall x\ P(x)$, where the domain consists of all Americans.

The negation of this statement is $\neg \forall x\ P(x)$, which is equivalent to $\exists x \neg P(x)$.

This negation can be expressed as “Some American does not eat cheeseburgers” or “There is an American who does not eat cheeseburgers.”

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Converse, Inverse and Contrapositive for Quantified Statements:

For open statements $p(x), q(x)$ — defined for a prescribed universe — and the universally quantified statement $\forall x\,[p(x) \rightarrow q(x)]$, we define:

1) The *contrapositive* of $\forall x\,[p(x) \rightarrow q(x)]$ to be $\forall x\,[\neg q(x) \rightarrow \neg p(x)]$.

2) The *converse* of $\forall x\,[p(x) \rightarrow q(x)]$ to be $\forall x\,[q(x) \rightarrow p(x)]$.

3) The *inverse* of $\forall x\,[p(x) \rightarrow q(x)]$ to be $\forall x\,[\neg p(x) \rightarrow \neg q(x)]$.

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PROBLEMS

1. For the universe of all integers let $p(x)$: $x > 0$, $q(x)$: x is even, $r(x)$: x is a perfect square, $s(x)$: x is divisible by 3, $t(x)$: x is divisible by 7. Write the following statements in symbolic form:

(a) At least one integer is even,

$$\exists x, q(x)$$

(b) There exists a positive integer that is even.

$$\forall x, [p(x) \wedge q(x)]$$

(c) Some even integers are divisible by 3.

$$\exists x, [q(x) \wedge s(x)]$$

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(d) Every integer is either even or odd.

$$\forall \boldsymbol{x}, [\boldsymbol{q}(\boldsymbol{x}) \vee \neg \boldsymbol{q}(\boldsymbol{x})]$$

(e) If x is even and a perfect square then x is not divisible by 3.

$$\forall \boldsymbol{x}, [\boldsymbol{q}(\boldsymbol{x}) \wedge \boldsymbol{r}(\boldsymbol{x})] \rightarrow \neg \boldsymbol{s}(\boldsymbol{x})$$

(f) If x is odd or is not divisible by 7 then x is divisible by 3.

$$\forall \boldsymbol{x}, [\boldsymbol{q}(\boldsymbol{x}) \wedge \boldsymbol{t}(\boldsymbol{x})] \rightarrow \boldsymbol{s}(\boldsymbol{x})$$

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2. Consider the universe of all polygons with 3 or 4 sides and define the following open statements for this universe. a(x): All interior angles of x are equal, e(x): x is an equilateral triangle, i(x): x is an isosceles triangle, p(x): x has an interior angle that exceeds 180^0, q(x): x is a quadrilateral, r(x): x is a rectangle, s(x): x is a square, t(x): x is a triangle. Write the following in verbal form:

(a) $\forall x, [q(x) \vee t(x)]$

For any x, x is a quadrilateral or a triangle.

Or Every polygon with 3 or more sides is a quadrilateral or a triangle

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(b) $\forall x, [i(x) \vee e(x)]$

For any x, x is an isosceles triangle or an equilateral triangle.

Or

Every polygon with 3 or 4 sides is an isosceles triangle or an equilateral triangle.

(c) $\exists x, [t(x) \wedge p(x)]$

For some x, x is a triangle and a quadrilateral.

Or

Some polygons with 3 or 4 sides are a triangle and a quadrilateral.

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(d) $\exists x, [q(x) \wedge \neg \mathbf{r(x)}]$

For some x, x is a quadrilateral and not a rectangle.

Or

There is a polygon with 3 or 4 sides is a quadrilateral and not a rectangle.

(e) $\forall x, \{[a(x) \wedge t(x)] \leftrightarrow e(x)\}$

For any x, interior angles of x are equal and are a triangle if and only if x is an equilateral triangle.

(f) $\forall x, t(x) \rightarrow \neg p(x)$

For any x, if x is a triangle then x does not have an interior angle that exceeds180^0.

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Consider the following statements with a set of all real numbers as the universe.

$\boldsymbol{p(x): x \geq 0, q(x): x^2 \geq 0, r(x): x^2 - 3x - 4 = 0,}$

$\boldsymbol{s(x): x^2 - 3 > 0.}$

Determine the truth values of the following:

(a) $\boldsymbol{\exists x, p(x) \land q(x)}$

For $x = 1$, $p(x): x \geq 0 \; and \; q(x): x^2 \geq 0$ are true.

Therefore, $\exists x, p(x) \land q(x)$ is true.

(b) $\boldsymbol{\forall x, p(x) \rightarrow q(x)}$

$q(x): x^2 \geq 0$ cannot be false for any real number x.

Therefore,$p(x) \rightarrow q(x)$ cannot be false for any real number x.

Therefore, $\forall x, p(x) \rightarrow q(x)$ is true.

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(c) $\forall \boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x}) \rightarrow \boldsymbol{s}(\boldsymbol{x})$

For $x = 1, q(x): x^2 \geq 0$ is true but $s(x): x^2 - 3 > 0$ is false.

Therefore, $\forall x, q(x) \rightarrow s(x)$ is false.

(d) $\forall \boldsymbol{x}, \boldsymbol{r}(\boldsymbol{x}) \vee \boldsymbol{s}(\boldsymbol{x})$

For $x = 1, r(x): x^2 - 3x - 4 = 0$ *and* $s(x): x^2 - 3 > 0$ are false.

Therefore, $\forall x, r(x) \vee s(x)$is false.

$(\boldsymbol{e})\ \exists \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x}) \wedge \boldsymbol{r}(\boldsymbol{x})$**.**

For $x = 4, p(x): x \geq 0$ is true and$r(x): x^2 - 3x - 4 = 0$ is true.

Therefore, $\exists x, p(x) \wedge r(x)$ is true.

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Negate each of the following:

(a) $\exists x, p(x) \vee q(x)$

$$\forall x, \neg p(x) \wedge \neg q(x)$$

(b) $\forall x,\ p(x) \to q(x)$

$$u \equiv \forall x, p(x) \to q(x) \equiv \forall x, \neg p(x) \vee q(x)$$

Its negation is $\neg u \equiv \exists x, p(x) \wedge \neg q(x)$

(c) $\forall x, p(x) \to \neg q(x)$

$$u \equiv \forall x, p(x) \to \neg q(x) \equiv \forall x, \neg p(x) \vee \neg q(x)$$

Its negation is $\neg u \equiv \exists x, p(x) \wedge q(x)$

(d) $\exists x, \left[\left(p(x) \vee q(x)\right) \to r(x)\right]$

$$u \equiv \exists x, \left[\left(p(x) \vee q(x)\right) \to r(x)\right]$$

$$\equiv \exists x, \left[\neg\left(p(x) \vee q(x)\right) \vee r(x)\right]$$

Its negation is $\neg u \equiv \forall x, \left[\left(p(x) \vee q(x)\right) \wedge \neg r(x)\right]$

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Write down the following propositions in symbolic form and find their negation:

(a) For all integers, if n is not divisible by 2, then n is odd.

Let $p(n)$: n is divisible by 2 and $q(n)$: n is odd

$u \equiv$ For all integers, if n is not divisible by 2, then n is odd

$\equiv \forall\, n, \neg p(n) \rightarrow q(n)$

$\equiv \forall n, p(n) \vee q(n)$

$\neg u \equiv \exists n, \neg p(n) \wedge \neg q(n)$

$\equiv$ Some integers are neither divisible by 2 nor odd.

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(b) If l, m, n are any integers where $l - m$ and $m - n$ are odd then $l - n$ is even

Let $p(x): l - m$ is odd, $q(x): m - n$ is odd and

$r(x): l - n$ is odd.

$u \quad \equiv$ For any integers, if $l - m$ and $m - n$ are odd then $l - n$ is even

$$\equiv \forall x, [p(x) \wedge q(x)] \rightarrow r(x)$$

$$\equiv \forall x, \neg[p(x) \wedge q(x)] \vee r(x)$$

$$\neg u \equiv \exists x, \ [p(x) \wedge q(x)] \wedge \neg r(x)$$

For some $l, m, n, l - m$ and $m - n$ are odd and $l - n$ is not odd.

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(c) All rational numbers are real and some real numbers are not rational.

Let $p(x)$: x is real and $q(x)$: x is rational

$u \quad \equiv$ All rational numbers are real and some real numbers are not rational

$$\equiv \forall x \in Q, p(x) \wedge \exists x \in R, \neg q(x)$$

$$\neg u \equiv \exists x \in Q, \neg\, p(x) \wedge \forall\, x \in R, q(x)$$

Some rational numbers are not real or all real numbers are rational.

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