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COMPLEX ANALYSIS , PROBABILITY AND STATISTICAL METHOD

MODULE : 01

CALCULUS OF COMPLEX VARIABLES

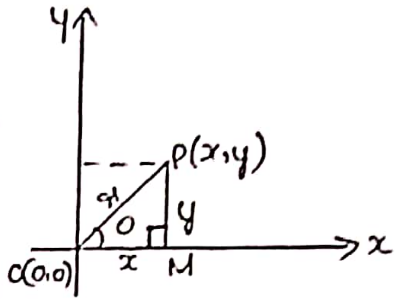
Definition:

let x, y be the two real values, then the number (or) variable $z = x+iy$ is called the complex variable (or) Complex number. where the first part of z is called real part and the second part is called imaginary part. But both x and y are the real values.

$\boxed{z = x+iy}$ is a complex number in the Cartesian form and its Conjugate complex is $\bar{z} = x-iy$ and $|z| = \sqrt{x^2+y^2}$

Geometrical representation:

Let $\vec{ox}$, $\vec{oy}$ be the real and imaginary axis, let 'p' be any point on the plane and 'M' be the foot of the perpendicular of 'p' on the real axis.



Let 'θ' be the angle b/n $\vec{op}$ and $\vec{ox}$. Then from the right angle Δ^{le} OPM.

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

Substitute the above in a complex variable $z = x+iy$

we get, $z = r\cos\theta + ir\sin\theta$

$$z = r(\cos\theta + i\sin\theta)$$

$\boxed{z = r\,e^{i\theta}}$ is called the complex variable in polar form.

Complex valued function:

1. Suppose $u(x,y)$ and $v(x,y)$ be the two real function in the variables 'x' and 'y', then the complex valued function in the Cartesian form can be defined as

$$w = f(z) = u(x,y) + iv(x,y) = u+iv$$

2. Suppose $u(r,\theta)$ and $v(r,\theta)$ be the two real function in the variables of 'r' and 'θ', then the Complex valued function in the polar form can be defined as

$$w = f(z) = u(r,\theta) + iv(r,\theta)$$

Some Important Results:

WKT,

$$e^{ix} = \cos x + i\sin x \rightarrow (1)$$

and $$e^{-ix} = \cos x - i\sin x \rightarrow (2)$$

from (1) + (2), $e^{ix} + e^{-ix} = 2\cos x$

$$\Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2} \rightarrow (3)$$

$\Rightarrow$ (1) − (2), $e^{ix} - e^{ix} = 2i\sin x$

$$\Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i} \rightarrow (4)$$

$\therefore$ (3) $\Rightarrow$ $$\cos ix = \frac{e^{i^2x} + e^{-i^2x}}{2}$$

$$= \frac{e^{-x} + e^{x}}{2}$$

$$\Rightarrow \boxed{\cos ix = \cosh x}$$

(4) $\Rightarrow$ $$\sin ix = \frac{e^{i^2x} - e^{-i^2x}}{2i}$$

$$= \frac{e^{-x} - e^{x}}{2i}$$

$$= \frac{-(e^{x} - e^{-x})}{2i}$$

$$= \frac{i^{\cancel{2}}(e^{x} - e^{-x})}{2\cancel{i}}$$

$$= \frac{i(e^{x} - e^{-x})}{2}$$

$$\Rightarrow \boxed{\sin ix = i\sinh x}$$

Some definitions:

* **Limit of a Complex variable:** Suppose 'z' be an Complex variable to the neighbourhood of 'z_0', then for any positive small quantity 'δ' (delta). $|z-z_0| \leq \delta$ is called the limit of a Complex variable.

* **Limit of a Complex valued function:** Suppose $w=f(z)$ be a complex valued function for any positive quantity 'ε', we have $|f(z)-f(z_0)| < \varepsilon$ (or) $|f(z)-l| < \varepsilon$ (or) $\lim_{z \to z_0} f(z) = l$ is called limit of a complex valved function

* **Differentiability of a function:** Suppose $w=f(z)$ be a complex valved function and is differentiable at z, then

$$f'(z) = \frac{dw}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z)-f(z)}{\delta z}, \ \delta z \neq 0,$$

* **Continuity:** A Complex valved function $f(z)$ is said to be a Continuous, if and only if $\lim_{z \to z_0} f(z) = f(z_0)$

* **Analytic of a function:**

let $w=f(z)$ be a Complex valved function and it is said to be an analytic function when $w=f(z)$ to be a differentiable at any point of z,

i.e., $$f'(z) = \frac{dw}{dz} = \lim_{\delta z \to 0}\left(\frac{f(z+\delta z)-f(z)}{\delta z}\right) \therefore \delta z \neq 0$$

It is also called regular (or) holomorphic function.

Theorem: Cauchy Rieman Equation in Cartesian form (or) C-R Equation.

Statement: The necessary Condition that the function $w=f(z)=u(x,y)+iv(x,y)=u+iv$ may be analytic at any point $z=x+iy$ is that, their exists 4 continuous first order partial derivatives,

i.e., $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ can satisfy the given equation

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

Proof: Given, $w=f(z)=u(x,y)+iv(x,y) \rightarrow (1)$ is a complex valued function in the Cartesian form, for the complex variable $z=x+iy$.

$$\therefore \text{eq}^n(1) \Rightarrow f(x+iy)=u(x,y)+iv(x,y) \rightarrow (2)$$

and given $w=f(z)$ is an analytic function.

differentiating eq^n(2) writ 'x' partially,

$$(2) \Rightarrow f'(x+iy).1=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\Rightarrow f'(x+iy)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \rightarrow (3)$$

diff. eq^n (2) writ 'y' partially,

$$(2) \Rightarrow f'(x+iy)(i)=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy)=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy)=\frac{i}{i^2}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy)=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \rightarrow (4)$$

$\therefore$ from (3) & (4)

$$\Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Hence proved.

Cauchy Rieman Equation in polar form.

Statement: If $w = f(z) = u(r, \theta) + iv(r, \theta)$ is an analytic function at $z = re^{i\theta}$, then their exists continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ can satisfy the equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof: Given $w = f(z) = u(r, \theta) + iv(r, \theta) \rightarrow (1)$ is an analytic function at $z = re^{i\theta}$.

$\therefore$ eq^n (1) $\Rightarrow w = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \rightarrow (2)$

differentiating eq^n (2) w.r.t 'r' partially,

$$(2) \Rightarrow f'(re^{i\theta})e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

$$\Rightarrow e^{i\theta}. f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \rightarrow (3)$$

diff. eq^n (2) w.r.t 'θ' partially,

$$(2) \Rightarrow f'(re^{i\theta})\, ri. e^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\Rightarrow e^{i\theta}. f'(re^{i\theta}) = \frac{1}{ri}\frac{\partial u}{\partial \theta} + \frac{i}{ir}\frac{\partial v}{\partial \theta}$$

$$\Rightarrow e^{i\theta}. f'(re^{i\theta}) = \frac{i}{i^2 r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\Rightarrow e^{i\theta}.\ f'(re^{i\theta}) = \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta} \rightarrow (4)$$

from eq^n (3) and (4)

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} - i\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Hence proved.

Derivation of analytic function:

Step 1: write the given Complex valued function $w=f(z)$

Step 2: write the $z=x+iy$ in the cartesian form and $z=re^{i\theta}$ in the polar form, we get $f(z)=u(x,y)+iv(x,y)$ in Cartesian form and $f(z)=u(r,\theta)+iv(r,\theta)$ in polar form.

Step 3: Identify u and v and verify the C-R equations in the respective forms.

Step 4: Find the derivative of $f(z)$ as $f'(z)=u_x+iv_x$ in the Cartesian form and $f'(z)=(u_r+iv_r).e^{-i\theta}$ in polar form.

Step 5: Substitute $x=z, y=0$ in cartesian form and $r=z, \theta=0$ in polar form in the required $f'(z)$.

Problems:

1. Verify the function $f(z)=z^2$ is analytic (or) not, hence find its derivative.

Sol^n Given, $f(z)=z^2 \rightarrow (1)$

let $z=x+iy$

$\therefore (1) \Rightarrow f(z)=(x+iy)^2$

$$f(z) = x^2 + i^2y^2 + 2ixy \qquad \because i^2 = -1$$
$$= x^2 - y^2 + 2ixy$$
$$f(z) = (x^2-y^2) + i(2xy)$$
$$\Rightarrow f(z) = u(x,y) + iv(x,y)$$
$$u = x^2 - y^2, \quad v = 2xy$$
$$\therefore \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$
$$\frac{\partial v}{\partial x} = +2y, \quad \frac{\partial v}{\partial y} = 2x$$
$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y$$

$\therefore$ The given $f(z)$ is satisfied C-R equation.

$\therefore f(z)$ is an analytic.

WKT, $f'(z) = u_x + iv_x$

$$\Rightarrow f'(z) = 2x + i2y$$
$$f'(z) = 2(x+iy) \rightarrow (2)$$

let $x = z, y = 0$

$$\therefore f'(z) = 2(z+0)$$
$$\therefore \underline{\underline{f'(z) = 2z}}$$

2. Verify $f(z) = z^3$ is analytic (or) not, hence find its derivative.

Soln: Given, $f(z) = z^3 \rightarrow (1)$

let $z = re^{i\theta}$

$$\therefore (1) \Rightarrow f(z) = (re^{i\theta})^3$$
$$f(z) = r^3 \cdot e^{i3\theta}$$
$$\Rightarrow f(z) = r^3(\cos 3\theta + i \sin 3\theta)$$
$$f(z) = r^3 \cos 3\theta + i r^3 \sin 3\theta$$
$$\therefore f(z) = u(r,\theta) + iv(r,\theta)$$

$u = r^3\cos 3\theta \quad , v = r^3\sin 3\theta$

$\therefore \frac{\partial u}{\partial r} = 3r^2\cos 3\theta \quad , \quad \frac{\partial u}{\partial \theta} = -3r^3\sin 3\theta$

$\frac{\partial v}{\partial r} = 3r^2\sin 3\theta \quad , \quad \frac{\partial v}{\partial \theta} = 3r^3\cos 3\theta$

$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

$\therefore$ $f(z)$ is an analytic function.

$\therefore$ WKT, $f'(z) = e^{-i\theta}\left[u_r + iv_r\right]$

$= e^{-i\theta}\left(3r^2\cos 3\theta + i\,3r^2\sin 3\theta\right)$

let $r = z, \theta = 0$

$\therefore f'(z) = 3z^2$

3. Verify $f(z) = z^n$ is an analytic function (or) not, hence find its derivative for any positive integer 'n'.

Given, $f(z) = z^n$

let $z = re^{i\theta}$

$f(z) = (re^{i\theta})^n = r^n . e^{in\theta}$

$\Rightarrow f(z) = r^n(\cos n\theta + i\sin n\theta)$

$f(z) = r^n\cos n\theta + ir^n.\sin n\theta$

$\therefore f(z) = u(r,\theta) + iv(r,\theta)$

$u = r^n\cos n\theta \quad , v = r^n.\sin n\theta$

$\therefore \frac{\partial u}{\partial r} = n.r^{n-1}\cos n\theta \quad , \quad \frac{\partial u}{\partial \theta} = -n.r^n.\sin n\theta$

$\frac{\partial v}{\partial r} = n.r^{n-1}\sin n\theta \quad , \quad \frac{\partial v}{\partial \theta} = n.r^n.\cos n\theta$

$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\cdot\frac{\partial u}{\partial \theta}$

$\therefore$ $f(z)$ is an analytic function.

$\therefore$ WKT, $f'(z) = e^{-i\theta}\left(u_r + i v_r\right)$

$$f'(z) = e^{-i\theta}\left(n.r^{n-1}\cos n\theta + i\,n\,r^{n-1}.\sin n\theta\right)$$

let $r = z$, $\theta = 0$

$$f'(z) = n.z^{n-1}$$

4. verify $f(z) = \sin z$ is analytic (or) not, hence find its derivative.

Soln: Given, $f(z) = \sin z \rightarrow (1)$

let $z = x + iy$

$$(1) \Rightarrow f(z) = \sin(x+iy) = \sin x.\cos(iy) + \cos x.\sin(iy)$$

$$\Rightarrow f(z) = \sin x.\cosh y + \cos x \sinh y$$

$$\Rightarrow f(z) = u(x,y) + iv(x,y)$$

$$u = \sin x.\cosh y, \quad v = \cos x.\sinh y$$

$$\frac{\partial u}{\partial x} = \cos x.\cosh y, \quad \frac{\partial u}{\partial y} = \sin x.\sinh y$$

$$\frac{\partial v}{\partial x} = -\sin x.\sinh y, \quad \frac{\partial v}{\partial y} = \cos x.\cosh y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore$ $f(z)$ is analytic

$\therefore$ WKT, $f'(z) = u_x + i v_x$

$$\Rightarrow f'(z) = \cos x.\cosh y - i \sin x.\sinh y$$

$$\Rightarrow f'(z) = \cos x.\cos(iy) - i \sin x.\sin(iy)$$

$$\Rightarrow f'(z) = \cos(x+iy)$$

$$f'(z) = \cos z$$

5. Verify $f(z) = \sin 2z$ is analytic (or) not, hence find its derivative.

Soln Given, $f(z) = \sin 2z \rightarrow (1)$

let $z = x+iy$

$(1) \Rightarrow f(z) = \sin 2(x+iy)$

$= \sin(2x+2iy)$

$f(z) = \sin(2x).\cos(2iy) + \cos(2x).\sin(2iy)$

$f(z) = \sin(2x).\cosh(2y) + \cos(2x).\sinh(2y)$

$\Rightarrow f(z) = u(x,y) + iv(x,y)$

$u = \sin(2x).\cosh(2y)$, $v = \cos 2x.\sinh(2y)$

$\therefore \frac{\partial u}{\partial x} = 2\cos 2x.\cosh(2y)$, $\frac{\partial u}{\partial y} = 2\sin(2x).\sinh(2y)$

$\frac{\partial v}{\partial x} = -2\sin(2x).\sinh(2y)$

$\frac{\partial v}{\partial y} = 2\cos(2x).\cosh(2y)$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore f(z)$ is an analytic function.

$\Rightarrow f'(z) = u_x + iv_x$

$f'(z) = 2\cos 2x.\cosh(2y) - i2\sin(2x).\sinh(2y)$

$\Rightarrow f'(z) = 2\cos 2x.\cos 2(iy) - i2\sin(2x).\sin 2(iy)$

$\Rightarrow f'(z) = 2[\cos 2x.\cos(2iy) - \sin(2x).\sin(2iy)]$

$\Rightarrow f'(z) = 2\cos(2x+2iy)$

$f'(z) = 2\cos 2(x+iy)$

$\Rightarrow f'(z) = 2\cos 2z$

6. verify the analytic function $f(z) = \log z$, hence find its derivative.

Soln: Given, $f(z) = \log z \rightarrow (1)$

let $z = re^{i\theta}$

$(1) \Rightarrow f(z) = \log(re^{i\theta})$

$\Rightarrow f(z) = \log r + \log e^{i\theta}$

$\Rightarrow f(z) = \log r + i\theta \cdot \log_e e \qquad \because \log_e e = 1$

$f(z) = \log r + i\theta$

$\Rightarrow f(z) = u(r,\theta) + iv(r,\theta)$

$u = \log r, \; v = \theta$

$\therefore \dfrac{\partial u}{\partial r} = \dfrac{1}{r}, \quad \dfrac{\partial u}{\partial \theta} = 0$

$\dfrac{\partial v}{\partial r} = 0, \quad \dfrac{\partial v}{\partial \theta} = 1$

$\therefore \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

$\therefore f(z)$ is analytic function.

$\Rightarrow f'(z) = (u_r + iv_r)e^{-i\theta}$

$f'(z) = e^{-i\theta}\left(\frac{1}{r} + i(0)\right)$

$\Rightarrow f'(z) = e^{-i\theta}\left(\frac{1}{r}\right)$

$\Rightarrow f'(z) = \dfrac{1}{re^{i\theta}}$

$\therefore f'(z) = \dfrac{1}{z}$

Harmonic property:

(1.) If $f(z) = u(x,y) + iv(x,y)$ is an analytic, then its real and imaginary parts are both harmonics.

i.e., $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

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(2) If $f(z) = u(r,\theta) + iv(r,\theta)$ is an analytic, then its real and imaginary parts are both harmonics,

i.e, $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

and $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$

Theorem :

If $f(z)$ is regular function of z, Show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(f|z|^2) = 4(f'(z)|^2).$$

Given, $f(z) = u(x,y) + iv(x,y) = u+iv$ is regular

$\Rightarrow f(z)$ is differentiable at any point of $z = x+iy$

$\Rightarrow f'(z) = u_x + iv_x$

and WKT, $f|z| = \sqrt{u^2+v^2}$

$$f(|z|^2) = u^2+v^2 = \phi \rightarrow (1)$$

$$\therefore f'|z| = \sqrt{u_x^2+v_x^2}$$

$$f'(|z|^2) = u_x^2 + v_x^2 \longrightarrow (2)$$

$$LHS = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot f(|z|^2)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \cdot \phi$$

$$LHS = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$\therefore \frac{\partial \phi}{\partial x} = 2u \cdot u_x + 2v \cdot v_x$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} = 2(u_x \cdot u_x + u \cdot u_{xx}) + 2(v_x \cdot v_x + v \cdot v_{xx})$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = 2\left(u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx}\right) \rightarrow (3)$$

$$\therefore \frac{\partial\phi}{\partial y} = 2\,u.u_y + 2v.v_y$$

$$\Rightarrow \frac{\partial^2\phi}{\partial y^2} = 2.\left(u_y.u_y + u.u_{yy}\right) + 2\left(v_y.v_y + v.v_{yy}\right)$$

$$\Rightarrow \frac{\partial^2\phi}{\partial y^2} = 2\left(u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right) \rightarrow (4)$$

from (3) & (4) (or) LHS (consider)

$$LHS = 2\left(u_x^2 + u.u_{xx} + v_x^2 + v.v_{xx} + u_y^2 + u.u_{yy} + v_y^2 + v.v_{yy}\right)$$

$$= 2\left[u_x^2 + v_x^2 + u_y^2 + v_y^2 + u\left(u_{xx} + u_{yy}\right) + v\left(v_{xx} + v_{yy}\right)\right]$$

$$= 2\left[u_x^2 + v_x^2 + u_y^2 + v_y^2 + u(0) + v(0)\right]$$

$$= 2\left[u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2\right]$$

$$= 2\left(u_x^2 + v_x^2 + v_x^2 + u_x^2\right)$$

$$= 2\left[2\left(u_x^2 + v_x^2\right)\right]$$

$$= 4\left(u_x^2 + v_x^2\right)$$

$$= 4\left(f'|z|^2\right)$$

$$= RHS$$

Theorem:- If $w = f(z)$ is regular, then show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$

Proof: Given $w = f(z) = u + iv$ is a regular (or) analytic.

$\therefore f(z)$ is differentiable at any point of $z = x + iy$.

$\Rightarrow f'(z) = u_x + iv_x$

and WKT $|f(z)| = \sqrt{u^2+v^2} = \phi$

$\Rightarrow u^2+v^2 = \phi^2 \rightarrow (1)$

and $|f'(z)| = \sqrt{u_x^2+v_x^2}$

$\Rightarrow |f'(z)|^2 = u_x^2+v_x^2 \rightarrow (2)$

diff. (1) wrt 'x' partially,

$\therefore (1) \Rightarrow 2u.u_x + 2v.v_x = 2\phi\phi_x$

$\Rightarrow uu_x + v.v_x = \phi\phi_x \rightarrow (3)$

|||ly diff. (1) wrt 'y' partially.

$(1) \Rightarrow 2uu_y + 2vv_y = 2\phi\phi_y$

$\Rightarrow uu_y + vv_y = \phi\phi_y \rightarrow (4)$

$\because$ funⁿ $f(z)$ is analytic.

$\therefore$ WKT $u_x = v_y$, $v_x = -u_y$

$u_y = -v_x$.

$\therefore (4) \Rightarrow -u.v_x + v.u_x = \phi\phi_y$

$vu_x - u.v_x = \phi\phi_y \rightarrow (5)$

$\therefore (3)^2+(5)^2$

$$(uu_x+v.v_x)^2 + (vu_x - uv_x)^2 = \phi^2\phi_x^2 + \phi^2\phi_y^2$$

$$\Rightarrow u^2u_x^2 + v^2v_x^2 + \cancel{2uv.u_xv_x} + v^2u_x^2 + u^2v_x^2 - \cancel{2uv\,u_xv_x} = \phi^2\phi_x^2 + \phi^2\phi_y^2$$

$$\Rightarrow u^2(u_x^2+v_x^2) + v^2(u_x^2+v_x^2) = \phi^2(\phi_x^2+\phi_y^2)$$

$$\Rightarrow (u^2+v^2)(u_x^2+v_x^2) = \phi^2(\phi_x^2+\phi_y^2)$$

$$\Rightarrow \cancel{\phi^2}(u_x^2+v_x^2) = \cancel{\phi^2}(\phi_x^2+\phi_y^2)$$

$$\Rightarrow \phi_x^2+\phi_y^2 = u_x^2+v_x^2$$

$$\Rightarrow \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = |f'(z)|^2$$

$$\Rightarrow \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

Theorem: If $f(z)$ is regular function with Constant Modulus, Show that $f(z)$ is also a Constant function.

Given, $\omega = f(z) = u + iv$ is regular function.

$\therefore f(z)$ is differentiable.

$\therefore f(z) = u_x + iv_x$

and $|f'(z)| = \sqrt{u_x^2 + v_x^2}$

$\Rightarrow |f'(z)|^2 = u_x^2 + v_x^2 \rightarrow (1)$

and given $|f(z)| = k$

$\Rightarrow \sqrt{u^2 + v^2} = k$

$\Rightarrow u^2 + v^2 = k^2 \rightarrow (2)$

diff. eqn (2) partially w.r.t 'x'.

(2) $\Rightarrow 2uu_x + 2vv_x = 0$

$\Rightarrow uu_x + vv_x = 0 \rightarrow (3)$

III'y (2) $\Rightarrow$ w.r.t 'y' partially

(2) $\Rightarrow 2uu_y + 2vv_y = 0$

$\Rightarrow uu_y + vv_y = 0 \rightarrow (4)$

$\therefore$ But $f(z)$ is analytic

$\therefore u_x = v_y$ and $v_x = -u_y$

$\Rightarrow u_y = -v_x$

$\therefore (4) \Rightarrow u(-v_x) + vu_x = 0$

$\Rightarrow vu_x - uv_x = 0 \rightarrow (5)$

$(3)^2+(5)^2$ $(uu_x+vv_x)^2+(vu_x-uv_x)^2=0$

$\Rightarrow u^2u_x^2+v^2v_x^2+2vu\cancel{v_xu_x}+v^2u_x^2+u^2v_x^2-2uv\cancel{u_xv_x}=0$

$\Rightarrow u^2(u_x^2+v_x^2)+v^2(u_x^2+v_x^2)=0$

$\Rightarrow (u^2+v^2)(u_x^2+v_x^2)=0$

$\Rightarrow K^2(u_x^2+v_x^2)=0$

$\Rightarrow K^2\neq 0,\ u_x^2+v_x^2=0$

$\Rightarrow |f'(z)|^2=0$

$\Rightarrow |f'(z)|=0$

$\therefore f(z)=c$

$\Rightarrow f(z)$ is Constant.

Problems:

1. Show that $f(z)=\cosh z$ is analytic and hence find its derivative.

Soln: $f(z)=\cosh z$

let $z=x+iy$

$\Rightarrow f(z)=\cosh(x+iy)\longrightarrow(1)$

WKT, $\cosh\theta=\cos i\theta$

$(1)\Rightarrow f(z)=\cosh i(x+iy)$

$=\cosh(ix+i^2y)$

$=\cosh(ix-y)$

$=\cos(ix).\cos y+\sin(ix).\sin y$

$=\cosh x.\cos y+i\sinh x.\sin y.$

$\Rightarrow f(z)=u(x,y)+iv(x,y)$

$\therefore u=\cosh x.\cos y\ ,\ v=\sinh x.\sin y$

$u_x=\sinh x.\cos y\ ,\ u_y=-\cosh x.\sin y$

$v_x=\cosh x.\sin y\ ,\ v_y=\sinh x.\cos y.$

$\therefore U_x = V_y, \quad V_x = -V_y$

$\therefore f(z)$ is analytic

$\therefore f'(z) = U_x + iV_x$

$\Rightarrow f'(z) = \sinh x \cdot \cos y + i \cosh x \cdot \sin y$

when $x = z, y = 0$

$\Rightarrow f'(z) = \sinh z$

2. Show that $f(z) = z + e^z$ is analytic, Hence find its derivative.

Soln: $f(z) = z + e^z$

let $z = x + iy$

$\Rightarrow f(z) = (x+iy) + e^{(x+iy)}$

$= (x+iy) + e^x \cdot e^{iy}$

$= (x+iy) + e^x \cdot (\cos y + i\sin y)$

$= x + e^x \cdot \cos y + iy + i\sin y \cdot e^x$

$= (x + e^x \cdot \cos y) + i(y + \sin y e^x)$

$\Rightarrow f(z) = u(x,y) + i\, v(x,y)$

$\Rightarrow u = x + e^x \cdot \cos y, \quad v = y + e^x \cdot \sin y$

$\therefore U_x = 1 + e^x \cdot \cos y, \quad U_y = -\sin y \cdot e^x$

$V_x = e^x \cdot \sin y, \quad V_y = 1 + e^x \cdot \cos y$

$\therefore U_x = V_y$ and $V_x = -U_y$.

$\therefore f(z)$ is analytic.

$\therefore f'(z) = U_x + iV_x$

$\Rightarrow f'(z) = 1 + e^x \cos y + i\, e^x \cdot \sin y$

$= 1 + e^x(\cos y + i\sin y)$

put $x = z, y = 0$ $\Rightarrow f'(z) = 1 + e^z$

3. Find the analytic function, whose real part is $u = \frac{x^4.y^4-2x}{x^2+y^2}$

Given, $u = \frac{x^4.y^4-2x}{x^2+y^2} \rightarrow (1)$

diff. (1) partially w.r.t 'x'.

$\therefore (1) \Rightarrow \frac{\partial u}{\partial x} = \frac{(x^2+y^2)(4x^3y^4-2)-(x^4.y^4-2x)(2x)}{(x^2+y^2)^2}$

when $x=z, y=0$

$\therefore \frac{\partial u}{\partial x} = \frac{(z^2+0)(4z^3(0)-2) - z^4.(0)-2(z)(2z)}{(z^2)^2}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{z^2(-2)+4z^2}{z^4} = \frac{-2z^2+4z^2}{z^4}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{2z^2}{z^4}$

$\therefore \frac{\partial u}{\partial x} = \frac{2}{z^2}$

$|||^{ly}$ $\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(4x^4y^3)-(x^4.y^4-2x)(2y)}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$

$\therefore \frac{\partial v}{\partial x} = \frac{2y(x^4.y^4-2x)-(x^2+y^2)(4x^4y^3)}{(x^2+y^2)^2}$

when $x=z, y=0$

$= \frac{2(0).(z^4(0)-2(z)) - z^2(4z^4(0))}{z^4}$

$\therefore \frac{\partial v}{\partial x} = 0$

$\therefore$ WKT $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\Rightarrow f'(z) = \frac{2}{z^2} + i(0)$

$\Rightarrow f'(z) = \frac{2}{z^2}$

$\Rightarrow \frac{df}{dz} = \frac{2}{z^2}$

$\Rightarrow \int df = \int \frac{2}{z^2}\, dz$

$\Rightarrow f(z) = 2 \cdot -\frac{1}{z} + C$

$\Rightarrow \boxed{f(z) = -\frac{2}{z} + C}$

4. Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

Sol: Given, $u = \frac{\sin 2x}{\cosh 2y - \cos 2x} \rightarrow (1)$

diff w.r.t 'x' (1) partially

$\therefore \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(0 + 2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{2\cos 2x\,(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$

when $x = z$, $y = 0$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{2\cos 2z\,(1 - \cos 2z) - 2\sin^2 2z}{(1 - \cos 2z)^2}$

$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$

$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1 - \cos 2z)^2}$

$= \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} = \frac{-2\cancel{(1 - \cos 2z)}}{(1 - \cos 2z)^{\cancel{2}}}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{-2}{1 - \cos 2z}$

Similarly $\frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sin 2hy - 0)}{(\cosh 2y - \cos 2x)^2}$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial v}{\partial x} = \frac{\sin 2x(2\sin 2hy) - 0}{(\cosh 2y - \cos 2x)^2}$$

when $z = x, y = 0$

$$\frac{\partial v}{\partial x} = 0$$

$\therefore$ WKT $f'(z) = u_x + iv_x$

$$\Rightarrow f'(z) = \frac{-2}{1-\cos 2z} + i(0)$$

$$\Rightarrow \frac{df}{dz} = \frac{-2}{1-\cos 2z}$$

$$\Rightarrow df = \frac{-2}{1-\cos 2z} dz$$

$$\Rightarrow df = \frac{-\cancel{2}}{\cancel{2}\sin^2 z} dz = \frac{-1}{\sin^2 z} dz$$

$$\Rightarrow \int df = \int \frac{-1}{\sin^2 z} dz$$

$$\Rightarrow f(z) = \int -\operatorname{cosec}^2 z \cdot dz$$

$$\Rightarrow \boxed{f(z) = \cot z + C}$$

5. Construct the Analytic function, whose real part is $\frac{\sin 2x}{\cosh 2y + \cos 2x}$.

Sol: Given, $u = \frac{\sin 2x}{\cosh 2y + \cos 2x} \rightarrow (1)$

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)(2\cos 2x) - \sin 2x(0 + (-2\sin 2x))}{(\cosh 2y + \cos 2x)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)\, 2\cos 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$

when $z = x,\ y = 0$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2z\,(1+\cos 2z) + 2\sin^2 2z}{(\cosh 2(0) + \cos 2z)^2}$$

$$= \frac{2\cos 2z\,(1+\cos 2z) + 2\sin^2 2z}{(1+\cos 2z)^2}$$

$$= \frac{2\cos 2z + 2\cos^2 2z + 2\sin^2 2z}{(1+\cos 2z)^2}$$

$$= \frac{2\cos 2z + 2(1)}{(1+\cos 2z)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\,\cancel{(1+\cos 2z)}}{(1+\cos 2z)^{\cancel{2}}} = \frac{2}{(1+\cos 2z)}$$

$$|||^{ly}\ \frac{\partial u}{\partial y} = \frac{(\cosh 2y + \cos 2x)(0) - \sin 2x\,(2\sinh 2y + 0)}{(\cosh 2y + \cos 2x)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{\sin 2x\,(2\sinh 2y)}{(\cosh 2y + \cos 2x)^2}$$

when $x = z,\ y = 0$

$$\frac{\partial v}{\partial x} = \frac{\sin 2z\,(2\sinh 2(0))}{(\cosh(0) + \cos 2z)^2}$$

$$\therefore \frac{\partial v}{\partial x} = 0$$

$\therefore$ WKT, $f'(z) = u_x + i v_x$

$$\Rightarrow f'(z) = \frac{2}{1+\cos 2z}$$

$$\Rightarrow \frac{df}{dz} = \frac{2}{1+\cos 2z}, \quad \int df = \int \frac{\cancel{2}}{\cancel{2}\cos^2 z}\, dz$$

$$f(z) = \int \frac{1}{\cos^2 z}\, dz = \int \sec^2 z\, dz$$

$$\boxed{f(z) = \tan z + C}$$

6. Construct the Analytic function, whose real part is

$$x^2-y^2+\frac{x}{x^2+y^2}$$

Given, $u = x^2-y^2+\frac{x}{x^2+y^2}$

$$u = \frac{(x^2+y^2)(x^2-y^2)+x}{(x^2+y^2)} = \frac{(x^4-y^4+x)}{(x^2+y^2)} \rightarrow (1)$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(4x^3+1) - (x^4-y^4+x)(2x)}{(x^2+y^2)^2}$$

when $x=z, y=0$

$$\frac{\partial u}{\partial x} = \frac{z^2(4z^3+1) - (z^4+z)(2z)}{(z^2+0)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{4z^5+z^2-2z^5-2z^2}{z^4}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{2z^5-z^2}{z^4}$$

$$= \frac{z^2(2z^3-1)}{z^4}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2z^3-1}{z^2} = 2z-\frac{1}{z^2}$$

$$|||^{ly}\ \frac{\partial u}{\partial y} = \frac{(x^2+y^2)(-4y^3) - (x^4-y^4+x)(2y)}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$$

when $z=x, y=0$

$$\boxed{v_x=0}$$

$$\therefore f'(z) = u_x + iv_x$$

$$\Rightarrow f'(z) = 2z - \frac{1}{z^2} + 0$$

$$\Rightarrow \frac{df}{dz} = 2z - \frac{1}{z^2}$$

$$\int df = \int\left(2z - \frac{1}{z^2}\right)dz$$

$$\Rightarrow f(z) = \frac{2z^2}{2} - \left(\frac{-1}{z}\right) + c$$

$$\boxed{\therefore f(z) = z^2 + \frac{1}{z} + c}$$

7. Find the analytic function $f(z) = u+iv$, whose imaginary part is $\frac{y}{x^2+y^2}$.

Given, $V = \frac{y}{x^2+y^2}$

$\Rightarrow V_x = \frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$

when $x = z$, $y = 0$

$\therefore \boxed{V_x = 0}$

$\therefore V_y = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}$

$\Rightarrow V_y = \frac{x^2-y^2}{(x^2+y^2)^2}$ when $x = z, y = 0$

$\therefore V_y = \frac{z^2}{z^4} = \frac{1}{z^2}$

$\therefore$ WKT, $f'(z) = u_x + iV_x$

$\Rightarrow f'(z) = V_y + iV_x$

$\Rightarrow f'(z) = \frac{1}{z^2}$

$\Rightarrow \frac{df}{dz} = \frac{1}{z^2}$

$\int df = \int \frac{1}{z^2} dz$

$\Rightarrow \boxed{f(z) = -\frac{1}{z} + c}$

8. Construct analytic function, whose real part is $u = e^{2x}(x\cos 2y - y\sin 2y)$.

Sol Given, $u = e^{2x}(x\cos 2y - y\cdot\sin 2y) \rightarrow (1)$

$\therefore u_x = 2e^{2x}(x.\cos 2y - y\sin 2y) + e^{2x}(\cos 2y - 0)$

$u_x = 2e^{2x}(x.\cos 2y - y\sin 2y) + e^{2x}(\cos 2y)$

when $x = z, y = 0$.

$\Rightarrow u_x = 2e^{2z}(z-0) + e^{2z}(1)$

$u_x = 2e^{2z}(z) + e^{2z}$

$\therefore v_y = e^{2x}(-2x.\sin 2y - 1.\sin 2y - 2y\cos 2y)$

when $x = z, y = 0$

$v_y = e^{2z}(0)$

$\therefore v_y = 0$

$\therefore$ WKT, $f'(z) = u_x + iv_x$

$\Rightarrow f'(z) = u_x - iv_y$

$\Rightarrow f'(z) = 2ze^{2z} + e^{2z} - 0$

$\Rightarrow f'(z) = (2z+1)e^{2z}$

$\Rightarrow \frac{df}{dz} = (2z+1)e^{2z}$

$\Rightarrow \int df = \int (2z+1)e^{2z}.dz$

$\therefore f(z) = (2z+1)\int e^{2z}dz - \int (2.\int e^{2z}dz).dz$

$= (2z+1)\frac{1}{2}.e^{2z} - \frac{1}{2}e^{2z} + c$

$= ze^{2z} + \cancel{\frac{1}{2}e^{2z}} - \cancel{\frac{1}{2}e^{2z}} + c$

$\therefore f(z) = ze^{2z} + c$

9. Find the analytic function, whose imaginary part is

$v = e^x(x.\sin y + y\cos y)$

Given, $v = e^x(x\sin y + y\cos y)$

$\therefore v_x = e^x(x.\sin y + y\cos y) + e^x(\sin y)$

$\Rightarrow v_x = e^x(x.\sin y + y\cos y) + e^x(\sin y)$

when $x = z,\ y = 0$

$$\Rightarrow v_x = e^z\,(z(0)+0) + e^z(0)$$

$$v_x = 0$$

$$\therefore v_y = e^x\,[x\cos y + \cos y + (-y\sin y)]$$

put $z = x,\ y = 0$

$$v_y = e^z\,(z+1+0)$$

$$= e^z\,(z+1)$$

$$\therefore f'(z) = u_x + i v_x$$

$$= e^z\,(z+1)$$

$$\therefore \frac{df}{dz} = e^z\,(z+1)$$

$$df = e^z\,(z+1)\cdot dz$$

$$\int df = \int e^z(z+1)\cdot dz$$

$$f(z) = (z+1)\,e^z - (1)\,e^z + c$$

$$= e^z\big((z+1)-1\big) + c$$

$$= z e^z + \cancel{e^z} - \cancel{e^z} + c$$

$$\therefore f(z) = z e^z + c$$

10. Find the analytic function, $u+iv$ whose imaginary part is
$$V=\left(r-\frac{1}{r}\right)\sin\theta,\ r\neq 0.$$

Given,
$$V=\left(r-\frac{1}{r}\right)\sin\theta,\ r\neq 0$$

$$\frac{d}{dr}\left(\frac{1}{r}\right)=\frac{-1}{r^2}$$

$$\therefore V_r=\left(1+\frac{1}{r^2}\right)\sin\theta\rightarrow(1)$$

$$\therefore V_\theta=\left(r-\frac{1}{r}\right)\cos\theta\rightarrow(2)$$

$$\therefore \text{WKT},\ f'(z)=e^{-i\theta}(u_r+iV_r)$$

$$\Rightarrow f'(z)=e^{-i\theta}\left(\frac{1}{r}V_\theta+iV_r\right)$$

$$\Rightarrow f'(z)=e^{-i\theta}\left(\frac{1}{r}\left(r-\frac{1}{r}\right)\cos\theta+i\left(1+\frac{1}{r^2}\right)\sin\theta\right)$$

when $z=r,\ \theta=0$

$$\Rightarrow f'(z)=\left[\frac{1}{z}\left(z-\frac{1}{z}\right)\right]$$

$$\Rightarrow f'(z)=1-\frac{1}{z^2}$$

$$\int\frac{1}{z^2}=\frac{-1}{z}$$

$$\Rightarrow \frac{df}{dz}=1-\frac{1}{z^2}$$

$$\Rightarrow \int df=\int\left(1-\frac{1}{z^2}\right)dz$$

$$\Rightarrow \boxed{f(z)=z+\frac{1}{z}+c}$$

11. Find the analytic function, $f(z)=u+iv$, if $(u-v)=e^x[\cos y-\sin y]$

Given, $f(z)=u+iv\rightarrow(1)$

and given, $u-v=e^x(\cos y-\sin y)\rightarrow(2)$

different wrt 'x' eqn (2) partially

$$(2)\Rightarrow u_x-v_x=e^x(\cos y-\sin y)\rightarrow(3)$$

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diff wrt 'y' partially,

$$(2) \Rightarrow U_y - V_y = -e^x(\cos y + \sin y) \rightarrow (4)$$

WKT, $U_y = -V_x$, $V_y = U_x$

$$\therefore (4) \Rightarrow -V_x - U_x = -e^x(\cos y + \sin y) \rightarrow (5)$$

(3)+(5),

$$U_x - V_x - V_x - U_x = e^x(\cos y - \sin y) - e^x(\cos y + \sin y)$$

$$-2V_x = e^x(\cos y - \cancel{\sin y} - \cos y + \cancel{\sin y})$$

$$\cancel{-2}V_x = \cancel{-2}e^x.\sin y$$

$$V_x = e^x.\sin y$$

when $x = z$, $y = 0$

$$\therefore V_x = 0.$$

(3)−(5), $2U_x = e^x(\cos y - \sin y) + e^x(\cos y + \sin y)$

$$2U_x = e^x(\cos y - \cancel{\sin y} + \cos y + \cancel{\sin y})$$

$$\cancel{2}U_x = \cancel{2}e^x.\cos y$$

$$U_x = e^x.\cos y$$

when $x = z$, $y = 0$

$$U_x = e^z$$

$\therefore$ WKT, $f'(z) = U_x + iV_x$

$$\Rightarrow f'(z) = e^z + 0 = e^z$$

$$\Rightarrow \frac{df}{dz} = e^z$$

$$\int df = \int e^z.dz$$

$$\Rightarrow f(z) = e^z + C$$

12. Find the analytic function $f(z) = u + iv$, if $(u - v) = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ and $f\left(\frac{\pi}{2}\right) = 0$.

Given, $f(z) = u + iv \rightarrow (1)$

Given, $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)} \rightarrow (2)$

diff. (2) partially w.r.t 'x'.

$$\therefore u_x - v_x = \frac{1}{2}\left[\frac{(\cos x - \cosh y)(-\sin x + \cos x) + (\cos x + \sin x - e^{-y})\sin x}{(\cos x - \cosh y)^2}\right]$$

when $x = z$, $y = 0$

$$\therefore u_x - v_x = \frac{1}{2}\left\{\frac{(\cos z - 1)(-\sin z + \cos z) + (\cos z + \sin z - 1)\sin z}{(\cos z - 1)^2}\right\}$$

$$\Rightarrow u_x - v_x = \frac{1}{2}\left\{\frac{\cos^2 z - \cos z \cancel{\sin z} - \cos z + \sin z + \cos z \cancel{\sin z} + \sin^2 z - \sin z}{(\cos z - 1)^2}\right\}$$

$$= \frac{-1}{2}\left|\frac{1 - \cancel{\cos z}}{(1 - \cos z)^2}\right|$$

$$\therefore u_x - v_x = \frac{-1}{2}\frac{1}{(1 - \cos z)} \rightarrow (3)$$

$$\text{III}^{ly} \quad u_y - v_y = \frac{1}{2}\left\{\frac{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{(\cos x - \cosh y)^2}\right\}$$

when $x = z$, $y = 0$

$$= \frac{1}{2}\left(\frac{\cos z - 1}{(\cos z - 1)^2}\right)$$

$$\Rightarrow -v_x - u_x = \frac{1}{2}\frac{1}{\cos z - 1} = \frac{-1}{2}\frac{1}{(1 - \cos z)} \rightarrow (4)$$

$(3) + (4) \Rightarrow -2v_x = 0$

$\Rightarrow v_x = 0$

$(3) - (4) \Rightarrow 2u_x = \dfrac{1}{1 - \cos z}$

$u_x = 1/2(1 - \cos z)$

$$\Rightarrow f'(z) = \frac{-1}{2}\left(\frac{1}{1-\cos z}\right)$$

$$= \frac{-1}{2}\left(\frac{1}{2\sin^2(z/2)}\right)$$

$$\Rightarrow \frac{df}{dz} = \frac{-1}{2}\left(\frac{1}{2}\frac{1}{\sin^2(z/2)}\right), \int df = \int \frac{-1}{2}\left(\frac{1}{2}\frac{1}{\sin^2(z/2)}\right) dz$$

$$\Rightarrow f(z) = \frac{1}{2}\int\left(-\operatorname{cosec}^2(z/2)\right) dz$$

$$\Rightarrow f(z) = +\frac{1}{2}\cot(z/2) + c \rightarrow (5)$$

By condition, $f(\pi/2) = +\frac{1}{2}\cot(\pi/4) + C$

$$0 = +\frac{1}{2}(1) + C$$

$\boxed{C = -1/2}$ $\quad \therefore (5) = f(z) = +\frac{1}{2}\cot(z/2) - \frac{1}{2}$

13. Find the analytic function of $f(z) = u+iv$, if
$u-v = (x-y)(x^2+4xy+y^2)$.

Given, $f(z) = u+iv$ and

Given, $u-v = (x-y)(x^2+4xy+y^2)$

$$\Rightarrow u-v = x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3$$

$$\Rightarrow u-v = x^3 + 3x^2y - 3xy^2 - y^3 \rightarrow (1)$$

diff (1) w.r.t 'x' partially

$$\therefore (1) \Rightarrow u_x - v_x = 3x^2 + 6yx - 3y^2 \rightarrow (2)$$

when $x = z, y = 0$

$$\therefore (2) \Rightarrow u_x - v_x = 3z^2 \rightarrow (3)$$

|||ly

$$u_y - v_y = 3x^2 - 6xy - 3y^2 \rightarrow (4)$$

when $x = z$, $y = 0$

$\therefore (3) \quad U_y - V_y = 3z^2 \rightarrow (5)$

WKT $U_y = -V_x$, $V_y = U_x$

$\therefore (4) \Rightarrow -V_x - U_x = 3z^2 \rightarrow (6)$

$\therefore (3) + (6) \Rightarrow -2V_x = 6z^2$

$\Rightarrow -V_x = 3z^2$

$\Rightarrow V_x = -3z^2$

$(3) - (6) \Rightarrow 2U_x = 0$

$\Rightarrow U_x = 0$

$\therefore$ WKT,

$f'(z) = U_x + iV_x$

$\Rightarrow f'(z) = 0 + i(-3z^2)$

$\Rightarrow f'(z) = -3iz^2$

$\Rightarrow \dfrac{df}{dz} = -3iz^2$

$\Rightarrow \int df = \int -3iz^2 . dz$

$\Rightarrow \int df = -3i \int z^2 . dz$

$\Rightarrow f(z) = -\cancel{3}i \dfrac{z^3}{\cancel{3}} + C$

$\Rightarrow f(z) = -iz^3 + C$

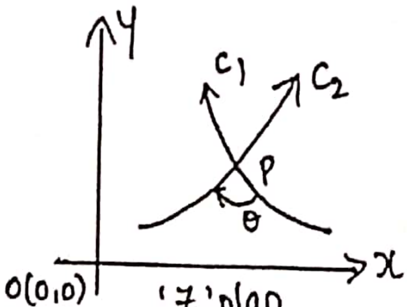
MODULE - 02

CONFORMAL MAPPING (OR) TRANSFORMATION.

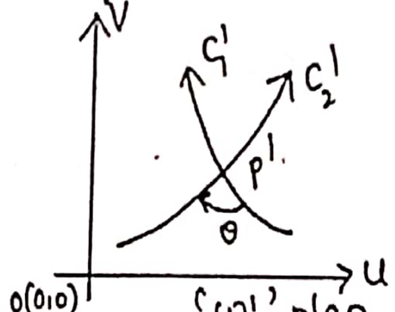
Definition:

If the angle between any 2 curves in the Magnitude and its direction is called conformal transformation(or)

Suppose 2 curves c_1, c_2 in the 'z' plan intersect at the point 'p', and the corresponding curves c_1', c_2' in the 'w'' plan intersect at 'p'', and if the angle of intersection of the curves 'p' is the same as the angle of intersection of the curves at 'p'' In Magnitude and direction, then this transformation is called the Conformal transformation.



'Z' plan



'w'' plan.

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Condition for $w=f(z)$ to represent a Conformal transformation:

1. Necessary Condition: If $w=f(z)$ represents a Conformal transformation of a domain (D) in the 'z' plan into a domain (D') of the 'w' plan, then $f(z)$ is analytic function of z in D.

2. Sufficient Condition: let $w=f(z)$ be an analytic function of z in a region (D) of the 'z' plan and let $f'(z) \neq 0$ inside 'D' then the mapping $w=f(z)$ is Conformal at the points of D.

1. Discussion of transformation $w=e^z$.

Given, $w=f(z)=e^z$ is analytic

$\therefore f(z)$ is differentiable.

$\Rightarrow f'(z)=e^z, \quad z\neq 0$

we have, $w=f(z)=u+iv=e^z \rightarrow (1)$

let $z=x+iy$

$\therefore (1) \Rightarrow u+iv=e^{x+iy}$

$\Rightarrow u+iv=e^x.e^{iy}$

$\Rightarrow u+iv=e^x(\cos y+i\sin y)$

$\Rightarrow u+iv=e^x\cos y+ie^x.\sin y$

$\therefore u=e^x\cos y \rightarrow (2), \quad v=e^x.\sin y \rightarrow (3)$

By eliminating 'y', we have

$(2)^2+(3)^2 = u^2+v^2 = (e^x\cos y)^2+(e^x\sin y)^2$

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$\therefore u^2+v^2 = e^{2x}(\cos^2 y + \sin^2 y)$

$\Rightarrow u^2+v^2 = e^{2x} \rightarrow (4)$

By eliminating 'x', we have

$\frac{(3)}{(2)} \Rightarrow \frac{v}{u} = \frac{e^x \sin y}{e^x \cos y} = \frac{\sin y}{\cos y} = \tan y.$

$\Rightarrow v = \tan y \cdot u \rightarrow (5)$

Case 1: let $x = c_1$ = constant, we have

eqn (4) $\Rightarrow u^2+v^2 = e^{2c_1} = r_1$ (Say)

$\Rightarrow u^2+v^2 = r^2 \rightarrow (6)$

'r' represents a circle having the centre as origin with radius 'r'.

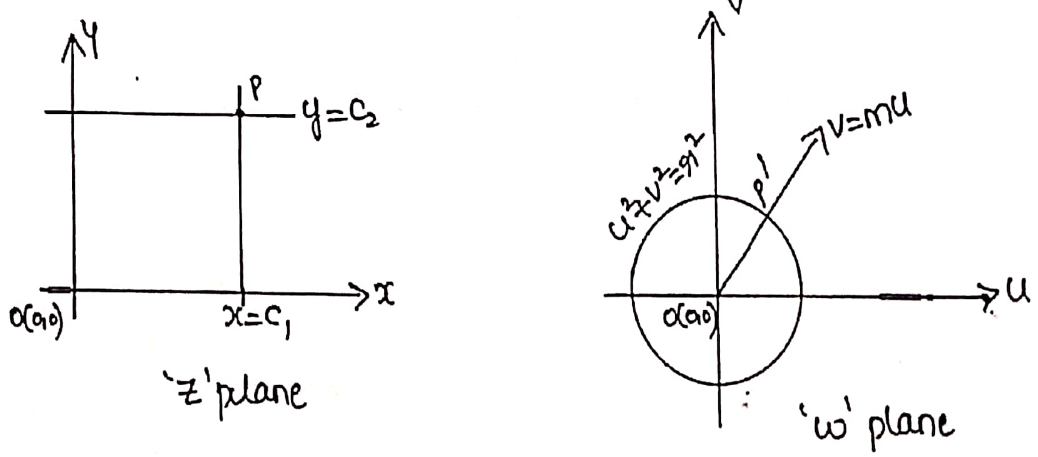
Case 2: let $y = c_2$ = constant, we have

eqn (5) $\Rightarrow v = \tan y \cdot u$

$\Rightarrow v = \tan c_2 \cdot u$

$\therefore v = m.u$ where $m = \tan c_2$ = Slope $\rightarrow (7)$

represents a straight line passing through the origin having the slope 'm'.



2. Discussion of transformation $w = z^2$.

Given, $w = f(z) = z^2$ is analytic

$\therefore f(z)$ is differentiable.

$\Rightarrow f'(z) = 2z,\ z \neq 0.$

we have, $w = f(z) = u + iv = z^2 \longrightarrow (1)$

let $z = x + iy$

$(1) \Rightarrow u + iv = z^2$

$\Rightarrow u + iv = (x+iy)^2 = x^2 + i^2y^2 + 2ixy$

$\Rightarrow u + iv = (x^2 - y^2) + i(2xy) \rightarrow *$

$\therefore u = x^2 - y^2 \rightarrow (2), \quad v = 2xy \rightarrow (3)$

By eliminating (u, v, x, y) continuously we have,

Case 1: let $u = c_1$ = Constant

$\therefore (2) \Rightarrow x^2 - y^2 = c_1 \rightarrow (4)$

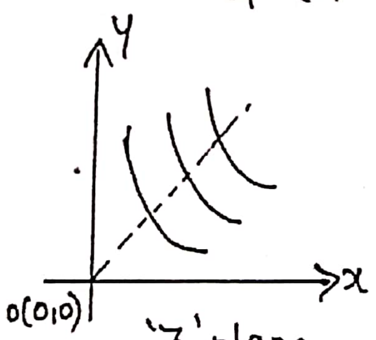
eqn (4) represents a hyperbola.

Case 2: let $v = c_2$

$\therefore (3) \Rightarrow 2xy = c_2$

$\Rightarrow xy = \frac{c_2}{2} \rightarrow (5)$

eqn (5) represents a rectangular hyperbola.



'z' plane

Case 3: let $x = c_3$

$\therefore (2) \Rightarrow u = c_3^2 - y^2$

$y^2 = c_3^2 - u \rightarrow (6)$

$\therefore (3) \Rightarrow v = 2xy$

$2c_3 y = v$

$y = \frac{v}{2c_3} \Rightarrow y^2 = \frac{v^2}{4c_3^2} \rightarrow (7)$

$\therefore$ From eq^n (6) & (7)

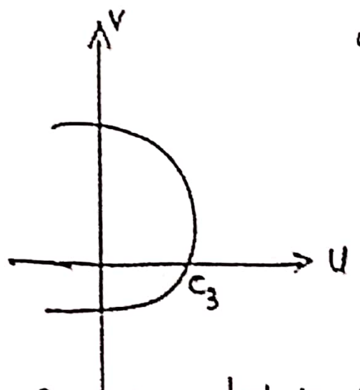
$$\frac{v^2}{4c_3^2} = c_3^2 - u$$

$$\Rightarrow v^2 = 4c_3^2 (c_3^2 - u)$$

$$\Rightarrow v^2 = -4c_3^2 (u - c_3^2)$$

$$(v-0)^2 = -4c_3^2 (u - c_3^2) \rightarrow (8)$$

$\therefore$ eq^n (8) represents a parabola, symmetric about 'u' axis in the -ve direction having the vertex at $(c_3^2, 0)$



Case 4: let $y = c_4$

$\therefore (2) \Rightarrow u = x^2 - c_4^2$

$x^2 = u + c_4^2 \rightarrow (9)$

$\therefore (3) \Rightarrow v = 2xy$

$v = 2x c_4$

$x = \frac{v}{2c_4} \Rightarrow x^2 = \frac{v^2}{4c_4^2} \rightarrow (10)$

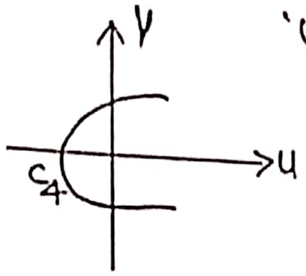
$\therefore$ From eqn (9) & (10)

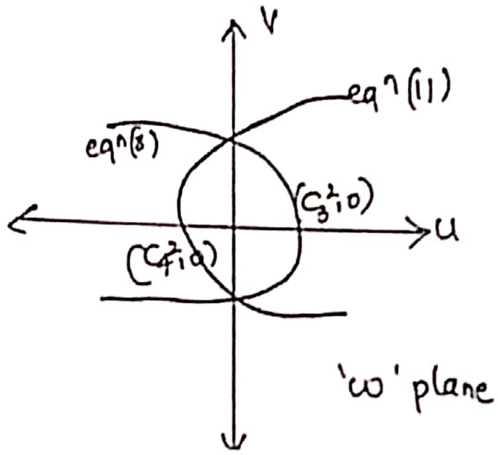
$$\frac{v^2}{4c_4^2} = u + c_4^2$$

$$v^2 = 4c_4^2 (u + c_4^2)$$

$$(v-0)^2 = 4c_4^2(u - (-c_4^2)) \rightarrow (11)$$

$\therefore$ eqn (11) represents a parabola symmetric about 'u' axis in the +ve direction having the vertex at $(-c_4^2, 0)$.





<u>Discussion of the transformation $w = z + \frac{1}{z}$, $z \neq 0$.</u>

Given, $w = f(z) = z + \frac{1}{z}$ is analytic,

$\therefore f(z)$ is differentiable.

$$\Rightarrow f'(z) = z + \frac{1}{z}, \quad z \neq 0$$

$$= 1 - \frac{1}{z^2}, \quad z \neq 1$$

let $z = re^{i\theta}$

$\therefore$ WKT, $w = f(z) = u + iv = z + \frac{1}{z}$

$$\Rightarrow u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$\Rightarrow u + iv = re^{i\theta} + \frac{1}{r} e^{-i\theta}$$

$\Rightarrow u+iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$

$= (r\cos\theta + ir\sin\theta) + \left(\frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta\right)$

$\Rightarrow u+iv = \left(r+\frac{1}{r}\right)\cos\theta + i\left(r-\frac{1}{r}\right)\sin\theta$

$\therefore u = \left(r+\frac{1}{r}\right)\cos\theta \rightarrow (1)$, $v = \left(r-\frac{1}{r}\right)\sin\theta \rightarrow (2)$

$\therefore (1) \Rightarrow \frac{u}{\left(r+\frac{1}{r}\right)} = \cos\theta \rightarrow (3)$ $\therefore (2) \Rightarrow \frac{v}{\left(r-\frac{1}{r}\right)} = \sin\theta \rightarrow (4)$

$\therefore (3)^2 + (4)^2 \quad \frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = \cos^2\theta + \sin^2\theta$

$$\frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\left(r-\frac{1}{r}\right)^2} = 1 \rightarrow (5)$$

From eqn (1) $\Rightarrow \frac{u}{\cos\theta} = r+\frac{1}{r} \rightarrow (6)$ | $\frac{v}{\sin\theta} = r-\frac{1}{r} \rightarrow (7)$

$\therefore (6)^2 - (7)^2$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2$$

$$\Rightarrow \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4r\left(\frac{1}{r}\right) \quad \Big| \ (a+b)^2-(a-b)^2 = 4ab$$

$$\Rightarrow \frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4$$

$$\Rightarrow \frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1$$

$$\Rightarrow \frac{u^2}{(2\cos\theta)^2} - \frac{v^2}{(2\sin\theta)^2} = 1 \rightarrow (8)$$

Case 1: when $r = c_1$, we can have eqn (5) $\Rightarrow \frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$ represents Ellipse at Centre origin & vertex @ $(\pm 2, 0)$

Case 2: If $\theta = c_2$ is a constant, we have

$$\text{eq}^n (8) \Rightarrow \frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$$

represents a hyperbola at the points $(\pm 2, 0)$

Also the Ellipse in the 'w' plane, we have $|z| = r$

$$= \sqrt{x^2+y^2} = r$$

$$\Rightarrow x^2+y^2 = r^2$$

represents a circle in the 'z' plane and for the hyperbola in the 'w' plane, we have θ = amplitude of z

$\tan\theta = m$ = Slope

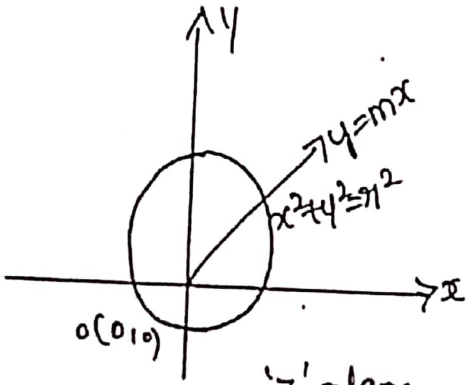
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$y/x = \tan\theta$$

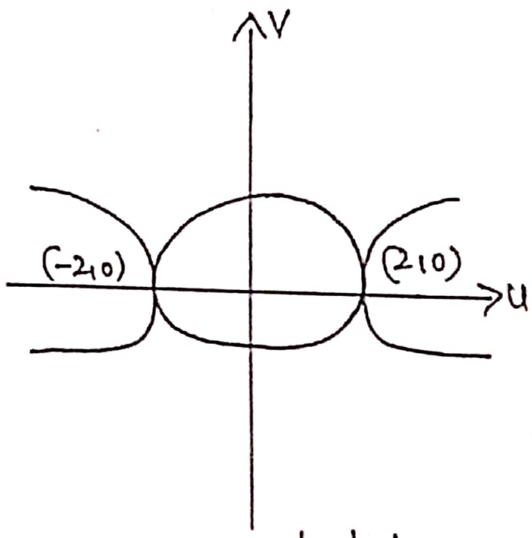
$$y = x\tan\theta$$

$$\boxed{y = mx}$$

represents a straight line.



'z' plane



'w' plane

Belinear Transformation:

A transformation of the form $w = \frac{az+b}{cz+d}$, where a, b, c, d are real (or) complex constants, such that $ad - bc \neq 0$ is called a Belinear transformation (or) Linear fractional transformation (or) Mobius transformation.

The ratio $w = \frac{az+b}{cz+d}$ Can also be expressed as

$cwz + dw - az - b = 0$ is linear both in 'z' & 'w', hence it is called a belinear transformation and if $\frac{dw}{dz} \neq 0$. hence it is the conformal.

Properties of Belinear transformation:

(1) The transformation $w = \frac{az+b}{cz+d}$ sets of a one-two-one Corresponds between the points of closed 'z'-plane and the closed w-plane.

(2) If $ad - bc = 0$, then 'w' is either a Constant (or) meaningless.

(3) Invarient points (or) fixed points:

If a point 'z' Maps into itself i.e., $w = z$ under the belinear transformation then the point is called invarient point @ fixed points of belinear transformation.

Ex: Suppose invarient of $w = z^2$ are the solutions of equations

$\Rightarrow z^2 - z = 0$

$\Rightarrow z(z-1) = 0$

$\therefore z = 0,\ z = 1.$

(4). Cross ratio: If z_1, z_2, z_3, z_4 are four distinct points then the ratio is called the cross ratio of these points and it's denoted by $(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$

(5) A bilinear transformation preserves the cross ratio of the 4 points (i.e.) Let w_1, w_2, w_3, w_4 be the images of 4 distinct points of z_1, z_2, z_3, z_4 in the z-plane under a bilinear transformation, then

$$\Rightarrow (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

$$\Rightarrow \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Problems:

1. Find the Bilinear transformation which map the points $z = 1, i, -1$ into $w = i, 0, -i$.

Given, $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and $w_1 = i$, $w_2 = 0$, $w_3 = -i$

WKT, From the one-two-one bilinear transformation

$$\Rightarrow (w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-i)(0+i)}{(i-0)(-i-w)} = \frac{(z-1)(i+1)}{(z-i)(-1-z)}$$

$$\Rightarrow \frac{\cancel{i}(w-i)}{\cancel{-i}(w+i)} = \frac{(z-1)(1+i)}{\cancel{-}(z+1)(1-i)}$$

$$\Rightarrow \frac{w-i}{w+i} = \frac{(z+1)(1+i)}{(z+1)(1-i)}$$

$\Rightarrow (w-i)(z+1)(1-i) = (w+i)(z-1)(1+i)$

$\Rightarrow (w-i)(z-iz+1-i) = (w+i)(z+iz-1-i)$

$\Rightarrow \cancel{wz} - iwz + w - i\cancel{w} - iz - \cancel{z} - \cancel{i} - 1 = \cancel{wz} + iwz - w - \cancel{iw} + iz - \cancel{z} - \cancel{i} + 1$

$\Rightarrow -iwz - iwz + w + w = iz + iz + 1 + 1$

$\Rightarrow -2iwz + 2w = 2iz + 2$

$\Rightarrow \cancel{2}w(1-iz) = \cancel{2}(1+iz)$

$\Rightarrow \boxed{w = \frac{(1+iz)}{(1-iz)}}$

2. Find the Belinear transformation, which map the points $z = 1, i, -1$ into $w = 2, i, -2$. Also find the invariant points of transformation.

Given, $z = 1, i, -1$, $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and

$w_1 = 2$, $w_2 = i$, $w_3 = -2$

WKT, Belinear transformation,

$$w = \frac{az+b}{cz+d} \rightarrow (1)$$

$\therefore$ when $z = 1$, $w = 2$

$\therefore (1) \Rightarrow 2 = \frac{a(1)+b}{c(1)+d} = \frac{a+b}{d+c}$

$\Rightarrow a+b = 2d+2c$

$\Rightarrow a+b-2c-2d = 0 \rightarrow (2)$

$\therefore$ when $z = i$, $w = i$

$\therefore (1) \Rightarrow i = \frac{ai+b}{ci+d} \Rightarrow ai+b = i(ci+d)$

$\Rightarrow ai+b = -c+id$

$\Rightarrow ai+b+c-id = 0 \rightarrow (3)$

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$\therefore$ when $z = -1$, $\omega = -2$

$$\therefore (1) \Rightarrow -2 = \frac{a(-1)+b}{c(-1)+d}$$

$$\Rightarrow -2 = \frac{-a+b}{-c+d}$$

$$\Rightarrow -a+b = -2(-c+d)$$

$$\Rightarrow -a+b = +2c-2d$$

$$\Rightarrow -a+b-2c+2d = 0 \rightarrow (4)$$

Now eqn (2) - (3)

$$\Rightarrow (a+\cancel{b}-2c-2d) - (ai+\cancel{b}+c-id) = 0$$

$$\Rightarrow a(1-i) - 3c + (i-2)d = 0$$

Now eqn (3) - (4)

$$\Rightarrow (ai+\cancel{b}+c-id) - (-a+\cancel{b}-2c+2d) = 0$$

$$\Rightarrow a(1+i) + 3c - d(i+2) = 0 \rightarrow (5)$$

a	c	d	a
$1-i$	-3	$i-2$	$1-i$
$1+i$	$+3$	$-(i+2)$	$1+i$

$$\Rightarrow \frac{a}{3(i+2)-3(i-2)} = \frac{c}{(i-2)(1+i)+(1-i)(i+2)} = \frac{d}{3(1-i)+3(1+i)}$$

$$\Rightarrow \frac{a}{3(\cancel{i}+2-\cancel{i}+2)} = \frac{c}{+\cancel{i}-\cancel{2}-\cancel{1}-2i+\cancel{i}+\cancel{2}+\cancel{1}-2i} = \frac{d}{3(1-\cancel{i}+1+\cancel{i})}$$

$$\Rightarrow \frac{a}{12} = \frac{c}{-2i} = \frac{d}{6} = k$$

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$\Rightarrow a = 12k, \quad c = -2ik, \quad d = 6k$

$\therefore$ eqⁿ(2) $\Rightarrow 1\cancel{2k} + b + 4ik - \cancel{12}k = 0$

$$\Rightarrow b = -4ik$$

$$\therefore (1) \Rightarrow w = \frac{12kz - 4ik}{-2ikz + 6k}$$

$$\Rightarrow w = \frac{12z - 4i}{-2iz + 6}$$

$$\Rightarrow w = \frac{6z - 2i}{-iz + 3} \rightarrow (6)$$

when $w = z$

$$(6) \Rightarrow z = \frac{6z - 2i}{-iz + 3}$$

$$\Rightarrow z(-iz + 3) = 6z - 2i$$

$$\Rightarrow z(-iz + 3) - 6z + 2i = 0$$

$$\Rightarrow -iz^2 + 3z - 6z + 2i = 0$$

$$\Rightarrow -i^2z^2 - 3zi + 2i^2 = 0 \quad (\times ly \text{ by } i)$$

$$\Rightarrow z^2 - 3zi - 2 = 0$$

$$\Rightarrow z = \frac{3i \pm \sqrt{9i^2 - 4(1)(2)}}{2(1)}$$

$$\Rightarrow z = \frac{3i \pm \sqrt{-9 + 8}}{2} = \frac{3i \pm \sqrt{-1}}{2} = \frac{3i \pm \sqrt{i^2}}{2}$$

$$\Rightarrow z = \frac{3i \pm i}{2}$$

Case 1: $z = \frac{3i + i}{2} = \frac{4i}{2} = 2i$

Case 2: $z = \frac{3i - i}{2} = \frac{2i}{2} = i$

$\therefore \boxed{z = 2i, i}$

3. Find the Belinear transformation, which maps $z = \infty, i, 0$ into $w = -1, -i, 1$. Also find the fixed points of the transformati-on.

Soln: For any real @ complex constants

WKT, Belinear transformation, $w = \frac{az+b}{cz+d} \rightarrow (1)$

$\therefore (1) \Rightarrow w = \frac{a+b/z}{c+d/z} \rightarrow (2)$

and given $z_1 = \infty, z_2 = i, z_3 = 0$ and $w_1 = -1, w_2 = -i, w_3 = 1$.

$\therefore$ when $z = \infty$, $\therefore w = -1$

$\therefore (2) \Rightarrow -1 = \frac{a+0}{c+0} = \frac{a}{c}$

$a = -c$

$a + c = 0 \rightarrow (3)$

$\therefore$ when $z = i$, $w = -i$

$\therefore (2) \Rightarrow -i = \frac{ai+b}{ci+d}, \Rightarrow -i(ci+d) = ai+b$

$\Rightarrow ai + b - c + id = 0 \rightarrow (4)$

$\therefore$ when $z = 0$, $w = 1$

$\therefore (2) \Rightarrow 1 = \frac{b}{d}$

$\Rightarrow b - d = 0 \rightarrow (5)$

$\therefore (3) + (4)$

$a(1+i) + b + di = 0 \rightarrow (6)$

$\therefore (5) \Rightarrow 0.a + b - d = 0 \rightarrow (7)$

a	b	c	a
$1+i$	1	i	$1+i$
0	1	-1	0

$\frac{a}{(-1-i)} = \frac{b}{(1+i)} = \frac{d}{(1+i)}$

$$\frac{a}{-(1+\lambda)} = \frac{b}{(1+\lambda)} = \frac{d}{(1+\lambda)} = K$$

$$\frac{a}{-1} = \frac{b}{1} = \frac{d}{1} = K$$

$a = -K$, $b = K$, $d = K$

and eq^n (3) $a+c=0$

$c = -a = -(-K) = \underline{\underline{K}}$

WKT,

$$w = \frac{az+b}{cz+d} = \frac{-Kz+K}{Kz+K} = \frac{-z+1}{z+1}$$

$$\boxed{w = \frac{-z+1}{z+1}} \rightarrow (8)$$

To find invariant points, let $w = z$

$(8) \Rightarrow z = \frac{-z+1}{z+1}$.

$z(z+1) = (-z+1)$

$\Rightarrow (z^2+z) + (z-1) = 0$

$\Rightarrow z^2+2z-1=0$

$$z = \frac{-2 \pm \sqrt{4-4(1)(-1)}}{2(1)} = \frac{-2\pm\sqrt{4+4}}{2} = \frac{-2\pm\sqrt{8}}{2}$$

$$\Rightarrow z = \frac{-2\pm 2\sqrt{2}}{2} = \frac{-2(-1\pm\sqrt{2})}{2} = -1\pm\sqrt{2}.$$

$\Rightarrow z = -1+\sqrt{2}$, $z = -1-\sqrt{2}$

4. Find the Belinear transformation, which map the points $z = 1, i, -1$ into $w = 0, 1, \infty$.

Sol: WKT, for any real (or) complex constants of a, b, c, d, the Belinear transfamation is $w = \frac{az+b}{cz+d} \rightarrow (1)$

$$\therefore (1) \Rightarrow \frac{1}{w} = \frac{cz+d}{az+b} \rightarrow (2)$$

and given, $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and $w_1 = 0$, $w_2 = +1$, $w_3 = \infty$

$\therefore$ when $z = 1$, $w = 0$ $\therefore \Rightarrow (1) \Rightarrow 0 = \frac{a+b}{c+d}$

$\Rightarrow a+b = 0 \rightarrow (3)$

$\therefore$ when $z = i$, $w = 1$

$$\Rightarrow (1) \Rightarrow 1 = \frac{ai+b}{ci+d}$$

$$ai + b - ci - d = 0 \rightarrow (4)$$

$\therefore$ when $z = -1$, $w = \infty$

$$\Rightarrow (3) \Rightarrow \frac{1}{\infty} = \frac{c(-1)+d}{d(-1)+b} = \frac{-c+d}{-d+b}$$

$$0 = -c + d \rightarrow (5)$$

eqn $(4) - (3)$

$$(i-1)a - ci - d = 0 \rightarrow (6)$$

$$\therefore (5) \Rightarrow 0.a - c + d = 0 \rightarrow (7)$$

$$\frac{a}{(i-1)} \quad \frac{c}{-i} \quad \frac{d}{-1} \quad \frac{a}{(i-1)}$$

$$0 \quad -1 \quad 1 \quad 0$$

$$\frac{a}{-i-1} = \frac{c}{-(i-1)} = \frac{d}{-(i-1)} = K$$

$\Rightarrow a = -K(i+1)$, $c = -K(i-1)$, $d = -K(i-1)$

$c = K(1-i)$ $\quad$ $d = K(1-i)$

$\Rightarrow$ eqn (3) $b = -a$

$$b = -(-K(1+i))$$

$$b = K(1+i)$$

$$\therefore w = \frac{-K(1+i)z + K(1+i)}{K(1-i)z + K(1-i)}$$

$$w = \frac{-(1+i)z + (1+i)}{(1-i)z + K(1-i)}$$

Complex Integration:

Suppose $w=f(z)$ be a Continuous complex valued function over a region 'R'. for any Complex variable $z=x+iy$ in the Curve 'C'. then the Complex integration of $f(z)$ from the point 'p' to 'Q' Can be defined as $\int_c f(z).dz$ and which will be evaluated by dividing the interval into 'n' no of parts.

Line Integral of a Complex valued function:

let $f(z)=u(x,y)+iv(x,y)$ be a continuous Complex valued function over a region 'R' of any Complex variable $z=x+iy$ in the Curve 'C', then the line of Integral of $f(z)$ Can be defined as $\int_c f(z).dz = \int_c (u+iv)(dx+idy)$

$$= \int_c (u.dx+iu.dy+ivdx-vdy)$$

$$= \int_c (udx-vdy)+i\int_c (vdx+udy)$$

Problems:

1. Evaluate $\int_c |z|^2.dz$, where 'c' is of square with the vertices $(0,0),(1,0),(1,1),(0,1)$.

Given, $\int_c |z|^2.dz$

$f(z)=|z|^2$, where $z=x+iy$

$\Rightarrow |z|=\sqrt{x^2+y^2}$

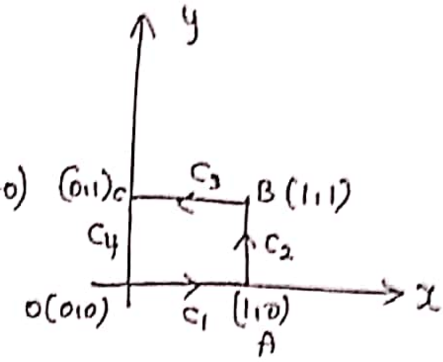
$\Rightarrow |z|^2=x^2+y^2$

and $dz=dx+idy$

$\therefore$ WKT $\int_C f(z).dz = \int_C (x^2+y^2)(dx+idy) \rightarrow (1)$

$$= \int_{C_1} (x^2+y^2)(dx+idy) + \int_{C_2} (x^2+y^2)(dx+idy) + \int_{C_3} (x^2+y^2)(dx+idy) + \int_{C_4} (x^2+y^2)(dx+idy) \rightarrow (2)$$

(i) Along the curves, $C_1 \Rightarrow y=0$
$dy=0$



$$\therefore \int_C (x^2+y^2)(dx+idy) = \int_{x=0}^{1} (x^2)(dx+0)$$

$$= \int_0^1 (x^2)(dx)$$

$$= \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

(ii) Along the curves, $C_2 \Rightarrow x=1, dx=0$

$$\therefore \int_C (x^2+y^2)(dx+idy) = \int_{y=0}^{1} (1+y^2)(idy)$$

$$= i\int_0^1 (1+y^2)\,dy \Rightarrow i\left(y+y^3/3\right)\Big|_0^1$$

$$= i\left(1+\frac{1}{3}\right) = \frac{4i}{3}$$

(iii) Along the curves, $C_3 \Rightarrow y=1, dy=0$

$$\therefore \int_C (x^2+y^2)(dx+idy) = \int_{x=1}^{0} (x^2+1)\,dx$$

$$= \left.\frac{x^3}{3}+x\right|_1^0 \Rightarrow 0-\left(\frac{1}{3}+1\right) = \frac{-4}{3}$$

(v) Along the curves, $C_4 \Rightarrow x=0, dx=0$

$$\therefore \int_C (x^2+y^2)(dx+idy) = \int_{y=1}^{0} y^2\, idy \Rightarrow i\left.\frac{y^3}{3}\right|_1^0 \Rightarrow 0-\frac{i}{3} = \frac{-i}{3}$$

$$\therefore \int_C f(z)\,dz = \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3}$$

$$= -1+1i$$

$$= -1+i$$

Cauchy's Integral Theorem (or) Fundamental Theorem.

Statement : If a function $f(z)$ is analytic at all the points within and on a closed contour 'c' then $\int_C f(z)dz = 0$.

Proof: Given, $w = f(z) = u(x,y) + i\,v(x,y)$ is analytic function over a region 'R' at any complex variable $z = x+iy$ of a curve 'c', we have $\int_C f(z).dz = \int_C (u+iv)(dx+idy)$.

$$\Rightarrow \int_C f(z).dz = \int_C (u\,dx - v\,dy) + i(v\,dx + u\,dy) \rightarrow (1)$$

By the Green's Theorem, WKT

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\therefore (1) \Rightarrow \int_C f(z).dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \rightarrow (2)$$

Given, $f(z) = u+iv$ is analytic and it can satisfy the C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\therefore (2) \Rightarrow \int_C f(z)dz = \iint_R \left(-\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

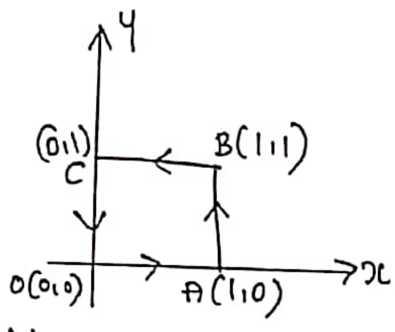
$$\Rightarrow \int_C f(z).dz = 0 + i0$$

$$\Rightarrow \int_C f(z)\,dz = 0$$

Problem:

Verifying Cauchy's Theorem for the function $f(z) = z^2$, where 'C' is the square having the vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$.

Given, The Complex valued function $f(z) = z^2$, where 'C' is a closed square having the vertices $O(0,0)$, $A(1,0)$, $B(1,1)$ and $C(0,1)$.



WKT, $z = x+iy$

$\Rightarrow dz = dx + idy$

$\therefore f(z)dz = z^2 . dz$

$\Rightarrow f(z).dz = (x+iy)^2 (dx+idy) \rightarrow (1)$

and $\int_C f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CO} f(z).dz \rightarrow (2)$

$\therefore$ Along OA $\Rightarrow y=0$
$dy = 0$

$$\int_{OA} f(z)dz = \int_{OA} (x+i(0))^2 . [dx+i(0)]$$

$$= \int_{OA} x^2 . dx = \int_{x=0}^{1} x^2 . dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$\therefore$ Along AB $\Rightarrow x = 1$
$dx = 0$

$$\int_{AB} f(z)dz = \int_{AB} (1+iy)^2 . (0+idy)$$

$$= i \int_{AB} (1+2iy-y^2)\, dy$$

$$= i \int_{y=0}^{1} (1+2iy-y^2)\, dy = i \left| y + iy^2 - \frac{y^3}{3} \right|_0^1$$

$$\int_{AB} f(z).dz = i\left|1+i-\frac{1}{3}\right| = i\left|1+\frac{2}{3}\right| = i^2+\frac{2i}{3}$$

$$= -1+\frac{2i}{3}$$

$$\therefore \int_{AB} f(z)dz = -1+\frac{2i}{3}$$

$\therefore$ Along BC $\Rightarrow y=1,\ dy=0$

$$\int_{BC} f(z).dz = \int_{BC} (x+i)^2.(dx+i(0))$$

$$= \int_{BC} (x^2+2ix-1)\,dx$$

$$= \int_{x=1}^{0} (x^2+2ix-1)\,dx$$

$$= \left(\frac{x^3}{3}+2i\frac{x^2}{2}-x\right|_1^0$$

$$= 0-\left(\frac{1}{3}+i-1\right)$$

$$= -\left(i-\frac{2}{3}\right) = \frac{2}{3}-i$$

$\therefore$ Along CO $\Rightarrow x=0,\ dx=0$

$$\int_{CO} f(z).dz = \int_{CO} (0+iy)^2.(0+idy)$$

$$= \int_{CO} i^2y^2.idy$$

$$= \int_{CO} -y^2.idy \Rightarrow -i\int_{y=1}^{0} y^2.dy$$

$$= -i\left|\frac{y^3}{3}\right|_1^0$$

$$= -i\left(0-\frac{1}{3}\right) = \frac{i}{3}$$

$\therefore (2) \Rightarrow \int_C f(z).\,dz = \frac{1}{3} - 1 + \frac{2i}{3} + \frac{2}{3} - i + \frac{i}{3}$

$$\int_C f(z)dz = 0$$

$\therefore$ Hence Cauchy's Theorem Verified.

Theorem : Cauchy's Integral formula.

Statement : If $f(z)$ is analytic inside and on a Simple closed Curve 'c' and if 'a' is anypoint within 'C', then $f(a) = \frac{1}{2\pi i} \int \frac{f(z)}{z-a} \,.\, dz$

Proof : Given $f(z)$ is an analytic function inside and on a Simple closed Curve 'c'.
Since 'a' is a point within 'C'. we shall enclose it by a circle 'c_1' and with $z=a$ as a Center and 'r' as a radius, Such that C_1 lies entirely within 'C'.
Therefore the function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the region b/n 'c' and 'c_1'.

we have $\int_C \frac{f(z)}{(z-a)} dz = \int_{c_1} \frac{f(z)}{(z-a)} dz \rightarrow (1)$

WKT, $|z-a| = r$

$\Rightarrow z-a = re^{i\theta}$

$\Rightarrow z = a+re^{i\theta}$

$\Rightarrow \frac{dz}{d\theta} = (0+rie^{i\theta})$

$\Rightarrow dz = ri.e^{i\theta}.d\theta$

$\therefore (1) \Rightarrow \int_C \frac{f(z)}{z-a} .dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} .rie^{i\theta}.d\theta$

$$\Rightarrow \int_C \frac{f(z)}{z-a} \cdot dz = i \int_0^{2\pi} f(a + re^{i\theta})\, d\theta \rightarrow (2)$$

when $r = 0$,

$$\therefore (2) \Rightarrow \int_C \frac{f(z)}{z-a}\, dz = i \int_{\theta=0}^{2\pi} f(a) \cdot d\theta.$$

$$\Rightarrow \int_C \frac{f(z)}{z-a} \cdot dz = i \cdot f(a) \cdot \int_0^{2\pi} 1 \cdot d\theta$$

$$= i \cdot f(a) \cdot |\theta|_0^{2\pi}$$

$$= i \cdot f(a)(2\pi)$$

$$\Rightarrow \int_C \frac{f(z)}{(z-a)}\, dz = i \cdot f(a) \cdot 2\pi$$

$$\therefore f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)}\, dz$$

Generalized Cauchy's Integral Formula:

If $f(z)$ is analytic inside and on a Simple closed curve 'c' and if 'a' is any point within 'C', then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} \cdot dz$$

also $f^n(a) \cdot \frac{2\pi i}{n!} = \int_C \frac{f(z)}{(z-a)^{n+1}} \cdot dz$

Problems:

1. Evaluate $\int_C \frac{e^{2z}}{(z+1)(z+2)} \cdot dz$, where 'c' is a circle $|z| = 3$.

Sol let $I = \int_C \frac{e^{2z}}{(z+1)(z+2)}\, dz \rightarrow (1)$

Since $c: |z| = 3$ is a circle with the Center '0' (zero) and radius '3'.

$z=-1$ and $z=-2$ are inside of the circle.

$\therefore f(z)=e^{2z}$ is analytic at all points except $z=-1$ and $z=-2$.

$$\therefore \frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$\Rightarrow 1 = A(z+2) + B(z+1) \rightarrow (2)$$

when $z=-2$, when $z=-1$

$\therefore (2) \Rightarrow 1=-B$ $\boxed{B=-1}$ $\therefore (2) \Rightarrow 1=A$ $\boxed{A=1}$

$$\therefore \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$\cdot 1 \frac{e^{2z}}{(z+1)(z+2)} = \frac{e^{2z}}{z+1} - \frac{e^{2z}}{z+2}$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)(z+2)} dz = \int_C \frac{e^{2z}}{z-(-1)} dz - \int_C \frac{e^{2z}}{z-(-2)} dz$$

$$= 2\pi i \, e^{2(-1)} - 2\pi i \, e^{2(-2)}$$

$$= 2\pi i \, e^{-2} - 2\pi i \, e^{-4}$$

$$= 2\pi i \left(e^{-2} - e^{-4}\right)$$

$$= 2\pi i \left(\frac{1}{e^2} - \frac{1}{e^4}\right) = 2\pi i \left(\frac{e^4 - e^2}{e^8}\right)$$

$$= 2\pi i \left(\frac{e^2 - 1}{e^4}\right)$$

2. Evaluate $\int_C \frac{e^{2z}}{(z+1)(z+2)} dz$, where 'c' is a circle $|z|=3$.

Sol: let $I = \int_C \frac{e^{2z}}{(z+1)(z+2)} \cdot dz \longrightarrow (1)$

Since $C : |z|=3$ is a circle with the Center '0'(zero) &

Radius 3.

$z=-1$ and $z=2$ are inside of the circle.

$\therefore f(z)=e^{2z}$ is analytic at all points except $z=-1$ and $z=2$.

$$\therefore \frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2} \rightarrow (2)$$

$$1 = A(z-2) + B(z+1)$$

when $z=-1$, when $z=+2$

$1 = A(-3)$ $\qquad$ $1 = B(3)$

$\boxed{A=-1/3}$ $\qquad$ $\boxed{B=1/3}$

$$\therefore \frac{1}{(z+1)(z-2)} = \frac{1}{3(z-2)} - \frac{1}{3(z+1)}$$

$$\therefore \frac{e^{2z}}{(z+1)(z-2)} = \frac{e^{2z}}{3(z-2)} - \frac{e^{2z}}{3(z+1)}$$

$$\therefore \int_c \frac{e^{2z}}{(z+1)(z-2)}\,dz = \int_c \frac{e^{2z}}{3(z-2)}\,dz - \int_c \frac{e^{2z}}{3(z+1)}\,dz$$

$$= \int_c \frac{e^{2z}}{3(z-(2))}\,dz - \int_c \frac{e^{2z}}{3(z-(-1))}\,dz$$

$$= \frac{2\pi i}{3} \cdot e^{2(2)} - \frac{2\pi i}{3} e^{2(-1)}$$

$$= \frac{2\pi i}{3} e^{4} - \frac{2\pi i}{3} e^{-2}$$

$$= \frac{2\pi i}{3}(e^4 - e^{-2}) = \frac{2\pi i}{3}\left(e^4 - \frac{1}{e^2}\right)$$

3. Evaluate $\int_c \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\,dz$ where 'c' is a circle $|z|=3$.

Sol let $I = \int_c \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\,dz \rightarrow (1)$

Since $C : |z|=3$ is a circle with a circle '0' and radius 3.

$\therefore z=1$ and $z=2$ are inside of the circle.

$\therefore f(z) = \sin(\pi z^2)+\cos(\pi z^2)$ is analytic at all points except $z=1$ and $z=2$.

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \longrightarrow (2)$$

$$1 = A(z-2) + B(z-1)$$

when $z=1$, when $z=2$

$1 = A(-1)$ $\quad$ $1 = B(1)$

$\boxed{A=-1}$ $\quad$ $\boxed{B=1}$

From eqn (2)

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\therefore \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)} dz = \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2} . dz - \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1} dz$$

where Substitute $z=2$ and $z=1$

$$= 2\pi i(\sin 4\pi + \cos 4\pi) - 2\pi i(\sin\pi + \cos\pi)$$

$$= 2\pi i(0+1) - 2\pi i(0-1)$$

$$= 2\pi i + 2\pi i$$

$$= \underline{\underline{4\pi i}}$$

4. Evaluate $\int_C \frac{\sin(\pi z^2)+\cos \pi z^2}{(z-1)^2(z-2)} dz$, where $C : |z| = 3$

Sol: let $I = \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2(z-2)} dz \longrightarrow (1)$

Since $C : |z|=3$ is a circle with centre '0' and radius '3'

$z=2$ and $z=1$ are inside of the circle

$\therefore f(z) = \sin(\pi z^2) + \cos(\pi z^2)$ is a analytic function at all the points except $z=1$ and $z=2$.

$$\therefore \frac{1}{(z-1)^2(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-2)}$$

$$1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2 \longrightarrow (2)$$

when $z=1$, when $z=2$

$1 = B(-1)$ $\quad$ $1 = C$

$\boxed{B=-1}$ $\quad$ $\boxed{C=1}$

$\therefore A+C=0$

$A = -C = -1$

$\boxed{A=-1}$

$$\therefore \frac{1}{(z-1)^2(z-2)} = \frac{-1}{(z-1)} - \frac{1}{(z-1)^2} + \frac{1}{(z-2)}$$

$$\therefore \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)\,dz}{(z-1)^2(z-2)} = \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)}\,dz - \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1}\,.dz - \int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2}\,dz$$

$$= 2\pi i(\sin 4\pi + \cos 4\pi) - 2\pi i(\sin\pi + \cos\pi) \cdots - \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^{1+1}}\,dz$$

$$= 2\pi i(0+1) - 2\pi i(0-1) - \frac{2\pi i}{1!} f'(1) \longrightarrow (3)$$

where $f(z) = \sin\pi z^2 + \cos\pi z^2$

$f'(z) = (\cos\pi z^2 - \sin\pi z^2)\,2\pi z$

$\therefore f'(1) = (\cos\pi - \sin\pi)\, 2\pi$

$= (-1-0)\, 2\pi$

$= -2\pi$

$\therefore$ from eqⁿ (3)

$$\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\, dz = 2\pi i + 2\pi i - \frac{2\pi i}{1!}(-2\pi)$$

$$= 4\pi i + \frac{4\pi^2 i}{1}$$

$$= 4\pi i\,(1+\pi)$$

5. Evaluate $\int_c \frac{z^2-z+1}{z-1}\, dz$ where 'c' is a circle (i) $|z|=1$ (ii) $|z| = 1/2$

Sol: let $I = \int_c \frac{z^2-z+1}{z-1}\, dz \rightarrow (1)$

(i) Since $c: |z|=1$ is a circle with the centre '0' and radius 1. (at origin)

$z=1$ is on the circle $|z|=1$.

Hence $f(z) = z^2-z+1$ is analytic except $z=1$.

$$\therefore \int_c \frac{z^2-z+1}{z-1}\, dz = 2\pi i\,(1^2-1+1)$$

$$= 2\pi i$$

(ii) Since $c: |z| = 1/2$ can represent the circle with centre as origin and radius $1/2$.

$z=1$ is outside of the circle $|z| = 1/2$

$$\therefore \int_c \frac{z^2-z+1}{z-1}\, dz = 0 \qquad (\because \text{outside boundary})$$