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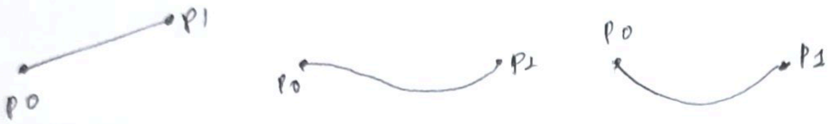
Module - 5

Bezier Curve

What is a curve?

It is a line / outline which gradually deviates from being straight for some or all its length.

A curve is a continuous and smooth flowing line without any sharp turns.



Representing curves

- Explicit: $y = f(x)$
- Implicit: $f(x, y) = 0$
- Parametric: $x = f(t)$, $y = f(t)$ — Most of the curve follows parametric form

Degree of polynomial defining the curve segment is 1 less than the number of polygon defining point

Degree n → (n-1) points.

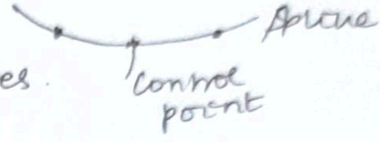
Degree 2 → (2-1) 1 point
Degree 3 → (3-1) 2 points
Degree 4 → (4-1) 3 points.

Spline

A spline curve is a mathematical representation for which it is easy to build an interface that allows user to design and control the shape of complex curves & surfaces.

A spline curve is defined, modified and manipulated by the control points

Control points are the set of coordinate positions which indicates the general shape of the curve.



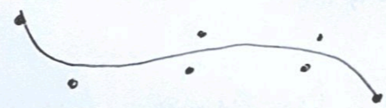
There are two types of spline curves.

→ Interpolation spline

→ Approximation Spline.

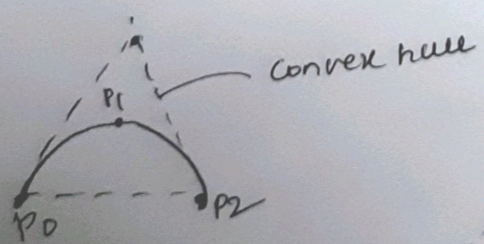
When the curve passes through all control points. ~~the~~ resulting curve is said as interpolating spline

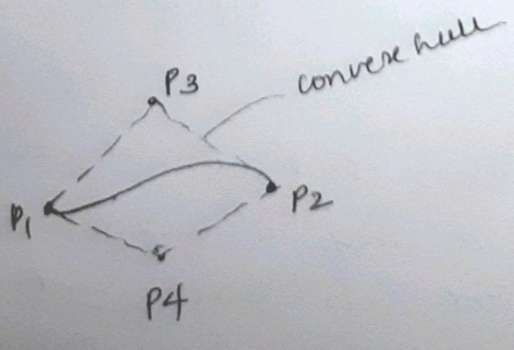
When the curve does not necessarily pass through all the control points then the resulting curve is said to be an approximating spline



Convex Hull: A polygon covering all the control points of a spline is called convex hull.

It ensures that curve will never cross the convex hull





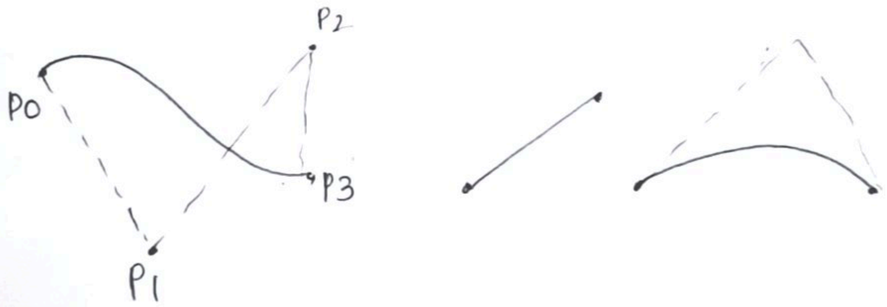
It is used to model smooth surface.

Bezier Curve

A Bezier curve is particularly a kind of spline generated from a set of control points by forming a set of polynomial function.

These functions are computed from the coordinates of the control points.

A Bezier curve is defined by defining polygon. It has no. of properties that makes them highly useful and convenient for curve and surface design.



** Properties of Bezier Curve

* Bezier curve is a polynomial of degree one less than the number of control points.
* The curve connects the first & last control points. Thus a basic characteristic of any Bezier curve is

$P(0) = P0$

$P(1) = Pn$

* A Bezier curve is defined by a set
values for the parametric first derivatives of Bezier curve at the end points calculated from control point coordinates as.

$$\boxed{\begin{aligned} p'(0) &= -n p_0 + n p_1 \\ p'(1) &= -n p_{n-1} + n p_n \end{aligned}}$$

The parametric second derivatives of a Bezier curve at the end points are calculated as

$$p''(0) = n(n-1)\left[(p_2 - p_1) - (p_1 - p_0)\right]$$

$$p''(1) = n(n-1)\left[(p_{n-2} - p_{n-1}) - (p_{n-1} - p_n)\right]$$

* Any Bezier curve is that it lies within the convex hull of control points.
Bezier blending functions are all positive & their sum is always 1

$$\boxed{\sum_{k=0}^{n} BEZ_{k,n}(u) = 1}$$

* The degree of the polynomial defining the curve segment is one less than the number of defining points.

* The curve generally follows the shape of defining polygon

* The tangent vectors at the end of the curve has same direction as the first and last polygon respectively

Bezier Spline Curves

Blending Function

$$\boxed{BEZ_{i,n}(u) = {}^{n}C_{i} \cdot u^{i}(1-u)^{n-i}}$$

or $Bez_{k,n}(u) = \underbrace{C(n,k)}_{\text{binomial co-efficients}} u^{k}(1-u)^{n-k}$

Let suppose we are given $(n+1)$ control points positions then $P_i = (x_i, y_i, z_i)$ $i \le n$

For individual coordinates

$$x(u) = \sum_{i=0}^{n} x_i \, BEZ_{i,n}(u)$$

$$y(u) = \sum_{i=0}^{n} y_i \, BEZ_{i,n}(u)$$

$$z(u) = \sum_{i=0}^{n} z_i \, BEZ_{i,n}(u)$$

These coordinates points can be blended to produce the position vector $P(u)$ which describes path of an approximation

So Bezier polynomial function between P_0 to P_n is.

$$P(u) = \sum_{i=0}^{n} P_i \underbrace{BEZ_{i,n}(u)}_{\text{Bezier Blending function}}$$

position vector

where $0 \le u \le 1$

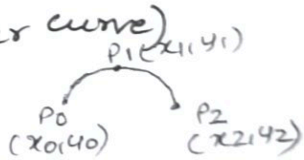
P_i - control points

$BEZ_{i,n}$ - Blending / Bezier function.

$$C(n,k) = \frac{n!}{n-k!\,k!}$$

$$^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$$

Imp. Bezier curve for 3 points. (Quadratic Bezier curve)



- 3 control points, polynomial degree 2

$$Q(u) = P_0 B_{0,2}(u) + P_1 B_{1,2}(u) + P_2 B_{2,2}(u)$$

$\rightarrow B_{0,2}(u) = {}^{2}C_{0} \, u^{0} (1-u)^{2-0}$

$= 1(1-u)^2$

$\rightarrow B_{1,2}(u) = {}^{2}C_{1} \, u^{1} (1-u)^{2-1}$

$= 2u(1-u)$

$\rightarrow B_{2,2}(u) = {}^{2}C_{2} \, u^{2} (1-u)^{2-2}$

$= u^2$

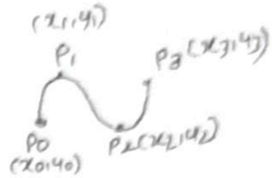
$$Q(u) = P_0 (1-u)^2 + P_1 2u(1-u) + P_2 u^2$$

$$Q(u) = P_0 (1-u)^2 + P_1\, 2u(1-u) + P_2 u^2$$

$$x(u) = (1-u)^2 x_0 + 2u(1-u)\, x_1 + u^2 x_2$$

$$y(u) = (1-u)^2 y_0 + 2u(1-u)\, y_1 + u^2 y_2$$

Imp **Bezier Curve for 4 points** (Cubic Bezier Curve)



$$Q(u) = P_0 B_{0,3}(u) + P_1 B_{1,3}(u) + P_2 B_{2,3}(u) + P_3 B_{3,3}(u)$$

$$B_{0,3}(u) = {}^3C_0\, u^0 (1-u)^{3-0}$$
$$= 1(1-u)^3$$
$$= (1-u)^3$$

$$B_{1,3}(u) = {}^3C_1\, u^1 (1-u)^{3-1}$$
$$= 3u(1-u)^2$$

$$B_{2,3}(u) = {}^3C_2\, u^2 (1-u)^{3-2}$$
$$= 3u^2(1-u)$$

$$P_{33}(u) = {}^3C_3\, u^3 (1-u)^{3-3}$$
$$= u^3$$

$$Q(u) = (1-u)^3 P_0 + 3u(1-u)^2 P_1 + 3u^2(1-u) P_2 + u^3 P_3$$

$$x(u) = (1-u)^3 x_0 + 3u(1-u)^2 x_1 + 3u^2(1-u) x_2 + u^3 x_3$$

$$y(u) = (1-u)^3 y_0 + 3u(1-u)^2 y_1 + 3u^2(1-u) y_2 + u^3 y_3$$

Matrix Representation

$$P(u) = [u^3 \;\; u^2 \;\; u \;\; 1] \cdot M_{Bez} \cdot \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$