SOLVING DIFFERENTIAL EQUATIONS

BY USING

LAPLACE TRANSFORM TECHNIQUE

LAPLACE TRANSFORMS OF DERIVATIVES

If $L[f(t)] = \overline{f}(s)$ then $L[f^n(t)] = s^n \overline{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \dots - f^{n-1}(0)$

In Particular, for $n = 1, 2, 3, 4$

i) If $L[f(t)] = \overline{f}(s)$ then $L[f'(t)] = s\overline{f}(s) - f(0)$

ii) If $L[f(t)] = \overline{f}(s)$ then $L[f''(t)] = s^2 \overline{f}(s) - s f(0) - f'(0)$

iii) If $L[f(t)] = \overline{f}(s)$ then $L[f'''(t)] = s^3 \overline{f}(s) - s^2 f(0) - s f'(0) - f''(0)$

iv) If $L[f(t)] = \overline{f}(s)$ then $L[f^{iv}(t)] = s^4 \overline{f}(s) - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$

Solve $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=e^{-t},\ y(0)=0,\ y'(0)=0$ **by using Laplace transforms.**

(VTU 2006, 2011)

Solution: Given, $y''+4y'+4y = e^{-t}$

$L[f'(t)] = s\overline{f}(s) - f(0)$

$L[f''(t)] = s^2\overline{f}(s) - s\,f(0) - f'(0)$

$L[e^{at}] = \frac{1}{s-a}$,

$y(0)=0,\ y'(0)=0$

Taking Laplace Transform on both side

$$L[y''] + 4L[y'] + 4L[y] = L[e^{-t}]$$

$$\Rightarrow \left[s^2\overline{y}(s) - s\,y(0) - y'(0)\right] + 4\left[s\overline{y}(s) - y(0)\right] + 4\overline{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow \left[s^2\overline{y}(s) - s(0) - (0)\right] + 4\left[s\overline{y}(s) - (0)\right] + 4\overline{y}(s) = \frac{1}{s+1}$$

$$s^2\overline{y}(s) + 4s\overline{y}(s) + 4\overline{y}(s) = \frac{1}{s+1}$$

$$\overline{y}(s)\left[s^2+4s+4\right] = \frac{1}{s+1}$$

$$\bar{y}(s)\left[(s+2)^2\right] = \frac{1}{s+1}$$

$$\bar{y}(s) = \frac{1}{(s+1)(s+2)^2}$$

we have, $y(t) = L^{-1}\left[\bar{y}(s)\right]$

$$y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

$$y(t) = L^{-1}\left[\frac{A}{s+1} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2}\right]$$

$$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Put $s = -1 \Rightarrow 1 = A(1) \Rightarrow A = 1$

Put $s = -2 \Rightarrow 1 = C(-1) \quad \Rightarrow C = -1$

Put $s = 0 \Rightarrow 1 = 4A + 2B + C \quad \Rightarrow B = -1$

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$L^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}t^{n-1}}{(n-1)!}$$

$$y(t) = L^{-1}\left[\frac{1}{s+1} + \frac{-1}{(s+2)} + \frac{-1}{(s+2)^2}\right]$$

$$y(t) = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{(s+2)}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$y(t) = e^{-t} - e^{-2t} - te^{-2t}$$

Solve $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=12t^2e^{-3t}$, $y(0)=0$, $y'(0)=0$ by using Laplace transforms.

(VTU 2005)

Solution:

Given, $y''+6y'+9y = 12t^2e^{-3t}$

Taking Laplace Transform on both side

$$L[y'']+6L[y']+9L[y] = L\left[12t^2e^{-3t}\right]$$

$$\Rightarrow \left[s^2\overline{y}(s)-s\,y(0)-y'(0)\right]+6\left[s\overline{y}(s)-y(0)\right]+9\overline{y}(s) = 12\left(\frac{2}{(s+3)^3}\right)$$

$$\left[s^2\overline{y}(s)-s(0)-(0)\right]+6\left[s\overline{y}(s)-(0)\right]+9\overline{y}(s) = \frac{24}{(s+3)^3}$$

$$s^2\overline{y}(s)+6s\overline{y}(s)+9\overline{y}(s) = \frac{24}{(s+3)^3}$$

$$\overline{y}(s)\left[s^2+6s+9\right] = \frac{24}{(s+3)^3}$$

$L[f'(t)] = s\overline{f}(s)-f(0)$

$L[f''(t)] = s^2\overline{f}(s)-s\,f(0)-f'(0)$

$L\left[e^{at}t^n\right] = \frac{n!}{(s-a)^{n+1}}$,

$y(0)=0,\ y'(0)=0$

$$\bar{y}(s)\left[(s+3)^2\right] = \frac{24}{(s+3)^3}$$

$$\bar{y}(s) = \frac{24}{(s+3)^5}$$

$$y(t) = L^{-1}\left[\bar{y}(s)\right]$$

$$y(t) = L^{-1}\left[\frac{24}{(s+3)^3}\right]$$

$$y(t) = 24e^{-3t}\frac{t^4}{4!}$$

$$y(t) = e^{-3t}t^4$$

$$L^{-1}\left[\frac{1}{(s-a)^{n+1}}\right] = \frac{e^{at}t^n}{n!}$$

Solve $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 1 - e^{2t}$, $y(0) = 1$, $y'(0) = 1$ by using Laplace transforms.

(VTU 2004)

Solution:

Given, $y'' - 3y' + 2y = 1 - e^{2t}$

Taking Laplace Transform on both side

$$L[y''] - 3L[y'] + 2L[y] = L\left[1 - e^{2t}\right]$$

$$L[f'(t)] = s\bar{f}(s) - f(0)$$

$$L[f''(t)] = s^2\bar{f}(s) - s f(0) - f'(0)$$

$$\Rightarrow \left[s^2\overline{y}(s)-s\,y(0)-y'(0)\right]-3\left[s\overline{y}(s)-y(0)\right]+2\overline{y}(s) = \frac{1}{s}-\frac{1}{s-2}$$

$$\left[s^2\overline{y}(s)-s(1)-(1)\right]-3\left[s\overline{y}(s)-(1)\right]+2\overline{y}(s) = \frac{1}{s}-\frac{1}{s-2}$$

$$s^2\overline{y}(s)-s-1-3s\overline{y}(s)+3+2\overline{y}(s) = \frac{1}{s}-\frac{1}{s-2}$$

$$\overline{y}(s)\left[s^2-3s+2\right] = \frac{1}{s}-\frac{1}{s-2}+s-2$$

$$y(0)=1,\ y'(0)=1$$

$$\overline{y}(s)\left[(s-1)(s-2)\right] = \frac{s-2-s+s(s-2)^2}{s(s-2)}$$

$$\overline{y}(s) = \frac{-2+s\left(s^2+4-2s\right)}{s(s-1)(s-2)^2}=\frac{s^3-2s^2+4s-2}{s(s-1)(s-2)^2}$$

we have, $y(t) = L^{-1}\left[\bar{y}(s)\right]$

$$y(t) = L^{-1}\left[\frac{s^3 - 2s^2 + 4s - 2}{s(s-1)(s-2)^2}\right]$$

$$y(t) = L^{-1}\left[\frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-2)} + \frac{D}{(s-2)^2}\right]$$

$$s^3 - 2s^2 + 4s - 2 = A(s-1)(s-2)^2 + Bs(s-2)^2 + Cs(s-1)(s-2) + Ds(s-1)$$

Put $s = 0 \Rightarrow -2 = A(-1)(4) \Rightarrow A = \dfrac{1}{2}$

Put $s = 1 \Rightarrow 1 = B(1)(1) \quad \Rightarrow B = 1$

Put $s = 2 \Rightarrow 6 = D(2)(1) \quad \Rightarrow D = 3$

Put $s = -1 \Rightarrow -9 = -18A - 9B - 6C + 2D \Rightarrow C = \dfrac{5}{2}$

$$\therefore \qquad y(t) = L^{-1}\left[\frac{1/2}{s} + \frac{1}{(s-1)} + \frac{5/2}{(s-2)} + \frac{3}{(s-2)^2}\right]$$

$$\Rightarrow \qquad y(t) = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s-1}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-2}\right) + 3L^{-1}\left(\frac{1}{(s-2)^2}\right)$$

$$\Rightarrow \qquad y(t) = \frac{1}{2}(1) + e^{t} + \frac{5}{2}e^{2t} + 3\left(te^{2t}\right)$$

$$\Rightarrow \qquad y(t) = \frac{1}{2} + e^{t} + \frac{5}{2}e^{2t} + 3te^{2t}$$

Solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t,\ y(0)=0,\ y'(0)=0$ by using Laplace transforms.

(VTU 2004)

Solution:

Given, $y''+2y'-3y = \sin t$

Taking Laplace Transform on both side

$$L[y'']+2L[y']-3L[y] = L[\sin t]$$

$L[f'(t)]=s\overline{f}(s)-f(0)$

$L[f''(t)]=s^2\overline{f}(s)-s\,f(0)-f'(0)$

$$\Rightarrow \left[s^2\overline{y}(s)-s\,y(0)-y'(0)\right]+2\left[s\overline{y}(s)-y(0)\right]-3\overline{y}(s) = \frac{1}{s^2+1}$$

$$\left[s^2\overline{y}(s)-s(0)-(0)\right]+2\left[s\overline{y}(s)-(0)\right]-3\overline{y}(s) = \frac{1}{s^2+1}$$

$y(0)=0,\ y'(0)=0$

$$s^2\overline{y}(s)+2s\overline{y}(s)-3\overline{y}(s) = \frac{1}{s^2+1}$$

$$\bar{y}(s)\left[s^2+2s-3\right] = \frac{1}{s^2+1}$$

$$\bar{y}(s)\left[(s-1)(s+3)\right] = \frac{1}{s^2+1}$$

$$\bar{y}(s) = \frac{1}{(s^2+1)(s-1)(s+3)}$$

we have, $y(t) = L^{-1}\left[\bar{y}(s)\right]$

$$y(t) = L^{-1}\left[\frac{1}{(s^2+1)(s-1)(s+3)}\right]$$

$$y(t) = L^{-1}\left[\frac{As+B}{s^2+1}+\frac{C}{(s-1)}+\frac{D}{(s+3)}\right]$$

$$1 = (As+B)(s-1)(s+3) + C(s^2+1)(s+3) + D(s^2+1)(s-1)$$

Put $s = 1 \Rightarrow 1 = C(2)(4) \Rightarrow C = \frac{1}{8}$

Put $s = -3 \Rightarrow 1 = D(10)(-4) \quad \Rightarrow D = -\frac{1}{40}$

Put $s = 0 \Rightarrow 1 = -3B + 3C - D \quad \Rightarrow B = -\frac{1}{5}$

Put $s = -1 \Rightarrow 1 = 4A - 4B + 4C - 4D \Rightarrow A = -\frac{1}{5}$

$$\therefore \qquad y(t) = L^{-1}\left[\frac{(-1/5)s + (-1/5)}{s^2+1} + \frac{1/8}{(s-1)} + \frac{-1/40}{(s+3)}\right]$$

$$\Rightarrow \qquad y(t) = -\frac{1}{5}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{5}L^{-1}\left(\frac{1}{s^2+1}\right) + \frac{1}{8}L^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{40}L^{-1}\left(\frac{1}{s+3}\right)$$

$$\Rightarrow \qquad y(t) = -\frac{1}{5}\cos t - \frac{1}{5}\sin t + \frac{1}{8}e^t - \frac{1}{40}e^{-3t}$$

Solve $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4t + e^{3t}$, $y(0) = 1$, $y'(0) = -1$ by using Laplace transforms.

(VTU 2013)

Solution:

Given, $$y'' - 3y' + 2y = 4t + e^{3t}$$

Taking Laplace Transform on both side

$$L[y''] - 3L[y'] + 2L[y] = 4L[t] + L\left[e^{3t}\right]$$

$$\Rightarrow \left[s^2\overline{y}(s) - s\,y(0) - y'(0)\right] - 3\left[s\overline{y}(s) - y(0)\right] + 2\overline{y}(s) = 4\left(\frac{1}{s^2}\right) + \frac{1}{s-3}$$

$$\Rightarrow \left[s^2\overline{y}(s) - s(1) - (-1)\right] - 3\left[s\overline{y}(s) - (1)\right] + 2\overline{y}(s) = \frac{4}{s^2} + \frac{1}{s-3}$$

$$\Rightarrow s^2\overline{y}(s) - s + 1 - 3s\overline{y}(s) + 3 + 2\overline{y}(s) = \frac{4}{s^2} + \frac{1}{s-3}$$

$$\bar{y}(s)\left[s^2 - 3s + 2\right] = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$$

$$\bar{y}(s)\left[(s-1)(s-2)\right] = \frac{4(s-3) + s^2 + (s-4)s^2(s-3)}{s^2(s-3)}$$

$$\bar{y}(s) = \frac{4s - 12 + s^2 + s^4 - 7s^3 + 12s^2}{s^2(s-3)(s-1)(s-2)} = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s-1)(s-2)}$$

we have, $y(t) = L^{-1}\left[\bar{y}(s)\right]$

$$y(t) = L^{-1}\left[\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)}\right]$$

$$y(t) = L^{-1}\left[\frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D}{(s-2)} + \frac{E}{(s-3)}\right]$$

Now, $s^4 - 7s^3 + 13s^2 + 4s - 12 = As(s-1)(s-2)(s-3) + B(s-1)(s-2)(s-3)$
$+ Cs^2(s-2)(s-3) + Ds^2(s-1)(s-3) + Es^2(s-1)(s-2)$

Put $s = 0 \Rightarrow -12 = -6B \Rightarrow B = 2$

Put $s = 1 \Rightarrow -1 = C(2) \quad \Rightarrow C = -\frac{1}{2}$

Put $s = 2 \Rightarrow 8 = D(-4) \quad \Rightarrow D = -2$

Put $s = 3 \Rightarrow 9 = E(18) \quad \Rightarrow E = \frac{1}{2}$

Put $s = 4 \Rightarrow 20 = 24A + 6B + 32C + 48D + 96E$

$$\Rightarrow 20 = 24A + 6(2) + 32\left(-\frac{1}{2}\right) + 48(-2) + 96\left(\frac{1}{2}\right) \Rightarrow 72 = 24A \Rightarrow A = 3$$

$$\therefore \quad y(t) = L^{-1}\left[\frac{3}{s} + \frac{2}{s^2} + \frac{-1/2}{(s-1)} + \frac{-2}{(s-2)} + \frac{1/2}{(s-3)}\right]$$

$$\Rightarrow \quad y(t) = 3L^{-1}\left(\frac{1}{s}\right) + 2L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - 2L^{-1}\left(\frac{1}{s-2}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$\Rightarrow \quad y(t) = 3(1) + 2(t) - \frac{1}{2}\left(e^{t}\right) - 2\left(e^{2t}\right) + \frac{1}{2}\left(e^{3t}\right)$$

$$\Rightarrow \quad y(t) = 3 + 2t - \frac{1}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$$

$$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$