Module-3
Part - 2

Continuous Probability Distribution

For every continuous random variable X, the real number $f(x)$ is said to be continuous probability function or probability density function (p.d.f) if the following conditions are satisfied:

(i) $f(x) \geq 0$ and

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Note: The probability of x lies in the interval (a, b) is defined as

$$P(a \leq x \leq b) = \int_a^b f(x)dx$$

Note: If X is a continuous random variable with p.d.f $f(x)$ then the function $f(x)$ is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

Mean and variance of continuous probability distribution

If X is a continuous random variable with probability density function $f(x)$ where $-\infty < x < \infty$ then the mean, variance and standard deviation of X are given by

Mean $\mu = \int_{-\infty}^{\infty} x f(x) dx$

Variance $V = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$ or $V = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$

Standard deviation $\sigma = \sqrt{V}$

Example **Find the constant C such that**

$$f(x)=\begin{cases} Cx^2, & 0\le x\le 3 \\ 0, & \text{otherwise} \end{cases}$$

is a probability density function. Also compute $P(1 < X < 2)$.

$f(x)$ is (p.d.f) if

(i) $f(x)\ge 0$ and (ii) $\int_{-\infty}^{\infty} f(x)\,dx=1$

From (ii), $\int_{-\infty}^{\infty} f(x)\,dx=1$

$\Rightarrow \int_{0}^{3} Cx^2\,dx=1$

$$\Rightarrow \qquad \int_0^3 Cx^2\, dx = 1$$

$$\Rightarrow \qquad C\left[\frac{x^3}{3}\right]_0^3 = 1$$

$$\Rightarrow \qquad C\left[\frac{(3)^3 - 0}{3}\right] = 1$$

$$\Rightarrow \qquad C[9] = 1 \qquad \Rightarrow \quad C = \frac{1}{9}$$

We have, $$P(a \le x \le b) = \int_a^b f(x)\,dx$$

$$\Rightarrow \qquad P(1 < x < 2) = \int_1^2 f(x)\,dx$$

$\Rightarrow$

$$P(1 < x < 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{1}{9} x^2\,dx$$

$$= \frac{1}{9}\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{9}\left[\frac{(2)^3 - (1)^3}{3}\right] = \frac{7}{27}$$

Example **A random variable X has the density function:**

$$f(x) = \begin{cases} kx^2, & -3 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Find k and also find $P(X \le 2)$ and $P(X > 1)$.

$f(x)$ is (p.d.f) if

(i) $f(x) \ge 0$ and (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

From (ii), $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow$ $\int_{-3}^{3} kx^2\,dx = 1$

$$\Rightarrow \quad \int_{-3}^{3} kx^2\,dx = 1$$

$$\Rightarrow \quad k\left[\frac{x^3}{3}\right]_{-3}^{3} = 1$$

$$\Rightarrow \quad k\left[\frac{(3)^3-(-3)^3}{3}\right] = 1$$

$$\Rightarrow \quad k\left[9+9\right] = 1$$

$$\Rightarrow \quad k = \frac{1}{18}$$

We have, $$P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$$\Rightarrow \quad P(X \le 2) = \int_{-\infty}^{2} f(x)\,dx$$

$$= \int_{-3}^{2} \frac{1}{18} x^2\,dx$$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{2}$$

$$= \frac{1}{18}\left[\frac{(2)^3 - (-3)^3}{3}\right]$$

$$= \frac{35}{54}$$

We have, $$P(X > x) = \int_x^{\infty} f(x)\,dx$$

$$\Rightarrow \quad P(X > 1) = \int_1^{\infty} f(x)\,dx$$

$$= \int_1^{3} \frac{1}{18} x^2\,dx$$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_1^3$$

$$= \frac{1}{18}\left[\frac{(3)^3 - (1)^3}{3}\right] = \frac{13}{27}$$

Example 3.2.13: A random variable X has the density function:

$$f(x)=\begin{cases} k(x+1), & -1<x<1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) k, (ii) mean and variance of X.

$f(x)$ is (p.d.f) if

(i) $f(x)\geq 0$ and (ii) $\int_{-\infty}^{\infty} f(x)\,dx=1$

From (ii), $\int_{-\infty}^{\infty} f(x)\,dx=1$

$\Rightarrow$ $\int_{-1}^{1} k(x+1)\,dx=1$

$$\Rightarrow \quad \int_{-1}^{1} k(x+1)\,dx = 1$$

$$\Rightarrow \quad k\left[\frac{x^2}{2}+x\right]_{-1}^{1} = 1$$

$$\Rightarrow \quad k\left[\left(\frac{(1)^2}{2}+1\right)-\left(\frac{(-1)^2}{2}-1\right)\right]=1$$

$$\Rightarrow \quad 2k = 1$$

$$\Rightarrow \quad k = \frac{1}{2}$$

We have, Mean $\mu = \int_{-1}^{1} x f(x) dx$

$$= \int_{-1}^{1} x \left[\frac{1}{2} (x+1) \right] dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(x^2 + x \right) dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\left(\frac{1}{3} + \frac{1}{2} \right) - \left(\frac{-1}{3} + \frac{1}{2} \right) \right] = \frac{1}{3}$$

We have, Variance $V=\int_{-1}^{1}(x-\mu)^2 f(x)\,dx$

$$=\int_{-1}^{1}\left(x-\frac{1}{3}\right)^2\left[\frac{1}{2}(x+1)\right]dx$$

$$=\frac{1}{2}\int_{-1}^{1}\left(x^2-\frac{2x}{3}+\frac{1}{9}\right)(x+1)\,dx$$

$$=\frac{1}{2}\int_{-1}^{1}\left(x^3+\frac{x^2}{3}-\frac{5x}{9}+\frac{1}{9}\right)dx$$

$$=\frac{1}{2}\left[\frac{x^4}{4}+\frac{x^3}{9}-\frac{5x^2}{18}+\frac{1}{9}x\right]_{-1}^{1}$$

$$=\frac{1}{2}\left[\left(\frac{1}{4}+\frac{1}{9}-\frac{5}{18}+\frac{1}{9}\right)-\left(\frac{1}{4}-\frac{1}{9}-\frac{5}{18}-\frac{1}{9}\right)\right] \quad =\frac{2}{9}$$

Example : The function $f(x)$ is defined as

$$f(x)=\begin{cases} e^{-x}, & x \geq 0 \\ 0, & x<0 \end{cases}.$$

Is $f(x)$ a probability density function? If so, determine the probability that the variate having this density will fall in the interval (1, 2). Also find the cumulative probability function F (2).

(i) Clearly, $f(x) \geq 0$ and

(ii) $$\int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{0}(0)dx+\int_{0}^{\infty} e^{-x}\,dx$$

$$= 0 + \int_0^\infty e^{-x}\,dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty = -\left[e^{-\infty} - e^{0}\right] = -(0-1) = 1$$

Hence the function $f(x)$ satisfies the requirements for a density function.

$\therefore$ $f(x)$ is a probability density function

We have, $P(a \le x \le b) = \int_a^b f(x)\,dx$

$\Rightarrow$ $P(1 < x < 2) = \int_1^2 f(x)\,dx$

$$\Rightarrow \qquad P(1 < x < 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^2 e^{-x}\,dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_1^2$$

$$= -\left(e^{-2} - e^{-1}\right) = 0.233$$

Also we have, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$

$$\Rightarrow \qquad F(2) = P(X \le 2) = \int_{-\infty}^{2} f(x)\,dx$$

$\Rightarrow$

$$F(2) = P(X \le 2) = \int_{-\infty}^{2} f(x)\,dx$$

$$= \int_{-\infty}^{0} (0)\,dx + \int_{0}^{2} e^{-x}\,dx$$

$$= 0 + \int_{0}^{2} e^{-x}\,dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_{0}^{2}$$

$$= -\left[e^{-2} - e^{0}\right] = 0.865$$

EXERCISE

1. If a function $f(x)$ defined by $f(x)=\begin{cases}\left(\dfrac{x+1}{8}\right), & 2<x<4\\ 0, & \text{otherwise}\end{cases}$

   (i) Prove that $f(x)$ is a p.d.f

   (ii) Find $P(X < 3.5)$ and $P(X \geq 3.5)$

2. Find the value of k such that $f(x)=\begin{cases}\dfrac{x}{6}+k, & 0\leq x\leq 3\\ 0, & \text{otherwise}\end{cases}$

   is a p.d.f. Also find $P(1 \leq X \leq 2)$.

3. Find the value of k such that $f(x)=\begin{cases}k\,x\,e^{-x}, & 0<x<1\\ 0, & \text{otherwise}\end{cases}$

   is a p.d.f. Also find its mean.

4. Find the value of k such that $f(x)=\begin{cases}k\,e^{-3x}, & x>0\\ 0, & \text{otherwise}\end{cases}$

   is a p.d.f. Also find $P(0.5 \leq X \leq 1)$ and $P(-2 \leq X \leq 1.5)$