Computational Astrophysics - Taller 6

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1. **Newton-Cotes Integration**

   (a) Integration of $\sin x$ from 0 to π. Computing the integral, it yields

$$\int_0^{\pi} \sin x \; dx = 2 \tag{1}$$

The results using the midpoint rule, the trapezoidal rule and the Simpson's rule with decreasing step h (or increasing number of intervals N, considering that $h \approx$(b-a)/N), as well as the percenterror is shown in Table 1

N	Step	Midpoint		Trapezoidal		Simpson	
		Value	% error	Value	% error	Value	% error
1	π	3.14159	57.079	1.92367	99.999	2.09439	4.719
2	$\pi/2$	2.22144	11.072	1.57079	21.460	2.00455	0.227
5	0.628	2.03328	1.664	1.93375	3.312	2.00010	5.4×10^{-3}
10	0.314	2.00824	0.412	1.98352	0.823	2.00000	3.3×10^{-4}
20	0.157	2.00205	0.102	1.99588	0.205	2.00000	2.1×10^{-5}
50	0.062	2.00032	0.016	1.99934	0.032	2.00000	5.4×10^{-7}
100	0.031	2.00008	4.1×10^{-3}	1.99983	8.2×10^{-3}	2.00000	3.3×10^{-8}
1000	0.003	2.00000	4.1×10^{-5}	1.99999	8.2×10^{-5}	2.00000	3.3×10^{-12}

Table 1: Integral Methods

   (b) Compute the integral of

$$I = \int_0^{\pi} x \sin x \; dx \tag{2}$$

The integral $I = \pi$. Similar to Table 1, the values are presented on Table 2

N	Step	Midpoint		Trapezoidal		Simpson	
		Value	% error	Value	% error	Value	% error
1	π	4.93480	57.079	6.04338	99.999	3.28986	4.719
2	$\pi/2$	3.48956	11.076	2.46715	21.468	3.14876	0.228
5	0.628	3.19390	1.665	3.03749	3.313	3.14176	5.4×10^{-3}
10	0.314	3.15455	0.412	3.11569	0.824	3.14160	3.3×10^{-4}
20	0.157	3.14482	0.102	3.13512	0.205	3.14159	2.1×10^{-5}
50	0.062	3.14210	0.016	3.14055	0.032	3.14159	5.4×10^{-7}
100	0.031	3.14172	4.1×10^{-3}	3.14133	8.2×10^{-3}	3.14159	3.3×10^{-8}
1000	0.003	3.14159	4.1×10^{-5}	3.14159	8.2×10^{-5}	3.14159	3.3×10^{-12}

Table 2: Integral Values

In both integrals, the three methods converge with a decreasing step size, and according to the decreasing error, they approach to the actual result of the integral. The method which converges faster is the Simpson's rule, with a percent error of 5.4×10^{-3} % with a step size of $\pi/5$ (N=5). The slower one is the Trapezoidal rule, reaching an approximate percent error of 10^{-3} % at $h = 0.031$ (N=100 intervals). The error of the Midpoint rule was almost the half of the Trapezoidal for $N \geq 5$.

2. **Number Density of Electrons in an Astrophysical Environment**

(a) The equation to solve is

$$n_{e^\pm} = \frac{8\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{p^2\ dp}{e^{\beta cp} + 1} \tag{3}$$

with $\beta = 1/(k_B T)$. Making the substitution $x = \beta cp$, the integral becomes:

$$n_{e^\pm} = \frac{8\pi(k_B T)^3}{(2\pi\hbar c)^3} \int_0^\infty \frac{x^2\ dx}{e^x + 1} \tag{4}$$

The function before making the integration decays too fast, so the integral can be performed from 0 to 20. The result for $k_B T = 20$ Mev is $n_{e^\pm} = 1.902 \times 10^{41}$ part m^{-3}, and for $T = 1 \times 10^9$ K is $n_{e^\pm} = 1.512 \times 10^{34}$ part m^{-3}

(b) Spectral distribution $\frac{dn_{e^\pm}}{dE}$: $n_{e^\pm} = \int \frac{dn_{e^\pm}}{dE}\ dE$. From equation (3), and since $E_{e^\pm} \approx pc$, the density is

$$n_{e^\pm} = \frac{8\pi}{(2\pi\hbar c)^3} \int_0^\infty \frac{E^2}{e^{\beta E} + 1}\ dE \tag{5}$$

thus

$$\frac{dn_{e^\pm}}{dE} = \frac{8\pi}{(2\pi\hbar c)^3} \frac{E^2}{e^{\beta E} + 1} \tag{6}$$

so

$$\sum_{i=0}^{\infty} \left[\frac{dn_{e^\pm}}{dE} \right]_i \times \Delta E = n_{e^\pm} \tag{7}$$

With $\Delta E = 5$ (and $k_B T = 20$ Mev), the electronic/positronic density is 3.804×10^{40} part m^{-3}, while for $\Delta E = 1$, the result is 1.902×10^{41} part m^{-3}