# Bézier Spline Curves

# Spline curve

- A spline is a flexible strip used to produce a smooth curve through a designated set of points.
- Several small weights are distributed along the length of the strip to hold it in position on the drafting table as the curve is drawn.
- The term *spline curve* originally referred to a curve drawn in this manner.
- In computer graphics, the term **spline curve** now refers to any composite curve formed with polynomial sections satisfying any specified continuity conditions at the boundary of the pieces.

- Splines are used to design curve and surface shapes, to digitize drawings, and to specify animation paths for the objects or the camera position in a scene.
- Typical computer-aided design (CAD) applications for splines include the design of automobile bodies, aircraft and spacecraft surfaces, ship hulls, and home appliances.
- We specify a spline curve by giving a set of coordinate positions, called **control points,** which indicate the general shape of the curve.
- These coordinate positions are then fitted with piecewise-continuous, parametric polynomial functions in one of two ways.
  - **Interpolation Splines**
  - **Approximation Splines**

- When polynomial sections are fitted so that all the control points are connected, as in Figure 1, the resulting curve is said to **interpolate** the set of control points.
- Interpolation methods are commonly used to digitize drawings or to specify animation paths.

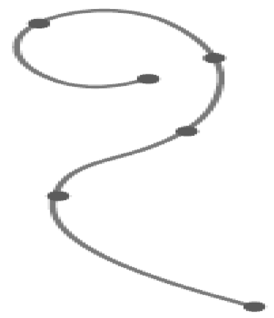


FIGURE 1
A set of six control points interpolated with piecewise continuous polynomial sections.

- when the generated polynomial curve is plotted so that some, or all, of the control points are not on the curve path, the resulting curve is said to **approximate** the set of control points (Figure 2).
- Approximation methods are used primarily as design tools to create object shapes.

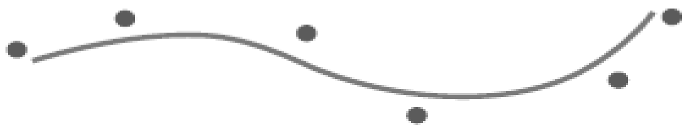


FIGURE 2
A set of six control points approximated with piecewise continuous polynomial sections.

# Bézier Spline Curves

- Developed by the French engineer **Pierre Bézier** for use in the design of Renault automobile bodies.
- **Bézier splines** have a number of properties that make them highly useful and convenient for curve and surface design.
- They are also easy to implement.
- For these reasons, Bézier splines are widely available in various CAD systems, in general graphics packages, and in assorted drawing and painting packages.

- In general, a Bézier curve section can be fitted to any number of control points, although some graphic packages limit the number of control points to four.
- The degree of the Bézier polynomial is determined by the number of control points to be approximated and their relative position.
- we can specify the Bézier curve path in the vicinity of the control points using blending functions, a characterizing matrix, or boundary conditions.

# Bézier Curve Equations

Consider $n + 1$ control-point positions, denoted as $\mathbf{p}_k = (x_k, y_k, z_k)$, with $k$ varying from 0 to $n$. These coordinate points are blended to produce the following position vector $\mathbf{P}(u)$, which describes the path of an approximating Bézier polynomial function between $\mathbf{p}_0$ and $\mathbf{p}_n$:

$$\mathbf{P}(u) = \sum_{k=0}^{n} \mathbf{p}_k \, \mathrm{BEZ}_{k,n}(u), \qquad 0 \le u \le 1 \tag{1}$$

The Bézier blending functions $\mathrm{BEZ}_{k,n}(u)$ are the *Bernstein polynomials*

$$\mathrm{BEZ}_{k,n}(u) = C(n, k) u^k (1 - u)^{n-k} \tag{2}$$

where parameters $C(n, k)$ are the binomial coefficients

$$C(n, k) = \frac{n!}{k!(n-k)!} \tag{3}$$

Equation 1 represents a set of three parametric equations for the individual curve coordinates:

$$x(u) = \sum_{k=0}^{n} x_k \, \mathrm{BEZ}_{k,n}(u)$$

$$y(u) = \sum_{k=0}^{n} y_k \, \mathrm{BEZ}_{k,n}(u)$$

$$z(u) = \sum_{k=0}^{n} z_k \, \mathrm{BEZ}_{k,n}(u)$$

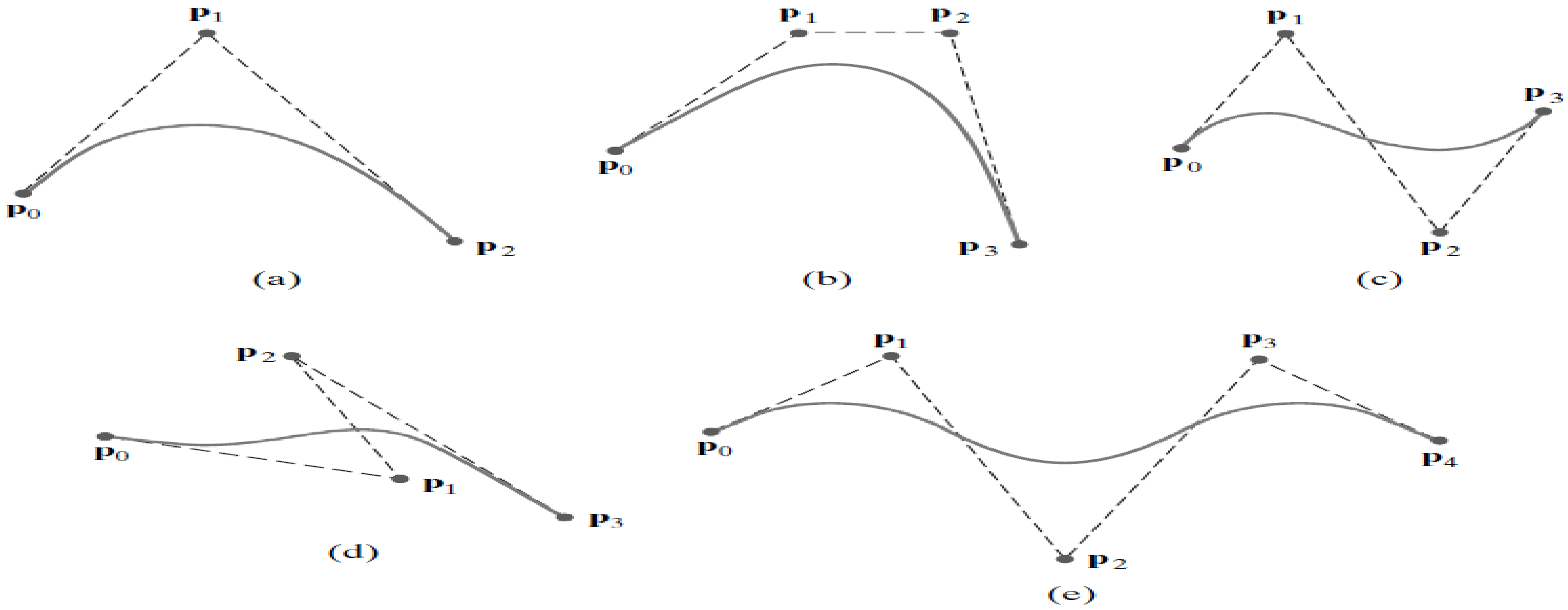
Recursive calculations can be used to obtain successive binomial-coefficient values as

$$C(n,k) = \frac{n-k+1}{k} C(n,k-1)$$

for $n \geq k$. Also, the Bézier blending functions satisfy the recursive relationship

$$\mathrm{BEZ}_{k,n}(u) = (1-u)\mathrm{BEZ}_{k,n-1}(u) + u\,\mathrm{BEZ}_{k-1,n-1}(u), \qquad n > k \geq 1$$

with $\mathrm{BEZ}_{k,k} = u^k$ and $\mathrm{BEZ}_{0,k} = (1-u)^k$.




Examples of two-dimensional Bézier curves generated with three, four, and five control points. Dashed lines connect the control-point positions.

# Properties of Bézier Curves

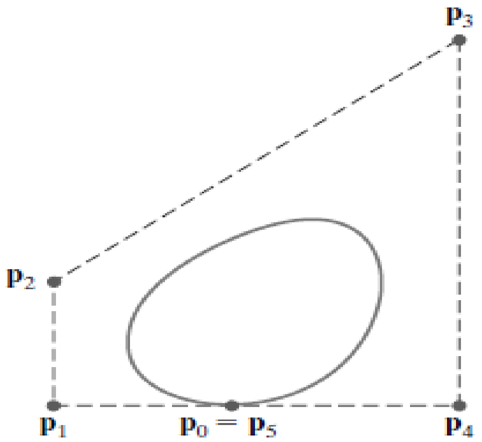
- A very useful property of a Bézier curve is that the curve connects the first and last control points. Thus, a basic characteristic of any Bézier curve is that

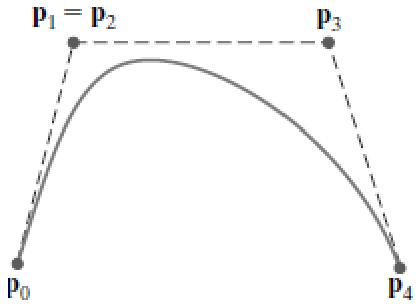
$$\mathbf{P}(0) = \mathbf{p}0 \qquad \mathbf{P}(1) = \mathbf{p}n$$

- Any Bézier curve lies within the convex hull (convex polygon boundary) of the control points. This follows from the fact that the Bézier blending functions are all positive and their sum is always 1. so that any curve position is simply the weighted sum of the control point positions.

$$\sum_{k=0}^{n} \mathrm{BEZ}_{k,n}(u) = 1$$

- The convex hull property for a Bézier curve ensures that the polynomial smoothly follows the control points without erratic oscillations.
- They generally follow the shape of the control polygon, which consists of the segments joining the control points.
- The degree of the polynomial defining the curve segment is one less that the number of defining control points.



$$\mathbf{p}_k = (x_k, y_k, z_k), \qquad k = 0, 1, 2, \ldots, n$$

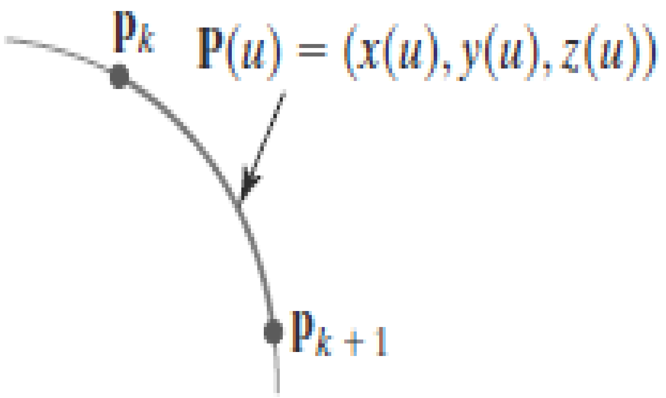



**FIGURE 10**
Parametric point function $\mathbf{P}(u)$ for a Hermite curve section between control points $\mathbf{p}_k$ and $\mathbf{p}_{k+1}$.

$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$$
$$y(u) = a_y u^3 + b_y u^2 + c_y u + d_y, \qquad (0 \leq u \leq 1)$$
$$z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$$