2.2 The rules of sum and product

The sum rule: Suppose that k tasks $T_1, T_2, T_3, \ldots, T_k$ are to be performed such that no two tasks can be performed at the same time. If T_i can be performed in n_i different ways, then one of the k tasks can be performed in $n_1 + n_2 + \cdots + n_k$ different ways.

Example: Suppose there are 16 boys and 18 girls in a class and if we select one of these students as the class representative. The no. of ways of selecting a boy = 16, The no. of ways of selecting a girl = 18. By sum rule, the no. of ways of selecting a student= 16+18

The product rule: Suppose that k tasks $T_1, T_2, T_3, \ldots, T_k$ are to be performed in a sequence. If T_i can be performed in n_i different ways then the sequence of tasks $T_1, T_2, T_3, \ldots, T_k$ can be performed in $n_1 n_2 \ldots n_k$ different ways.

Problems

1. There are 20 married couple in a party. Find the number of ways of choosing one woman and one man from the party such that the two are not married to each other. One woman can be selectedin 20 ways.

After neglectingher husbandthereare 19 men remaining.

One man can be selectedin 19 ways.

By productrule,required number $= 20 \times 19 = 380$.

2.A license plate consists of two English letters followed by four digits. If repetitions are allowed, how many of the plates have only vowels and even integers ? Vowels are a, e, i, o, u totally5 and even integers are 0,2,4,6,8 totally5.

Each of the first two positionscan be filled in 5 ways.

Each of the next four positionscan be filled in 5 ways.

By productrule,required number $= 5^2 \times 5^4 = 5^6 = 15{,}625$

3. Cars of a particular manufacturer come in 4 models, 12 colours, 3 Engine sizes and two transition types. How many distinct cars can be manufactured? Of these how many have the same colour?

Number of distinct cars that can be manufactured $= 4 \times 12 \times 3 \times 2 = 288$.

Number of distinct cars with the same colourthatcan be manufactured $= 4 \times 3 \times 2 = 24$.

4. How many 3 digit numbers can be formed by using the 6 digits 2, 3, 4, 5, 6, 8 if the number is to be even and repetitionsare not allowed.

Since the number is even, unit place can be filled by 2 or 4 or 6 or 8, totally4 ways.

Tenth place can be filled in$6 - 1 = 5$ ways.

Hundredthplacecan be filled in $6 - 2 = 4$ ways.

By productrule,the requirednumber $= 4 \times 5 \times 4 = 80$.

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5. A bit is either 0 or 1. A byte is a sequence of 8 bits. Find (i) The no. of bytes. (ii) The no. of bytes that begin with 11 and end with11.

(i) Each bit can be filled in 2 ways, either 0 or 1.

Each byte contains8 bits.

By productrule,the requirednumber $= 2 \times 2 \times ... 8$ times $= 2\text{^}8 = 256$

(ii) In a byte beginning and endingwith 11, there are 4 open positionsto fill.

By productrule,the requirednumber $= 2 \times 2 \times 2 \times 2 = 2^4 = 16$.

1	1					1	1

(iii) **The no. of bytes that begin with 11 and do not end with11. (iv) The no.of bytes that begin with 11 or end with11.**

(iii) In a byte beginning with 11, there are 6 positions to fill.
By product rule, this can be done in $2^6 = 64$ ways.

Therefore, the no. of bytes that begin with 11 and do not end with11 $=$ No. of bytes beginning with 11$-$No. of bytes beginning and ending with 11

$= 64 - 16 = 48.$

(iv) No. of bytes beginning with 11 or ending with 11

$=$ No. of bytes beginning with 11$+$No. of bytes ending with 11
$-$ No. of bytes beginning and ending with 11

$= 64 + 64 - 16 = 112.$

6. Suppose that a valid computer password consists of 7 characters, the first of which is one of the letters A, B, C, D, E, F, G and the remaining 6 characters are letters chosen from the Englishalphabetor a digit. How many differentpasswordsare possible?

First charactercan be chosen in 7 ways.

Each of the remaining 6 characterscan be chosen in $26 + 10 = 36$ ways.

By productrule,required number $= 7 \times 36^6$

7. Find the total no. of positive integers that can be formed from the digits 1, 2, 3, 4 if no digit is repeatedin any one integer.

By productrule,

Number of integers containingone digit $= 4$

Number of integers containingtwo digits $= 4 \times 3 = 12$

Number of integers containingthreedigits $= 4 \times 3 \times 2 = 24$

Number of integers containingfour digits $= 4 \times 3 \times 2 \times 1 = 24$

By sum rule,

Total number of integers $= 4 + 12 + 24 + 24 = 64$.

8 Find the no. of proper divisors of 441000.

$441000 = 2^3 3^2 5^3 7^2$

Every divisor is of the form $d = 2^p 3^q 5^r 7^s$

p, q, r and s can be selected in 4,3,4,3 ways respectively.

By product rule, n umber of divisors $= 4 \times 3 \times 4 \times 3 = 144.$

Out of which, 2 of them are improper.

Therefore, the total number of proper divisors $= 142.$

PRACTICE WORK

1. Find the total number of positive integers that can be formed from the digits 1, 2, [1], [23] if no digit is repeatedin any one integer. <u>Answer</u>: 64

2. A sports committee of 3 in a college is to be formed consisting of one representative each from boy students, girl students and teachers. if there are 3 possible representatives from boy students, 2 from girl students and 4 from teachers, determine how many different committees can be formed. Answer: $3 \times 2 \times 4 = 24$

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[1] . A label identifier for a computer programme consists of one letter of the Englishalphabet followed by 2 digits. If repetitions are allowed, how many distinct label identifiers are possible? Answer: 260

[2] . Find the number of 3 digit even numbers with no repeated digits.

Answer: $[(\quad) \quad]1 \times 9 \; (\quad) \; (\quad)+ \; 4 \times 8[\quad (\quad)] \times 8 = 328$

[3] . How many among the first 100,000 positive integers contain exactly one 3, one 4 and one 5in their decimal representations? Answer: $5 \times 4 \times 3 \times 7 \times 7 = 2940$

Permutations

- A permutation is the number of possible arrangements in a set when the order of the arrangements matters.

- The number of permutations of n distinct objects is $n!$ (Taken all at a time)
- The number of circular permutations of n distinct objects is $(n-1)!$

- The number of permutations of size r of n distinct objects is $\frac{n!}{(n-r)!}$

✞The number of permutations of n objects of which n_1 are of the first type and n_2 are of the second type is $\frac{n!}{n_1!n_2!}$

Problems:

1. In how many ways can 6 men and 6 women be seated in a row (i) if any person may sit next to any other? (ii) If men and women must occupy alternative seats?

(i)If there is no restriction, 12 persons in a row can sit in 12! Ways.

(ii) 6 men in odd places and 6 women in even places can be seated in $6! \times 6!$ ways.

6 men in even places and 6 women in odd places can be seated in $6! \times 6!$ ways.

Therefore, total number of arrangements $= 2 \times 6! \times 6!$

2. In how many ways can three men and three women be seated at a round table if

(i) No restriction is imposed?

(ii) Two particular women must not sit together?

(iii) Each women is to be between two men?

(i) If no restriction imposed, 6 persons can be seated in a round table in $(6-1)! = 5! = 120$ ways.

(ii)Two women can sit together in 2 ways. Consider this as 1 unit.

One unit and 4 remaining persons can sit in $(5-1)! = 4! = 24$ ways.

Therefore, If two women can sit together, total no. of arrangements is $2 \times 24 = 48$

If two women can't sit together, total no. of arrangements is $120 - 48 = 72$

(iii) Three men can be seated in $(3-1)! = 2!$ ways by leaving one seat between them.

Three women can be seated in the remaining 3 seats in $3!$ ways.

Therefore, total number of arrangements is $2! \times 3! = 12$.

3. A student has three books on C++ and four books on Java. In how many ways can he arrange three books on a shelf (i) If there are no restrictions? (ii) If the languages should alternate? (iii) If all the C++ books must be next to each other? (iv) If all the C++ books must be next to each other and all the Java books must be next to each other?

- If there are no restrictions, three books on C++ and four books on Java, totally 7 books can be arranged in $7!$ ways.

- Three C++ books in even places and four Java books in odd places can be arranged in

  $3! \times 4! = 144$ ways.

- Three C++ books together can be arranged in 3! Ways. Consider this as one unit. Now one unit and four Java books can be arranged in 5! ways.

  Therefore, total number of arrangements is $3! \times 5! = 720$.

- Three C++ books can be arranged in 3! Ways. Consider this as one unit.

  Four Java books can be arranged in 4! Ways. Consider this as one unit. Two units can be arranged in 2 ways.

  Therefore, total number of arrangements is $2 \times 3! \times 4!$.

4. Find the number of permutations of the letters of the word MASSASAUGA. In how many of these, all four A's together? How many of them begin with S?

(i)In 10 alphabets, 'S' repeated three times and 'A' repeated 4 times. Therefore,

total no. of permutations == 25,200 $\frac{10!}{3!\times 4!}$

(ii) Consider four A's together as one unit. Consider the remaining 6 letters as 6 units. Now, we have 7 units.Out of 7 units, 'S' repeated three times.

Therefore, total no. of permutations = $\frac{7!}{3!} = 840$

(iii)First alphabet is fixed as S. Now, 9 alphabets remaining. In 9 alphabets, 'S' repeated twice and 'A' repeated 4 times. Therefore, total no. of permutations == $\frac{9!}{2!\times 4!}$ = 7560

5. (i) How many arrangements are there for all letters in the word SOCIOLOGICAL? In how many of these arrangements (ii) A and G are adjacent? (iii) All the vowels are adjacent ?

(i) In 12 letters, 'O' repeated thrice and 'C', 'I', 'L' repeated twice each.

Therefore, total number of $\frac{12!}{3!\times 2!\times 2!\times 2!}$ arrangements== 99,79,200

(ii) A and G together can be arranged in 2 ways. Consider this as one unit.Consider the remaining 10 letters as 10 units. Now we have 11 units

In 11 units, ‘O’ repeated thrice and ‘C’, ‘I’, ‘L’ repeated twice each.

Therefore, total number of arrangements= 2 ×= $\frac{11!}{3!\times2!\times2!\times2}$ 16,63,200

!

(iii) Two I’s and three O’s together can be arranged $\frac{5!}{2!\times3}$ in = 10 ways.

!

Consider this as a single unit. Consider the remaining 6 letters as 6 units.

In 7 units, ‘C’ and ‘L’ repeated twice each.

Therefore, total number of arrangements= 10 $\frac{7!}{2!\times2}$

×

!

6. (i) Find the number of permutations of the letters of the word MISSISSIPPI

(ii) How many of these begin with I?

(iii) How many of these begin and end with an S?

(i)In 11 letters, ‘S’ and ‘I’ repeated 4 times each and ‘P’ repeated twice.

Therefore, total no. of permutations $\frac{11!}{4!\times 4!\times 2}$ == 34,650

!

(ii) First letter is fixed as I. Now,10 letters remaining. In 10 letters, ‘S’ repeated four times, ‘I’ repeated thrice and ‘P’ repeated twice.

Therefore, total no. of permutations $\frac{10!}{4!\times 3!\times 2}$ == 12.600

!

(iii) Starting and ending letters are fixed as ‘S’. Now, 9 letters remaining. In 9 letters, ‘S’ and ‘P’ repeated twice each and ‘I’ repeated four times.

Therefore, total no. of permutations $\frac{9!}{2!\times 2!\times 4}$ == 3780

!

7. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

We have 7 digits, out of which there are two 4’s and two 5’s.

Let $n = x_1x_2x_3x_4x_5x_6x_7$. x_1 must be 5 or 6 or 7.

Suppose $x_1 = 5$, remaining 6 digits can be arranged in $\frac{6!}{2!} = 360$ ways.

Suppose $x_1 = 6$, remaining 6 digits can be arranged in $\frac{6!}{2!\times 2\,!} = 180$ ways.

Suppose $x_1 = 7$, remaining 6 digits can be arranged in $\frac{6!}{2!\times 2\,!} = 180$ ways.

Therefore total no. of arrangements$= 360 + 180 + 180 = 720$.

8. How many different three-digit numbers can be formed with 3 four's, 4 two's and 2 three's?

We have 9 digits, out of which there are three 4's, four 2's and two 3's. let $n = x_1x_2x_3$

Suppose $x_1 = 3, x_2 \neq 3$, remaining 2 digits can be arranged in $2 \times 3 = 6$ ways.

Suppose $x_1 = 3, x_2 = 3$, remaining digit can be arranged in 2 ways.

Suppose $x_1 = 4$, remaining 2 digits can be arranged in $3 \times 3 = 9$ ways.

Suppose $x_1 = 2$, remaining 2 digits can be arranged in $3 \times 3 = 9$ ways.

Therefore, total no. of arrangements $= 6 + 2 + 9 + 9 = 26$

PRACTICE WORK

1.How many 8 digit numbers have one or more repeated digits? Answer: $10^8 - \binom{10}{8}$

2.How many different strings (sequences) of length four can be formed using the letters of the word FLOWER? Answer: $\binom{6}{4}$

3.How many nine letter words can be formed by using the letters of the word DIFFICULT?

$\frac{9!}{2!\times 2!}$ Answer:

4. Find the number of permutations of all letters of the word BASEBALL if the words are begin and end with a vowel? Answer: 540

5.How many four digit numbers can be formed with the 10 digits 0,1,2,3,4,5,6,7,8,9

a) If repetitions are allowed?

b) Repetitions are not allowed? c) The last digit must be zero and repetitions are not allowed?

Answer: (a) 9000 (b) 4536 (c) 504

6. In how many ways can 7 books be arranged on a shelf if

(a) Any arrangement is allowed

(b) Three particular books must always be together?

(c) Two particular books must occupy the ends? Answer: (a) 5040 (b) 720 (c) 240