Computational Astrophysics

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Outline

Round-off Error

Round-off error arises from the error inherent in representing a floating point number with a finite number of bits in the computer.

Round-off Error

```
"""
Round-off Error
We find the value of epsilon for which 1. + epsilon = 1.

This gives the machine epsilon value, representing the error inherent to
representing a floating point number
"""

epsilon = 1. # Initial value for epsilon

while (1. + epsilon != 1.):
    epsilon = epsilon /2.

print (epsilon)

#Iterates, halving epsilon until 1 + epsilon = 1
# Prints the value of the machine epsilon
```

Truncation Error

Truncation error is a feature of an algorithm.

Typically we expand the expressions about some small quantity. When throwing away higher-order terms, there is a truncation in the expression.

This introduces an error in the representation !!.

If the quantity we expand about truly is small, then the error is small.

Convergence Test

Example:

Function:

$$f(x) = \sin(x) \tag{1}$$

Taylor series representation:

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \tag{2}$$

Truncation for $|x| \ll 1$:

$$f(x) = x - \frac{x^3}{6} + \mathcal{O}(x^5) \tag{3}$$

Convergence Test

```
"""
Truncation Error - Convergence Test
We implement a 5th order accurate approximation of the function Sin(x)

We define a procedure that calculates the difference between the value of
the Sin(x) function and the truncated approximation.

Calculating the value of this epsilon for x<1 and then taking half of
this value of x we show that the epsilon reduces by 2**5 = 32,
demonstrating 5th-order accuracy
"""

import math as m

# Definition of the function giving the truncation error
def epsilon(x):
    return m.sin(x) - (x - (x**3)/6)

# Value to calculate the function
x = 0.1

# Results. We use the formating in the print function to show the results
print("\nFor x = %f the value of the truncation error is epsilon = %e" %( x, epsilon(x)))
print("For x = %f the value of the truncation error is epsilon = %e" %( x/2, epsilon(x/2)))
print("")
print("The ratio of these values is %f " %(epsilon(x)/epsilon(x/2)))
```

Differentiation of a Discrete Function

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$$\left.\frac{df}{dx}\right|_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x}. \tag{6}$$

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Taylor expansion

$$\begin{aligned} f(x_0 + \delta x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \delta x^n && (8) \\ &= f(x_0) + f'(x_0)\delta x + \mathcal{O}(\delta x^2) && (9) \end{aligned}$$

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$$\left.\frac{df}{dx}\right|_{x_0} = \frac{f(x_0 + \delta x) - f(x_0)}{\delta x} + \mathcal{O}(\delta x) \tag{10}$$

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Forward difference estimate for $f'(x_0)$

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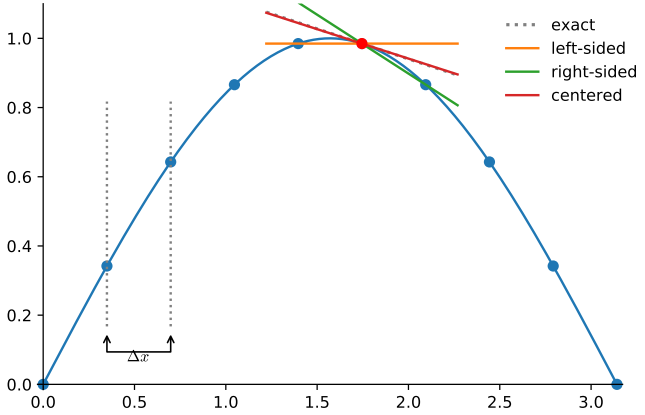

Differentiation of an Analytic Function

$$f'(x_0) = \frac{f(x_0 + \delta x) - f(x_0)}{\delta x} + \mathcal{O}(\delta x) \tag{11}$$

First order *forward difference* estimate for $f'(x_0)$

$$f'(x_0) = \frac{f(x_0) - f(x_0 - \delta x)}{\delta x} + \mathcal{O}(\delta x) \tag{12}$$

First order *backward difference* estimate for $f'(x_0)$

$$f'(x_0) = \frac{f(x_0 + \delta x) - f(x_0 - \delta x)}{2\delta x} + \mathcal{O}(\delta x^2) \tag{13}$$

Second order *central difference* estimate for $f'(x_0)$

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A rule-of-thumb for the election is $\delta \approx \sqrt{\epsilon}$.

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For example, when modeling the stellar collapse of an iron core to a neutron star it is useful to resolve the steep density gradients near the neutron star. In this process, a resolution of order 100 m is required within $\sim 30\,\mathrm{km}$ of the origin, while a cell size of order 10 km is sufficient at radii above $\sim$1000 km.

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Eliminating $f''(x_i)$ and solving for $f'(x_i)$,

$$f'(x_i) = \frac{\delta x_1}{\delta x_2(\delta x_1 + \delta x_2)} f(x_{i+1}) - \frac{\delta x_1 - \delta x_2}{\delta x_2 \delta x_1} f(x_i) - \frac{\delta x_2}{\delta x_1(\delta x_1 + \delta x_2)} f(x_{i-1})\,. \tag{15}$$

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If $\delta x_1 = \delta x_2 = \delta x$ this reduces to the standard difference equation (13).