

# Computational Astrophysics

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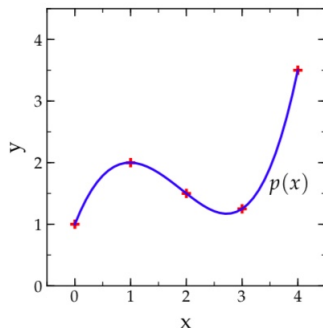
# Outline

## 1 Interpolation

- Linear Interpolation
- Quadratic Interpolation
- Lagrange Interpolation

# Interpolation

We know the values of a function  $f$  only at discrete locations  $x_i$ , but want to know its values at general points  $x$ .



# Interpolation

We look for an approximation  $p(x)$  that uses the discrete information about  $f(x)$  at  $x_i$  to interpolate  $f(x)$  between the  $x_i$  with  $p(x_i) = f(x_i)$ .

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If  $x$  is outside  $[x_{\min}, x_{\max}]$ , where  $x_{\min} = \min\{x_i; \forall i\}$  and  $x_{\max} = \max\{x_i; \forall i\}$ ,  $p(x)$  extrapolates  $f(x)$ .

# Linear Interpolation

We obtain the linear approximation  $p(x)$  for  $f(x)$  in the interval  $[x_i, x_{i+1}]$  by

$$p(x) = f(x_i) + \underbrace{\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}}_{\text{1st-order forward difference}} (x - x_i) + \mathcal{O}(h^2), \quad (1)$$

where  $h = x_{i+1} - x_i$ .

# Linear Interpolation

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where  $h = x_{i+1} - x_i$ .

The interpolated function can be differentiated once, but the derivative will be discontinuous at  $x_{i+1}$  and  $x_i$ .

# Quadratic Interpolation

The quadratic approximation  $p(x)$  for  $f(x)$  in the interval  $[x_i, x_{i+1}]$  is given by

$$\begin{aligned} p(x) = & \frac{(x - x_{i+1})(x - x_{i+2})}{(x_i - x_{i+1})(x_i - x_{i+2})} f(x_i) \\ & + \frac{(x - x_i)(x - x_{i+2})}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})} f(x_{i+1}) \\ & + \frac{(x - x_i)(x - x_{i+1})}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} f(x_{i+2}) \\ & + \mathcal{O}(h^3), \end{aligned} \tag{2}$$

where  $h = \max\{x_{i+2} - x_{i+1}, x_{i+1} - x_i\}$ .



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where  $h = \max\{x_{i+2} - x_{i+1}, x_{i+1} - x_i\}$ .

$p(x)$  is twice differentiable. Its first derivative will be continuous, but  $p''(x)$  will have finite-size steps.

# Lagrange Interpolation

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# Lagrange Interpolation

*Lagrange Interpolation* provides a means of constructing general interpolating polynomials of degree  $n$  using data at  $n + 1$  points. Consider linear interpolation rewritten as

$$\begin{aligned} f(x) \approx p(x) &= \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1}) + \mathcal{O}(h^2), \\ &= \sum_{j=i}^{i+1} f(x_j) L_{1j}(x) + \mathcal{O}(h^2), \end{aligned} \tag{3}$$

$$\text{where } L_{1j}(x) = \left. \frac{x - x_k}{x_j - x_k} \right|_{k \neq j}.$$

# Lagrange Interpolation

This is generalized this to an  $n$ -th degree polynomial that passes through all  $n + 1$  data points:

$$p(x) = \sum_{j=0}^n f(x_j) L_{nj}(x) + \mathcal{O}(h^{n+1}), \quad (4)$$

with

$$L_{nj}(x) = \prod_{k \neq j}^n \frac{x - x_k}{x_j - x_k}. \quad (5)$$

Note that this clearly satisfies the interpolating condition,  $p(x_i) = f(x_i)$ .

# Lagrange Interpolation

