Computational Astrophysics

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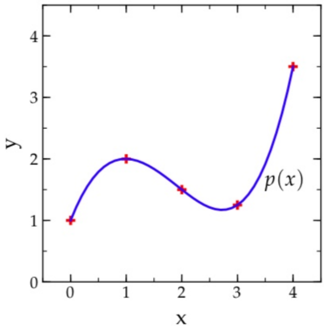
April 3, 2019

Outline

Interpolation

We know the values of a function f only at discrete locations x_i, but want to know its values at general points x.



Interpolation

We look for an approximation $p(x)$ that uses the discrete information about $f(x)$ at x_i to interpolate $f(x)$ between the x_i with $p(x_i) = f(x_i)$.

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If x is outside $[x_{\min}, x_{\max}]$, where $x_{\min} = \min\{x_i; \forall i\}$ and $x_{\max} = \max\{x_i; \forall i\}$, $p(x)$ extrapolates $f(x)$.

Linear Interpolation

We obtain the linear approximation $p(x)$ for $f(x)$ in the interval $[x_i, x_{i+1}]$ by

$$p(x) = f(x_i) + \underbrace{\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}}_{\text{1st-order forward difference}} (x - x_i) + \mathcal{O}(h^2)\,, \qquad (1)$$

where $h = x_{i+1} - x_i$.

Linear Interpolation

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where $h = x_{i+1} - x_i$.

The interpolated function can be differentiated once, but the derivative will be discontinuous at x_{i+1} and x_i.

Quadratic Interpolation

The quadratic approximation $p(x)$ for $f(x)$ in the interval $[x_i, x_{i+1}]$ is given by

$$
\begin{aligned}
p(x) = {} & \frac{(x - x_{i+1})(x - x_{i+2})}{(x_i - x_{i+1})(x_i - x_{i+2})} f(x_i) \\
& + \frac{(x - x_i)(x - x_{i+2})}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} f(x_{i+1}) \\
& + \frac{(x - x_i)(x - x_{i+1})}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} f(x_{i+2}) \\
& + \mathcal{O}(h^3)\,,
\end{aligned} \tag{2}
$$

where $h = \max\{x_{i+2} - x_{i+1}, x_{i+1} - x_i\}$.

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\end{aligned}
\tag{2}
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where $h = \max\{x_{i+2} - x_{i+1}, x_{i+1} - x_i\}$.

$p(x)$ is twice differentiable. Its first derivative will be continuous, but $p''(x)$ will have finite-size steps.

Lagrange Interpolation

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Lagrange Interpolation provides a means of constructing general interpolating polynomials of degree n using data at $n+1$ points. Consider linear interpolation rewritten as

$$
\begin{aligned}
f(x) \approx p(x) &= \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1}) + \mathcal{O}(h^2)\,, \\
&= \sum_{j=i}^{i+1} f(x_j) L_{1j}(x) + \mathcal{O}(h^2)\,, \qquad (3) \\
\text{where } L_{1j}(x) &= \left.\frac{x - x_k}{x_j - x_k}\right|_{k \neq j}\,.
\end{aligned}
$$

Lagrange Interpolation

This is generalized this to an n-th degree polynomial that passes through all $n+1$ data points:

$$p(x) = \sum_{j=0}^{n} f(x_j) L_{nj}(x) + \mathcal{O}(h^{n+1}) \,, \tag{4}$$

with

$$L_{nj}(x) = \prod_{k \neq j}^{n} \frac{x - x_k}{x_j - x_k} \,. \tag{5}$$

Note that this clearly satisfies the interpolating condition, $p(x_i) = f(x_i)$.