Computational Astrophysics
2019
Solution Exercise 6.2

6.2 Number Density of Electrons in an Astrophysical Environment

In high temperature astrophysical environments ($T \gtrsim 10^9\,\mathrm{K} \sim 80\,\mathrm{keV}$), the reaction

$$\gamma \leftrightarrow e^- + e^+ \tag{1}$$

comes into equilibrium. We will assume the limit of $k_B T = 20\,\mathrm{MeV} \gg m_e c^2$, when the electrons become relativistic, i.e. $E_{e^-} \sim E_{e^+} = pc$ where p is the momentum of the electrons and positrons and c is the speed of light.

In such an environment, the number density of electrons/positrons is given by

$$n_{e^\pm} = \frac{2}{(2\pi\hbar)^3} \int \frac{d^3\vec{p}}{e^{\beta c p} + 1} = \frac{8\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{p^2 dp}{e^{\beta c p} + 1}, \tag{2}$$

where $\beta = 1/(k_B T)$. Making the substitution of $x = \beta p c$ to make the integral dimensionless, we obtain

$$n_{e^\pm} = \frac{8\pi (k_B T)^3}{(2\pi\hbar c)^3} \int_0^\infty \frac{x^2 dx}{e^x + 1}. \tag{3}$$

Note that in this deduction we have considered that the chemical potential of the electrons (and positrons) is 0.

(a) Use any of the presented integration methods to determine what is the total number density of electrons in this environment. Try different conditions to ensure convergence.

(b) This formula not only has the total number of electrons (and positrons) but also encodes the spectral distribution ($\frac{dn_{e^\pm}}{dE}$, i.e. $n_{e^\pm} = \int \frac{dn_{e^\pm}}{dE} dE$).
Such distributions are used in computational astrophysics all the time, but must be discretized into a finite number of energy groups. Hence, create energy groups with $\Delta E = 5\,\mathrm{MeV}$ and evaluate $[dn_{e^\pm}/dE]_i = [n_{e^\pm}]_i/\Delta E$ for each bin i, using any method you like. Verify your method by confirming that,

$$\sum_{i=0}^{\infty} \left[\frac{dn_{e^\pm}}{dE}\right]_i \times \Delta E = n_{e^\pm}\,. \tag{4}$$

Note that you will not have to calculate an infinite number of $\left[\frac{dn_{e^\pm}}{dE}\right]_i$, but rather high enough until the values are negliable, $E \sim 150\,\mathrm{MeV}$.

Solution

a) The behavior of the function $f(x) = \frac{x^2 dx}{e^x + 1}$ is shown in Figure 1, obtained with the file plotIntegrand.py.

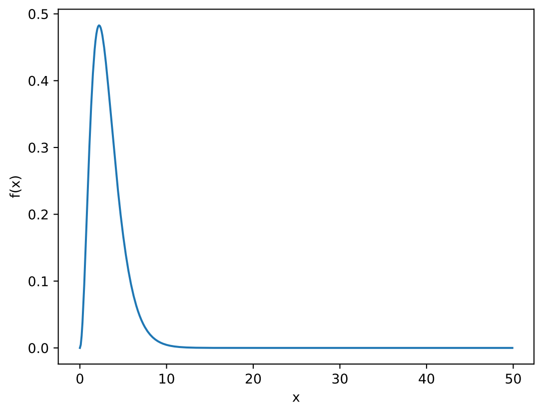

Figure 1: Behavior of the integrand in equation (3) as function of x.

Note that values of $x > 20$ give values of $f(x)$ so small that they will not contribute greatly to the value of the integral. Hence, the infinite upper limit in the integral in equation (3) will be replaced by a finite value x_{max}, for which the value of the integrand falls below the epsilon of the machine ($f(x_{max}) < \epsilon \sim 10^{-16}$).

In order to calculate the constant in front of the integral,

$$C = \frac{8\pi(k_B T)^3}{(2\pi\hbar c)^3}, \tag{5}$$

we need the values

$$\begin{aligned} k_B T &= 20\ \text{MeV} = 3.20435 \times 10^{-12}\ \text{J} &(6)\\ \hbar &= 1.0545718 \times 10^{-34}\ \text{m}^2\text{kg/s} &(7)\\ c &= 3 \times 10^{8}\ \text{m/s}, &(8)\end{aligned}$$

giving the result

$$C = 1.052756 \times 10^{41}\ \text{m}^{-3}. \tag{9}$$

The code in the file solution6.2a.py solves the adimensional integral using the midpoint rule with the parameter $h = 1 \times 10^{-5}$ and the upper limit as described above. The resulting value for the integral is

$$\int_0^\infty \frac{x^2 dx}{e^x + 1} = 1.803085354731 \tag{10}$$

and the corresponding number density of electrons gives

$$n_{e^\pm} = 1.8982095515123674 \times 10^{41}\ \text{m}^{-3}. \tag{11}$$

b) In order to write the integral in the adequate terms, remember that the energy of the photons is $E = pc$ and therefore $x = \beta E$. Hence, the number density of electrons is written as

$$\begin{aligned} n_{e^\pm} &= \frac{8\pi(k_B T)^3}{(2\pi\hbar c)^3} \int_0^\infty \frac{x^2 dx}{e^x + 1} &(12)\\ &= \frac{8\pi}{(2\pi\hbar c)^3} \int_0^\infty \frac{E^2 dE}{e^{\beta E} + 1}. &(13)\end{aligned}$$

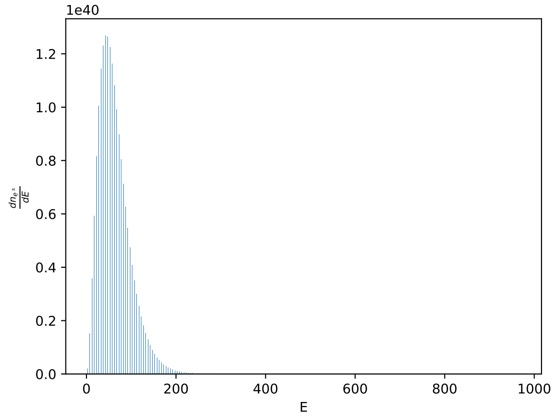

Figure 2: Behavior of the spectrum of electrons from equation (16) as function of the energy E.

This equation let us write

$$[n_{e^\pm}]_i = \frac{8\pi}{(2\pi\hbar c)^3} \int_{E_i}^{E_{i+1}} \frac{E^2 dE}{e^{\beta E} + 1} \tag{14}$$

and then

$$\left[\frac{dn_{e^\pm}}{dE}\right]_i = \frac{[n_{e^\pm}]_i}{\Delta E}. \tag{15}$$

The code in the file solution6.2b.py takes the width of the intervals as $\Delta E = E_{i+1} - E_i = 5$ MeV to calculate the integral. The resulting spectrum is represented in the Figure 2, which has exactly the same form as that in Figure 1.

To check our integration, the file solution6.2b.py also calculates the total density of electrons as

$$n_{e^\pm} = \sum_i \left[\frac{dn_{e^\pm}}{dE}\right]_i \Delta E, \tag{16}$$

and the result is

$$n_{e^\pm} = 1.8982348403614643 \times 10^{41} \text{ m}^{-3} \tag{17}$$

which differs from the result in equation (11) by a 1.3×10^{-3} %