Module-3
Part – 3
Binomial Distribution

Binomial Distribution

Trials of a random experiment are called **Bernoulli trials**, if they satisfy the following conditions:

(i) There should be a finite number of trials.

(ii) The trials should be independent.

(iii) Each trial has exactly two outcomes: **success or failure.**

(iv) The probability of success remains the same in each trial.

If p is the probability of success and q is the probability of failure then the probability distribution of number of successes in an experiment consisting of n Bernoulli trials may be obtained by the binomial expansion of $(q+p)^n$.

Hence, this distribution of number of successes $\boldsymbol{X}$ can be written as

X	0	1	2	…	r	…	n
$P(X)$	${}^nC_0\, q^n$	${}^nC_1\, q^{n-1} p$	${}^nC_2\, q^{n-2} p^2$	…	${}^nC_r\, q^{n-r} p^r$		${}^nC_n\, p^n$

The above probability distribution is known as binomial distribution with parameters n and p, because for given values of n and p, we can find the complete probability distribution. A binomial distribution with n-Bernoulli trials and probability of success in each trial as p, is denoted by $B(n, p)$.

If ***p*** **is the probability of success and** ***q*** **is the probability of failure** then the probability of ***r*** **successes out of** ***n*** **trails** is given by

$$P(X = r) = P(r) = {}^{n}C_{r}\, q^{n-r} p^{r} ,$$

$r = 0, 1, \ldots, n$ and $q = 1 - p$.

Mean and variance of Binomial distribution

The mean and variance of the binomial distribution are given by

Mean $\mu = np$

Variance $V = npq$

Standard deviation $\sigma = \sqrt{V} = \sqrt{npq}$

Example **The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that (i) exactly two will be defective, (ii) at least two will be defective, (iii) none will be defective.**

Here $n = 12$ pens are manufactured

Let p = Probability of defective = $\frac{1}{10}$

Hence, $q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

We have, $P(r) = {}^{n}C_r\, q^{n-r} p^r$

$$\Rightarrow \qquad P(r) = {}^{12}C_r \left(\frac{9}{10}\right)^{12-r} \left(\frac{1}{10}\right)^r$$

$$P(r) = {}^{12}C_r\left(\frac{9}{10}\right)^{12-r}\left(\frac{1}{10}\right)^r$$

(i) P (exactly two will be defective) $= P(r=2)$

$$= {}^{12}C_2\left(\frac{9}{10}\right)^{12-2}\left(\frac{1}{10}\right)^2 \quad = (66)\left(\frac{9}{10}\right)^{10}\left(\frac{1}{10}\right)^2$$

$$= 0.2301$$

(ii) P (at least two will be defective) $= P(r \geq 2)$

$$= 1 - P(r < 2)$$

$$= 1 - \{P(r=0) + P(r=1)\}$$

$$=1-\left\{{}^{12}C_0\left(\frac{9}{10}\right)^{12-0}\left(\frac{1}{10}\right)^0+{}^{12}C_1\left(\frac{9}{10}\right)^{12-1}\left(\frac{1}{10}\right)^1\right\}$$

$$=1-\left\{(1)\left(\frac{9}{10}\right)^{12}(1)+(12)\left(\frac{9}{10}\right)^{11}\left(\frac{1}{10}\right)\right\}$$

$$=1-\left(\frac{9}{10}\right)^{11}\left(\frac{21}{10}\right)=0.3412$$

(iii) P (none will be defective) $=P(r=0)$

$$={}^{12}C_0\left(\frac{9}{10}\right)^{12-0}\left(\frac{1}{10}\right)^0 \quad =(1)\left(\frac{9}{10}\right)^{12}(1)=0.2824$$

Example The number of telephone lines busy at an instant of time is a binomial variate with probability 0.2. If at an instant 10 lines are chosen at random, find the probability that (i) 5 lines are busy, (ii) at most 2 lines are busy, (iii) all lines are busy.

Here n = 10 lines are chosen

Let p = Probability of busy = 0.2

Hence, $q = 1 - p = 1 - 0.2 = 0.8$

We have, $P(r) = {}^{n}C_{r}\, q^{n-r} p^{r}$

$$\Rightarrow \quad P(r) = {}^{10}C_{r}\,(0.8)^{10-r}(0.2)^{r}$$

$$P(r) = {}^{10}C_r (0.8)^{10-r} (0.2)^r$$

(i) P (5 lines are busy) $= P(r=5)$

$$= {}^{10}C_5 (0.8)^{10-5} (0.2)^5 = (252)(0.8)^5 (0.2)^5$$

$$= 0.02642$$

(ii) P (at most 2 lines are busy) $= P(r \le 2)$

$$= P(r=0) + P(r=1) + P(r=2)$$

$$= {}^{10}C_0 (0.8)^{10-0} (0.2)^0 + {}^{10}C_1 (0.8)^{10-1} (0.2)^1 + {}^{10}C_2 (0.8)^{10-2} (0.2)^2$$

$$= 0.6778$$

$$P(r) = {}^{10}C_r (0.8)^{10-r} (0.2)^r$$

(iii) P (all lines are busy) $= P(r = 10)$

$$= {}^{10}C_{10} (0.8)^{10-10} (0.2)^{10}$$

$$= (1)(1)(0.2)^{10} = \frac{1}{5^{10}}$$

Example **The probability of a newly generated virus attacked to the computer will corrupt the 4 files out of 20 files opened in an hour. If 12 files are opened in an hour, find the probability that (i) at least 10 files are corrupted, (ii) exactly 3 files are corrupted, (iii) all the files are corrupted, (iv) all the files are safe, (v) more than 2 but not more than 5 files are corrupted.**

Here $n = 12$ files are opened in an hour

Let p = Probability of corrupted files = $\dfrac{4}{20} = \dfrac{1}{5}$

Hence, $q = 1 - p = 1 - \dfrac{1}{5} = \dfrac{4}{5}$

We have, $P(r) = {}^{n}C_{r}\, q^{n-r} p^{r}$

$$\Rightarrow \qquad P(r) = {}^{12}C_{r} \left(\frac{4}{5}\right)^{12-r} \left(\frac{1}{5}\right)^{r}$$

$$P(r) = {}^{12}C_r \left(\frac{4}{5}\right)^{12-r} \left(\frac{1}{5}\right)^{r}$$

(i) P (at least 10 files are corrupted) $= P(r \geq 10)$

$$= P(r=10) + P(r=11) + P(r=12)$$

$$= {}^{12}C_{10} \left(\frac{4}{5}\right)^{12-10} \left(\frac{1}{5}\right)^{10} + {}^{12}C_{11} \left(\frac{4}{5}\right)^{12-11} \left(\frac{1}{5}\right)^{11} + {}^{12}C_{12} \left(\frac{4}{5}\right)^{12-12} \left(\frac{1}{5}\right)^{12}$$

$= 0.000004526$

(ii) P (3 files are corrupted) $= P(r=3)$

$$= {}^{12}C_3 \left(\frac{4}{5}\right)^{12-3} \left(\frac{1}{5}\right)^{3} = 0.001074$$

$$P(r) = {}^{12}C_r \left(\frac{4}{5}\right)^{12-r} \left(\frac{1}{5}\right)^r$$

(iii) P (all files are corrupted) $= P(r = 12)$

$$= {}^{12}C_{12} \left(\frac{4}{5}\right)^{12-12} \left(\frac{1}{5}\right)^{12} \quad = \left(\frac{1}{5}\right)^{12}$$

(iv) P (all files are safe) = 1 – P (all files are corrupted)

$$= 1 - \left(\frac{1}{5}\right)^{12}$$

(v) P (more than 2 files but not more than 5 are corrupted)

$$= P(2 < r \leq 5)$$

$$= P(r = 3) + P(r = 4) + P(r = 5)$$

$$= {}^{12}C_3\left(\frac{4}{5}\right)^{12-3}\left(\frac{1}{5}\right)^3 + {}^{12}C_4\left(\frac{4}{5}\right)^{12-4}\left(\frac{1}{5}\right)^4 + {}^{12}C_5\left(\frac{4}{5}\right)^{12-5}\left(\frac{1}{5}\right)^5$$

$$= 0.42225$$

Example A box contains 100 transistors, 20 of which are defective and 10 are selected at random, find the probability that (i) all are defective, (ii) at least one is defective, (iii) all are good, (iv) at most 3 are defective.

Here $n = 10$ are selected at random

Let p = Probability of defective = $\frac{20}{100} = \frac{1}{5}$

Hence, $q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

We have, $P(r) = {}^{n}C_{r}\, q^{n-r}\, p^{r}$

$$\Rightarrow \quad P(r) = {}^{10}C_{r}\left(\frac{4}{5}\right)^{10-r}\left(\frac{1}{5}\right)^{r}$$

(i) P (all are defective) $= P(r = 10)$

$$= {}^{10}C_{10}\left(\frac{4}{5}\right)^{10-10}\left(\frac{1}{5}\right)^{10} = (1)(1)\left(\frac{1}{5}\right)^{10} = \frac{1}{5^{10}}$$

(ii) P (at least one is defective) $= P(r \geq 1)$

$$= 1 - P(r < 1) = 1 - P(r = 0)$$

$$=1-{}^{10}C_0\left(\frac{4}{5}\right)^{10-0}\left(\frac{1}{5}\right)^0 \qquad =1-\left(\frac{4}{5}\right)^{10}$$

(iii) P (all are good) = P (none is defective) $= P(r=0)$

$$={}^{10}C_0\left(\frac{4}{5}\right)^{10-0}\left(\frac{1}{5}\right)^0 =(1)\left(\frac{4}{5}\right)^{10}(1)=\left(\frac{4}{5}\right)^{10}$$

(iv) P (at most 3 are defective) $= P(r\le 3)$

$$=P(r=0)+P(r=1)+P(r=2)+P(r=3)$$

$$= {}^{10}C_0\left(\frac{4}{5}\right)^{10-0}\left(\frac{1}{5}\right)^{0} + {}^{10}C_1\left(\frac{4}{5}\right)^{10-1}\left(\frac{1}{5}\right)^{1} + {}^{10}C_2\left(\frac{4}{5}\right)^{10-2}\left(\frac{1}{5}\right)^{2} + {}^{10}C_3\left(\frac{4}{5}\right)^{10-3}\left(\frac{1}{5}\right)^{3}$$

$$=(1)\left(\frac{4}{5}\right)^{10}(1)+(10)\left(\frac{4}{5}\right)^{9}\left(\frac{1}{5}\right)+(45)\left(\frac{4}{5}\right)^{8}\left(\frac{1}{5}\right)^{2}+(120)\left(\frac{4}{5}\right)^{7}\left(\frac{1}{5}\right)^{3}$$

$$=\left(\frac{4}{5}\right)^{7}\left[\left(\frac{4}{5}\right)^{3}+2\left(\frac{4}{5}\right)^{2}+\left(\frac{4}{5}\right)\left(\frac{9}{5}\right)+(120)\left(\frac{1}{5}\right)^{3}\right]$$

$$=\left(\frac{4}{5}\right)^{7}\left[\frac{64}{125}+\frac{32}{25}+\frac{36}{25}+\frac{120}{125}\right]=\left(\frac{4}{5}\right)^{7}\left(\frac{524}{125}\right)$$

Example Out of 800 families with 5 children each, how many would you expect to have (i) 3 boys, (ii) 5 girls, (iii) either 2 or 3 boys? (Assume equal probabilities for boys and girls)

Here $n = 5$ children

Let p = Probability = $\frac{1}{2}$

Hence, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

We have, $P(r) = {}^nC_r\, q^{n-r} p^r$

$$\Rightarrow \qquad P(r) = {}^5C_r \left(\frac{1}{2}\right)^{5-r} \left(\frac{1}{2}\right)^r$$

$$P(r) = {}^5C_r\left(\frac{1}{2}\right)^5$$

(i) $P\text{ (3 boys)} = P(r=3)$

$$= {}^5C_3\left(\frac{1}{2}\right)^5 \quad = \frac{5}{16}$$

$\therefore$ For 800 families, the probability of number families having 3 boys is

$$= \frac{5}{16} \times 800 = 250$$

(ii) $P\text{ (5 girls)} = P\text{ (no boys)} = P(r=0)$

$$= {}^5C_0\left(\frac{1}{2}\right)^5 \quad = \frac{1}{32}$$

$\therefore$ For 800 families, the probability of number families having 5 girls is

$$= \frac{1}{32} \times 800 = 25$$

(iii) P (either 2 or 3 boys) $= P(2 \le r \le 3)$

$$= P(2) + P(3)$$

$$= \frac{1}{2^5}\left\{ {}^5C_2 + {}^5C_3 \right\} = \frac{5}{8}$$

$\therefore$ For 800 families, the probability of number families having either 2 or 3 boys is $\frac{5}{8} \times 800 = 500$

Example The probability that a man aged 60 will live upto 70 is 0.65. Out of 10 men, now at the age of 60, find the probability that (i) at least 7 will live upto 70, (ii) exactly 9 will live upto 70, (iii) at most 9 will live upto 70.

Here n = 10 men

Let p = Probability that a man aged 60 will live upto 70 = 0.65

Hence, $q = 1 - p = 1 - 0.65 = 0.35$

We have, $P(r) = {}^{n}C_{r}\, q^{n-r} p^{r}$

$\Rightarrow$ $P(r) = {}^{10}C_{r}\,(0.35)^{10-r}(0.65)^{r}$

$$P(r) = {}^{10}C_r (0.35)^{10-r} (0.65)^r$$

(i) P (at least 7 will live upto 70) $= P(r \geq 7)$

$$= P(r=7) + P(r=8) + P(r=9) + P(r=10)$$

$$= {}^{10}C_7 (0.35)^{10-7} (0.65)^7 + {}^{10}C_8 (0.35)^{10-8} (0.65)^8$$
$$+ {}^{10}C_9 (0.35)^{10-9} (0.65)^9 + {}^{10}C_{10} (0.35)^{10-10} (0.65)^{10}$$

= 0.5138

(ii) P (exactly 9 will live upto 70) $= P(r=9)$

$$= {}^{10}C_9 (0.35)^{10-9} (0.65)^9 \quad = 0.07249$$

(iii) P (at most 9 will live upto 70) $= P(r \le 9)$

$$= 1 - P(r > 9)$$

$$= 1 - P(r = 10)$$

$$= 1 - {}^{10}C_{10}(0.35)^{10-10}(0.65)^{10}$$

$$= 0.9865$$

Example In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.

Here $n = 20$

Given, Mean = 2

We have, Mean $\mu = np$

$$\Rightarrow \quad p = \frac{\mu}{n} = \frac{2}{20} = 0.1$$

Hence, $q = 1 - p = 1 - 0.1 = 0.9$

We have, $P(r) = {}^{n}C_r\, q^{n-r} p^r$

$\Rightarrow$ $P(r) = {}^{20}C_r\,(0.9)^{20-r}(0.1)^r$

P (at least 3 defective parts) $= P(r \geq 3)$

$= 1 - P(r < 3)$

$= 1 - \{P(r=0) + P(r=1) + P(r=2)\}$

$= 1 - \left\{ {}^{20}C_0\,(0.9)^{20-0}(0.1)^0 + {}^{20}C_1\,(0.9)^{20-1}(0.1)^1 + {}^{20}C_2\,(0.9)^{20-2}(0.1)^2 \right\}$

= 0.323

$\therefore$ For 1000 such samples, $P(r \geq 3) = 1000 \times 0.323 = 323$

Example **Fit a binomial distribution for the data**

x	0	1	2	3	4	5
f	2	14	20	34	22	8

Here $n = 5$ [since 6 values of x_i are given, $n + 1 = 6$]

$$\sum f_i = 2 + 14 + 20 + 34 + 22 + 8 = 100$$

$$\sum x_i f_i = 0 + 14 + 40 + 102 + 88 + 40 = 284$$

$$\therefore \text{ Mean } \mu = \frac{\sum x_i f_i}{\sum f_i} = \frac{284}{100} = 2.84$$

We have, Mean $\mu = np$

$$\Rightarrow \quad p = \frac{\mu}{n} = \frac{2.84}{5} = 0.568$$

Hence, $q = 1 - p = 1 - 0.568 = 0.432$

We have, $P(r) = {}^{n}C_{r}\, q^{n-r} p^{r}$

$$P(r) = {}^{5}C_{r}(0.432)^{5-r}(0.568)^{r}$$

Now, $F(x_i) = \left(\sum f_i\right) P(x_i)$

$$F(0) = (100)P(0) = (100)\left[{}^{5}C_{0}(0.432)^{5-0}(0.568)^{0}\right] = 1.505$$

$$F(1) = (100)P(1) = (100)\left[{}^{5}C_{1}(0.432)^{5-1}(0.568)^{1}\right] = 9.891$$

$$F(2) = (100)P(2) = (100)\left[{}^{5}C_{2}(0.432)^{5-2}(0.568)^{2}\right] = 26.01$$

$$F(3) = (100)P(3) = (100)\left[{}^{5}C_{3}(0.432)^{5-3}(0.568)^{3}\right] = 34.199$$

$$F(4)=(100)P(4)=(100)\left[{}^{5}C_{4}(0.432)^{5-4}(0.568)^{4}\right]=22.483$$

$$F(5)=(100)P(5)=(100)\left[{}^{5}C_{5}(0.432)^{5-5}(0.568)^{5}\right]=5.912$$

$\therefore$ The theoretical frequencies are

x	0	1	2	3	4	5
f	1.505	9.89	26.01	34.199	22.483	5.912

Fit a binomial distribution to the following frequency distribution:

x	0	1	3	4
f	28	62	10	4

Fit a binomial distribution to the following frequency distribution:

x	0	1	2	3	4	5	6
f	13	25	52	58	32	16	4

EXERCISE

1. Determine the binomial distribution for which mean is 2 and mean + variance = 3. Also find $P(X \le 3)$.
2. If the probability that a new born child is a male is 0.6, find the probability that in a family of 5 children there are exactly 3 boys.
3. A die is thrown five times. If getting an odd number is a success, find the probability of getting at least four successes?
4. The probability that a bomb dropped from a plane will strike the target is 1/5. If six bombs are dropped, find the probability that (i) exactly two will strike the target, (ii) at least two will strike the target.
5. If 10% of the rivets produced by a machine are defective, find the probability that out of 5 rivets chosen at random (i) none will be defective, (ii) one will be defective and (iii) at least two will be defective.
6. A product is 0.5% defective and is packed in cartons of 100. What percentage contains not more than 3 defectives?
7. If in a lot of 500 solenoids 25 are defective, find the probability of 0, 1, 2, 3 defective solenoids in a random sample of 20 solenoids.

8. A sortie of 20 aeroplanes is sent on an operational flight. The chance that an aeroplane fails to return is 5%. Find the probability that (i) one plane does not return, (ii) at the most 5 planes do not return.
9. If on an average one vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive, at least 4 will arrive safely.
10. Five hundred articles were selected at random out of a batch containing 10,000 articles and 30 were found to be defective. How many defective articles would you reasonably expect to have in the whole batch?
11. Fit a binomial distribution to the following frequency distribution:

x	0	1	3	4
f	28	62	10	4

12. Fit a binomial distribution to the following frequency distribution:

x	0	1	2	3	4	5	6
f	13	25	52	58	32	16	4