Computational Astrophysics

2019

Exercises 06

1. **Newton-Cotes Integration**

   (a) Integrate $f(x) = \sin x$ from $x = 0$ to $x = \pi$ using (1) the midpoint rule, (2) the trapezoidal rule, and (2) Simpson's rule. Show how the three converge with decreasing step size h and compare their errors. (You may need to take special care of boundary points!)

   (b) Compute the integral
   $$I = \int_0^\pi x \, \sin x \, dx$$
   using (1) the midpoint rule, (2) the trapezoidal rule, and (3) Simpson's rule. Compare the errors and show how the three converge with decreasing step size h.

2. **Number Density of Electrons in an Astrophysical Environment**

   In high temperature astrophysical environments ($T \gtrsim 10^9\,\mathrm{K} \sim 80\,\mathrm{keV}$), the reaction
   $$\gamma \leftrightarrow e^- + e^+ \tag{1}$$
   comes into equilibrium. We will assume the limit of $k_B T = 20\,\mathrm{MeV} \gg m_e c^2$, when the electrons become relativistic, i.e. $E_{e^-} \sim E_{e^+} = pc$ where p is the momentum of the electrons and positrons and c is the speed of light.

   In such an environment, the number density of electrons/positrons is given by
   $$n_{e^\pm} = \frac{2}{(2\pi\hbar)^3} \int \frac{d^3\vec{p}}{e^{\beta c p} + 1} = \frac{8\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{p^2 dp}{e^{\beta c p} + 1}, \tag{2}$$
   where $\beta = 1/(k_B T)$. Making the substitution of $x = \beta p c$ to make the integral dimensionless, we obtain
   $$n_{e^\pm} = \frac{8\pi (k_B T)^3}{(2\pi\hbar c)^3} \int_0^\infty \frac{x^2 dx}{e^x + 1}. \tag{3}$$
   Note that in this deduction we have considered that the chemical potential of the electrons (and positrons) is 0.

   (a) Use any of the presented integration methods to determine what is the total number density of electrons in this environment. Try different conditions to ensure convergence.

   (b) This formula not only has the total number of electrons (and positrons) but also encodes the spectral distribution ($\frac{dn_{e^\pm}}{dE}$, i.e. $n_{e^\pm} = \int \frac{dn_{e^\pm}}{dE} dE$).
   Such distributions are used in computational astrophysics all the time, but must be discretized into a finite number of energy groups. Hence, create energy groups with $\Delta E = 5\,\mathrm{MeV}$ and evaluate $[dn_{e^\pm}/dE]_i = [n_{e^\pm}]_i / \Delta E$ for each bin i, using any method you like. Verify your method by confirming that,
   $$\sum_{i=0}^{\infty} \left[\frac{dn_{e^\pm}}{dE}\right]_i \times \Delta E = n_{e^\pm}\,. \tag{4}$$
   Note that you will not have to calculate an infinite number of $\left[\frac{dn_{e^\pm}}{dE}\right]_i$, but rather high enough until the values are negliable, $E \sim 150\,\mathrm{MeV}$.