The operator K for contact Lagrangian systems

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XVII International ICMAT Summer School on Geometry, Dynamics and Field theory • Miraflores de la Sierra, Madrid, Spain, June 23-27, 2025

Abstract Many mechanical systems with dissipation can be described using Herglotz's variational principle, which leads to a modified form of the Euler–Lagrange equations. These equations admit a geometric formulation within the framework of contact geometry. In order to study singular Lagrangian systems, we define the evolution operator K in the context of contact mechanics. We illustrate how the main properties of this operator in ordinary mechanics naturally extend to the contact case. Specifically, we show how it provides a geometric description of the evolution equations and how it relates the Hamiltonian and Lagrangian constraints.

Singular Lagrangian and Hamiltonian systems

A Lagrangian is *singular* [1] if its Euler—Lagrange equations cannot be written in normal form. In this case, existence and uniqueness of solutions are not guaranteed. These systems are also characterized by the fact that the Legendre transformation is not a local diffeomorphism, which complicates the formulation of a Hamiltonian formalism. The study of singular Lagrangians began in the 1950s, when P.A.M. Dirac and P. Bergmann independently developed a *constraint algorithm*. The aim of the constraint algorithm is twofold:

- Find the set of points where solutions to the equations of motion exist. Typically, this subset is defined by the vanishing of a collection of functions, the *constraints* of the system.
- Characterize the multiplicity of the solutions, which is described by the so-called *gauge transformations*. This non-uniqueness is a key feature of many modern physical theories, such as electromagnetism, Yang–Mills theory, or general relativity.

The evolution operator K for singular Lagrangian systems

A key object to study the relation between the constraint algorithms of the Lagrangian and the Hamiltonian formalisms is the so-called *evolution operator* K. This geometrical object was first presented in coordinates in [2], and later given a geometric characterization in [3]. Some of its most relevant properties are:

- It gives a geometric description of the Euler–Lagrange equations (with second-order condition included)
- It gives a geometric description of the Hamilton–Dirac equations.
- Relates the Hamiltonian constraint functions with the Lagrangian ones.

Characterization of the evolution operator K The evolution operator K provides an unambiguous and well-defined time evolution in the Lagrangian formulation, when applied to functions defined on the cotangent bundle T^*Q. Its coordinate expression, in local coordinates, is

$$K(q,v) = v^i \frac{\partial}{\partial q^i}\Big|_{\mathcal{F}L} + \frac{\partial L}{\partial q^i}\frac{\partial}{\partial p_i}\Big|_{\mathcal{F}L}.$$

The evolution operator K is a vector field along the Legendre transformation, that is, the diagram

$$\begin{tikzcd} & \mathrm{T}\mathrm{T}^*Q \arrow[d, "\tau_{\mathrm{T}^*Q}"] \\ \mathrm{T}Q \arrow[ur, "K"] \arrow[r, "\mathcal{F}L"'] & \mathrm{T}^*Q \end{tikzcd}$$

commutes, or, equivalently, we have that $\mathcal{F}L = \tau_{\mathrm{T}^*Q} \circ K$.

Proposition

The evolution operator K is the only vector field along $\mathcal{F}L$ that satisfies the two following conditions:

1. $\mathcal{F}L^*(i_K(\omega_Q \circ \mathcal{F}L)) = \mathrm{d}E_L$*, where $\omega_Q \in \Omega^2(\mathrm{T}^*Q)$ is the canonical 2-form in the cotangent bundle and E_L is the Lagrangian energy function.*
2. $\mathrm{T}(\pi_Q) \circ K = \mathrm{Id}_{\mathrm{T}Q}$*, with π_Q the canonical projection $\pi_Q\colon \mathrm{T}^*Q \to Q$.*

These two conditions are referred to in the literature as the *dynamical condition* and the *second-order condition*, respectively. That is because there is a clear analogy between these conditions and those satisfied by a second-order Lagrangian vector field.

Properties relating the Lagrangian and Hamiltonian formalisms Let us now see the importance of this evolution operator, by stating some of its most remarkable properties.

Proposition

Let $\xi\colon I \to \mathrm{T}Q$ be a path in the tangent bundle and $\dot{\xi}\colon I \to \mathrm{T}(\mathrm{T}Q)$ its canonical lift. Then, ξ is a solution of the Euler–Lagrange equations, if and only if

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi\,.$$

Note that, therefore, we can write the Euler–Lagrange equations intrinsically (with the second-order condition included) with the evolution operator in this way. Also, as an immediate consequence of the last proposition, we have that

$$\mathrm{T}(\mathcal{F}L) \circ X_L \circ \xi = K \circ \xi\,.$$

The following proposition relates the Hamiltonian formalism and the evolution operator K.

Proposition

Let $\psi\colon I \to \mathrm{T}^*Q$ be a path on the cotangent bundle, and $\dot{\psi}\colon I \to \mathrm{T}(\mathrm{T}^*Q)$ its canonical lift. Then, ψ is a solution to Hamilton–Dirac's equations for L if and only if

$$\dot{\psi} = K \circ \mathrm{T}(\pi_Q) \circ \dot{\psi}\,.$$

Now, suppose that we have a Hamiltonian vector field X_H, defined on the image of the Legendre transformation $\mathcal{F}L(\mathrm{T}Q) \subseteq \mathrm{T}^*Q$, for the Hamiltonian formalism associated to the Lagrangian system $(\mathrm{T}Q, E_L)$. Then, the solutions to Hamilton–Dirac's equations can be written as the integral curves $\dot{\psi} = X_H \circ \psi$, and we can write

$$K \circ \xi = \dot{\psi} = X_H \circ \psi = X_H \circ \mathcal{F}L \circ \xi\,.$$

Additionally, we have the following proposition relating the Hamiltonian and Lagrangian constraint algorithms:

Proposition

Suppose that $f \in \mathscr{C}^\infty(\mathrm{T}^*Q)$ is a Hamiltonian constant of motion, such as a Hamiltonian constraint. Then, $(K \cdot f)$ is a Lagrangian constraint.

A complete classification of Lagrangian and Hamiltonian constraints appearing in the constraint algorithms for singular Lagrangian systems can also achieved using the K operator. In fact, *all* the Lagrangian constraints can be obtained from the Hamiltonian ones using the time-evolution operator [2].

Contact mechanics

The tools and structures of contact geometry have proven to be very effective to model dissipative systems. In particular, for mechanical systems, contact geometry has been successfully applied to describe non-conservative Lagrangian and Hamiltonian dynamics [4].

Take the manifold $\mathrm{T}Q \times \mathbb{R}$, with local coordinates (q^i, v^i, s), and a given *contact Lagrangian* $L \in \mathscr{C}^\infty(\mathrm{T}Q \times \mathbb{R})$. Through the so-called Herglotz's principle, one can derive the *generalized Euler–Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial v^i}\frac{\partial L}{\partial s}\,, \qquad \dot{s} = L\,.$$

We can write these equations geometrically, using the Lagrangian 1-form $\eta_L = \mathrm{d}s - {}^tJ \circ \mathrm{d}L$ and the Lagrangian energy $E_L = \Delta(L) - L$, as

$$i_{X_L}\mathrm{d}\eta_L = \mathrm{d}E_L + \left(\frac{\partial L}{\partial s}\right)\eta_L\,, \qquad i_{X_L}\eta_L = -E_L\,, \qquad J \circ X_L = \Delta\,.$$

Here, Δ is the Liouville vector field and J is the vertical endomorphism, in coordinates

$$\Delta = v^i\frac{\partial}{\partial v^i}\,, \qquad J = \frac{\partial}{\partial v^i} \otimes \mathrm{d}q^i\,.$$

This Lagrangian formulation of dissipative mechanics also allows for *singular* Lagrangians. In that case:

- The generalized Euler–Lagrange equations cannot be written in normal form. Hence, existence and uniqueness of solutions are not guaranteed, and we need to develop constraint algorithms.
- The Legendre transformation is not a local diffeomorphism. And so the associated Hamiltonian formalism is not well-defined in general.

This demands the study of geometrical structures and objects which are well-suited for this singular case.

The evolution operator K for contact systems

The contact evolution operator K can be characterized in a way similar to the ordinary case. It is the *unique vector field along the contact Legendre map* $\mathcal{F}L\colon \mathrm{T}Q \times \mathbb{R} \to \mathrm{T}^*Q \times \mathbb{R}$ which satisfies:

1. $\mathrm{T}\pi_1 \circ K = \mathrm{Id}_{\mathrm{T}Q}$, where $\pi_1 : \mathrm{T}^*Q \times \mathbb{R} \to Q$ is the canonical projection.
2. The *dynamical conditions*

$$\mathcal{F}L^*(i_K(\mathrm{d}\eta_Q \circ \mathcal{F}L)) = \mathrm{d}E_L + \frac{\partial L}{\partial s}\eta_L\,, \qquad i_K(\eta_Q \circ \mathcal{F}L) = -E_L\,,$$

where η_Q is the canonical contact form on $\mathrm{T}^*Q \times \mathbb{R}$. Alternatively, they can be written as

$$\mathcal{F}L^*(B \circ K) = \mathrm{d}E_L + \left(\frac{\partial L}{\partial s} - E_L\right)\eta_L\,,$$

where $B(X) = i_X\mathrm{d}\eta_Q + (i_X\eta_Q)\eta_Q$ is the canonical isomorphism defined by the contact form η_Q.

In local coordinates, we have that

$$K(q,v,s) = v^i\frac{\partial}{\partial q^i}\Big|_{\mathcal{F}L} + \left(\frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i}\right)\frac{\partial}{\partial p_i}\Big|_{\mathcal{F}L} + L\frac{\partial}{\partial s}\Big|_{\mathcal{F}L}.$$

Properties of the contact evolution operator We present here some relevant properties of the contact evolution operator K.

Proposition

Let $\xi\colon I \to \mathrm{T}Q \times \mathbb{R}$ be a path, and $\dot{\xi}\colon I \to \mathrm{T}(\mathrm{T}Q \times \mathbb{R})$ its canonical lift. Then, ξ is a solution of the generalized Euler–Lagrange equations, if and only if

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi\,.$$

Hence, also within this formalism, the equations of motion defined by the evolution operator K incorporate the second-order condition independently of the regularity of the Lagrangian function.

From last proposition we can write

$$\mathrm{T}(\mathcal{F}L) \circ X_L \circ \xi = K \circ \xi\,.$$

Proposition

Let $\psi\colon I \to \mathrm{T}^*Q \times \mathbb{R}$ be a path on the extended cotangent bundle, and let $\dot{\psi}\colon I \to \mathrm{T}(\mathrm{T}^*Q \times \mathbb{R})$ be its canonical lift. Then, ψ is a solution to the Herglotz–Dirac equations for L if and only if

$$\dot{\psi} = K \circ \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi}\,,$$

where $\pi_0\colon \mathrm{T}^*Q \times \mathbb{R} \to Q \times \mathbb{R}$ and $\rho\colon \mathrm{T}(Q \times \mathbb{R}) \to \mathrm{T}Q \times \mathbb{R}$ are the canonical projections.

Note that, if there exists a solution to Hamilton's equation $\psi\colon I \to \mathrm{T}^*Q \times \mathbb{R}$, then it can be expressed as

$$\psi = \mathcal{F}L \circ \xi$$

where $\xi\colon I \to \mathrm{T}Q \times \mathbb{R}$ solves the generalized Euler–Lagrange equations. Also, we have that

$$\dot{\psi} = \mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi = K \circ \rho \circ \mathrm{T}(\pi_0) \circ \dot{\psi}.$$

Now, if we have a Hamiltonian vector field X_H, defined on the image of the Legendre transformation $\mathcal{F}L(\mathrm{T}Q \times \mathbb{R})$, for the Hamiltonian formalism associated to the contact Lagrangian system $(\mathrm{T}Q \times \mathbb{R}, E_L)$. Then,

$$K \circ \xi = \dot{\psi} = X_H \circ \psi = X_H \circ \mathcal{F}L \circ \xi\,.$$

In this context, using these last results, it is also easy to prove the following proposition.

Proposition

Suppose that $f \in \mathscr{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$ is a contact Hamiltonian constant of motion, such as a contact Hamiltonian constraint. Then, $K \cdot f$ is a contact Lagrangian constraint.

Example Here, we introduce a velocity-dependent dissipation term to the Cawley Lagrangian, an academic model introduced by R. Cawley to study singular Lagrangians in Dirac's theory of constraint systems. Consider the manifold $\mathrm{T}\mathbb{R}^3 \times \mathbb{R}$ with canonical coordinates $(x, y, z; v_x, v_y, v_z; s)$, and the Lagrangian function

$$L = v_x v_z + \frac{1}{2}yz^2 - \gamma s v_y\,.$$

where γ is a non-zero damping coefficient. The Legendre map is

$$\mathcal{F}L\colon (x,y,z;v_x,v_y,v_z;s) \longmapsto (x,y,z;\, p_x = v_z,\, p_y = -\gamma s,\, p_z = v_x;\, s)$$

Hence, the first Hamiltonian constraint is $\phi_0 = p_y + \gamma s$.

The Lagrangian energy is $E_L = v_x v_z - \frac{1}{2}yz^2$, therefore, we can take as a Hamiltonian $H = p_x p_z - \frac{1}{2}yz^2$.

Now, we can compute the corresponding Hamiltonian vector fields, with respect to the precontact structure defined on $P_0 := \mathcal{F}L(\mathrm{T}Q \times \mathbb{R})$ by taking $\eta_0 := j^*(\eta_Q)$, where $j\colon P_0 \hookrightarrow \mathrm{T}^*Q \times \mathbb{R}$. In coordinates, we have

$$\eta_0 = \mathrm{d}s - p_x\mathrm{d}x + \gamma s\mathrm{d}y - p_z\mathrm{d}z\,,$$

and

$$X_H = p_z\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + p_x\frac{\partial}{\partial z} - \gamma b p_x\frac{\partial}{\partial p_x} + c\frac{\partial}{\partial p_y} + (yz - \gamma b p_z)\frac{\partial}{\partial p_z} + \left(p_x p_z + \frac{1}{2}yz^2 - \gamma b s\right)\frac{\partial}{\partial s}\,,$$

where b and c are arbitrary functions. The equations also yield the constraint $\phi_1 := \frac{1}{2}z^2 + \gamma\left(p_x p_z + \frac{1}{2}yz^2\right)$.

Now, demanding the tangency to the first constraint submanifold $(X_H \cdot \phi_0) = 0$ we obtain an expression for c, in terms of b. The condition $(X_H \cdot \phi_1) = 0$ will determine the function b. The constraint algorithm ends here.

The evolution operator is given by

$$K = v_x\frac{\partial}{\partial x}\Big|_{\mathcal{F}L} + v_y\frac{\partial}{\partial y}\Big|_{\mathcal{F}L} + v_z\frac{\partial}{\partial z}\Big|_{\mathcal{F}L} - \gamma v_y v_z\frac{\partial}{\partial p_x}\Big|_{\mathcal{F}L} + \left(\frac{1}{2}z^2 + \gamma^2 s v_y\right)\frac{\partial}{\partial p_y}\Big|_{\mathcal{F}L} + (yz - \gamma v_x v_y)\frac{\partial}{\partial p_z}\Big|_{\mathcal{F}L} + L\frac{\partial}{\partial s}\Big|_{\mathcal{F}L}.$$

We can apply the evolution operator to the Hamiltonian constraints to obtain Lagrangian ones. Namely,

$$\chi_1 := (K \cdot \phi_0) = \frac{1}{2}z^2 + \gamma\left(v_x v_z + \frac{1}{2}yz^2\right)\,, \qquad \chi_2 := (K \cdot \phi_1) = z v_z(1 + 2\gamma y) + \gamma v_y\left(\frac{1}{2}z^2 - 2\gamma v_x v_z\right)\,,$$

are two Lagrangian constraints. It is possible to see that these are all the Lagrangian constraints.

Acknowledgements

Partial financial support from • Ministerio de Ciencia, Innovación y Universidades (Spain), project PID2021-125515NB-C21 • Secretary of University and Research of the Ministry of Business and Knowledge of the Catalan Government, project 2021 SGR 00603

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