

COMPLEX ANALYSIS , PROBABILITY AND STATISTICAL METHOD

MODULE : 01

CALCULUS OF COMPLEX VARIABLES

Definition:

let x, y be the two real values, then the number (or) variable $z = x + iy$ is called the complex variable (or) complex number. where the first part of z is called real part and the second part is called imaginary part. But both x and y are the real values.

$\boxed{z = x + iy}$ is a complex number in the Cartesian form and its conjugate complex is $\bar{z} = x - iy$ and $|z| = \sqrt{x^2 + y^2}$

Geometrical representation:

let \vec{ox} , \vec{oy} be the real and imaginary axes, let 'p' be any point on the plane and 'M' be the foot of the perpendicular of 'p' on the real axis. $O(0,0)$

let ' θ ' be the angle b/w \vec{op} and \vec{ox} . Then from the right angle \triangle opM.

$$\sin \theta = \frac{y}{r} \Rightarrow y = r \sin \theta$$

$$\cos \theta = \frac{x}{r} \Rightarrow x = r \cos \theta$$

Substitute the above in a complex variable $z = x + iy$
we get, $z = r \cos \theta + i r \sin \theta$

$$z = r (\cos \theta + i \sin \theta)$$

$[z = r e^{i\theta}]$ is called the complex variable
in polar form.

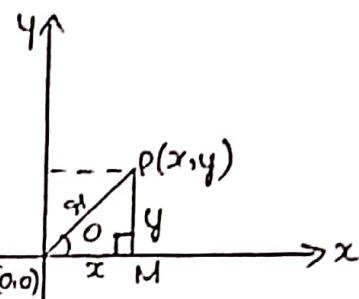
Complex valued function:

- Suppose $u(x,y)$ and $v(x,y)$ be the two real functions in the variables 'x' and 'y', then the complex valued function in the Cartesian form can be defined as

$$w = f(z) = u(x,y) + iv(x,y) = u + iv$$

- Suppose $u(r,\theta)$ and $v(r,\theta)$ be the two real functions in the variables of 'r' and ' θ ', then the complex valued function in the polar form can be defined as

$$w = f(z) = u(r,\theta) + iv(r,\theta)$$



Some Important Results:

WKT,

$$e^{ix} = \cos x + i \sin x \rightarrow (1)$$

$$\text{and } e^{-ix} = \cos x - i \sin x \rightarrow (2)$$

$$\text{from (1) + (2), } e^{ix} + e^{-ix} = 2 \cos x$$

$$\Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2} \rightarrow (3)$$

$$\Rightarrow (1) - (2), e^{ix} - e^{-ix} = 2i \sin x$$

$$\Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i} \rightarrow (4)$$

$$\begin{aligned} \therefore (3) \Rightarrow \cos ix &= \frac{e^{i\alpha x} + e^{-i\alpha x}}{2} \\ &= \frac{e^{-x} + e^x}{2} \end{aligned}$$

$$\Rightarrow \boxed{\cosh x}$$

$$\begin{aligned} (4) \Rightarrow \sin ix &= \frac{e^{i\alpha x} - e^{-i\alpha x}}{2i} \\ &= \frac{e^{-x} - e^x}{2i} \\ &= -\frac{(e^x - e^{-x})}{2i} \\ &= \frac{i(e^x - e^{-x})}{2i} \\ &= \frac{i(e^x - e^{-x})}{2} \end{aligned}$$

$$\Rightarrow \boxed{\sinh x = i \sinh x}$$

Some definitions:

* Limit of a Complex variable: Suppose z be an complex variable to the neighbourhood of z_0 , then for any positive small quantity ' δ ' (delta), $|z - z_0| \leq \delta$ is called the limit of a Complex variable.

* Limit of a Complex valued function: Suppose $w = f(z)$ be a complex valued function for any positive quantity ' ϵ ', we have $|f(z) - f(z_0)| < \epsilon$ (or) $|f(z) - l| < \epsilon$ (or)
 $\lim_{z \rightarrow z_0} f(z) = l$ is called limit of a complex valued function

* Differentiability of a function: Suppose $w = f(z)$ be a complex valued function and is differentiable at z , then
 $f'(z) = \frac{dw}{dz} = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z}$, $\delta z \neq 0$,

* Continuity: A Complex valued function $f(z)$ is said to be a continuous, if and only if $\lim_{z \rightarrow z_0} f(z) = f(z_0)$.

Analytic of a function:

let $w = f(z)$ be a Complex valued function and it is said to be an analytic function when $w = f(z)$ to be a differentiable at any point of z ,

$$\text{i.e., } f'(z) = \frac{dw}{dz} = \lim_{\delta z \rightarrow 0} \left(\frac{f(z + \delta z) - f(z)}{\delta z} \right) \therefore \delta z \neq 0$$

It is also called regular (or) holomorphic function.

Theorem: Cauchy Riemann Equation in Cartesian form (or)
C-R Equation.

Statement: The necessary Condition that the function $w=f(z) = u(x,y) + iv(x,y) = u+iv$ may be analytic at any point $z=x+iy$ is that, there exists 4 continuous first order partial derivatives,

i.e., $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ can satisfy the given equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Proof: Given, $w=f(z) = u(x,y) + iv(x,y) \rightarrow (1)$ is a complex valued function in the Cartesian form, for the complex variable $z=x+iy$.

$$\therefore \text{eqn (1)} \Rightarrow f(x+iy) = u(x,y) + iv(x,y) \rightarrow (2)$$

and given $w=f(z)$ is an analytic function.

Differentiating eqn (2) wrt 'x' partially,

$$(2) \Rightarrow f'(x+iy) \cdot 1 = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\Rightarrow f'(x+iy) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \rightarrow (3)$$

Diff. eqn (2) wrt 'y' partially,

$$(2) \Rightarrow f'(x+iy) (i) = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy) = \frac{i}{i^2} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow f'(x+iy) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow f'(z+iy) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \rightarrow (4)$$

\therefore from (3) & (4)

$$\Rightarrow \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Hence proved.

Cauchy Riemann Equation in polar form.

Statement: If $w = f(z) = u(\Re, \theta) + iv(\Re, \theta)$ is an analytic function at $z = \Re e^{i\theta}$, then there exists continuous first order partial derivatives $\frac{\partial u}{\partial \Re}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial \Re}, \frac{\partial v}{\partial \theta}$ can satisfy the equations

$$\frac{\partial u}{\partial \Re} = \frac{1}{\Re} \cdot \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial \Re} = -\frac{1}{\Re} \frac{\partial u}{\partial \theta}$$

Proof: Given $w = f(z) = u(\Re, \theta) + iv(\Re, \theta) \rightarrow (1)$ is an analytic function at $z = \Re e^{i\theta}$.

$$\therefore \text{eqn (1)} \Rightarrow w = f(\Re e^{i\theta}) = u(\Re, \theta) + iv(\Re, \theta) \rightarrow (2)$$

Differentiating eqn (2) w.r.t ' \Re ' partially,

$$(2) \Rightarrow f'(\Re e^{i\theta}) e^{i\theta} = \frac{\partial u}{\partial \Re} + i \frac{\partial v}{\partial \Re}$$

$$\Rightarrow e^{i\theta} \cdot f'(\Re e^{i\theta}) = \frac{\partial u}{\partial \Re} + i \frac{\partial v}{\partial \Re} \rightarrow (3)$$

Diff. eqn (2) w.r.t ' θ ' partially,

$$(2) \Rightarrow f'(\Re e^{i\theta}) \Re i \cdot e^{i\theta} = \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta}$$

$$\Rightarrow e^{i\theta} \cdot f'(\Re e^{i\theta}) = \frac{1}{\Re} \frac{\partial u}{\partial \theta} + \frac{i}{\Re} \frac{\partial v}{\partial \theta}$$

$$\Rightarrow e^{i\theta} \cdot f'(\Re e^{i\theta}) = \frac{1}{i^2 \Re} \frac{\partial u}{\partial \theta} + \frac{1}{\Re} \frac{\partial v}{\partial \theta}$$

$$\Rightarrow e^{i\theta} \cdot f'(re^{i\theta}) = \frac{1}{r} \frac{\partial v}{\partial \theta} - i \frac{1}{r} \frac{\partial u}{\partial \theta} \rightarrow (4)$$

from eqn (3) and (4)

$$\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial \theta} = \frac{1}{r} \frac{\partial v}{\partial \theta} - i \frac{1}{r} \frac{\partial u}{\partial \theta}$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial \theta} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

Hence proved.

Derivation of analytic function:

Step 1: Write the given complex valued function $w=f(z)$

Step 2: Write the $z=x+iy$ in the cartesian form and

$z=re^{i\theta}$ in the polar form, we get

$f(z) = u(x, y) + iv(x, y)$ in cartesian form and

$f(z) = u(r, \theta) + iv(r, \theta)$ in polar form.

Step 3: Identify u and v and verify the C-R equations in
the respective forms.

Step 4: Find the derivative of $f(z)$ as

$f'(z) = u_x + iv_x$ in the cartesian form and

$f'(z) = (u_r + iv_r) \cdot e^{-i\theta}$ in polar form.

Step 5: Substitute $x=\bar{z}$, $y=0$ in cartesian form and
 $r=z$, $\theta=0$ in polar form in the required $f'(z)$.

Problems:

- Verify the function $f(z) = z^2$ is analytic (or) not, hence find its derivative.

Sol Given, $f(z) = z^2 \rightarrow (1)$

Let $z = x+iy$

$$\therefore (1) \Rightarrow f(z) = (x+iy)^2$$

$$f(z) = x^2 + i^2 y^2 + 2ixy \quad \therefore i^2 = -1$$

$$= x^2 - y^2 + 2ixy$$

$$f(z) = (x^2 - y^2) + i(2xy)$$

$$\Rightarrow f(z) = u(x,y) + i v(x,y)$$

$$u = x^2 - y^2, v = 2xy$$

$$\therefore \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = +2y, \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y$$

\therefore The given $f(z)$ is satisfied C-R equation.

$\therefore f(z)$ is an analytic.

$$\text{WKT, } f'(z) = u_x + i v_x$$

$$\Rightarrow f'(z) = 2x + i 2y$$

$$f'(z) = 2(x+iy) \rightarrow (2)$$

$$\text{let } x = z, y = 0$$

$$\therefore f'(z) = 2(z+0)$$

$$\therefore f'(z) = 2z$$

Q. Verify $f(z) = z^3$ is analytic or not, hence find its derivative.

Soln Given, $f(z) = z^3 \rightarrow (1)$

$$\text{let } z = r e^{i\theta}$$

$$\therefore (1) \Rightarrow f(z) = (r e^{i\theta})^3$$

$$f(z) = r^3 \cdot e^{i3\theta}$$

$$\Rightarrow f(z) = r^3 (\cos 3\theta + i \sin 3\theta)$$

$$f(z) = r^3 \cos 3\theta + i r^3 \sin 3\theta$$

$$\therefore f(z) = u(r, \theta) + i v(r, \theta)$$

$$U = \pi^3 \cos 3\theta, V = \pi^3 \sin 3\theta$$

$$\therefore \frac{\partial U}{\partial x} = 3\pi^2 \cos 3\theta, \quad \frac{\partial U}{\partial \theta} = -3\pi^3 \sin 3\theta$$

$$\frac{\partial V}{\partial x} = 3\pi^2 \sin 3\theta, \quad \frac{\partial V}{\partial \theta} = 3\pi^3 \cos 3\theta$$

$$\therefore \frac{\partial U}{\partial x} = \frac{1}{\pi} \frac{\partial V}{\partial \theta}, \quad \frac{\partial V}{\partial x} = -\frac{1}{\pi} \frac{\partial U}{\partial \theta}$$

$\therefore f(z)$ is an analytic function.

$$\therefore \text{WKT, } f'(z) = e^{-i\theta} [U_x + iV_x]$$

$$= e^{-i\theta} (3\pi^2 \cos 3\theta + i 3\pi^2 \sin 3\theta)$$

$$\text{let } z = x, \theta = 0$$

$$\therefore f'(z) = 3z^2$$

=

3. Verify $f(z) = z^n$ is an analytic function or not, hence find its derivative for any positive integer 'n'.

$$\text{Given, } f(z) = z^n$$

$$\text{let } z = re^{i\theta}$$

$$f(z) = (re^{i\theta})^n = r^n \cdot e^{in\theta}$$

$$\Rightarrow f(z) = r^n (\cos n\theta + i \sin n\theta)$$

$$f(z) = r^n \cos n\theta + ir^n \sin n\theta$$

$$\therefore f(z) = u(r, \theta) + iv(r, \theta)$$

$$U = r^n \cos n\theta, V = r^n \sin n\theta$$

$$\therefore \frac{\partial U}{\partial x} = n \cdot r^{n-1} \cos n\theta, \quad \frac{\partial U}{\partial \theta} = -n \cdot r^n \sin n\theta$$

$$\frac{\partial V}{\partial x} = n \cdot r^{n-1} \sin n\theta, \quad \frac{\partial V}{\partial \theta} = n \cdot r^n \cos n\theta$$

$$\therefore \frac{\partial U}{\partial x} = \frac{1}{\pi} \frac{\partial V}{\partial \theta}, \quad \frac{\partial V}{\partial x} = -\frac{1}{\pi} \cdot \frac{\partial U}{\partial \theta}$$

$\therefore f(z)$ is an analytic function.

$$\therefore \text{WKT, } f(z) = e^{i\theta} (u_n + i v_n)$$

$$f(z) = e^{i\theta} (n \cdot z^{n-1} \cos \theta + i n \cdot z^{n-1} \sin \theta)$$

$$\text{let } n=2, \theta=0$$

$$f(z) = n \cdot z^{n-1}$$

4. Verify $f(z) = \sin z$ is analytic or not, hence find its derivative.

Soln Given, $f(z) = \sin z \rightarrow (1)$

$$\text{let } z = x+iy$$

$$(1) \Rightarrow f(z) = \sin(x+iy) = \sin x \cdot \cos(iy) + \cos x \cdot \sin(iy)$$

$$\Rightarrow f(z) = \sin x \cdot \cosh y + i \cos x \sinh y$$

$$\Rightarrow f(z) = u(x,y) + i v(x,y)$$

$$u = \sin x \cdot \cosh y, \quad v = \cos x \cdot \sinh y$$

$$\frac{\partial u}{\partial x} = \cos x \cdot \cosh y, \quad \frac{\partial u}{\partial y} = \sin x \cdot \sinh y$$

$$\frac{\partial v}{\partial x} = -\sin x \cdot \sinh y, \quad \frac{\partial v}{\partial y} = \cos x \cdot \cosh y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore f(z)$ is analytic

$$\therefore \text{WKT, } f'(z) = u_x + i v_x$$

$$\Rightarrow f'(z) = \cos x \cdot \cosh y - i \sin x \cdot \sinh y$$

$$\Rightarrow f'(z) = \cos x \cdot \cos(iy) - i \sin x \cdot \sin(iy)$$

$$\Rightarrow f'(z) = \cos(x+iy)$$

$$f'(z) = \cos z$$

=====

5. Verify $f(z) = \sin 2z$ is analytic or not, hence find its derivative.

Soln Given, $f(z) = \sin 2z \rightarrow (1)$

$$\text{let } z = x+iy$$

$$(1) \Rightarrow f(z) = \sin 2(x+iy)$$

$$= \sin(2x+2iy)$$

$$f(z) = \sin(2x) \cdot \cos(2iy) + \cos(2x) \cdot \sin(2iy)$$

$$f(z) = \sin(2x) \cdot \cosh(2y) + \cos(2x) \cdot \sinh(2y)$$

$$\Rightarrow f(z) = u(x,y) + iv(x,y)$$

$$u = \sin(2x) \cdot \cosh(2y), v = \cos(2x) \cdot \sinh(2y)$$

$$\therefore \frac{\partial u}{\partial x} = 2\cos(2x) \cdot \cosh(2y), \frac{\partial u}{\partial y} = 2\sin(2x) \cdot \sinh(2y)$$

$$\frac{\partial v}{\partial x} = -2\sin(2x) \cdot \sinh(2y)$$

$$\frac{\partial v}{\partial y} = 2\cos(2x) \cdot \cosh(2y)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore f(z)$ is an analytic function.

$$\Rightarrow f'(z) = u_x + iv_x$$

$$f'(z) = 2\cos(2x) \cdot \cosh(2y) - i2\sin(2x) \cdot \sinh(2y)$$

$$\Rightarrow f'(z) = 2\cos(2x) \cdot \cos(2iy) - i2\sin(2x) \cdot \sin(2iy)$$

$$\Rightarrow f'(z) = 2[\cos(2x) \cdot \cos(2iy) - \sin(2x) \cdot \sin(2iy)]$$

$$\Rightarrow f'(z) = 2\cos(2x+2iy)$$

$$f'(z) = 2\cos 2(x+iy)$$

$$\Rightarrow f'(z) = 2\cos 2z$$

6. Verify the analytic function $f(z) = \log z$, hence find its derivative.

Soln Given, $f(z) = \log z \rightarrow (1)$

$$\text{let } z = r_1 e^{i\theta}$$

$$(1) \Rightarrow f(z) = \log(r_1 e^{i\theta})$$

$$\Rightarrow f(z) = \log r_1 + \log e^{i\theta}$$

$$\Rightarrow f(z) = \log r_1 + i\theta \cdot \log e \quad \because \log e = 1$$

$$f(z) = \log r_1 + i\theta$$

$$\Rightarrow f(z) = u(r_1, \theta) + i v(r_1, \theta)$$

$$u = \log r_1, v = \theta$$

$$\therefore \frac{\partial u}{\partial r_1} = \frac{1}{r_1}, \quad \frac{\partial u}{\partial \theta} = 0$$

$$\frac{\partial v}{\partial r_1} = 0, \quad \frac{\partial v}{\partial \theta} = 1$$

$$\therefore \frac{\partial u}{\partial r_1} = \frac{1}{r_1} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r_1} = -\frac{1}{r_1} \frac{\partial u}{\partial \theta}$$

$\therefore f(z)$ is analytic function.

$$\Rightarrow f'(z) = (u_r + i v_r) e^{-i\theta}$$

$$f'(z) = e^{-i\theta} \left(\frac{1}{r_1} + i(0) \right)$$

$$\Rightarrow f'(z) = e^{-i\theta} \left(\frac{1}{r_1} \right)$$

$$\Rightarrow f'(z) = \frac{1}{r_1 e^{i\theta}}$$

$$\therefore f'(z) = \frac{1}{z}$$

Harmonic property:

(1) If $f(z) = u(x, y) + i v(x, y)$ is an analytic, then its real and imaginary parts are both harmonics.

$$\text{i.e., } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

(2) If $f(z) = u(x, y) + i v(x, y)$ is an analytic, then its real and imaginary parts are both harmonic,

$$\text{i.e., } \frac{\partial^2 u}{\partial x^2} + \frac{1}{y^2} \frac{\partial^2 u}{\partial y^2} + \frac{1}{y^2} \frac{\partial^2 v}{\partial y^2} = 0$$

$$\text{and } \frac{\partial^2 v}{\partial x^2} + \frac{1}{y^2} \frac{\partial^2 v}{\partial y^2} + \frac{1}{y^2} \frac{\partial^2 u}{\partial y^2} = 0$$

Theorem :

If $f(z)$ is singular function of z , show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (f'(z)^2) = 4(f'(z)^2).$$

Given, $f(z) = u(x, y) + i v(x, y) = u + iv$ is singular

$\Rightarrow f(z)$ is differentiable at any point of $z = x+iy$

$$\Rightarrow f'(z) = U_x + i V_x$$

$$\text{and w.r.t, } f'(z) = \sqrt{U^2 + V^2}$$

$$f'(z)^2 = U^2 + V^2 = \phi \rightarrow (1)$$

$$\therefore f'(z) = \sqrt{U_x^2 + V_x^2}$$

$$f'(z)^2 = U_x^2 + V_x^2 \rightarrow (2)$$

$$\text{LHS} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \cdot f'(z)^2$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \cdot \phi$$

$$\text{LHS} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$\therefore \frac{\partial \phi}{\partial x} = 2U \cdot U_x + 2V \cdot V_x$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} = 2(U_x \cdot U_x + U \cdot U_{xx}) + 2(V_x \cdot V_x + V \cdot V_{xx})$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \Re (U_x^2 + U \cdot U_{xx} + V_x^2 + V \cdot V_{xx}) \rightarrow (3)$$

$$\therefore \frac{\partial \phi}{\partial y} = \Re U \cdot V_y + \Im V \cdot V_y$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = \Re (V_y \cdot U_y + U \cdot U_{yy}) + \Im (V_y \cdot V_y + V \cdot V_{yy})$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = \Re (V_y^2 + U \cdot U_{yy} + V_y^2 + V \cdot V_{yy}) \rightarrow (4)$$

from (3) & (4) (or) LHS (consider)

$$\begin{aligned} \text{LHS} &= \Re (U_x^2 + U \cdot U_{xx} + V_x^2 + V \cdot V_{xx} + U_y^2 + U \cdot U_{yy} + V_y^2 + V \cdot V_{yy}) \\ &= \Re [U_x^2 + V_x^2 + U_y^2 + V_y^2 + U(U_{xx} + U_{yy}) + V(V_{xx} + V_{yy})] \\ &= \Re [U_x^2 + V_x^2 + U_y^2 + V_y^2 + U(0) + V(0)] \\ &= \Re [U_x^2 + V_x^2 + (-V_x)^2 + (U_x)^2] \\ &= \Re (U_x^2 + V_x^2 + V_x^2 + U_x^2) \\ &= \Re (4U_x^2) \\ &= 4 (U_x^2 + V_x^2) \\ &= 4 |f'(z)|^2 \\ &\stackrel{=} {=} \text{RHS} \\ &= \end{aligned}$$

Theorem: If $w=f(z)$ is regular, then show that

$$\left| \frac{\partial}{\partial x} |f(z)|^2 \right|^2 + \left| \frac{\partial}{\partial y} |f(z)|^2 \right|^2 = |f'(z)|^2.$$

Proof: Given $w=f(z) = u+iv$ is a regular @ analytic.

$\therefore f(z)$ is differentiable at any point of $z=x+iy$.

$$\Rightarrow f'(z) = u_x + i v_x$$

$$\text{and WKT } |f'(z)| = \sqrt{u^2 + v^2} = \phi$$

$$\Rightarrow u^2 + v^2 = \phi^2 \rightarrow (1)$$

$$\text{and } |f'(z)| = \sqrt{u_x^2 + v_x^2}$$

$$\Rightarrow |f'(z)|^2 = u_x^2 + v_x^2 \rightarrow (2)$$

diff. (1) w.r.t 'x' partially,

$$\therefore (1) \Rightarrow 2u \cdot u_x + 2v \cdot v_x = 2\phi \phi_x$$

$$\Rightarrow uu_x + vv_x = \phi \phi_x \rightarrow (3)$$

likewise diff. (1) w.r.t 'y' partially.

$$(1) \Rightarrow 2u u_y + 2v v_y = 2\phi \phi_y$$

$$\Rightarrow uu_y + vv_y = \phi \phi_y \rightarrow (4)$$

\therefore fun $f(z)$ is analytic.

$$\therefore \text{WKT } u_x = v_y, v_x = -u_y \\ u_y = -v_x.$$

$$\therefore (4) \Rightarrow -u \cdot v_x + v \cdot u_x = \phi \phi_y$$

$$vu_x - uv_x = \phi \phi_y \rightarrow (5)$$

$$\therefore (3)^2 + (5)^2$$

$$(uu_x + v \cdot v_x)^2 + (vu_x - uv_x)^2 = \phi^2 \phi_x^2 + \phi^2 \phi_y^2$$

$$\Rightarrow u^2 u_x^2 + v^2 v_x^2 + 2uv \cdot v_x u_x + v^2 u_x^2 + u^2 v_x^2 - 2uv \cdot u_x v_x = \phi^2 \phi_x^2 + \phi^2 \phi_y^2$$

$$\Rightarrow u^2 (u_x^2 + v_x^2) + v^2 (u_x^2 + v_x^2) = \phi^2 (\phi_x^2 + \phi_y^2)$$

$$\Rightarrow (u^2 + v^2) (u_x^2 + v_x^2) = \phi^2 (\phi_x^2 + \phi_y^2)$$

$$\Rightarrow \phi^4 (u_x^2 + v_x^2) = \phi^4 (\phi_x^2 + \phi_y^2)$$

$$\Rightarrow \phi_x^2 + \phi_y^2 = u_x^2 + v_x^2$$

$$\Rightarrow \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 = |f'(z)|^2$$

$$\Rightarrow \left\{ \frac{\partial}{\partial x} |f(z)| \right\}^2 + \left\{ \frac{\partial}{\partial y} |f(z)| \right\}^2 = |f'(z)|^2$$

\equiv

Theorem: If $f(z)$ is regular function with Constant Modulus,
Show that $f(z)$ is also a constant function.

Given, $w=f(z)=u+iv$ is regular function.

$\therefore f(z)$ is differentiable.

$$\therefore f(z) = u_x + i v_x$$

$$\text{and } |f'(z)| = \sqrt{u_x^2 + v_x^2}$$

$$\Rightarrow |f'(z)|^2 = u_x^2 + v_x^2 \rightarrow (1)$$

$$\text{and given } |f(z)| = k$$

$$\Rightarrow \sqrt{u^2 + v^2} = k$$

$$\Rightarrow u^2 + v^2 = k^2 \rightarrow (2)$$

diff. eqn (2) partially w.r.t 'x'.

$$(2) \Rightarrow 2uu_x + 2vv_x = 0$$

$$\Rightarrow uu_{xx} + vv_{xx} = 0 \rightarrow (3)$$

111⁶⁴ (2) \Rightarrow w.r.t 'y' partially

$$(2) \Rightarrow 2uuy + 2vvy = 0$$

$$\Rightarrow uu_y + vv_y = 0 \rightarrow (4)$$

\therefore But $f(z)$ is analytic

$$\therefore u_x = v_y \text{ and } v_x = -u_y$$

$$\Rightarrow u_y = -v_x$$

$$\therefore (4) \Rightarrow u(-v_x) + v u_x = 0$$

$$\Rightarrow vu_x - uv_x = 0 \rightarrow (5)$$

$$(3)^2 + (5)^2 (uu_x + vv_x)^2 + (vu_x - uv_x)^2 = 0$$

$$\Rightarrow u^2 u_x^2 + v^2 v_x^2 + 2uv/v_x u_x + v^2 u_x^2 + u^2 v_x^2 - 2uv/u_x v_x = 0$$

$$\Rightarrow u^2 (u_x^2 + v_x^2) + v^2 (u_x^2 + v_x^2) = 0$$

$$\Rightarrow (u^2 + v^2)(u_x^2 + v_x^2) = 0$$

$$\Rightarrow k^2(u_x^2 + v_x^2) = 0$$

$$\Rightarrow k^2 \neq 0, u_x^2 + v_x^2 = 0$$

$$\Rightarrow |f'(z)|^2 = 0$$

$$\Rightarrow |f'(z)| = 0$$

$$\therefore f(z) = C$$

$\Rightarrow f(z)$ is Constant.

Problems:

- Show that $f(z) = \cosh z$ is analytic and hence find its derivative.

Soln $f(z) = \cosh z$

let $z = x+iy$

$$\Rightarrow f(z) = \cosh(x+iy) \rightarrow (1)$$

WKT $\cosh \theta = \cos i\theta$

$$(1) \Rightarrow f(z) = \cosh i(x+iy)$$

$$= \cosh(ix+i^2y)$$

$$= \cosh(ix-y)$$

$$= \cos(ix) \cdot \cosh y + \sin(ix) \cdot \sinh y$$

$$= \cosh x \cdot \cosh y + i \sinh x \cdot \sinh y.$$

$$\Rightarrow f(z) = u(x,y) + iv(x,y)$$

$$\therefore u = \cosh x \cdot \cosh y, v = \sinh x \cdot \sinh y$$

$$u_x = \sinh x \cdot \cosh y, v_y = -\cosh x \cdot \sinh y$$

$$u_x = \cosh x \cdot \sinh y, v_y = \sinh x \cdot \cosh y.$$

$$\therefore U_x = V_y, V_x = -V_y$$

$\therefore f(z)$ is analytic

$$\therefore f'(z) = U_x + iV_x$$

$$\Rightarrow f'(z) = \sinhx \cdot \cosy + i \coshx \cdot \siny$$

when $x=z, y=0$

$$\Rightarrow f'(z) = \sinh z$$

Q. Prove that $f(z) = z + e^z$ is analytic, hence find its derivative.

Soh $f(z) = z + e^z$

let $z = x + iy$

$$\Rightarrow f(z) = (x+iy) + e^{(x+iy)}$$

$$= (x+iy) + e^x \cdot e^{iy}$$

$$= (x+iy) + e^x \cdot (\cos y + i \sin y)$$

$$= x + e^x \cdot \cos y + iy + i e^x \cdot \sin y$$

$$= (x + e^x \cdot \cos y) + i(y + e^x \cdot \sin y)$$

$$\Rightarrow f(z) = u(x, y) + i v(x, y)$$

$$\Rightarrow u = x + e^x \cdot \cos y, v = y + e^x \cdot \sin y$$

$$\therefore U_x = 1 + e^x \cdot \cos y, V_y = -e^x \cdot \sin y \cdot e^x$$

$$U_x = e^x \cdot \sin y, V_y = 1 + e^x \cdot \cos y$$

$$\therefore U_x = V_y \text{ and } V_x = -U_y.$$

$\therefore f(z)$ is analytic.

$$\therefore f'(z) = U_x + iV_x$$

$$\Rightarrow f'(z) = 1 + e^x \cos y + i e^x \sin y$$

$$= 1 + e^x (\cos y + i \sin y)$$

$$\text{put } z=z, y=0 \Rightarrow f'(z) = \underline{\underline{1 + e^z}}$$

3. Find the analytic function, whose real part is $u = \frac{x^4 \cdot y^4 - 2x}{x^2 + y^2}$

Given, $u = \frac{x^4 \cdot y^4 - 2x}{x^2 + y^2} \rightarrow (1)$

diff. (1) partially w.r.t 'x'.

$$\therefore (1) \Rightarrow \frac{\partial u}{\partial x} = \frac{(x^2 + y^2)(4x^3 y^4 - 2) - (x^4 \cdot y^4 - 2x)(4y^3)}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(\bar{z}^2 + 0)(4\bar{z}^3(0) - 2) - \bar{z}^4(0) - 2(\bar{z})(2\bar{z})}{(\bar{z}^2)^2} \quad \text{when } x = \bar{z}, y = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\bar{z}^2(-2) + 4\bar{z}^3}{\bar{z}^4} = \frac{-2\bar{z}^3 + 4\bar{z}^3}{\bar{z}^4}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2\bar{z}^3}{\bar{z}^4}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{2}{\bar{z}^2}$$

$$\text{Now } \frac{\partial u}{\partial y} = \frac{(\bar{x}^2 + \bar{y}^2)(4x^4 y^3) - (x^4 \cdot y^4 - 2x)(2y)}{(\bar{x}^2 + \bar{y}^2)^2} = -\frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial v}{\partial x} = \frac{2y(x^4 \cdot y^4 - 2x) - (\bar{x}^2 + \bar{y}^2)(4x^4 y^3)}{(\bar{x}^2 + \bar{y}^2)^2} \quad \text{when } x = \bar{z}, y = 0$$

$$= \frac{2(0) \cdot (\bar{z}^4(0) - 2(\bar{z})) - \bar{z}^2(4\bar{z}^4(0))}{\bar{z}^4}$$

$$\therefore \frac{\partial v}{\partial x} = 0$$

$$\therefore \text{WKT } f'(z) = \frac{\partial v}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\Rightarrow f'(z) = \frac{2}{\bar{z}^2} + i(0)$$

$$\Rightarrow f'(z) = \frac{2}{z^2}$$

$$\Rightarrow \frac{df}{dz} = \frac{2}{z^2}$$

$$\Rightarrow \int df = \int \frac{2}{z^2} dz$$

$$\Rightarrow f(z) = 2 \cdot -\frac{1}{z} + C$$

$$\Rightarrow f(z) = -\frac{2}{z} + C$$

4. Find the analytic function whose real part is $\frac{\sin \vartheta x}{\cosh 2y - \cos 2x}$

Sol: Given, $u = \frac{\sin \vartheta x}{\cosh 2y - \cos 2x}$ $\rightarrow (1)$

diff wrt 'x' (1) partially

$$\therefore \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(\vartheta \cos \vartheta x) - \sin \vartheta x (0 + 2 \sin \vartheta x)}{(\cosh 2y - \cos 2x)^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2 \cos \vartheta x (\cosh 2y - \cos 2x) - 2 \sin^2 \vartheta x}{(\cosh 2y - \cos 2x)^2}$$

when $x=z, y=0$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2 \cos \vartheta z (1 - \cos 2z) - 2 \sin^2 \vartheta z}{(1 - \cos 2z)^2}$$

$$= \frac{2 \cos \vartheta z - 2 \cos^2 \vartheta z - 2 \sin^2 \vartheta z}{(1 - \cos 2z)^2}$$

$$= \frac{2 \cos \vartheta z - 2 (\cos^2 \vartheta z + \sin^2 \vartheta z)}{(1 - \cos 2z)^2}$$

$$= \frac{2 \cos \vartheta z - 2}{(1 - \cos 2z)^2} = \frac{-2(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{-2}{1 - \cos 2z}$$

$$\text{Similarly } \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x (2 \sin 2y - 0)}{(\cosh 2y - \cos 2x)^2}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial v}{\partial x} = \frac{\sin 2x (2 \sin 2y) - 0}{(\cosh 2y - \cos 2x)^2}$$

when $z=x, y=0$

$$\frac{\partial v}{\partial x} = 0$$

$$\therefore \text{OKT } f'(z) = u_x + i v_x$$

$$\Rightarrow f'(z) = \frac{-2}{1 - \cos 2z} + i(0)$$

$$\Rightarrow \frac{df}{dz} = \frac{-2}{1 - \cos 2z}$$

$$\Rightarrow df = \frac{-2}{1 - \cos 2z} dz$$

$$\neq df = \frac{-2}{2 \sin^2 z} dz = \frac{-1}{\sin^2 z} dz$$

$$\Rightarrow \int df = \int \frac{-1}{\sin^2 z} dz$$

$$\Rightarrow f(z) = \int -\operatorname{cosec}^2 z \cdot dz$$

$$\Rightarrow f(z) = \cot z + C$$

5. Construct the analytic function, whose real part is

$$\frac{\sin 2x}{\cosh 2y + \cos 2x}$$

Sol Given, $u = \frac{\sin 2x}{\cosh 2y + \cos 2x} \rightarrow (1)$

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y + \cos 2x)(2 \cos 2x) - \sin 2x (0 + (-2 \sin 2x))}{(\cosh 2y + \cos 2x)^2}$$

$$\therefore \frac{\partial U}{\partial x} = \frac{(\cosh 2y + \cos 2x) \sin 2x + 2 \sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$

when $z=x, y=0$

$$\frac{\partial U}{\partial x} = \frac{2 \cos 2z (1 + \cos 2z) + 2 \sin^2 2z}{(\cosh 2z + \cos 2z)^2}$$

$$= \frac{2 \cos 2z (1 + \cos 2z) + 2 \sin^2 2z}{(1 + \cos 2z)^2}$$

$$= \frac{2 \cos 2z + 2 \cos^2 2z + 2 \sin^2 2z}{(1 + \cos 2z)^2}$$

$$= \frac{2 \cos 2z + 2(1)}{(1 + \cos 2z)^2}$$

$$\frac{\partial U}{\partial x} = \frac{2(1 + \cos 2z)}{(1 + \cos 2z)^2} = \frac{2}{(1 + \cos 2z)}$$

$$\text{iii) } \frac{\partial U}{\partial y} = \frac{(\cosh 2y + \cos 2x)(0) - \sin 2x (2 \sin 2hy + 0)}{(\cosh 2y + \cos 2x)^2}$$

$$\frac{\partial U}{\partial y} = -\frac{\partial U}{\partial x}$$

$$\Rightarrow \frac{\partial V}{\partial x} = \frac{-\sin 2x (2 \sin 2hy)}{(\cosh 2y + \cos 2x)^2} \quad \text{when } x=z, y=0$$

$$\frac{\partial V}{\partial x} = \frac{-\sin 2z (2 \sin 2h(0))}{(\cosh(0) + \cos 2z)^2}$$

$$\therefore \frac{\partial V}{\partial x} = 0$$

$$\therefore \text{OKT, } f'(z) = U_x + iV_x$$

$$\Rightarrow f'(z) = \frac{2}{1 + \cos 2z}$$

$$\Rightarrow \frac{df}{dz} = \frac{2}{1 + \cos 2z}, \int df = \int \frac{2}{1 + \cos^2 z} dz$$

$$f(z) = \int \frac{1}{\cos^2 z} dz = \int \sec^2 z dz$$

$$f(z) = \tan z + C$$

6. Construct the analytic function, whose real part is

$$x^2 - y^2 + \frac{x}{x^2 + y^2}$$

$$\text{Given, } u = x^2 - y^2 + \frac{x}{x^2 + y^2}$$

$$u = \frac{(x^2 + y^2)(x^2 - y^2) + x}{(x^2 + y^2)} = \frac{(x^4 - y^4 + x)}{(x^2 + y^2)} \rightarrow (1)$$

$$\frac{\partial u}{\partial x} = \frac{(x^2 + y^2)(4x^3 + 1) - (x^4 - y^4 + x)(2x)}{(x^2 + y^2)^2}$$

when $x = z, y = 0$

$$\frac{\partial u}{\partial x} = \frac{z^2(4z^3 + 1) - (z^4 + z)(2z)}{(z^2 + 0)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{4z^5 + z^3 - 2z^5 - 2z^2}{z^4}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{2z^5 - z^2}{z^4}$$

$$= \frac{2z^2(z^3 - 1)}{z^4}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2z^3 - 1}{z^2} = 2z - \frac{1}{z^2}$$

$$\text{III}^{(4)} \frac{\partial u}{\partial y} = \frac{(x^2 + y^2)(-4y^3) - (x^4 - y^4 + x)(2y)}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}$$

when $z = x, y = 0$

$$v_x = 0$$

$$\therefore f'(z) = u_x + i v_x$$

$$\Rightarrow f'(z) = 2z - \frac{1}{z^2} + 0$$

$$\Rightarrow \frac{df}{dz} = 2z - \frac{1}{z^2}$$

$$\int df = \int \left(2z - \frac{1}{z^2}\right) dz$$

$$\Rightarrow f(z) = \frac{2z^2}{2} - \left(\frac{-1}{z}\right) + C$$

$$\boxed{f(z) = z^2 + \frac{1}{z} + C}$$

7. Find the analytic function $f(z) = u + iv$, whose imaginary part is $\frac{y}{x^2+y^2}$.

Given, $v = \frac{y}{x^2+y^2}$

$$\Rightarrow v_x = \frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

when $x=z, y=0$

$$\therefore v_x = 0$$

$$\therefore v_y = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}$$

$$\Rightarrow v_y = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{when } x=z, y=0$$

$$\therefore v_y = \frac{z^2}{z^4} = \frac{1}{z^2}$$

$$\therefore \text{OKT}, f'(z) = u_x + iv_x$$

$$\Rightarrow f'(z) = v_y + iv_x$$

$$\Rightarrow f'(z) = \frac{1}{z^2}$$

$$\Rightarrow \frac{df}{dz} = \frac{1}{z^2}$$

$$\int df = \int \frac{1}{z^2} dz$$

$$\Rightarrow f(z) = -\frac{1}{z} + c$$

8. Construct analytic function, whose real part is $u = e^{2x}(\cos 2y - 4 \sin 2y)$.

Given, $u = e^{2x}(\cos 2y - 4 \sin 2y) \rightarrow (1)$

$$\therefore u_x = 2e^{2x} (x \cos 2y - y \sin 2y) + e^{2x} (\cos 2y - 0)$$

$$u_x = 2e^{2x} (x \cos 2y - y \sin 2y) + e^{2x} (\cos 2y)$$

when $x=z, y=0$.

$$\Rightarrow u_x = 2e^{2z} (z-0) + e^{2z} (1)$$

$$u_x = 2e^{2z} (z) + e^{2z}$$

$$\therefore u_y = e^{2x} (-2x \cdot \sin 2y - 1 \cdot \sin 2y - 2y \cos 2y)$$

when $x=z, y=0$

$$u_y = e^{2z} (0)$$

$$\therefore \underline{\underline{u_y = 0}}$$

$$\therefore \text{WKT, } f'(z) = u_x + i u_y$$

$$\Rightarrow f'(z) = u_x - i u_y$$

$$\Rightarrow f'(z) = 2z e^{2z} + e^{2z} - 0$$

$$\Rightarrow f'(z) = (2z+1) e^{2z}$$

$$\Rightarrow \frac{df}{dz} = (2z+1) e^{2z}$$

$$\Rightarrow \int df = \int (2z+1) e^{2z} dz$$

$$\therefore f(z) = (2z+1) \int e^{2z} dz - \int (2 \cdot \int e^{2z} dz) dz$$

$$= (2z+1) \frac{1}{2} e^{2z} - \frac{1}{2} e^{2z} + C$$

$$= z e^{2z} + \frac{1}{2} e^{2z} - \frac{1}{2} e^{2z} + C$$

$$\therefore f(z) = z e^{2z} + C$$

9. find the analytic function, whose imaginary part is

$$V = e^x (x \sin y + y \cos y)$$

$$\text{Given, } V = e^x (x \sin y + y \cos y)$$

$$\therefore V_x = e^x (x \sin y + y \cos y) + e^x (\sin y)$$

$$\Rightarrow V_x = e^x (x \sin y + y \cos y) + e^x (\sin y)$$

when $x=z, y=0$

$$\Rightarrow u_x = e^z (z(0)+0) + e^z(0)$$

$$u_x = 0$$

$$\therefore u_y = e^z [x \cos y + \cos y + (y \sin y)]$$

$$u_y = e^z (z+1+0)$$

put $z=x, y=0$

$$= e^z (z+1)$$

$$\therefore f'(z) = u_x + i u_y$$

$$= e^z (z+1)$$

$$\therefore \frac{df}{dz} = e^z (z+1)$$

$$df = e^z (z+1) \cdot dz$$

$$\int df = \int e^z (z+1) \cdot dz$$

$$f(z) = (z+1) e^z - (1) e^z + c$$

$$= e^z ((z+1)-1) + c$$

$$= z e^z + e^z - e^z + c$$

$$\therefore f(z) = z e^z + c$$

=====

10. Find the analytic function, $u+iv$ whose imaginary part is
 $v = \left(\frac{1}{r} - \frac{1}{r^2}\right) \sin\theta, r \neq 0.$

Given, $v = \left(\frac{1}{r} - \frac{1}{r^2}\right) \sin\theta, r \neq 0$ $\left| \frac{d}{dr} \left(\frac{1}{r}\right) = -\frac{1}{r^2} \right.$

$$\therefore V_r = \left(1 + \frac{1}{r^2}\right) \sin\theta \rightarrow (1)$$

$$\therefore V_\theta = \left(\frac{1}{r} - \frac{1}{r^2}\right) \cos\theta \rightarrow (2)$$

$$\therefore \text{WKT}, f'(z) = e^{-i\theta} (U_r + iV_r)$$

$$\Rightarrow f'(z) = e^{-i\theta} \left(\frac{1}{r} V_\theta + iV_r\right)$$

$$\Rightarrow f'(z) = e^{-i\theta} \left(\frac{1}{r} \left(\frac{1}{r} - \frac{1}{r^2}\right) \cos\theta + i \left(1 + \frac{1}{r^2}\right) \sin\theta\right)$$

when $z=r, \theta=0$

$$\Rightarrow f'(z) = \left[\frac{1}{z} \left(z - \frac{1}{z}\right)\right]$$

$$\Rightarrow f'(z) = 1 - \frac{1}{z^2}$$

$$\left| \int \frac{1}{z^2} dz = -\frac{1}{z} \right.$$

$$\Rightarrow \frac{df}{dz} = 1 - \frac{1}{z^2}$$

$$\Rightarrow \int df = \int \left(1 - \frac{1}{z^2}\right) dz$$

$$\Rightarrow \boxed{f(z) = z + \frac{1}{z} + C}$$

11. Find the analytic function, $f(z) = u+iv$,

$$\text{if } (u-v) = e^x [\cos y - \sin y]$$

$$\text{Given, } f(z) = u+iv \rightarrow (1)$$

$$\text{and given, } u-v = e^x (\cos y - \sin y) \rightarrow (2)$$

differentiate eqn (2) partially

$$(1) \Rightarrow U_{,x} - V_{,x} = e^x (\cos y - \sin y) \rightarrow (3)$$

diff w.r.t 'y' partially,

$$(2) \Rightarrow u_y - v_y = -e^x (\cos y + \sin y) \rightarrow (4)$$

w.k.t, $u_y = -v_x$, $v_y = u_x$

$$\therefore (4) \Rightarrow -v_x - u_x = -e^x (\cos y + \sin y) \rightarrow (5)$$

(3)+(5),

$$u_x - v_x - v_x - u_x = e^x (\cos y - \sin y) - e^x (\cos y + \sin y)$$

$$-2v_x = e^x (\cos y - \sin y - \cos y + \sin y)$$

$$\therefore v_x = \frac{1}{2} e^x \cdot \sin y$$

$$v_x = \underline{\underline{e^x \cdot \sin y}}$$

when $x=z, y=0$

$$\therefore v_x = 0.$$

$$(3)-(5), \quad \underline{\underline{2u_x}} = e^x (\cos y - \sin y) + e^x (\cos y + \sin y)$$

$$\underline{\underline{2u_x}} = e^x (\cos y - \sin y + \cos y + \sin y)$$

$$\underline{\underline{2u_x}} = e^x \cdot \cos y$$

$$u_x = \underline{\underline{e^x \cdot \cos y}}$$

when $x=z, y=0$

$$u_x = e^z$$

$$\therefore \text{w.k.t, } f'(z) = u_x + i v_x$$

$$\Rightarrow f'(z) = e^z + 0 = e^z$$

$$\Rightarrow \frac{df}{dz} = e^z$$

$$\int df = \int e^z \cdot dz$$

$$\Rightarrow f(z) = e^z + C$$

Q. Find the analytic function $f(z) = \underline{\underline{u+iV}}$, if $(u-v) =$

$$\frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cos y)} \text{ and } f\left(\frac{\pi}{2}\right) = 0.$$

Given, $f(z) = u + iv \rightarrow (1)$

$$\text{Given, } u - v = \frac{\cos x + \sin x - e^{-y}}{\omega(\cos x - \cosh y)} \rightarrow (2)$$

diff. (2) partially w.r.t 'x'.

$$\therefore u_x - v_x = \frac{1}{2} \left[\frac{(\cos x - \cosh y)(-\sin x + \cos x) + (\cos x + \sin x - e^{-y}) \sin x}{(\cos x - \cosh y)^2} \right]$$

when $x=z, y=0$

$$\therefore u_x - v_x = \frac{1}{2} \left[\frac{(\cos z - 1)(-\sin z + \cos z) + (\cos z + \sin z - 1) \sin z}{(\cos z - 1)^2} \right]$$

$$\Rightarrow u_x - v_x = \frac{1}{2} \left\{ \frac{\cos^2 z - \cos z \sin z - \cos z + \sin z + \cos z \sin z +}{\sin^2 z - \sin z} \right\}$$
$$= \frac{-1}{2} \left| \frac{1 - \cos z}{(1 - \cos z)^2} \right|$$

$$\therefore u_x - v_x = \frac{-1}{2} \frac{1}{(1 - \cos z)} \rightarrow (3)$$

$$\text{IIIly } u_y - v_y = \frac{1}{2} \left\{ \frac{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y}) \sinh y}{(\cos x - \cosh y)^2} \right\}$$

when $x=z, y=0$

$$= \frac{1}{2} \left(\frac{\cos z - 1}{(\cos z - 1)^2} \right)$$

$$\Rightarrow -v_x - u_x = \frac{1}{2} \frac{1}{\cos z - 1} = -\frac{1}{2} \frac{1}{(1 - \cos z)} \rightarrow (4)$$

$$(3) + (4) \Rightarrow -2v_x = 0$$

$$\Rightarrow v_x = 0$$

$$(3) - (4) \Rightarrow 2u_x = \frac{1}{1 - \cos z}$$

$$u_x = \frac{1}{2} \frac{1}{(1 - \cos z)}$$

$$\Rightarrow f'(z) = \frac{1}{2} \left(\frac{1}{1 - \cos z} \right)$$

$$= \frac{1}{2} \left(\frac{1}{2 \sin^2(z/2)} \right)$$

$$\Rightarrow \frac{df}{dz} = \frac{1}{2} \left(\frac{1}{2 \sin^2(z/2)} \right), \int df = \int \frac{1}{2} \left(\frac{1}{2 \sin^2(z/2)} \right) dz$$

$$\Rightarrow f(z) = \frac{1}{2} \int (-\operatorname{cosec}^2(z/2)) dz$$

$$\Rightarrow f(z) = \frac{1}{2} \cot(z/2) + C \rightarrow (5)$$

$$\text{By condition, } f(\pi/2) = \frac{1}{2} \cot(\pi/2) + C$$

$$0 = \frac{1}{2}(1) + C$$

$$\boxed{C = -1/2} \quad \therefore (5) = f(z) = \frac{1}{2} \underline{\cot(z/2)} - \frac{1}{2}$$

13. find the analytic function of $f(z) = u+iv$, if

$$u-v = (x-y)(x^2+4xy+y^2).$$

Given, $f(z) = u+iv$ and

$$\text{Given, } u-v = (x-y)(x^2+4xy+y^2)$$

$$\Rightarrow u-v = x^3+4x^2y+2y^2-x^2y-4xy^2-4y^3$$

$$\Rightarrow u-v = x^3+3x^2y-3xy^2-y^3 \rightarrow (1)$$

diff (1) wrt 'x' partially

$$\therefore (1) \Rightarrow u_x - v_x = 3x^2 + 6xy - 3y^2 \rightarrow (2)$$

when $x=z, y=0$

$$\therefore (2) \Rightarrow u_x - v_x = 3z^2 \rightarrow (3)$$

IIIrd

$$u_y - v_y = 3x^2 - 6xy - 3y^2 \rightarrow (4)$$

when $x=z, y=0$

$$\therefore (3) \quad U_y - V_y = 3z^2 \rightarrow (5)$$

$$\text{WKT } U_y = -V_x, \quad V_y = U_x$$

$$\therefore (4) \Rightarrow -V_x - U_x = 3z^2 \rightarrow (6)$$

$$\therefore (3) + (6) \Rightarrow -2V_x = 6z^2$$

$$\Rightarrow -V_x = 3z^2$$

$$\Rightarrow V_x = -3z^2$$

$$(3) - (6) \Rightarrow 2U_x = 0$$

$$\Rightarrow U_x = 0$$

\therefore WKT,

$$f'(z) = U_x + iV_x$$

$$\Rightarrow f'(z) = 0 + i(-3z^2)$$

$$\Rightarrow f'(z) = -3iz^2$$

$$\Rightarrow \frac{df}{dz} = -3iz^2$$

$$\Rightarrow \int df = \int -3iz^2 \cdot dz$$

$$\Rightarrow \int df = -3i \int z^2 \cdot dz$$

$$\Rightarrow f(z) = -\frac{3i}{3} \frac{z^3}{3} + C$$

$$\Rightarrow f(z) = -iz^3 + C$$

