Probability distribution of a random variable

A description giving the values of the random variable X along with the corresponding probabilities is called the probability distribution of the random variable X.

X	x_1	x_2	x_3	…	x_n
$P(X)$	p_1	p_2	p_3	…	p_n

Discrete Probability Distribution

Let a random variable X assume values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ such that

(i) $P(X = x_i) = p_i \geq 0$, for each x_i and

(ii) $\sum_{i=1}^{n} p_i = p_1 + p_2 + \ldots + p_n = 1$.

Then the probability distribution is called discrete probability distribution.

Mean and variance of discrete probability distribution

Let X be a random variable whose possible values $x_1, x_2, ..., x_n$ occur with probabilities $p_1, p_2, ..., p_n$, respectively. The mean of X, denoted by μ, is the number $\sum_{i=1}^{n} x_i p_i$, i.e., the mean of X is the weighted average of the possible values of X, each value being weighted by its probability with which it occurs. The mean of a random variable X is also called the expectation of X, denoted by $E(X)$.

Thus,
$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + ... + x_n p_n$$

In other words, the mean or expectation of a random variable X is the sum of the products of all possible values of X by their respective probabilities.

The variance of X is denoted by $V(X)$ and is defined by

$$V(X) = \sum_{i=1}^{n} x_i^2 p_i - \mu^2 = E(X^2) - [E(X)]^2$$

The non-negative number

$$\sigma_X = \sqrt{V(X)}$$

is called the standard deviation of the random variable X.

Example A random variable X has the following probability distribution:

X	-3	-2	-1	0	1	2	3
$P(X)$	k	$2k$	$3k$	$4k$	$3k$	$2k$	k

(i) **Find k**

(ii) **Evaluate $P(X \le 1)$, $P(X > 1)$, $P(-1 < X \le 2)$**

(iii) **Find its mean and standard deviation.**

(i) We have, $\sum_{i=1}^{n} p_i = 1$

$$\Rightarrow \quad p_1 + p_2 + \ldots + p_8 = 1$$

$$\Rightarrow \quad k + 2k + 3k + 4k + 3k + 2k + k = 1$$

$$\Rightarrow \quad 16k = 1$$

$$\Rightarrow \quad k = \frac{1}{16}$$

(ii) $$P(X \le 1) = 1 - P(X > 1)$$

$$= 1 - [P(X = 2) + P(X = 3)]$$

$$= 1 - (2k + k)$$

$$= 1 - 3k$$

$$= 1 - \frac{3}{16} = \frac{13}{16}$$

$$P(X > 1) = P(X = 2) + P(X = 3)$$

$$= 2k + k = 3k = \frac{3}{16}$$

$$P(-1 < X \le 2) = P(X=0)+P(X=1)+P(X=2)$$

$$= 4k + 3k + 2k = 9k = \frac{9}{16}$$

(iii) Mean $\mu = \sum x_i p_i$

$$=(-3)(k)+(-2)(2k)+(-1)(3k)+0+(1)(3k)+(2)(2k)+(3)k$$

$$=-3k-4k-3k+3k+4k+3k=0$$

Variance $V = \sum x_i^2 p_i - \mu^2$

$$=(-3)^2(k)+(-2)^2(2k)+(-1)^2(3k)+0+(1)^2(3k)+(2)^2(2k)+(3)^2 k$$

$$=9k+8k+3k+3k+8k+9k=40k=\frac{40}{16}=\frac{5}{2}$$

Standard deviation $\sigma = \sqrt{V} = \sqrt{\frac{5}{2}} = 1.581$

Example A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6
$P(X)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(i) **Find k**

(ii) **Evaluate $P(X < 4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$**

(iii) **Find the minimum value of k so that $P(X \leq 2) > 0.3$**

(i) We have, $\sum_{i=1}^{n} p_i = 1$

$\Rightarrow$ $p_1 + p_2 + \ldots + p_8 = 1$

$\Rightarrow$ $k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$\Rightarrow$ $49k = 1$

$\Rightarrow$ $k = \frac{1}{49}$

(ii) $P(X<4)=1-P(X\geq 4)$

$$=1-\left[P(X=4)+P(X=5)+P(X=6)\right]$$

$$=1-(9k+11k+13k)$$

$$=1-33k$$

$$=1-\frac{33}{49}=\frac{16}{49}$$

$$P(X\geq 5)=P(X=5)+P(X=6)$$

$$=11k+13k=24k=\frac{24}{49}$$

$$P(3<X\leq 6)=P(X=4)+P(X=5)+P(X=6)$$

$$=9k+11k+13k=33k=\frac{33}{49}$$

(iii) $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$$= k + 3k + 5k = 9k > 0.3$$

$$= k > \frac{0.3}{9} = \frac{1}{30}$$

$\therefore$ The minimum value of k is $\frac{1}{30}$.

Example **A random variable X has the following probability distribution:**

X	0	1	2	3	4	5	6	7
$P(X)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k

(ii) Evaluate $P(X < 6)$, $P(X \ge 6)$, $P(3 < X \le 6)$

(iii) Find the minimum value of x so that $P(X \le x) > \frac{1}{2}$.

(i) We have, $\sum_{i=1}^{n} p_i = 1$

$\Rightarrow \quad p_1 + p_2 + ... + p_8 = 1$

$\Rightarrow \quad 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow \quad 10k^2 + 9k - 1 = 0$

$\Rightarrow \quad (10k - 1)(k + 1) = 0$

$\Rightarrow \quad k = \frac{1}{10} \qquad [\because p_i \geq 0]$

(ii) $P(X < 6) = 1 - P(X \geq 6)$

$$= 1 - [P(X = 6) + P(X = 7)]$$

$$= 1 - (2k^2 + 7k^2 + k) = 1 - 9k^2 - k = 1 - \frac{9}{100} - \frac{1}{10} = \frac{81}{100}$$

$$P(X \geq 6) = P(X = 6) + P(X = 7)$$

$$= 2k^2 + 7k^2 + k \quad = 9k^2 - k \quad = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

$$P(3 < X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6)$$

$$= 3k + k^2 + 2k^2 \quad = 3k + 3k^2 \quad = \frac{3}{10} + \frac{3}{100} = \frac{33}{100}$$

(iii) $$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= 0 + k = k = \frac{1}{10} < \frac{1}{2}$$

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= 0 + k + 2k = 3k = \frac{3}{10} < \frac{1}{2}$$

$$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 0 + k + 2k + 2k = 5k = \frac{5}{10} = \frac{1}{2}$$

$$P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= 0 + k + 2k + 2k + 3k = 8k = \frac{8}{10} > \frac{1}{2}$$

$\therefore$ The minimum value of x so that $P(X \le x) > \frac{1}{2}$ is 4.

Example A random variable X has the following probability distribution:

X	-2	-1	0	1	2	3
$P(X)$	0.1	k	0.2	$2k$	0.3	k

(i) Find k

(ii) Evaluate $P(X < 1)$, $P(X > -1)$

(iii) Find its mean and standard deviation.

(i) We have, $\sum_{i=1}^{n} p_i = 1$

$\Rightarrow \quad p_1 + p_2 + \ldots + p_8 = 1$

$\Rightarrow \quad 0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$\Rightarrow \quad 4k + 0.6 = 1$

$\Rightarrow \quad 4k = 0.4$

$\Rightarrow \quad k = 0.1$

(ii) $$P(X<1)=1-P(X\geq 1)$$

$$=1-\left[P(X=1)+P(X=2)+P(X=3)\right]$$

$$=1-(2k+0.3+k)$$

$$=0.7-3k$$

$$=0.7-3(0.1)=0.4$$

$$P(X>-1)=1-P(X\leq -1)$$

$$=1-\{P(X=-2)+P(X=-1)\}$$

$$=1-\{0.1+k\}$$

$$=1-\{0.1+0.1\}=1-0.2=0.8$$

(iii) Mean $\mu = \sum x_i p_i$

$$= (-2)(0.1) + (-1)(k) + 0 + (1)(2k) + (2)(0.3) + (3)k$$

$$= -0.2 - k + 2k + 0.6 + 3k$$

$$= 4k + 0.4 = 4(0.1) + 0.4 = 0.8$$

Variance $V = \sum x_i^{\,2} p_i - \mu^2$

$$= 0.4 + k + 2k + 1.2 + 9k - 0.64$$

$$= 12k + 0.96 = 12(0.1) + 0.96 = 2.16$$

Standard deviation $\sigma = \sqrt{V} = \sqrt{2.16} = 1.47$

EXERCISE

1. A random variable X has the following probability distribution:

X	0	1	2	3	4	5
$P(X)$	k	$5k$	$10k$	$10k$	$5k$	k

(i) Find k

(ii) Find mean and standard and deviation

2. Find the standard deviation for the following probability distribution:

X	8	12	16	20	24
$P(X)$	1/8	1/6	3/8	1/4	1/12