Module-3
Part – 4
Exponential Distribution

Exponential Distribution

The continuous probability distribution having the probability density function ***f*** (***x***) given by

$$f(x)=\begin{cases}\alpha\, e^{-\alpha x}, & 0\le x<\infty\\ 0, & \text{otherwise}\end{cases}$$

where $\alpha > 0$, is known as the exponential distribution.

Mean and variance of Exponential distribution

The mean and variance of the binomial distribution are given by

Mean $\mu = \dfrac{1}{\alpha}$

Variance $V = \dfrac{1}{\alpha^2}$

Standard deviation $\sigma = \sqrt{V} = \dfrac{1}{\alpha}$

Note: Since the function $f(x)$ is zero for $-\infty < x < 0$, the probabilities of the exponential distribution for various cases are as follows:

(i) $P(0 \le x < a) = \int_0^a f(x)\,dx, \quad a > 0$

(ii) $P(x \ge a) = 1 - P(x < a) = 1 - \int_0^a f(x)\,dx, \quad a > 0$

Example **If x is an exponential variate with mean 5, evaluate the following:**

(i) $P(0<x<1)$, **(ii)** $P(x>2)$,

(iii) $P(-\infty<x<10)$ **(iv)** $P(x\le 0 \text{ or } x\ge 1)$

The p.d.f of the exponential distribution is $f(x)=\begin{cases}\alpha e^{-\alpha x}, & 0\le x<\infty\\ 0, & \text{otherwise}\end{cases}$

Given, Mean = 5 i.e., Mean $\mu=\frac{1}{\alpha}=5$

$$\Rightarrow \qquad \alpha=\frac{1}{5}$$

(i) $P(0<x<1)=\int_0^1 f(x)\,dx=\int_0^1 \alpha e^{-\alpha x}\,dx$

$$=\int_0^1\left(\frac{1}{5}\right)e^{-\frac{1}{5}x}\,dx=\frac{1}{5}\left[\frac{e^{-\frac{1}{5}x}}{-\frac{1}{5}}\right]_0^1=-\left[e^{-0.2}-e^{0}\right]=0.1813$$

(ii) $$P(x>2)=1-P(x\le 2)=1-\int_0^2 f(x)\,dx$$

$$=1-\int_0^2 \alpha e^{-\alpha x}\,dx$$

$$=1-\int_0^2 \left(\frac{1}{5}\right) e^{-\frac{1}{5}x}\,dx$$

$$=1-\frac{1}{5}\left[\frac{e^{-\frac{1}{5}x}}{-\frac{1}{5}}\right]_0^2$$

$$=1+\left[e^{-2/5}-e^{0}\right]:=0.6703$$

(iii) $P(-\infty < x < 10) = P(-\infty < x < 0) + P(0 \le x < 10)$

$$= 0 + \int_0^{10} f(x)\,dx$$

$$= \int_0^{10} \alpha\, e^{-\alpha x}\, dx$$

$$= \int_0^{10} \left(\frac{1}{5}\right) e^{-\frac{1}{5}x}\, dx \quad = \frac{1}{5}\left[\frac{e^{-\frac{1}{5}x}}{-\frac{1}{5}}\right]_0^{10}$$

$$= -\left[e^{-2} - e^{0}\right]$$

$$= 0.8647$$

(iv) $$P(x \le 0 \text{ or } x \ge 1) = \int_1^\infty f(x)\,dx$$

$$= \int_1^\infty \alpha\, e^{-\alpha x}\, dx$$

$$= \int_1^\infty \left(\frac{1}{5}\right) e^{-\frac{1}{5}x}\, dx \quad = \frac{1}{5}\left[\frac{e^{-\frac{1}{5}x}}{-\frac{1}{5}}\right]_1^\infty$$

$$= -\left[0 - e^{-0.2}\right]$$

$$= 0.8187$$

Example **The length of a telephone conversation has an exponential distribution with a mean of 3 minutes. Find the probability that a call (i) ends in less than 3 minutes, (ii) takes between 3 and 5 minutes.**

The p.d.f of the exponential distribution is $f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 \le x < \infty \\ 0, & \text{otherwise} \end{cases}$

Given, Mean = 3 i.e., Mean $\mu = \frac{1}{\alpha} = 3$

$$\Rightarrow \qquad \alpha = \frac{1}{3}$$

(i) P (less than 3 minutes) $= P(x < 3)$

$$= P(-\infty < x < 3)$$

$$= P(-\infty < x < 0) + P(0 \le x < 3)$$

$$= 0 + \int_0^3 f(x)\,dx$$

$$= \int_0^3 \alpha\, e^{-\alpha x}\, dx \quad = \int_0^3 \left(\frac{1}{3}\right) e^{-\frac{1}{3}x}\, dx$$

$$= \frac{1}{3}\left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}}\right]_0^3$$

$$= -\left[e^{-1} - e^{0}\right]$$

$$= 0.6321$$

(ii) P (between 3 and 5 minutes) $= P(3 < x < 5)$

$$= \int_3^5 f(x)\,dx \quad = \int_3^5 \alpha e^{-\alpha x}\,dx$$

$$= \int_3^5 \left(\frac{1}{3}\right) e^{-\frac{1}{3}x}\,dx$$

$$= \frac{1}{3}\left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}}\right]_3^5$$

$$= -\left[e^{-\frac{5}{3}} - e^{-1}\right] \quad = 0.179$$

Example **In a certain town, the duration of a shower is exponentially distributed with mean 5 minutes, what is the probability that a shower will last for (i) less than 10 minutes, (ii) 10 minutes or more?**

The p.d.f of the exponential distribution is $f(x)=\begin{cases}\alpha e^{-\alpha x}, & 0\le x<\infty \\ 0, & \text{otherwise}\end{cases}$

Given, Mean = 5 i.e., Mean $\mu=\dfrac{1}{\alpha}=5$

$$\Rightarrow \quad \alpha=\frac{1}{5}$$

(i) P (less than 10 minutes) $=P(x<10)$

$$=P(-\infty<x<10)=P(-\infty<x<0)+P(0\le x<10)$$

$$=0+\int_0^{10} f(x)\,dx$$

$$= \int_0^{10} \alpha\, e^{-\alpha x}\, dx \quad = \int_0^{10} \left(\frac{1}{5}\right) e^{-\frac{1}{5}x}\, dx$$

$$= \frac{1}{5}\left[\frac{e^{-\frac{1}{5}x}}{-\frac{1}{5}}\right]_0^{10} = -\left[e^{-2} - e^{0}\right] = 0.8647$$

(ii) P (10 minutes or more) $= P(x \geq 10)$

$$= 1 - P(x < 10)$$

$$= 1 - 0.8647 = 0.1353$$

Example 3.5.4: The sale per day in a shop is exponentially distributed with the average sales amounting to Rs.100 and net profit is 8%. Find the probability that the net profit exceeds Rs.30 on two consecutive days.

The p.d.f of the exponential distribution is $f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 \le x < \infty \\ 0, & \text{otherwise} \end{cases}$

Given, Mean = 100 i.e., Mean $\mu = \dfrac{1}{\alpha} = 100$

$$\Rightarrow \qquad \alpha = \frac{1}{100}$$

Let A be the amount for which profit is 8%

Given, profit = Rs.30

$\therefore$ 8% of A = 30

$$\Rightarrow \qquad A = \frac{30}{8\%} = \frac{30}{0.08} = 375$$

Now, P (profit exceeding Rs.30) = 1 − P (profit ≤ Rs.30)

= 1 − P (sales ≤ Rs.375)

$$=1-\int_{0}^{375} f(x)\,dx$$

$$=1-\int_{0}^{375} \alpha\, e^{-\alpha x}\, dx$$

$$=1-\int_{0}^{375} (0.01)e^{-0.01x}\, dx$$

$$= 1 - (0.01)\left[\frac{e^{-0.01x}}{-0.01}\right]_0^{375}$$

$$= 1 + \left[e^{-3.75} - e^{0}\right]$$

$$= 0.0235$$

EXERCISE

1. The length of a telephone conversation has an exponential distribution with a mean of 5 minutes. Find the probability that a call (i) ends in less than 5 minutes, (ii) takes between 5 and 10 minutes. **(VTU 2019)**
2. The average lifetime of a car is 15 years and it is exponentially decreases. If you buy a 10 years old car, what is the probability that it is in service after 10 years of purchase from your side.
3. The mileage (in thousands of kilometres) which car owners get with a certain kind of radial tyre is a random variable having an exponential distribution with mean 40. Find the probabilities that one of these tyres will last (i) at least 20,000 kms, (ii) at most 30,000 kms.
4. The sale per day in a shop is exponentially distributed with mean is Rs.100. If sales tax is levied at the rate of 8%, what is the probability that the sales tax return from that shop will not exceed Rs.60 per day?