Computational Astrophysics

2019

Solution Exercise 3.3

Second Derivative of a Function

Using the Taylor expansions for $f(x_0 + \delta x)$ and $f(x_0 - \delta x)$, find the difference approximation to the second derivative of f at x_0.

Solution

Consider the Taylor expansion of function $f(x)$ around $x_0 + \delta x$ and $x_0 - \delta x$,

$$f(x_0 + \delta x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0} \delta x + \frac{1}{2} \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} \delta x^2 + ... \tag{1}$$

$$f(x_0 - \delta x) = f(x_0) - \left.\frac{\partial f}{\partial x}\right|_{x_0} \delta x + \frac{1}{2} \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} \delta x^2 + ... \tag{2}$$

Adding these expression we get

$$f(x_0 + \delta x) + f(x_0 - \delta x) = 2f(x_0) + \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} \delta x^2 + \mathcal{O}(\delta x^4) \tag{3}$$

from which we can obtain the second derivative of f as

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} \delta x^2 = f(x_0 + \delta x) + f(x_0 - \delta x) - 2f(x_0) + \mathcal{O}(\delta x^4) \tag{4}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} = \frac{f(x_0 - 2f(x_0) + \delta x) + f(x_0 - \delta x)}{\delta x^2} + \mathcal{O}(\delta x^2) \tag{5}$$